KfK 3935
Juni 1985

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PWA 16/85

Test Procedures to Detect a Loss of Material in a Sequence of Balance Periods

by

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Kernforschungszentrum Karlsruhe GmbH
ISSN 0303-4003

Testverfahren zur Entdeckung eines Materialverlustes in einer Folge von Bilanzperioden

Zusammenfassung

Ein Workshop über realzeitnahe Bilanzierung wurde im KfK veranstaltet und kam im Dezember 1982 zu einem vorläufigen Ende. Im Rahmen dieses Workshops wurde eine Zahl sequentieller statistischer Tests vorgeschlagen, die in einem Safeguards-Konzept mit realzeitnaher Bilanzierung herangezogen werden können. In der vorliegenden Arbeit werden die erfolgversprechenden Tests einer genaueren Untersuchung unterzogen. Die Analyse basiert auf dem chemischen Trennprozeß einer 1000-Tonnen Modell-Wiederaufarbeitungsanlage.

Abstract

A workshop on Near-Real-Time Accountancy (NRTA) was held in KfK which came to a preliminary end in December 1982. In the framework of this workshop a number of sequential statistical test procedures were proposed which can be used in the case of a NRTA based safeguards regime. In the report the most promising test procedures are investigated. The analysis is based on the chemical separation porcess of a large model reprocessing facility with a throughput of 1000 tonnes per year.

Contents

1. Introduction

At the end of an International Workshop on the Near-Real-Time Accountancy Measure, which was organized by the Nuclear Materials Safeguards Project of the Nuclear Research Center Karlsruhe (Gupta 1983) and whose members were active from December 1980 until July 1982, it was agreed that several of the more important statistical evaluation procedures should be analysed theoretically and numerically in order to arrive at concrete conclusions about their usefulness in the connection considered here. In this report the results of these analyses are laid down in major detail; a summary report has already been presented at the ESARDA Symposium in Venice (Avenhaus, Beedgen, Sellinschegg 1984).

The principle of material accountancy which is applied in the framework of international nuclear material safeguards in partial fulfillment of the Non-Proliferation Treaty may be described as follows: At the beginning of an inventory period (t_0,t_1) the real or physical inventory I_0 of the material balance area under consideration is measured. In the interval of time (t_0,t_1) the net transfers D_1, i.e., the sum of the input minus the sum of the outputs are measured, which yield together with I_0 the so-called book inventory $B_1 = I_0+D_1$ at time t_1, i.e., the amount of material which should be in the material balance area. This book inventory is compared to the realinventory I_1 at time t_1, i.e., that amount of material which really is found in the area. If no material was lost or diverted this assumption is called the null hypothesis H_0, and if there were no measurement errors, then the difference MUF_1 (Material Unaccounted For)

$$MUF_1 = B_1-I_1 = I_0+D_1-I_1$$

is zero. If on the contrary the amount of material M_1 was lost or diverted this assumption is called the alternative hypothesis H_1 - this differences between book and real ending inventory were just M_1. Since however random and systematic measurement errors cannot be avoided, these two alternatives are smoothed, and a test of significance has to be performed by means of which it can be decided, whether a non-zero value of MUF_1 can be explained by measurement errors, or the alternative hypothesis has to be assumed to be valid. This is done in such a way that a significance threshold s is chosen and that, H_0 resp. H_1 is taken to be true if MUF_1 is smaller resp. larger then s. The value of this significance threshold is determined by means of the false alarm probability α, i.e., the probability that H_1 is taken to be true when in fact H_0 is true. A measure for the efficiency of this procedure is the probability of detection $1-\beta$, i.e. the probability that H_1 taken to be true when in fact H_1 is true.

Two statements should be made at this point: First, a decision between H_0 and H_1 can be made only at the end of an inventory period. In a concrete situation this may mean that this time is considered to be too long which means that intermediate inventories become necessary. Second, in case of large material balance areas the measurement errors may become so large, that the probability of detecting a given loss or diversion is no longer considered to be sufficient which led to the idea to improve this situation with the help of additional inventories. Both observations led to the proposal to investigate sequences of inventory periods; the whole problem area has become known under the name Near-Real-Time-Accountancy (NRTA).

Let us consider first the idea to improve the probability of detecting a loss or diversion by introducing intermediate inventories: Given the reference time interval (t_o, t_n) which is partitioned into the n inventory periods

$$(t_o, t_1)\ ,\ (t_1, t_2)\ ,\ \ldots,\ (t_{n-1},\ t_n)\ ,$$

and given the n material balance test statistics MUF_1, MUF_2, ..., MUF_n for the n inventory periods which are defined as in the case of only one inventory period. Then that test procedure is of interest which leads for a given false alarm probability to the highest achievable probability of detection.

As pointed out in more detail in the third chapter of this report, the statistical theory provides a solution to this problem in form of the Lemma of Neyman and Pearson. If one now assumes that losses or diversion of a given total amount occur in such a way that the total probability of detection is mimimized - this is called the guaranteed probability of detection - then one is led to the test statistic

$$MUF_1+MUF_2+...+MUF_n = I_o+D_1+...+D_n-I_n,$$

which means that the optimal test procedure consists in testing the overall balance which means to ignore the intermediate inventories (Avenhaus and Jaech 1981).

This result however, means that a decision is taken only at the end of the reference time interval (t_o,t_n). Therefore, we are confronted with a conflict situation: In the interest of a high probability of detection the material balance test should be performed as late as possible. In the interest of a short detection time, however, intermediate inventories should be taken and the corresponding intermediate balances should be tested.

In addition, a mathematical difficulty has to be considered which, by the way, plays no role in the determination of the Neyman-Pearson test. In two subsequent material balance test statistics the intermediate inventory occurs twice, namely as ending inventory of the first and as beginning inventory of the second period. This and also the fact that there may exist persistent systematic errors mean that these test statistics are correlated. One way out of this difficulty, which will be described in the fifth chapter of this report, is to use instead of the original MUF statistics linear combinations of the form

$$MUFR_1 = MUF_1$$

$$MUFR_2 = MUF_2+a_{21}.MUF_1$$

$$MUFR_3 = MUF_3+a_{32}.MUF_2+a_{31}.MUF_1$$

where the coefficients a_{ij} are determined in such a way that the new statistics $MUFR_i$ are uncorrelated. This way, which also has become known as the Kalman-Filter approach, does not solve all problems connected with the correlations as we will see.

Let us ignore for the moment the objective of a high probability of detecting a loss or diversion, and let us look for that test procedure which leads for given lengths of the single inventory periods - which naturally have to be agreed upon by all parties involved - to the shortest possible detection time. In so doing a new problem arises: Since at the end of one inventory period an eventual loss or diversion cannot be detected with certainty, one has to take as the objective the expected detection time, i.e., the weighted sum of all possible detection times with the appropriate probabilities. Now, since these probabilities have to add to one, and since this is true only for an infinite time horizon, we have to consider an infinite sequence of inventory periods. This, in turn, has the consequence that for most test procedures - except for the power-one-test which also will be discussed in this report - the total false alarm probability becomes one and can therefore no longer be used as a boundary condition, but must be replaced by the expected "detection" time under H_0, i.e., the expected time until a false alarm.

For this statistical decision problem - minimization of the expected detection time under H_1 for a given expected detection time under H_0 - there is no solution which would correspond to the Neyman-Pearson test. But even if there existed such a test we would not yet have solved all problems, because the total probability of detection as another objective to be minimizied was ignored and furthermore, because the infinite time horizon causes a conceptual difficulty: In the interest of all parties of the Non-Proliferation Treaty the safeguards authority should declare in regular intervals of time that in those material balance areas, in which the tests of the balances resulted in a confirmation of the null hypothesis H_0, in fact no material was lost or diverted. Such a procedure however, in general is not in agreement with a test procedure which in principle extends over an infinite time horizon.

As a way out of these difficulties in the last years a series of test procedures were proposed which had been proven useful in other areas, e.g., quality control. According to the agreement at the end of the before mentioned NRTA workshop only a small number of those procedures were investigated and compared (see Table 8.1 of this report). Since two of these procedures, the CUMUF-test and the CUSUM or Page's test, played a special role in the international discussion and furthermore, since they have not yet been described in sufficient detail in commonly used statistical textbooks, they are analyzed in some detail in the fourth and sixth chapter of this report.

It should be mentioned here that it is not meaningful to use all these procedures simultaneously - this has been proposed several times - since in such a case the false alarm probability resp. the expected detection time under H_0 would get out of control.

Since all these problems are characterized by many parameters, only very limited analytical investigations are possible. Therefore, one had to look for an appropriate model plant, a useful measurement model and reasonable loss or diversion scenarios with the help of which numerical investigation could be performed.

It turned out that the reference reprocessing plant developed by Kluth et al. (1981) served the purpose of this study best. It is described in the seventh chapter of this report, together with a measurement model and a set of loss patterns.

Even though, it would have been meaningful, as outlined above, in those cases, where the expected detection time is the objective, to use the expected detection time under H_0 as a boundary condition, it was decided for several technical reasons to use instead the total false alarm probability for the reference time.

The results of the numerical investigations are collected in the 8 chapter of this report, together with some conclusions drawn from these results. It should be mentioned here, however, that no procedure turns out to be the very best among all circumstances. Therefore, these results have to be presented to all responsible practitioners in the governments and safeguards authorities so that they can evaluate their relative merits and take their decision, i.e., select an appropriate test procedure.

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2. Multiple Balance Model

Let us assume a discrete number of balance periods $N=\{1,2,....,n,.....\}$ for a well defined class of material. For each period $k \in N$ we build the difference between book and physical inventory, which is defined as

$$MUF_k = I_{k-1} + D_k - I_k \quad , \qquad (2.1)$$

where D_k is defined as

$$D_k = R_k - S_k \quad . \qquad (2.2)$$

In Eqs. (2.1) and (2.2) we have the following meaning:

I_{k-1}: beginning inventory of period k

I_k : ending inventory of period k and beginning inventory of period k+1

R_k : increase to inventory during balance period k

S_k : decrease from inventory during balance period k.

We assume that I_k, R_k and S_k are random variables that can be written as

$$I_k = E(I_k) + ZI_k + SI_k \qquad (2.3)$$

for $k \in N$. $E(I_k)$ is the true value of inventory, ZI_k is the random error of measurements and SI_k is the systematic measurement error. Furthermore, we define

$$T_k = R_k - S_k = E(T_k) + ZT_k + ST_k \qquad (2.4)$$

for all k, where $E(T_k)$ are the true values, ZT_k the random measurement errors and ST_k the systematic measurement errors.

A further assumption is that all measurement errors are distributed normally with zero means and that all measurement errors are stochastically independent.

The variances are defined as

$$Var(I_k) = Var(ZI_k) + Var(SI_k) \qquad \text{and} \qquad (2.5)$$

$$Var(T_k) = Var(ZT_k) + Var(ST_k) \qquad (2.6)$$

for $k \in N$.

For $i,j \in N$ we define the covariance of T_i and T_j as

$$\sigma_{ij} = \text{cov}(T_i,T_j). \qquad (2.7)$$

The concept of multiple balances is primarily used for detection of possible nuclear materials losses in a bulk handling facility. The detection has to be timely and with a sufficient high probability. The true MUF_k values are zero in the ideal situation of no losses and no measurement errors. In actual practice, however, nonzero MUF_k's may occur for a number of reasons, e.g. (a) measurement errors (b) loss of material. Measurement errors are included in our model by using the concept of random variables in determining the materials balance. Given a sequence of nonzero MUF values we have to decide whether the reason for nonzero values is due to measurement errors or loss. In our case we use the theory of statistical hypotheses testing to decide at the hand of a given sequence of MUF values whether the situation of no loss or loss of nuclear material is given. Loss of material may occur in a variety of pattern and we have to take into account that the control authority has no knowledge of the actual loss/diversion situation.

One essential part for designing statistical tests for nuclear materials accounting data is their expected performance in detecting losses of such material. Performance measures embody the concepts of loss-detection probability and loss-detection time. The performance of a special test has to be studied under a variety of loss patterns, which have to be selected according to reasonable assumptions. We try to analyse the performance of different test procedures for selected loss patterns.

We assume two hypotheses for the mean values of the random variables MUF_k, $k \in N$. If there is no loss/diversion of material all materials balances have zero mean. This situation is described by the null hypothesis:

$$H_0 : \quad E(MUF_k) = 0 \text{ for all } k \in N. \qquad (2.8)$$

A loss/diversion of material can take place in one or more balance periods.

Taking this into account, we formulate the alternative hypothesis:

$$H_1 : \quad E(MUF_k) = m_k \geq 0 \qquad (2.9)$$

$$\text{with } m_k > 0 \text{ for at least one } k.$$

Hypothesis H_1 means that we have a loss l_k of material during balance period k. This loss can be zero or positive, but at least in one balance period we have a positive loss. In our considerations we are not restricted to a fixed number of inventory periods.

The basic problem is to find test procedures that enable a decision between H_0 and H_1. The further problem is to find test procedures with a small probability of Type II error (decision for H_0 if H_1 is true, i.e. we have a loss and we do not detect it). And an even further problem is to find test procedures which indicate a loss/diversion almost immediately after it has happened.

Finally, a few words about the role of statistical test procedures in international safeguards: Some concern exists about the situation that a statistical test procedure leads to the decision of a loss/diversion of nuclear materials if in fact this is not the case, i.e. a false alarm has happened. Before the inspector makes a final statement, follow-up procedures are undertaken. Follow-up procedures can include e.g. controls of errors made while acquiring the data. No generally accepted operational follow-up procedures exist at the present time.

The statistical test procedures that are applied in this study assume that the materials accounting data which are delivered by the plant operator are not falsified and an inspector verification procedure is not necessary. This is a very important point, because under a general safeguards situation a verification of operator's data has to be performed in some way.

3. Properties of Neyman-Pearson Test Statistics

In this chapter we assume an arbitrary but fixed number of n balance periods, i.e. $|N| = n$. This assumption is necessary for developing the Neyman-Pearson test statistic (Lehmann 1959). The analysis in this part is only concerned with detection probability of a loss not at all with detection time. This fact has to be pointed out.

3.1 Neyman-Pearson-Test

The random vector $\underline{MUF} = (MUF_1, MUF_2, \ldots\ldots, MUF_n)'$ (3.1)

has a multivariate normal distribution with variance covariance matrix $\underline{\underline{\Sigma}}$.

In case of no loss/diversion we get corresponding to Eq. (2.8) the null hypothesis

$$H_0 : \quad E(\underline{MUF}) = (0,0,0,\ldots\ldots,0)' \tag{3.2}$$

In case of loss/diversion we consider a fixed quantity M, which is somehow distributed among all n balance periods, i.e.

$$M = \sum_{i=1}^{n} m_i \quad . \tag{3.3}$$

For the n-dimensional vector we write

$$\underline{M} = (m_1, m_2, \ldots, m_n)' \quad . \tag{3.4}$$

Corresponding to Eq. (2.9) we get for the alternative hypothesis

$$H_1 : \qquad E(\underline{MUF}) = \underline{M} \tag{3.5}$$

$$\text{with } m_i \geq 0 \text{ for all } i = 1,2,\ldots,n \text{ and } \sum_{i=1}^{n} m_i = M > 0 \; .$$

For a fixed vector $(m_1, m_2, \ldots, m_n)'$ H_0 and H_1 are simple hypotheses, i.e. both sets consist of only one element. This is an important assumption to determine the best test for H_0 against H_1 according to the lemma of Neyman and Pearson.

The Neyman-Pearson test is defined as

$$f_1(\underline{muf})/f_0(\underline{muf}) \begin{cases} > k_\alpha \; , & \text{accept } H_1 \\ \leq k_\alpha \; , & \text{accept } H_0 \end{cases} \tag{3.6}$$

In Eq. (3.6) muf_i are realizations of MUF_i for i=1,2,....,n and $\underline{muf} = (muf_1, muf_2, \ldots, muf_n)'$. $f_0(\underline{muf})$ resp. $f_1(\underline{muf})$ is the density of $\underline{MUF}$ under hypotheses H_0 resp. H_1 and α is the Type I error (false alarm) probability.

After some calculations we get an equivalent expression for Eq. (3.6)

$$\underline{M}' \underline{\underline{\Sigma}}^{-1} \underline{muf} \quad \begin{cases} > k_\alpha' , & \text{accept } H_1 \\ \leq k_\alpha' , & \text{accept } H_0 \end{cases} \qquad (3.7)$$

Obviously the Neyman-Pearson test statistic $M'\underline{\underline{\Sigma}}^{-1}\ \underline{MUF}$ is a linear combination of the single MUF_i variables.

The probability of Typ II error (no detection) for the Neyman-Pearson test in our situation is

$$\beta_{NP}(\underline{M}) = \Phi(U_{1-\alpha} - \sqrt{\underline{M}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{M}}) \qquad (3.8)$$

where Φ is the standard normal distribution function and U its inverse.

In Eq. (3.8) a special simple hypotheses is assumed. But there are many loss/diversion patterns that split the amount M among the n balance periods. We must assume that an inspector does not know which diversion scenario a diverter will choose. But we know, that for each possible diversion scenario a test statistic exists that leads to the highest detection probability. Unfortunately we do not know its properties under different scenarios.

We continue our analysis with the reasonable assumption that a divertor will choose the diversion strategy that has the lowest detection probability when the inspector uses Neyman-Pearson tests. Therefore, we have to solve the optimization problem

$$\min_{\underline{M}} \{1-\beta_{NP}(\underline{M})\} = \min_{\underline{M}} \max_{\delta} \{1-\beta(\delta,\underline{M})\} \qquad (3.9)$$

where δ is a test of size α for H_0 against H_1 and $\beta(\delta,\underline{M})$ is the nondetection probability of δ under alternative hypotheses H_1.

The optimization problem (3.9) is solved by the test

$$\sum_{i=1}^{n} muf_i \quad \begin{cases} > k_\alpha , & \text{accept } H_1 \\ \leq k_\alpha , & \text{accept } H_0 \end{cases} \qquad (3.10)$$

which is the materials balance test for the whole time period. Furthermore, it can be shown that (3.10) is a saddle point solution (Avenhaus, Jaech 1981), i.e.

$$\min_{\underline{M}} \max_{\delta} \{1-\beta(\delta,\underline{M})\} = \max_{\delta} \min_{\underline{M}} \{1-\beta(\delta,\underline{M})\} \tag{3.11}$$

$$= \Phi\left(\frac{M}{\sqrt{e\Sigma e}} - U_{1-\alpha}\right) .$$

It has to be pointed out that the test (3.10) leads to the highest guaranteed detection probability taking all diversion strategies into account.

The optimal loss/diversion pattern according to Eq. (3.11) is

$$\underline{M}^{*} = \frac{M}{e'\cdot\underline{\underline{\Sigma}}\cdot e} \underline{\underline{\Sigma}}\, e \tag{3.12}$$

with $e' = (1,1,\ldots,1)$.

It is already mentioned that the Neyman-Pearson test statistics are linear combinations of the single MUF_i values. For the rest of this chapter we are looking for the minimum variance unbiased linear estimate of the total amount of loss/diversion M (Jaech 1978). In addition to a choice between the null hypothesis and its alternative the control authority may also require some statement as to the size of the nonnull effect. To answer the question of the apparent magnitude of the effect, the safeguards authority needs at least point estimates; tests of significance will not suffice to talk about further consequences. The point estimate is given by

$$\hat{\Theta} = \sum_{i=1}^{n} a_i \cdot MUF_i \quad . \tag{3.13}$$

The a_i's are to be determined from

$$\min_{a_1,\ldots,a_n} \{var(\hat{\Theta})\} \tag{3.14}$$

under the boundary condition

$$E\left(\sum_{i=1}^{n} a_i \cdot MUF_i\right) = \sum_{i=1}^{n} a_i \cdot m_i = M. \tag{3.15}$$

With the definitions

$$q_i = m_i/M \qquad \text{for } i = 1,2,\ldots,n \tag{3.16}$$

$$\underline{q}' = (q_1,q_2,\ldots\ldots,q_n) \qquad \text{and}$$

$$\underline{a}' = (a_1,a_2,\ldots\ldots,a_n) \tag{3.17}$$

and using Lagrange's multiplier method we get as solution of the optimization problem (3.14)

$$\underline{a} = \underline{\underline{\Sigma}}^{-1}\underline{q} \;/\; (\underline{q}'\cdot\underline{\underline{\Sigma}}^{-1}\cdot\underline{q}) \tag{3.18}$$

$$= \frac{1}{M}\,\underline{\underline{\Sigma}}^{-1}\cdot\underline{M} \;/\; (\frac{1}{M^2}\quad \underline{M}'\;\underline{\underline{\Sigma}}^{-1}\;\underline{M})$$

$$= M\cdot\Sigma^{-1}\;\underline{M} \;/\; (\underline{M}'\;\underline{\underline{\Sigma}}^{-1}\underline{M}) \quad .$$

So with

$$\hat{\Theta} = \underline{a}'\cdot\;\underline{MUF} = \frac{M}{M'\;\underline{\underline{\Sigma}}^{-1}M}\;\cdot\;\underline{M}'\cdot\underline{\underline{\Sigma}}^{-1}\cdot\;\underline{MUF}$$

we recognize that the minimum variance unbiased estimate is up to a constant factor the Neyman-Pearson test statistic.

It should be mentioned that Frick (1979) has determined the optimal guaranteed probability of detection for the test statistic (3.13) taking into account all loss patterns with total loss M before the general test problem (3.1 and 3.2) was solved with the help of the Neyman Pearson Lemma.

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4. CUMUF Statistic

We know that in case of a finite number of balance periods the materials balance test for the whole time period (3.10) is the inspector's saddle point strategy. The test statistic in (3.10) is the cumulative sum of the MUF realizations. The cumulative sum of random variables is a often used procedure in statistics. Following these ideas we define the model of cumulative MUF values:

$$CUMUF_k = MUF_1 + MUF_2 + \ldots. + MUF_k \qquad (4.1)$$

for all k N. In case of n balance periods the random variable $CUMUF_n$ is a minimum variance linear unbiased estimate for the amount of loss if the loss happens according to the saddle point strategy.

The random variables $CUMUF_k$ in Eq. (4.1) have a normal distribution but are stochastically dependent.

For the hypotheses in Eqs. (2.9) and (4.10) we get

$$H_0 : \; E(CUMUF_k) = 0 \quad \text{for all } k \quad N \qquad (4.2)$$

and

$$H_1 : \; E(CUMUF_k) = m_1+m_2+..+m_k \geq 0 \qquad (4.3)$$

$$\text{with } m_k > 0 \text{ for at least one } k .$$

For all k the random variable $CUMUF_k$ is an unbiased linear estimate for the amount of loss/diversion in the first k balance periods. That means if a test for loss or no loss of material gives an alarm, the CUMUF statistic can be used to get a quantitative idea about the amount of loss.

4.1 Truncated Sequential CUMUF Test

We define a sequential test procedure using the CUMUF statistic and give the boundary condition for a truncation at the nth balance period. (Beedgen 1983a,b)

The reason for looking at sequential test procedures is that the materials balance test for n balance periods is a fixed sample size test, that allows a decision only at the end of balance period n, whether the loss/diversion takes place in the first or last balance period. This fact causes problems with the requirement of a timely detection in international safeguards.

A truncated test is considered because

- limitation of the probability of Type I error (false alarm)
- standard sequential tests can occasionally lead to very large sample sizes
- the safeguards authorities are used to have definite inspection periods.

The truncated test can be described as a mixture of a sequential test and a fixed sample size test.

The CUMUF statistic is considered because of its properties as a point estimate. We now describe a sequential test with boundaries s_k and truncation performed at the end of balance period n as follows :

at each observation $k < n$ test

$$\sum_{i=1}^{k} muf_i \begin{cases} > s_k \text{ , accept } H_1 \\ \\ \leq s_k \text{ , take another sample} \end{cases} \qquad (4.4a)$$

and at $k = n$, test

$$\sum_{i=1}^{n} muf_i \begin{cases} > s_n \text{ , accept } H_1 \\ \\ \leq s_n \text{ , accept } H_0 \end{cases} \qquad (4.4b)$$

where $cumuf_i$ are realizations of the random variables $CUMUF_i$. Fig. 4.1 illustrates the test procedure. To accept the hypothesis of no loss we have to use the information of all n balance periods whereas the acceptance of the hypothesis of loss can happen from period one to n.

For the false alarm probability α of the truncated sequential CUMUF test, we get

$$1-\alpha = \text{Prob}_{H_0} \{CUMUF_1 \leq s_1, \ldots\ldots, CUMUF_n \leq s_n\} \qquad (4.5)$$

and with the assumption that $(CUMUF_1, \ldots\ldots, CUMUF_n)$ has a multivariate normal distribution with covariance matrix $\underline{\underline{\Gamma}} = (\gamma_{ij})$ we get

$$1-\alpha = \frac{1}{(2\pi)^{n/2} |\underline{\underline{\Gamma}}|^{1/2}} \int_{-\infty}^{s_1} \cdots \int_{-\infty}^{s_n} \exp\left(-\frac{1}{2}\, \underline{x}'\, \underline{\underline{\Gamma}}^{-1} \underline{x}\right) d\underline{x} \qquad (4.6)$$

$$= \frac{1}{(2\pi)^{n/2} |\underline{\underline{\Gamma}}|^{1/2}} \int_{-\infty}^{\sigma_1 U_{1-\alpha_1}} \cdots \int_{-\infty}^{\sigma_n U_{1-\alpha_n}} \exp\left(-\frac{1}{2}\, \underline{x}'\, \underline{\underline{\Gamma}}^{-1} \underline{x}\right) d\underline{x}$$

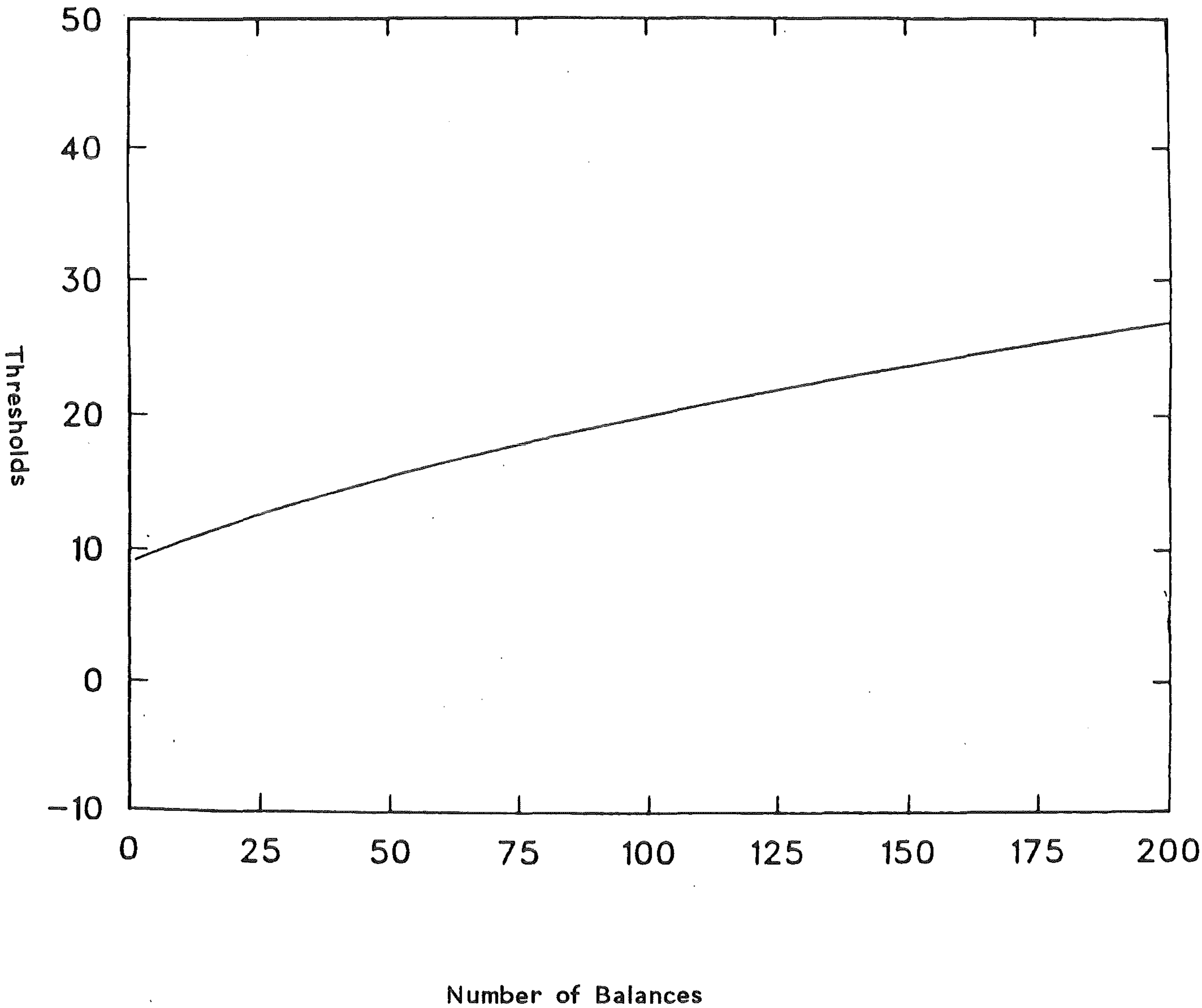

Fig. 4.1

Example of the thresholds for the CUMUF test; as long as the $CUMUF_i$ results are below the threshold line, H_0 is accepted. The first $CUMUF_i$ result that is above the line leads to the rejection of H_0.

with γ_{kk}^2 = var($CUMUF_k$)

and $1-\alpha_k = P_{H_0}\{CUMUF_k \le s_k\}$, k=1,2,...,n.

For the probability of Type II error (nondetection) we get

$$\beta_{CUMUF}(\underline{\alpha},\underline{M}) = \frac{1}{(2\Pi)^{n/2}|\underline{\underline{\Gamma}}|^{1/2}} \int_{-\infty}^{\sigma_1 U_{1-\alpha_1}} \cdots \int_{-\infty}^{\sigma_n U_{1-\alpha_n}} \exp(-\tfrac{1}{2}(\underline{x}-\underline{M})'\underline{\underline{\Gamma}}^{-1}(\underline{x}-\underline{M}))d\underline{x} \tag{4.7}$$

where $\underline{\alpha} = (\alpha_1,\ldots,\alpha_n)'$.

Using the sequential truncated CUMUF test, the inspector wants a guaranteed detection probability for a loss/diversion of amount M regardless the actual loss/diversion scenario. Therefore, he wants to solve the optimization problem

$$\max_{\underline{\alpha}} \ \min_{\underline{M}}(1-\beta_{CUMUF}(\underline{\alpha},\underline{M})) \ , \tag{4.8}$$

where $\underline{\alpha}$ obeys boundary condition Eq. (4.6) .

The solution of (4.8) exists and is

$$\underline{\alpha} = (0,0,\ldots,0,\alpha)' \tag{4.9a}$$

for the inspector and

$$\underline{M} = (0,\ldots,0,M) \tag{4.9b}$$

for the divertor if the condition $U_{1-\alpha} \ge M/\gamma_{nn}$ is fulfilled. We sketch the proof of (4. 9a,b) for the case n=2. We get for Eq. (4.7)

$$\beta((\alpha_1,\alpha_2)\ ,\ (m_1,m_1+m_2)) = \frac{1}{2\Pi\ \sqrt{1-\rho^2}} \int_{-\infty}^{U_{1-\alpha_1}-m_1/\gamma_{11}} dx_1 \int_{-\infty}^{U_{1-\alpha_2}-(m_1-m_2)/\gamma_{22}} dx_2$$

$$\exp\left(-\frac{x_1^2 \cdot 2\rho x_1 x_2 + x_2{}^2}{2(1-\rho^2)}\right) \tag{4.10}$$

where ρ is correlation coefficient of $CUMUF_1$ and $CUMUF_2$. From Eq. (4.10) it follows immediately that for fixed $M=m_1+m_2$, α_1 and α_2 the function β is strictly decreasing in m_1 , $0 \le m_1 \le M$, because $U_{1-\alpha_1} - m_1/\gamma_{11}$ is a strictly descreasing function in m_1. Therefore, we get the result

$$\min_{\substack{m_1 \quad 0, M=m_1+m_2}} \{1-\beta((\alpha_1,\alpha_2),(m_1,M)\} = 1-\beta((\alpha_1,\alpha_2),(0,M)) \tag{4.11}$$

Equation (4.11) implies that if the inspector performs the truncated sequential CUMUF test, a loss of amount M will have the lowest detection probability if the whole quantity is lost during the second balance period.

Next we need to solve

$$\max_{\alpha_1} \{1-\beta((\alpha_1,\alpha_2),(0,M))\}.$$

We have

$$\frac{\delta}{\delta\alpha_1}\beta((\alpha_1,\alpha_2),\ (0,M))$$

$$= -\ \Phi\left(\frac{U_{1-\alpha_2}-M/\sigma_2 - U_{1-\alpha_1}}{\sqrt{1-\rho^2}}\right) + \exp\left(-\frac{1}{2}(U_{1-\alpha_2}-M/\gamma_{22})^2\right)\ \exp(U_{\alpha_2}^2/2).$$

$$\cdot\ \Phi\left(\frac{U_{1-\alpha_1}-\rho(U_{1-\alpha_2}-M/\gamma_{22})}{\sqrt{1-\rho^2}}\right)\cdot \frac{\Phi\left(\frac{U_{1-\alpha_2}-\rho U_{1-\alpha_1}}{\sqrt{1-\rho^2}}\right)}{\Phi\left(\frac{U_{1-\alpha_1}-\rho U_{1-\alpha_2}}{\sqrt{1-\rho^2}}\right)}$$

$$\geq \Phi\left(\frac{U_{1-\alpha_2}-\rho U_{1-\alpha_1}}{\sqrt{1-\rho^2}}\right)\cdot\left[-1+\exp\left(-\frac{1}{2}(U_{1-\alpha_2}-M/\gamma_{22})^2\right)\ \exp(U_{\alpha_2}^2/2)\cdot \frac{\Phi\left(\frac{U_{1-\alpha_1}-\rho(U_{1-\alpha_2}-M/\gamma_{22})}{\sqrt{1-\rho^2}}\right)}{\Phi\left(\frac{U_{1-\alpha_1}-\rho U_{1-\alpha_2}}{\sqrt{1-\rho^2}}\right)}\right]$$

and the value in brackets is greater or equal zero if $U_{1-\alpha} \geq M/\gamma_{22}$ what means that β is increasing for α_1 in this case. Therefore,

$$\max_{\alpha_1} \{1-\beta((\alpha_1,\alpha_2),(0,M))\} = 1-\beta((0,\alpha),(0,M)) \tag{4.12}$$

if $U_{1-\alpha} \geq M/\gamma_{22}$.

We conclude that for the two-dimensional case the solution of the optimization problem (4.8) leads to the materials balance test (3.10).

The proof for the case $n > 2$ can be accomplished by the method of mathematical induction. Eq. (4.9a) describes the materials balance test for the whole time period and we already know that this test does not meet the requirement of timeliness. As a consequence, we calculate the test thresholds that

$$\alpha_k = \alpha^* \qquad , \; k = 1,2,\ldots,n \, , \qquad (4.13)$$

in Eq. (4.6) is fulfilled. Now, we give the final description of the truncated sequential CUMUF test:

at each observation $k < n$ test

$$\sum_{i=1}^{k} \mathrm{muf}_i \begin{cases} > \sigma_k \, U_{1-\alpha^*} \, , & \text{accept } H_1 \\ \leq \sigma_k \, U_{1-\alpha^*} \, , & \text{take another sample} \end{cases} \qquad (4.14a)$$

and at k=n, test

$$\sum_{i=1}^{n} \mathrm{muf}_i \begin{cases} > \sigma_n \, U_{1-\alpha^*} \, , & \text{accept } H_1 \\ < \sigma_n \, U_{1-\alpha^*} \, , & \text{accept } H_0 \, . \end{cases} \qquad (4.14b)$$

For this test the loss/diversion scenario in Eq. (4.9b) leads to the lowest detection probability. That means an inspector has a lower boundary for the detection capabilities of this test. A reason for selecting the false alarm probabilities according Eq. (4.13) is that in this situation it is comfortable to get a desired overall false alarm probability by simulation.

A consequence of Eq. (4.8) is the fact that it is possible for the truncated CUMUF test to calculate guaranteed detection probabilities for a certain amount of loss no matter what the diversion pattern might be. That is a very attractive ability from the inspector point of view.

4.2 References:

R. Beedgen (1983a), Truncated Sequential Test Procedure Using the CUMUF Statistic for a Timely Detection of Diversion. Nuclear Safeguards Technology 1982, Vol. II, Vienna, 383-392.

R. Beedgen (1983b), Statistical Considerations Concerning Multiple Materials Balance Models. LA-9645-MS, Los Alamos.

R. Avenhaus, R. Beedgen, D. Sellinschegg (1984), Comparison of Test Procedures for Near-Real Time Accountancy. Proc. 6th. ESARDA Symp., Venice, 555-560.

5. The Independence Transformation

Whereas in the case of the Neyman-Pearson Test, i.e. that test which maximizes the probability of detection for a finite series of balance periods and a given total false alarm probability, the stochastic dependency of the MUF variables did not cause any analytical problems, it does so in the case of sequential test procedures. Therefore, it is a natural idea, to transform the original MUF variables into uncorrelated - and consequently, because of their normality independent - random variables.

This idea was formulated for the first time by K.B. Stewart (1958) who started, however, with a different motivation. Later, R. Avenhaus and H. Frick (1977) used this independency transformation in order to determine the guaranteed probability of detection for a finite number of balance periods and a given false alarm probability.

In 1977, D.H. Pike and G.W. Morrison presented this Kalman Filter approach which turned out to be exactly the same as Stewart's approach. D.J. Pike, A.J. Wood and coworkers (1980) finally interpreted it in terms of conditional expectations.

In this chapter, three approaches to the independence transformation are presented: Stewart's approach, diagonalization of the covariance matrix of the MUF-vector, and the use of conditional expectations. The equivalent to the latter one, namely the Kalman Filter approach, is not presented here because of its completely different terminology. In addition, only the most simple sequential test procedure based on the independently transformed MUF's will be discussed. Their use in connection with further test procedures will be the subject of different chapters of this paper.

5.1 Stewart's Starting Inventory

K.B. Stewart started as follows: Let us assume that the inspector has accepted the null hypothesis (no loss or diversion of material) at the end of the first balance period. Then the question arises how to choose the initial inventory for the subsequent period. It would be natural to take the value of the real inventory, I_1. Since, however, the variances of the real inventory, $var(I_1)$, and of the book inventory, $var(B_1)$, may be very different the inspector better will take a linear combination of both inventories and give that inventory the more weight, the smaller its variance is.

According to this procedure, the starting inventory of a second period is

$$S_1 = c_2 \cdot B_1 + (1-c_2) \cdot I_1 \ , \tag{5-1}$$

and its variance is in the case, that all inventories and flow measurements are uncorrelated, given by

$$var(S_1) = c_2^2 \cdot var(B_1) + (1-c_2)^2 \cdot var(I_1) \ . \tag{5-2}$$

The weighting factor c_2 is determined in such a way that the variance $var(S_1)$ is minimized. This leads to the following determinant for c_2:

$$\frac{d}{d_c} var(S_1) = 2 \cdot [c_2 \cdot var(B_1) - (1-c_2) \cdot var(I_1)] = 0 \ , \tag{5-3}$$

which gives

$$c_2^* = \frac{var(I_1)}{var(I_1)+var(B_1)} = \frac{var(I_1)}{var(MUF_1)} \ . \tag{5-4}$$

The variance of the optimal starting inventory S_1^* is given by the relation

$$\frac{1}{var(S_1^*)} = \frac{1}{var(I_1)} + \frac{1}{var(B_1)} \ , \tag{5-5}$$

which is smaller than both variances of I_1 and B_1: Even if one of the two inventories has a much larger variance than the other, it is useful in the sense of a small variance, to take it into account, with a small weighting factor, naturally.

The covariance of MUF_1 and the new material balance test statistic $MUFR_2^{(1)}$ for the second period, defined by

$$MUFR_2^{(1)} = S_1^* + D_2 - I_2 \quad , \qquad (5\text{-}6)$$

is given by

$$cov(MUF_1,\ MUFR_2^{(1)}) = cov(I_o+B_1,\ c_2^*\cdot B_1+(1-c_2^*)\cdot I_1+D_2-I_2) =$$

$$= cov(B_1,\ c_2^*\cdot B_1+(1-c_2^*)\cdot I_1) =$$

$$= c_2^*\cdot var(B_1)-(1-c_2^*)\cdot var(I_1) = 0 \qquad (5\text{-}7)$$

because of the determinant (5-3) for c_2^* which means that MUF_1 and $MUFR_2$ are independent because of our normality assumptions.

Let us now consider n inventory periods with

$$MUF_i = I_{i-1}+D_i-I_i, \quad i = 1,2,\dots\ n\ . \qquad (5\text{-}8)$$

In analogy to (5-1) we define the starting inventory S_{i-1} for the i-th inventory period as

$$S_{i-1} = c_i\cdot BR_{i-1}+(1-c_i)\cdot I_{i-1},\ i = 1,2,\dots\ n, \qquad (5\text{-}9)$$

$$S_o = I_o \quad ,$$

where the transformed book inventory BR_i of the i-th inventory period is given by

$$BR_i = S_{i-1}+D_i,\ i = 1,2,\dots,\ n,$$

$$BR_i = B_1\ . \qquad (5\text{-}10)$$

The transformed MUF variables then are given by

$$MUFR_i^{(1)} = S_{i-1} + D_i - I_i, \quad i = 1,2,\ldots$$

$$MUFR_1^{(1)} = MUF_1 \quad . \qquad (5\text{-}11)$$

For these transformed variables one obtains, as one can see immediately, the following recursive relation:

$$MUFR_1^{(1)} = MUF_1 \ ,$$

$$MUFR_i^{(1)} = c_i \cdot MUFR_{i-1}^{(1)} + MUF_i, \quad i = 2,\ldots,n \quad . \qquad (5\text{-}12)$$

The optimal values of the weighting factors c_i, $i = 2,\ldots, n$, are determined as before:

$$\frac{d}{dc_{i+1}} var(S_i) = 2\cdot\left[c_{i+i}(var(S_{i-1})+var(D_i))-(1-c_{i+1})var(I_i)\right] = 0 \ , \qquad (5\text{-}13)$$

which leads to

$$c_{i+1}^{*} = \frac{var(I_i)}{var(D_i)+var(I_i)+var(S_{i-1}^{*})} \ , \quad i = 1,2,\ldots n-1,$$

$$S_0^{*} = I_0 \qquad (5\text{-}14a)$$

and furthermore, to

$$\frac{1}{var(S_i^{*})} = \frac{1}{var(I_i)} + \frac{1}{var(S_{i-1}^{*})+var(D_i)} \ , \quad i = 1,2,\ldots n. \qquad (5\text{-}14b)$$

From relation (5-14a) we see that the optimal weighting factors c_i^{*} can be determined by a continued fraction development, using the recursive relation (5-14b) for the optimal starting inventories S_i^{*}, $i = 0,1,2, \ldots$.

Again, two subsequent $MUFR^{(1)}$'s are uncorrelated which means that all $MUFR_i^{(1)}$, $i = 1,\dots,n$, are uncorrelated: We have with (5-12) and (5-13)

$$\mathrm{cov}(MUFR_i^{(1)}, MUFR_{i+1}^{(1)} =$$

$$= \mathrm{cov}(MUFR_i^{(1)}, c_{i+1}^* \cdot MUFR_i^{(1)} + MUF_{i+1}) =$$

$$= \mathrm{cov}(S_{i-1}+D_i-I_i, c_{i+1}^*(S_{i-1}+D_i-I_i)+I_i+D_{i+1}-I_{i+1}) =$$

$$= \mathrm{cov}(S_{i-1}+D_i-I_i, c_{i+1}^*(S_{i+1}+D_i)+(1-c_{i+1}^*)\cdot I_i =$$

$$= c_{i+1}^* \cdot (\mathrm{var}(S_{i-1})+\mathrm{var}(D_i)-(1-c_{i+1}^*)\cdot \mathrm{var}(I_i) =$$

$$= 0 \qquad (5\text{-}15)$$

which means again that the $MUFR^{(1)}$'s are independent because of our normality assumptions.

For later purposes, we determine explicitly the coefficients c_2^* and c_3^*. From (5-14a) and (5-14b) we get

$$c_2^* = \frac{\mathrm{var}(I_1)}{\mathrm{var}(I_0)+\mathrm{var}(D_1)+\mathrm{var}(I_1)} = \frac{\mathrm{var}(I_1)}{\mathrm{var}(MUF_1)} , \qquad (5\text{-}14a')$$

and furthermore,

$$c_3^* = \frac{\mathrm{var}(I_2)}{\mathrm{var}(D_2)+\mathrm{var}(I_2)+\dfrac{1}{\dfrac{1}{\mathrm{var}(B_1)}+\dfrac{1}{\mathrm{var}(I_1)}}} =$$

$$= \frac{\mathrm{var}(MUF_1)\cdot \mathrm{var}(I_2)}{\mathrm{var}(MUF_1)\cdot \mathrm{var}(MUF_2)-\mathrm{var}(I_1)^2} . \qquad (5\text{-}14a'')$$

So far, we have assumed that all inventories and flow measurements are mutually uncorrelated. If we assume that inventories and flow measurements within one inventory period are correlated, but that inventories and flow measurements of different periods are uncorrelated, then we get results in analogy to those given above: if we determine the starting inventories such that their variances are minimal, then the resulting $MUFR^{(1)}$ variables are uncorrelated and thus, independent. If, however, inventories and flow measurements of different inventory periods are correlated - which may happen in practice, if, e.g., measurement instruments are not recalibrated after each period, then the starting inventory with minimal variance does not lead any more to uncorrelated MUF variables.

5.2 Diagonalization of the covariance matrix

Whereas Stewart's original intention was to construct starting inventories with minimal variance, and uncorrelated transformed MUFR's were a by-product, we now directly try to determine transformed MUFR's which are uncorrelated for any covariance structure of the original MUF's.

We define new material balance test statistics by the following linear transformations:

$$MUFR_1^{(2)} = MUF_1$$
$$MUFR_2^{(2)} = a_{21} \cdot MUF_1 + MUF_2$$
$$MUFR_3^{(2)} = a_{31} \cdot MUF_1 + a_{32} \cdot MUF_2 + MUF_3$$
$$\vdots$$
$$MUFR_i^{(2)} = a_{i1} \cdot MUF_1 + a_{i2} \cdot MUF_2 + \ldots + MUF_i$$
$$\vdots$$
$$MUFR_n^{(2)} = a_{n1} \cdot MUF_1 + a_{n2} \cdot MUF_2 + \ldots + MUF_n \,, \tag{5-16}$$

and we want to determine the coefficients of the transformation in such a way that the transformed MUFR's are uncorrelated:

$$\mathrm{cov}(MUFR_i^{(2)}, MUFR_j^{(2)}) = 0 \text{ for } i \neq j \,. \tag{5-17}$$

The set (5-17) of equations consists of $\frac{1}{2}\cdot n\cdot(n-1)$ linear independent equations, this is just the number of coefficients to be determined.
If we compare (5-16) with (5-12), assuming that we will get the same transformed variables, i.e., $MUFR_i^{(1)} = MUFR_i^{(2)}$, $i = 1,2,\dots n$, then we get

$$a_{21} = c_2^*$$

and for $i \geq 3$

$$\begin{aligned} a_{i1} &= c_i^* \cdot c_{i-1}^* \cdot c_{i-2}^* \cdot \dots \cdot c_2^* \\ a_{i2} &= c_{i-1}^* \cdot c_{i-2}^* \dots c_2^* , \\ &\vdots \\ a_{i,i-1} &= c_2^* . \end{aligned} \qquad (5\text{-}18)$$

For the purpose of illustration, we determine the first three coefficients. With the notation

$$\begin{aligned} var(MUF_i) &= \sigma_i^2, \quad i = 1,2,\dots , \\ cov(MUF_i, MUF_j) &= \sigma_{ij} = \rho_{ij}\cdot\sigma_i\sigma_j, \quad i \neq j \end{aligned} \qquad (5\text{-}19)$$

we get

$$a_{21} = -\rho_{12}\cdot\frac{\sigma_2}{\sigma_1}$$

$$a_{32} = \frac{\rho_{12}\rho_{13}-\rho_{23}}{1-\rho_{12}^2}\cdot\frac{\sigma_3}{\sigma_2}$$

$$a_{31} = \frac{\rho_{12}\rho_{23}-\rho_{13}}{1-\rho_{12}^2}\cdot\frac{\sigma_3}{\sigma_1} . \qquad (5\text{-}18')$$

Furthermore, we get

$$var(MUFR_2^{(2)}) = \sigma_2^2\cdot(1-\rho_{12}^2) < \sigma_2^2 = var(MUF_2) , \qquad (5\text{-}20a)$$

and also

$$\text{var}(\text{MUFR}_3^{(2)}) = \sigma_3^2 \cdot (1-\rho_{13}^2 - \frac{(\rho_{23}-\rho_{12}\rho_{13})^2}{1-\rho_{12}^2}) < \sigma_3^2 = \text{var}(\text{MUF}_3), \qquad (5\text{-}20b)$$

which means that both the variances of $\text{MUFR}_2^{(2)}$ and $\text{MUFR}_3^{(2)}$ are smaller than those of MUF_2 and MUF_3. In fact, it can be shown generally that among all linear transformations of the form (5-16) the coefficients a_{ij}, which satisfy the conditions (5-17) minimize the variances of the transformed variables as we will see in the next section. In our example, we get from the conditions

$$\frac{\partial}{\partial a_{21}} \text{var}(a_{21}\cdot\text{MUF}_1 + \text{MUF}_2) = 0$$

$$\frac{\partial}{\partial a_{31}} \text{var}(a_{31}\cdot\text{MUF}_1 + a_{32}\cdot\text{MUF}_2 + \text{MUF}_3) = 0$$

$$\frac{\partial}{\partial a_{32}} \text{var}(a_{31}\cdot\text{MUF}_1 + a_{32}\cdot\text{MUF}_2 + \text{MUF}_3) = 0 \qquad (5\text{-}21)$$

again exactly the coefficients (5-18).

In the special case considered before,

$$\text{cov}(\text{MUF}_1, \text{MUF}_2) = -\text{var}(I_1)$$

$$\text{cov}(\text{MUF}_2, \text{MUF}_3) = -\text{var}(I_2)$$

$$\text{cov}(\text{MUF}_1, \text{MUF}_3) = 0$$

we get from (5-18')

$$a_{21} = \frac{\text{var}(I_1)}{\text{var}(\text{MUF}_1)}$$

$$a_{32} = -\frac{\text{var}(\text{MUF}_1)\cdot\text{var}(I_2)}{(\text{var}(I_1))^2-\text{var}(\text{MUF}_1)\cdot\text{var}(\text{MUF}_2)}$$

$$a_{31} = \frac{\text{var}(I_1)\cdot\text{var}(I_2)}{(\text{var}(I_1))^2-\text{var}(\text{MUF}_1)\cdot\text{var}(\text{MUF}_2)} \quad .$$

The comparison with formulae (5-14a') and (5-14a") gives

$$c_2^* = a_{21},\ c_3^* = a_{32},\ c_3^*\cdot c_2^* = a_{31} \ ,$$

which is not surprising.

In the following section we will present a statistical interpretation of the diagonalization of the covariance matrix of the MUF_i, $i = 1,2,\dots,n$, i.e., of the transformation coefficients c_i^* resp. a_{ij} in terms of (partial) regression coefficients, and we will see that the $\text{MUF}_i^{(2)}$ have a minimum variance among all transformed MUF's of the form (5-16). It should be mentioned, however, that this interpretation is based on the normality of the MUF_i variables, whereas Stewart's approach and also the diagonalization of the covariance matrix did not require such an assumption.

5.3 Conditional expectations

Quite generally, let us consider a (p+q)-dimensional random vextor $\underline{X}$, which is normally distributed with expectation vector and covariance matrix

$$E(\underline{X}) = \underline{0},\ \text{cov}(\underline{X}) := E(\underline{X}\cdot\underline{X}') = \underline{\underline{\Sigma}} \ . \qquad (5\text{-}22)$$

We partition this p+q-dimensional random vector into the p-dimensional random vector $\underline{X}^{(1)}$ and into the q-dimensional random vector $\underline{X}^{(2)}$:

$$\underline{X} = \begin{pmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{pmatrix} . \qquad (5\text{-}23)$$

Accordingly, we partition the covariance matrix $\underline{\underline{\Sigma}}$ in the following form

$$\underline{\underline{\Sigma}} = \begin{pmatrix} \underline{\underline{\Sigma}}_{11} & \underline{\underline{\Sigma}}_{12} \\ \underline{\underline{\Sigma}}_{21} & \underline{\underline{\Sigma}}_{22} \end{pmatrix} \tag{5-24}$$

where the submatrices $\underline{\underline{\Sigma}}_{11}$, $\underline{\underline{\Sigma}}_{12}$, $\underline{\underline{\Sigma}}_{21} = \underline{\underline{\Sigma}}_{12}'$ and $\underline{\underline{\Sigma}}_{22}$ are given by the relations

$$\underline{\underline{\Sigma}}_{11} = \mathrm{cov}(\underline{X}_1), \qquad \underline{\underline{\Sigma}}_{12} = \mathrm{cov}(\underline{X}_1, \underline{X}_2'),\ \underline{\underline{\Sigma}}_{22} = \mathrm{cov}(\underline{X}_2). \tag{5-25}$$

We consider the following linear transformation:

$$\underline{Y} = \begin{pmatrix} \underline{Y}^{(1)} \\ \underline{Y}^{(2)} \end{pmatrix} = \begin{pmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} - \underline{\underline{\Sigma}}_{12} \cdot \underline{\underline{\Sigma}}_{11}^{-1} \cdot \underline{X}^{(1)} \end{pmatrix} = \begin{pmatrix} \underline{\underline{I}}_p & \underline{\underline{0}} \\ -\underline{\underline{\Sigma}}_{12} \Sigma_{11}^{-1} & \underline{\underline{I}}_p \end{pmatrix} \cdot \begin{pmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{pmatrix}, \tag{5-26}$$

where $\underline{\underline{I}}_p$ and $\underline{\underline{I}}_q$ are unity matrices with ranks p and q.

According to Anderson (1957), p. 23 the two random vectors $\underline{Y}^{(1)}$ and $\underline{Y}^{(2)}$ are uncorrelated,

$$\mathrm{cov}(\underline{Y}^{(1)}, \underline{Y}^{(2)}) = \underline{\underline{0}} \tag{5-27}$$

and therefore, because of our normality assumptions, also independent.

Furthermore, we consider the conditional distribution of $\underline{X}^{(1)}$, given $\underline{X}^{(2)} = \underline{x}^{(2)}$. It is again a normal distribution (see, e.g. Anderson (1957), p.28, eq. (5)) the expected value and variance of which are

$$E(\underline{X}^{(2)} | \underline{x}^{(1)}) = \underline{\underline{\Sigma}}_{21} \cdot \underline{\underline{\Sigma}}_{11}^{-1} \cdot \underline{x}^{(1)} \tag{5-28}$$

$$\mathrm{cov}(\underline{X}^{(2)} | \underline{x}^{(1)}) = \underline{\underline{\Sigma}}_{22} - \underline{\underline{\Sigma}}_{21} \cdot \underline{\underline{\Sigma}}_{11}^{-1} \cdot \underline{\underline{\Sigma}}_{12} =: \underline{\underline{\Sigma}}_{22 \cdot 1}\ . \tag{5-29}$$

Because of (5-28), the matrix

$$\underline{\underline{\Sigma}}_{21} \circ \Sigma_{11}^{-1}$$

is called the *matrix of (partial) regression coefficients* of $\underline{X}^{(1)}$ on $\underline{X}^{(2)}$.

If we consider in (5-28) $\underline{x}^{(2)}$ again as the random variable $\underline{X}^{(2)}$, then we see with (5-26) and (5-27), that the random vectors

$$\underline{X}^{(1)} \text{ and } \underline{X}^{(2)}-E(\underline{X}^{(1)}|\underline{X}^{(2)}) = \underline{X}^{(2)}-\underline{\underline{\Sigma}}_{12}\cdot\underline{\underline{\Sigma}}_{11}^{-1}\cdot\underline{X}^{(1)} \qquad (5\text{-}30)$$

are uncorrelated and therefore independent. The second random variable, as a linear combination of normally distributed random variables, is again normally distributed with expected vector zero and variance

$$\mathrm{var}(\underline{X}^{(2)}-\underline{\underline{\Sigma}}_{12}\cdot\underline{\underline{\Sigma}}_{11}^{-1}\cdot\underline{X}^{(1)}) = \underline{\underline{\Sigma}}_{22} - \underline{\underline{\Sigma}}_{12}\cdot\underline{\underline{\Sigma}}_{11}^{-1}\cdot\underline{\underline{\Sigma}}_{21} = \underline{\underline{\Sigma}}_{11\cdot 2} \quad . \qquad (5\text{-}31)$$

In addition, it can be shown (Anderson (1957), p. 32) that among all linear combinations $\underline{\underline{A}}\cdot\underline{X}^{(2)}$ that linear combination which minimizes the variance of the random variable

$$X_i^{(1)}-\underline{\underline{A}}\cdot\underline{X}^{(2)} \quad ,$$

is just given by the linear combination

$$\underline{\underline{A}}\cdot\underline{X}^{(2)} = \underline{\underline{\Sigma}}_{12}\cdot\underline{\underline{\Sigma}}_{11}^{-1}\cdot\underline{X}^{(2)} \; .$$

The numerical calculation of the transformation, i.e. of the (partial) regression coefficients may be achieved with the help of some general formulae given by Anderson (1957) on page 34ff: Let

$$\underline{X} = \begin{pmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \\ \underline{X}^{(3)} \end{pmatrix} \qquad (5\text{-}32)$$

be a normally distributed random vector, where $\underline{X}^{(1)}$ is of p_1 components, $\underline{X}^{(2)}$ of p_2 components, and $\underline{X}^{(3)}$ of p_3 components. Then we have

$$E(X^{(1)}|\underline{X}^{(2)}, \underline{X}^{(3)}) = E(\underline{X}^{(1)}|\underline{X}^{(3)}) + \underline{\underline{\Sigma}}_{12\cdot 3}\cdot\underline{\underline{\Sigma}}_{22\cdot 3}^{-1}\cdot(\underline{X}^{(2)} - E(\underline{X}^{(2)}|\underline{X}^{(3)})), \qquad (5\text{-}33)$$

where $\underline{\underline{\Sigma}}_{12\cdot 3}$ and $\underline{\underline{\Sigma}}_{22\cdot 3}$ are given by

$$\underline{\underline{\Sigma}}_{12\cdot 3} = \underline{\underline{\Sigma}}_{12} - \underline{\underline{\Sigma}}_{13}\cdot\underline{\underline{\Sigma}}_{33}^{-1}\cdot\underline{\underline{\Sigma}}_{32} \tag{5-33a}$$

$$\underline{\underline{\Sigma}}_{22\cdot 3} = \underline{\underline{\Sigma}}_{22} - \underline{\underline{\Sigma}}_{23}\cdot\underline{\underline{\Sigma}}_{33}^{-1}\cdot\underline{\underline{\Sigma}}_{32} \quad . \tag{5-33b}$$

In particular, one obtains for $p_1=p, p_2=1, p_3=p-q-1$, the components

$$\sigma_{ij\cdot q+1,\ldots,p} = \sigma_{ij\cdot q+2,\ldots,p} - \frac{\sigma_{i,q+1\cdot q+2,\ldots,p}\cdot\sigma_{j,q+1\cdot q+2,\ldots,p}}{\sigma_{q+1,q+1\cdot q+2,\ldots,p}} \tag{5-34}$$

$$i,j = 1,\ldots,q,$$

$$\sigma_{ij\cdot 0} = \sigma_{ij}$$

$$\sigma_{ii} = \sigma_i^2 \quad .$$

For $p_1=p_2=p_3=1$ we obtain

$$E(X^{(3)}|X^{(2)}X^{(1)}) = E(X^{(3)}|X^{(1)}) + \frac{\sigma_{32\cdot 1}}{\sigma_{22\cdot 1}^2}\cdot(X^{(2)}-E(X^{(2)}|X^{(1)})) \quad , \tag{5-34a}$$

$$E(X^{(2)}|X^{(1)}) = \frac{\sigma_{12}}{\sigma_1^2}\cdot X^{(2)} \quad . \tag{5-34b}$$

Let us apply these general results to our concrete problem. Under the null hypothesis H_0 the MUF_i-variables have zero expectation values, therefore the transformed variables

$$MUFR_i^{(3)} = MUF_i - E(MUF_i|MUF_1,\ldots MUF_{i-1}) \tag{5-35}$$

are independent of the $MUF_1^{(3)}\ldots MUF_{i-1}^{(3)}$. Since, however, the transformed variables are linear combinations of the original variables, the $MUFR_i^{(3)}$ are also independent of the $MUFR_{i-1}^{(3)}\ldots MUFR_1^{(3)}$. Furthermore, since these properties hold also for random variables with non-zero expectations, the transformed variables, defined according to (5-30) for MUF variables with zero expectations, are also independent if the MUF_i have non-zero expectations. Finally, because of the minimum variance property of the random variables

$$X_i^{(1)} - (\Sigma_{12}\Sigma_{11}^{-1}X^{(2)})_i \quad , \quad i = 1,2,\ldots n,$$

which we mentioned before, we know that the $MUFR_i^{(3)}$ have a minimum variance among all transformed MUF's of the form

$$MUF_i - \sum_{j=1}^{i-1} a_{ij}\cdot MUF_j \quad , \qquad i = 1\ldots n \; ;$$

in formulae (5-21) we showed for the purpose of illustration for i=1 and 2 that the variances of the $MUF_i^{(3)}$ are smaller than those of the MUF_i.

If we apply the general formulae (5-33) resp. (5-34) to the problem of the determination of the coefficients c_{ij} of the transformation (5-16), we get the recursive relations (Sellinschegg, 1982)

$$a_{ij} = -\frac{\sigma_{ij\cdot 1\ldots.j-1}}{\sigma_{jj\cdot 1\ldots.j-1}} - \frac{\sigma_{i,j+1\cdot i\ldots.j}}{\sigma_{j+1,j+1\cdot i\ldots.j}} - \frac{\sigma_{i,i-1\cdot 1\ldots i-2}}{\sigma_{i-1,i-1\cdot 1\ldots j-1}} \cdot a_{i-1,j} \quad ,$$

or in a somewhat different notation,

$$a_{ij} = -\frac{\sigma_{ij\cdot 1\ldots j-1}}{\sigma_{jj\cdot 1\ldots j-1}} - \sum_{k=1}^{i-j-1} \frac{\sigma_{i,j+k\cdot 1\ldots j+k-1}}{\sigma_{j+k,j+k\cdot 1\ldots.j+k+1}} \cdot a_{j+k,j} \quad \text{for } j<i \quad ,$$

where

$$a_{ii} = 1 \quad . \tag{5-36}$$

As before, we determine explicitly the first transformation coefficients. We have with (5-34b)

$$MUF_2^{(3)} = MUF_2 - E(MUF_2|MUF_1) = MUF_2 - \frac{\sigma_{12}}{\sigma_1^2}\cdot MUF_1 \; ,$$

where

$$\frac{\sigma_{12}}{\sigma_1^2} = \frac{cov(MUF_1,MUF_2)}{var(MUF_1)} = -\,a_{21} \quad ,$$

and furthermore, with (5-34a),

$$MUF_3^{(3)} = MUF_3 - E(MUF_3 | MUF_1, MUF_2) =$$

$$= MUF_3 - E(MUF_3 | MUF_1) - \frac{\sigma_{32\cdot 1}}{\sigma_{22\cdot 1}} \cdot (MUF_2 - E(MUF_2 | MUF_1)) =$$

$$= (-\frac{\sigma_{31}}{\sigma_{11}^2} + \frac{\sigma_{32\cdot 1}}{\sigma_{22\cdot 1}^2} \cdot \frac{\sigma_{12}}{\sigma_{22}}) \cdot MUF_1 - \frac{\sigma_{32\cdot 1}}{\sigma_{22\cdot 1}^2} \cdot MUF_2 + MUF_3 \quad ,$$

where

$$\sigma_{32\cdot 1} = cov(MUF_2, MUF_3) - \frac{cov(MUF_1,\ MUF_3)}{var(MUF_1)} \quad ,$$

$$\sigma_{22\cdot 1} = var(MUF_2) - \frac{cov(MUF_1, MUF_2)}{var(MUF_1)}$$

and therefore,

$$-\frac{\sigma_{31}}{\sigma_{11}^2} + \frac{\sigma_{32\cdot 1}}{\sigma_{22\cdot 1}^2} \cdot \frac{\sigma_{12}}{\sigma_{22}} = -\frac{cov(MUF_1, MUF_3)}{var(MUF_1)} +$$

$$- \frac{cov(MUF_1, MUF_2) \cdot cov(MUF_2, MUF_3)}{var(MUF_1) \cdot var(MUF_2) - cov(MUF_1, MUF_2)^2} =$$

$$= a_{31} \quad ,$$

$$-\frac{\sigma_{32\cdot 1}}{\sigma_{22\cdot 1}^2} = \frac{var(MUF_1) cov(MUF_2, MUF_3) - cov(MUF_1, MUF_3)}{var(MUF_1) var(MUF_2) - cov(MUF_1, MUF_2)}$$

$$= a_{32}$$

in accordance with relations (5-18').

5.4 Test procedures based on the single transformed MUF's

Let us assume now that the inspector obtains sequentially the observed data muf_1, muf_2,...,muf_n, and that he has to decide whether or not material has been lost or diverted. This means that he has to perform a statistical test with the two hypothesis

$$\begin{aligned} H_o &: E(MUF_1) = \ldots = E(MUF_n) = 0 \\ H_1 &: E(MUF_i) = M_i, \quad i = 1\ldots n . \end{aligned} \tag{5-37}$$

If he decides on the basis of the single MUF_i or the single transformed MUF_i, which from now on we will call $MUFR_i$, then he has to determine an acceptance region for each MUF_i resp. $MUFR_i$, i.e., a region for an observed muf_i resp. $mufr_i$ which leads to the rejection of the alternative hypothesis H_1.

Naturally we assume under the alternative hypothesis H_1

$$E(MUF_i) \geq 0 \text{ for } i = 1\ldots n . \tag{5-37'}$$

Therefore a one-sided test procedure seems to be reasonable. This means that the acceptance regions for the single test, now based on the $MUFR_i$, are given by the sets

$$\{mufr_i : mufr_i \leq k_i\} \quad , \; i = 1\ldots n . \tag{5-38}$$

The significance thresholds k_i are determined with the help of the single false alarm probabilities α_i, given by

$$1-\alpha_i = prob\{MUFR_i \leq k_i/H_0\} = \Phi\left(\frac{k_i}{\sqrt{var(MUFR_i)}}\right), \quad i = 1\ldots n, \tag{5-39}$$

where Φ is the normal distribution function, which leads to

$$k_i = \sqrt{var(MUFR_i)} \cdot U_{1-\alpha_i}, \quad i = 1\ldots n, \tag{5-39'}$$

where U is the inverse of Φ; the variances of the transformed MUF variables are

$$\mathrm{var}(MUFR_i) = \mathrm{var}(S^*_{i-1})+\mathrm{var}(D_i)+\mathrm{var}(I_i) \quad , \qquad (5\text{-}39'')$$

where the variances $\mathrm{var}(S^*_i)$ are given by equations (5-14a and b).

The overall false alarm probability α is given by the relation

$$1-\alpha = \mathrm{prob}\{MUFR_1 \leq k_1 \wedge \ldots \wedge MUFR_n \leq k_n | H_o\} \quad . \qquad (5\text{-}40)$$

Because of the independence of the $MUFR_i$, $i = 1\ldots n$, we get

$$1-\alpha = \prod_{i=1}^{n} \mathrm{prob}\{MUFR_i \leq k_i | H_o\}$$

or, with (5-39) simply

$$1-\alpha = \prod_{i=1}^{n} (1-\alpha_i) \quad . \qquad (5\text{-}41)$$

Pragmatically, the inspector will fix the value of the overall false alarm probability and take the same values for the single false alarm probabilities,

$$1-\alpha_i = \sqrt[n]{1-\alpha} \quad , \quad i = 1\ldots n,$$

thus we get for the significance thresholds

$$k_i = \sqrt{\mathrm{var}(MUFR_i)} \cdot U_{n\sqrt{1-\alpha}} \quad , \quad i = 1\ldots n \quad . \qquad (5\text{-}39'')$$

The single probabilities $1-\beta_i$ of detection are given by the relations

$$\beta_i = \mathrm{prob}\{MUFR_i < k_i | H_1\} \quad , \quad i = 1\ldots n \quad , \qquad (5\text{-}42)$$

or with (5-39")

$$\beta_i = \Phi\left(\frac{E(MUFR_i)}{\sqrt{\mathrm{var}(MUFR_i)}} - U_{n\sqrt{1-\alpha}}\right) \quad , \quad i = 1\ldots n \quad . \qquad (5\text{-}42')$$

According to (5-16), the expected values $E(MUFR_i)$ are given by the relations

$$E(MUFR_1) = M_1$$

$$E(MUFR_2) = a_{21} \cdot M_1 + M_2$$

$$\vdots$$

$$E(MUFR_n) = \sum_{i=1}^{n=1} a_{n_i} \cdot M_i + M_n \qquad (5\text{-}43a)$$

or by the recursive relation derived from (5-12),

$$E(MUFR_1) = M_1$$

$$E(MUFR_i) = c_i^* \cdot E(MUFR_{i-1}) + M_i, \quad i = 2,\ldots n, \qquad (5\text{-}43b)$$

where the relation between c_i^* and a_{ij} is given by (5-18). The overall probability of detection $1-\beta$ then is given by

$$1-\beta = 1- \prod_{i=1}^{n} \beta_i \quad . \qquad (5\text{-}44)$$

In the special case $n = 2$ we obtain with (5-19)

$$1-\beta = 1-\Phi\left(U_{\sqrt{1-\alpha}} - \frac{M_1}{\sigma_1}\right) \cdot \Phi\left(U_{\sqrt{1-\alpha}} - \frac{1}{1-\rho_{12}^2} \cdot \frac{M_2}{\sigma_2} + \frac{\rho_{12}}{1-\rho_{12}^2} \cdot \frac{M_1}{\sigma_1}\right) .$$

Let us compare this total probability of detection with that based on the original MUF data. The single probabilities of detection $1-\tilde{\beta}_i$, $i=1,2$, as functions of the single false alarm probabilities $\tilde{\alpha}_i$ are given by the relations

$$1-\tilde{\beta}_i = \Phi\left(\frac{M_i}{\sigma_i} - U_{1-\tilde{\alpha}_i}\right), \quad i = 1,2 . \qquad (5\text{-}45)$$

The overall detection probability $1-\tilde{\beta}$ and false alarm probability α are given by

$$1-\tilde{\beta} = 1-B\left(U_{1-\tilde{\alpha}_1} - \frac{M}{\sigma_1} , U_{1-\tilde{\alpha}_2} - \frac{1}{1-\rho_{12}^2} \cdot \frac{M_2}{\sigma_2} + \frac{\rho_{12}}{1-\rho_{12}^2} \cdot \frac{M_1}{\sigma_1} , \rho_{12}\right)$$

$$(5\text{-}46a)$$

$$\alpha = B(U_{1-\tilde{\alpha}_1}, U_{1-\tilde{\alpha}_2}, \rho_{12}) , \qquad (5\text{-}46b)$$

where

$$B(h,k,q) = \frac{1}{2\pi\sqrt{1-\rho^2}} \cdot \int_{-\infty}^{h} dt_1 \int_{-\infty}^{k} dt_2 \; \exp\left(-\frac{1}{2(1-\rho^2)}\cdot(t_1^2-2\rho t_1\cdot t_2+t_2^2)\right) \qquad (5\text{-}46c)$$

is the biavariate normal distribution.

Between $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ and α we have the relations (Avenhaus 1979)

$$\begin{aligned} 1-\tilde{\alpha} &\geq (1-\tilde{\alpha}_1)(1-\tilde{\alpha}_2) && \rho_{12}\geq 0 \\ && \text{for} \\ (1-\tilde{\alpha}_1)(1-\tilde{\alpha}_2) &\geq 1-\alpha\geq 1-\tilde{\alpha}_1-\tilde{\alpha}_2 && \rho_{12}\leq 0 \; . \end{aligned} \qquad (5\text{-}47a)$$

Between $\tilde{\beta}$ and $\tilde{\beta}_i$, i=1,2, we have the relations

$$\begin{aligned} \tilde{\beta} &\geq \tilde{\beta}_1\cdot\tilde{\beta}_2 && \rho_{12}\geq 0 \\ && \text{for} \\ \tilde{\beta}_1\cdot\tilde{\beta}_2 &\geq \tilde{\beta} \geq \tilde{\beta}_1 + \tilde{\beta}_{2-1} && \rho_{12}\leq 0 \; . \end{aligned} \qquad (5\text{-}47b)$$

Now we have for $\rho_{12}<0$ and $\tilde{\alpha}_1=\tilde{\alpha}_2$ the relations

$$1-\alpha = (\sqrt{1-\alpha})^2 < (1-\tilde{\alpha})^2 , \quad \frac{1}{1-\rho_{12}^2}\cdot\frac{M_2}{\sigma_2} - \frac{\rho_{12}}{1-\rho_{12}^2}\cdot\frac{M_i}{\sigma_1} > \frac{M_2}{\sigma_2} ,$$

therefore

$$\beta = \beta_1\beta_2<\tilde{\beta}_1\tilde{\beta}_2 \quad ,$$

however, we do not arrive at a relation between β and $\tilde{\beta}$.

We see, even though the variances of the transformed MUF's are smaller than those of the original ones, we cannot decide analytically that the test which is based on the transformed MUF's, leads to a smaller overall probability of detection. Our numerical studies, however, will demonstrate that in all classes considered this is true, indeed, for a wide class of parameter values.

So far, we have considered a one-sided test procedure for the transformed MUF variables. Since the transformation coefficients a_{ij}, however, are not necessarily positive, we can get negative expected transformed MUF-values for the alternative hypothesis H_1 which means that we should use a two sided test, i.e. a test the acceptance regions of which are given by the sets

$$\{mufr_i : -k_i < mufr_i < k_i\} \quad , \; i = 1 \ldots n \quad . \tag{5-48}$$

In this case, the significance thresholds k_i are given by the relations

$$k_i = \sqrt{var(MUFR_i) \cdot U_{1-\alpha_i/2}} \; . \tag{5-49}$$

The overall false alarm probability is again given by (5-41) thus we get, if we put all the single false alarm probabilities the same,

$$1-\alpha_i/2 = 1 - \frac{1}{2} \cdot (1 - \sqrt[n]{1-\alpha}) = \frac{1}{2}(1 + \sqrt[n]{1-\alpha}) \quad .$$

The single probabilities of detection are

$$1-\beta_i = \Phi\left(\frac{E(MUFR_i)}{\sqrt{var(MUFR_i)}} - U_{1-\alpha_i/2}\right) - \Phi\left(-\frac{E(MUFR_i)}{\sqrt{var(MUFR_i)}} - U_{1-\alpha/2}\right) \quad ,$$

and the overall probability of detection is again given by formula (5-44).

5.5 References

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6. Cumulative Sum Tests

Cumulative sum tests, or shortly CUSUM tests, which play an important role in quality control, are constructed in the following way: One assumes that a given null hypothesis, e.g. a production plan, is correct. As long as the test, which is repeated sequentially in regular time distances, confirms this null hypothesis, the test is continued. If, however, the null hypothesis is rejected, the test stops (and with it, e.g., the production).

These tests serve the purpose that the system to be observed is not interfered with as long as the null hypothesis is maintained, however, that a deviation from this hypothesis is to be detected as soon as possible. Therefore, the criteria for these tests are no longer false alarm and detection probabilities, but average run lengths, i.e. average time distances between rejections of the null hypothesis.

In order to illustrate these ideas, first a simple sequential test procedure for independent variables is discussed. Thereafter, CUSUM tests for identically and independently distributed variables, and finally CUSUM tests for identically distributed variables with a special covariance structure are considered. In both cases, general properties will be analyzed, and integral equations for average run lengths will be established.

In the ninth chapter, we will present numerical results for CUSUM tests which are based on the original material balance test statistics MUF_i, i=1...n, as well as on the independently transformed statistics $MUFR_i$, i=1...n, which we introduced in the foregoing chapter. For an analytical treatment the former ones are better suited, since the independence transformation leads to variables which are no longer identically distributed therefore, the integral equation does not hold any more, and one has to use simulation methods. From the efficiency point of view, however, it is much better to use the transformed variables.

6.1 Simple sequential test for independently and identically normally distributed random variables

Given the independently and identically normally distributed random variables $X_1, X_2, \dots$. Let the null hypothesis H_0 be

$$H_0: X_i \sim n(\mu_0, \sigma_0^2), \quad i = 1,2,\dots , \qquad (6\text{-}1)$$

and let the alternative hypothesis H_1 be

$$\begin{aligned} &H_1: \text{There exists a point T in time such that} \\ &\quad X_i \sim n(\mu_0, \sigma_0^2), \quad i = 1,2,\dots \tau-1, \\ &\quad X_i \sim n(\mu_1, \sigma_1^2), \quad i = \tau, \tau+1, \dots\dots \end{aligned} \qquad (6\text{-}2)$$

After the i-th observation x_i it is decided

$$\begin{aligned} &H_0 \text{ is not rejected, if } x_i \leq s, \\ &H_0 \text{ is rejected, if } x_i > s, \quad i = 1,2,\dots \ . \end{aligned} \qquad (6\text{-}3)$$

The *significance threshold* is connected with the *single false alarm probability* via

$$1-\alpha = \Phi\left(\frac{s-\mu_0}{\sigma_0}\right) \ . \qquad (6\text{-}4)$$

Since, however, in principle an infinite number of single tests can be performed, the total false alarm probability is equal to one, i.e. not a reasonable criterion for the determination of s. Instead, the *run length* RL, i.e., the number of observations until the rejection of H_0, is taken, the distribution under H_0 of which is given by

$$\text{prob}_0(RL=i) = \alpha\cdot(1-\alpha)^{i-1}, \quad i = 1,2,\dots , \qquad (6\text{-}5)$$

and the expected value ARL_0 of which is given by

$$ARL_0 = \sum_{i=1}^{\infty} i\cdot\text{prob}_0(RL=i) = \frac{1}{\alpha} \qquad (6\text{-}6)$$

the value of which we now use for the determination of s.

Under H_1, the single probabilities of detection $1-\beta$ are given by

$$1-\beta = \Phi\left(\frac{\mu_1-\mu_0-\sigma_0 \cdot U_{1-\alpha}}{\sigma_1}\right) \quad , \tag{6-7}$$

i.e. the run length distribution is

$$\text{prob}_1(RL=i) = (1-\beta)\cdot\beta^{i-1} \quad , \tag{6-8}$$

and therefore its expected value

$$ARL_1 = \sum_{i=1}^{\infty} i\cdot\text{prob}_1(RL=1) = \frac{1}{1-\beta} \quad . \tag{6-9}$$

In case of the CUSUM test procedure which will be treated in the following, the corresponding relations can no longer be given explicitly.

6.2 Definition of CUSUM tests and general properties

Given the independently and identically distributed random variables $X_1, X_2, \ldots$. Let the two hypotheses H_o and H_1 be

H_o: X_i, i=1,2,..., are identically distributed with distribution function F_o (6-10a)

H_1: There exists a point τ in time such that $X_1,\ldots X_{i-1}$ are identically distributed with distribution function F_o and that $X_\tau, X_{\tau+1}, \ldots$ are identically distributed with distribution function F_1. (6-10b)

The CUSUM test procedure for this problem is given by

Definition 6.1 (Page 1954, 1955)

Given the test problem (6-10). With $y_i = x_i-k$, i=1,2,..., and

$$s'_n = \sum_{k=1}^{n} y_2, \; s'_o = 0 \tag{6-11}$$

the null hypothesis H_o is rejected if

$$s_n' - \min_{0 \leq i \leq n} s_i' \geq h, \; h > 0 \quad . \qquad (6\text{-}12)$$

k is called *reference value*, h *decision value*. □

The CUSUM test procedure, as defined above, can be formulated alternatively, as can be proven easily:

Theorem 6.2

Let us define the sequence S_n of random variables by the following recursive relation

$$S_n = \max(0, S_{n-1}+Y_n), \; n = 1,2,\ldots, \quad S_o = 0 \quad , \qquad (6\text{-}13)$$

and let us decide that H_o is rejected after the n-th observation, if

$$s_n \geq h, \; h > 0 \quad . \qquad (6\text{-}14)$$

Then H_o is rejected after n observations if H_o is rejected after n observations with the test procedure given by Definition 6.1. □

In the following we introduce some characteristic quantities of the CUSUM test (see Figure 6.1),

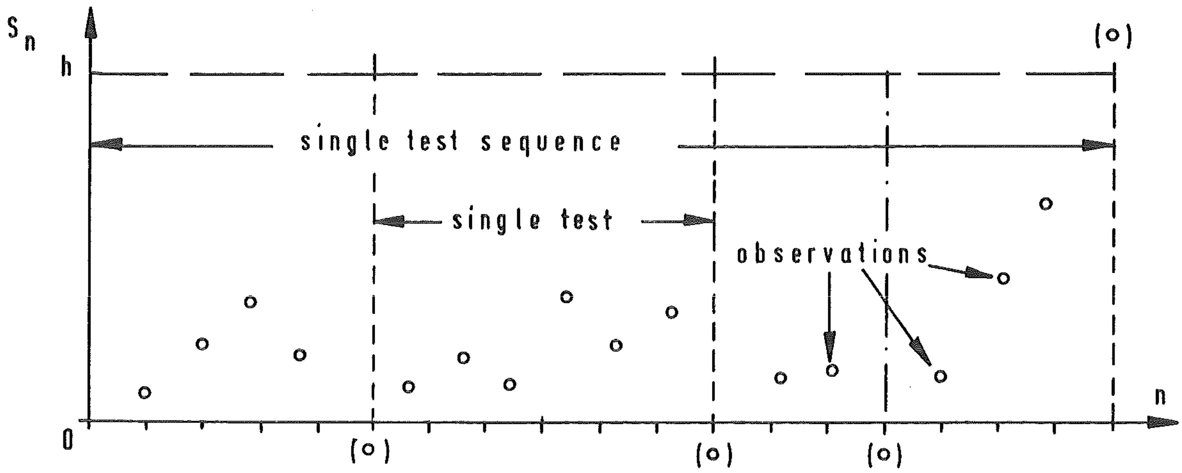

Figure 6.1: Observations, single tests and single test sequences for CUSUM tests

Definition 6.3

A *single test* is a sequence of observations which starts with $S_0=z$, $0\leq z\leq h$, and which ends at the lower (0) or at the upper (h) limit. In the extreme case the single test may consist of one single observation.

A *single test sequence* is a sequence of single tests, which starts with $S_0=z$, $0\leq z\leq h$, and which ends at the upper limit (h). □

Quite generally we characterize CUSUM test by their run lengths:

Definition 6.4

The *run length* RL of a CUSUM test is given by

$$RL = \min\{n \in \mathbb{N} : S_n \geq h\} \quad . \tag{6-15}$$

The mass function of the run length of a CUSUM test, which starts at $S_0=S_0'=z$, is written as

$$p_n(z) = \text{prob}(RL=n|S_0=z) \quad . \tag{6-16}$$

The *expected* (or average) *run length* RL, i.e. the number of observations of a single test sequence, which starts at $S_0=z$, is written as

$$L(z) = \sum_{i=1}^{\infty} i \cdot p_i(z) \quad ; \tag{6-17}$$

furthermore,

$$L(0) = ARL \tag{6-17'}$$

is the expected run length of a single test sequence which starts at $S_0=0$.

Finally, we call

$$P(z) = \text{prob}(0<S_i<h \text{ for } i=1\ldots n-1 \quad S_n<0,\ n=1,2,\ldots|S_0=z)$$

the probability that a single test, which starts at $S_0=z$, ends below zero, and we call N(z) the expected number of observations of a single test. □

Relations between these quantities gives

Theorem 6.5

The quantities L(z), P(z) and N(z), which were introduced in Definition 6.4, fulfill the relations

$$L(0) = \frac{N(0)}{1-P(0)} \tag{6-18}$$

$$L(z) = N(z) + L(0)\cdot P(z) \quad . \tag{6-19}$$

Scetch of the proof

1) The probability that - if we start at $S_0=0$ - exactly s single tests are performed, is

$$P(0)^{s-1}\cdot(1-P(0)), \quad s = 1,2,\dots \quad .$$

Therefore the expected number of single tests is

$$\sum_{s=1}^{\infty} s\cdot P(0)^{s-1}\cdot(1-P(0)) = \frac{1}{1-P(0)} \quad .$$

Since the expected number of observations of a single test is just N(0), we get (6-18).

2) If we start at $S_0=z$, L(z) is given by the expected number N(z) of observations of the first single test, if the first single test ends at h, multiplied with its probability, plus the expected number of observations of the first single test and the number of observations of a sequence of tests which starts at zero, if the first single test ends at zero, multiplied with its probability:

$$L(z) = N(z)\cdot(1-P(z)) + (N(z) + L(0))\cdot(1-P(z)) =$$
$$= N(z) + L(0)\cdot P(z) \quad .$$

□

6.3 CUSUM tests for independent continuous random variables

In order to determine integral equations for the average run lengths for independently and continuously distributed random variables, we first determine recursive integral equations for the mass function of the run length RL:

Theorem 6.6

Let f(x) and F(x) be density and distribution function of the independently and continuously distributed random variables $X_i, i=1,2,\ldots$. Then there is the following recursive integral equation for the mass function $p_n(z)$ of the run length RL of the CUSUM test which starts at $S_0=S_0'=z$:

$$p_n(z) = p_{n-1}(0)\cdot F(k-z)+\int_0^h p_{n-1}(z)\cdot f(y+k-z)dy \quad \text{for } n=2,3,\ldots$$
$$p_1(z) = 1-F(h-z+k) \quad . \qquad (6\text{-}20)$$

Proof

Let us consider all cases which are possible for the first step of the test, see Table 6.1:

Table 6.1: Possible cases for the first step of a CUSUM test

Observation	New value of the test statistic (score)	Result
$x\leq k-z$	0	test is continued
$k-z\leq x\leq h+k-z$	$z+x-k$	test is continued
$x\geq h+k-z$	h	rejection of H_0

According to this table we have

$$\text{prob}(X\leq k-z) = F(k-z)$$

$$\text{prob}(x<X\leq x+dx\,|\,k-z<x<h+k-z) = f(x)dx$$

$$\text{prob}(X>h+k-z) = 1-F(h+k-z) \quad .$$

The probability that the run length RL of the test is $n>1$, is according to the Total Probability Theorem given by

$$p_n(z) = \text{prob}(RL=n|z) =$$

$$= \text{prob}(RL=n|X\leq k-z)\cdot\text{prob}(X\leq k-z) +$$

$$+ \int_{k-z}^{h+k-z} \text{prob}(RL=n|x<X\leq x+dx|k-z<x<h+k-z) \otimes$$

$$\otimes \ \text{prob}(x<X\leq x+dx|h-z<x<h+k-z)dx +$$

$$+ \ \text{prob}(RL=n|X>h+k-z)\cdot\text{prob}(X>h+k-z) \quad .$$

Now we have according to this table

$$\text{prob}(RL=n|X\leq k-z)=p_{n-1}(0) \ ,$$

$$\text{prob}(RL=n|x<X\leq x+dx|h-z<x<h+k-z)=p_{n-1}(z+x-k) \quad ,$$

$$\text{prob}(RL=n|X>h+k-z)=0,$$

therefore

$$p_n(z)=p_{n-1}(0)\cdot F(k-z) + \int_{k-z}^{h+k-z} p_{n-1}(z+x-k)\cdot f(x)dx \quad ,$$

which completes the proof with the transformation $y=z+x-k$. □

With the help of this Theorem we prove

<u>Theorem</u> 6.7

Let $f(x)$ and $F(x)$ be density and distribution function of the independently and continuously distributed random variables $X_i, i=1,2,\ldots$. Then the following integral equation holds for the expected run length $L(z)$ of the CUSUM test, which starts at $S_o=S_o'=z$:

$$L(z)=L(0)\cdot F(k-z)+ \int_0^h L(y)\cdot f(y-z+k)dy \quad . \qquad (6\text{-}21)$$

Proof

Because of

$$\sum_{n=1}^{\infty} p_n(z) = 1$$

we get immediately with (6-20)

$$L(z) = \sum_{n=1}^{\infty} n\cdot p_n(z) =$$

$$= 1+\sum_{n=2}^{\infty} (n-1)\cdot p_n(z) =$$

$$= 1+\sum_{n=2}^{\infty} (n-1)\cdot \left[p_{n-1}(0)\cdot F(k-z)+ \int_{k-z}^{h+k-z} p_{n-1}(z+x-k)\cdot f(x)dx\right] =$$

$$= 1+L(0)\cdot F(k-z)+ \int_{k-z}^{h+k-z} L(z+x-k)\cdot f(x)dx =$$

$$= 1+L(0)\cdot F(k-z)+ \int_{o}^{h} L(y)\cdot f(y-z+k)dy \ .$$

□

For the sake of completeness, we present similar formulae for P(z) and N(z) without proof:

Theorem 6.8

For the quantities P(z) and N(z) we have

$$P(z) = F(k-z)+\int_{k-z}^{k-z-h} f(x)\cdot P(z+x-k)dx = \int_{o}^{h} f(y+k-z)\cdot P(y)dy \qquad (6\text{-}22)$$

$$N(z) = 1+\int_{k-z}^{h-z-k} f(x)\cdot N(z+x-h)dx = 1+\int_{o}^{h} f(y-k-z)N(y)dy \ . \qquad (6\text{-}23)$$

□

Let us consider now the special case that the random variables X_i,l=1,2,... are independently and normally distributed:

$$H_o\colon X_i \sim n(\mu_o,\sigma^2) \quad \text{for} \quad i = 1\ldots\tau$$

$$H_1\colon X_i \sim n(\mu_1,\sigma^2) \quad \text{for} \quad i = \tau+1, \ldots. \qquad (6\text{-}24)$$

The question arises how to fix the values of the reference and decision values k and h. We answer this question by considering the analogy between CUSUM and Sequential probability Ratio (SPR) tests (Lehmann 1959).

In case of the CUSUM test a single test of the single test sequence can be interpreted as a SPR test in that sense, that the test procedure is continued as long as the observations s_i are between the limits 0 and h,

$$0 < \sum_i (x_i - k) < h \quad .$$

In case of the SPR test we have to continue the test procedure as long as we have for given σ^2, $H_o\colon\mu=\mu_o$ and $H_1\colon\mu=\mu_1$,

$$K_o < \sum_{i=1}^{\tau}\left(x_i - \frac{\mu_o+\mu_1}{2}\right) < K_1, \quad K_v = \frac{\sigma^2}{\mu_1-\mu_o}\cdot \ln k_v, \quad v = 0,1 \quad .$$

Furthermore, we have approximately

$$k_o \simeq \frac{\alpha_1}{1-\alpha_o}\ , \ k_1 \simeq \frac{1-\alpha_1}{\alpha_o} \qquad ,$$

i.e., for $\alpha_1 \ll 1$

$$k_1 \simeq \frac{1}{\alpha_o} \qquad ,$$

where α_o and α_1 are the error probabilities first and second kind.

This means, that we get for the single CUSUM test the same relations as for the SPR test, if with

$$\alpha_o = 1\text{-}P(0)$$

we identify

$$k = \frac{1}{2} \cdot (\mu_o + \mu_1), \; h = K_1 = - \frac{\sigma^2}{\mu_1 - \mu_o} \cdot \ln(1-P(0)) \; . \qquad (6\text{-}25)$$

It should be mentioned, however, that the probability P(0) does not give any information about the frequency of false alarms.

6.4 CUSUM tests for material balance test statistics

So far, we have considered CUSUM test procedures for independently and identically distributed random variables. We can apply them to our material balance test statistics MUF_i, i=1,2,..., if we perform the independence transformation (5-11) or (5-16) and divide the transformed variables by their standard deviations. This way, we can formulate at least an integral equation for the average run length under the null hypothesis H_o. It is not possible, however, to proceed in the same way in order to determine the average run length under the alternative hypothesis H_1 even if we assume constant loss or diversion since the transformed variables have different expected values.

Surprisingly enough it is possible to formulate integral equations for the average run lengths under the null as well as under the alternative hypothesis of constant loss or diversion, if one performs a CUSUM test with the help of the original material balance test statistics MUF_i, i=1,2,... . Generally, this is possible for various kinds of stochastic processes, namely so-called ARMA and MA processes (see, e.g., Montgomery and Johnson (1976)). In the following, we will derive those equations for our specific purposes.

Theorem 6.8

Let us consider the independently and identically distributed random variables D_i with distribution functions F_D and densities f_D and the independently and identically distributed random variables I_i, i=1,2,... with distribution functions F_I and densities f_I, and let us consider the CUSUM test, defined by (6-13) and (6-14), based on the test statistics

$$S_n := \max(0, S_{n-1}+Y_n), \; n=1,2,..., \; S_o=s_o \;, \qquad (6\text{-}13)$$

where the random variables Y_n are defined by

$$Y_n := I_{n-1}+D_n-I_n-k, \; I_o=i_o \; , \tag{6-26}$$

with reference value k and decision value h.

Then the probability $p_1(z)$ that the test is finished after one observation if it starts with $S_o=s_o$, $I_o=i_o$, $z=S_o+i_o$, is given by

$$p_1(z) = 1-F_{D-I}(h+k-z) \quad , \tag{6-27a}$$

where F_{D-I} is the distribution function of the random variable D-I.

Furthermore, the probability $p_n(z)$, that the test is finished after n observations if it starts at $z=s_o+i_o$, satisfies the recursive integral equation

$$\begin{aligned} p_n(z) &= \int_{-\infty}^{\infty} p_{n-1}(x)\cdot F_D(-z+x+k)\cdot f_I(x)dx \; + \\ &+ \int_{-\infty}^{\infty} p_{n-1}(x)\cdot f_D(x-z+k)\cdot (F_I(x)-F_I(x-h))dx, \; n=2,3,\ldots \; . \end{aligned} \tag{6-27b}$$

Finally, the average run length L(z) of a test which starts at $z=s_o+i_o$ satisfies the integral equation

$$\begin{aligned} L(z) &= 1+ \int_{-\infty}^{\infty} L(x)\cdot F_D(-z+x+k)f_I(x)dx \; + \\ &+ \int_{-\infty}^{\infty} L(x)\cdot f_D(x+z-k)\cdot (F_I(x)-F_I(x-h))dx \quad . \end{aligned} \tag{6-28}$$

Proof

According to eqs. (6-13) and (6-26) we have

$$\begin{aligned} p_1(s_o,i_o) &:= \text{prob}(S_1>h|S_o=s_o,I_o=i_o) = \\ &= \text{prob}(S_o+Y_1>h|S_o=s_o,I_o=i_o) = \\ &= \text{prob}(D_1-I_1+i_o+s_o-k>h) = \\ &= 1-F_{D-I}(h+k-(S_o+i_o) \quad . \end{aligned}$$

Furthermore, we have

$$p_n(s_o,i_o) := \text{prob}(S_k \le h \wedge S_n > h \mid S_o = s_o, I_o = i_o \text{ for } 1 \le k < n) =$$

$$= \int_{-\infty}^{\infty} p_{n-1}(0,i_1)\cdot\text{prob}(S_1 \le 0 \wedge i_1 \le I_1 \le i_1+di_1 \mid S_o=s_o, I_o=i_o)di_1 +$$

$$+ \int_{-\infty}^{\infty}\Big[\int_0^h p_{n-1}(s_1,i_1)\cdot\text{prob}(s_1 \le S_1 \le s_1+ds_1 \wedge i_1 \le I_1 \le i_1+di_1 \mid S_o=s_o, I_o=i_o)ds_1\Big]di_1 =$$

$$= \int_{-\infty}^{\infty} p_{n-1}(0,i_1)\cdot\text{prob}(S_1 \le 0 \mid i_1 \le I_1 \le i_1+di_1 \mid S_o=s_o, I_o=i_o) \otimes$$

$$\otimes\ \text{prob}(i_1 \le I_1 \le i_1+di_1)di_1 +$$

$$+ \int_{-\infty}^{\infty}\Big[\int_0^h p_{n-1}(s_1,i_1)\cdot\text{prob}(s_1 \le S_1 \le s_1+ds_1 \mid i_1 \le I_1 \le i_1+di_1 \mid S_o=s_o, I_o=i_o) \otimes$$

$$\otimes\ \text{prob}(i_1 \le I_1 \le i_1+di_1)ds_1\Big]di_1 =$$

$$= \int_{-\infty}^{\infty} p_{n-1}(0,i_1)\cdot\text{prob}(D_1+s_o+i_o-i_1-k \le 0)\cdot f_I(i_1)di_1 +$$

$$+ \int_{-\infty}^{\infty}\Big[\int_0^h p_{n-1}(s_1,i_1)\cdot\text{prob}(s_1 \le D_1+s_o+i_o-i_1-k \le s_1+ds_1)\cdot f_{I1}(i_1)ds_1\Big]di_1 =$$

$$= \int_{-\infty}^{\infty} p_{n-1}(0,i_1)\cdot F_D(-s_o-i_o+i_1+k)\cdot f_I(i_1)di_1 +$$

$$+ \int_{-\infty}^{\infty}\Big[\int_0^h p_{n-1}(s_1,i_1)\cdot f_D(s_1-s_o-i_o+i_1+k)\cdot f_{I1}(i_1)ds_1\Big]di_1 =$$

$$= p_n(s_o+i_o) \quad .$$

With $s_o+i_o=z$ and the transformation $s_1+i_1=x$ in the second integral we get

$$p_n(z) = \int_{-\infty}^{\infty} p_{n-1}(i)\cdot F_D(-z+i+k)\cdot f_I(i)di +$$

$$- \int_{-\infty}^{\infty}\Big[\bar{p}_{n-1}(x)\cdot f_D(x-z+k)\cdot\int_0^h f_{I1}(x-s_1)ds_1\Big]dx$$

which leads immediately to (6-27b).

Finally, we get

$$L(z) = 1+\sum_{n=2}^{\infty}(n-1)\cdot p_n(z) =$$

$$= 1+\sum_{n=2}^{\infty}(n-1)\cdot\Big[\int_{-\infty}^{\infty}p_{n-1}(x)F_D(-z+x+k)\cdot f_I(x)dx +$$

$$- \int_{-\infty}^{\infty}p_{n-1}(x)\cdot f_D(x-z+k)\cdot(F_I(x-h)-F_I(x))dx\Big] =$$

$$= 1+\int_{-\infty}^{\infty}L(x)\cdot F_D(-z+x+k)\cdot f_I(x)dx-\int_{-\infty}^{\infty}L(x)\cdot f_D(x-z+k)(F_I(x-h)-F_I(x))dx$$

which completes the proof. □

Let us consider the special case that the random variables I_n,n=1,2,... are degenerated, i.e., that their observations give zero with probability one:

$$F_I(x) = \begin{cases} 0 & \text{for } x<0 \\ 1 & \text{for } x\geq 0 \end{cases} .$$

Then we get from (6-28)

$$L(z) = 1+L(0)\cdot F(-z+k)+ \int_{0}^{\infty}L(x)f_D(x-z+k)dx-\int_{h}^{\infty}L(x)f_D(x-z+k)dx =$$

$$= 1+L(0)\cdot F(-z+k)+ \int_{0}^{h}L(x)f_D(x-z+k)dx$$

in accordance with (6-21); this is reasonable since under this assumption the random variables Y_n, n=1,2,..., defined by (6-26), are independent.

The application of Theorem 6.18 to our material balance test statistics MUF_i,i=1,2,... and the test problem defined by (6-10a) and (6-10b) is obvious: Under the null hypothesis H_o we assume

$$E(D_i) = E(I_i) = 0 \text{ for } i=1,2,\ldots .$$

Under the alternative hypothesis H_1 - constant loss or diversion μ - we assume

$$E(D_i) = \mu,\ E(I_i) = 0 \text{ for } i=1,2,\ldots$$

In some cases it is necessary to take into account persistent systematic errors which are of random origin. Also for these cases one can establish integral equations for the average run lengths of the CUSUM test:

Theorem 6.9

Let us consider the independently and identically distributed random variables D_i, the independently and identically distributed random variables I_i=i=1,2,... and the independently distributed random variable E with density f_E. Let us consider the CUSUM test, defined by eqs. (6-13) and (6-14), based on the test statistics

$$S_n := \max(0, S_{n-1}+Y_n), \quad n=1,2,\ldots, \quad S_0=s_0, \tag{6-13}$$

where the random variables Y_n are defined by

$$Y_n := I_{n-1}+D_n-I_n+E-k, \quad n=1,2,\ldots, \tag{6-29}$$

with reference value k and decision value h.

Then the probability $p_{E1}(z;k)$ that the CUSUM test with reference value k is finished after one observation if it starts with $S_0=s_0, I_0=i_0, z=s_0+i_0$, is given by

$$p_{E1}(z;k) = 1-F_{D-I+E}(h+k-z) \ , \tag{6-30a}$$

where $F_{D-I+E}(x)$ is the distribution function of the random variable D-I+E.

Furthermore, the probability $p_{En}(z;k)$ that the CUSUM test with reference value k is finished after n observations, if it starts at $z=s_0+i_0$, is given by

$$p_{En}(z;k) = \int_{-\infty}^{\infty} p_n(z;k-e)\cdot f_E(e)de, \tag{6-30b}$$

where $p_n(z;k-e)$ is the probability that the CUSUM test with reference value k-e for the random variables Y_n given by (6-29) which do *not* contain the random variable E, is finished after n observations.

Finally, the average run length $L_E(z;k)$ of a CUSUM test with reference value k, which starts at $z=s_0+i_0$, is given by

$$L_E(z;k) = \int_{-\infty}^{\infty} L(z;k-e)f_E(e)de,$$

where $L(z;k-e)$ is the average run length of the CUSUM test with reference value k-e for the random variables Y_n given by (6-29) which do *not* contain the random variable E.

Proof

According to (6-13) and (6-29) we have

$$\begin{aligned} p_{E1}(s_0,i_0;k)) &:= \text{prob}(S_1>h|S_0=s_0,I_0=i_0) = \\ &= \text{prob}(S_0+Y_1>h|S_0=s_0,I_0=i_0) = \\ &= \text{prob}(D_1-I_1+E+i_0+s_0-k>h) = \\ &= 1-F_{D-I+E}(h+k-(s_0+i_0)). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} p_{En}(z;k) &:= \text{prob}(S_k\leq h\wedge S_n>h|S_0+I_0=z \text{ for } 1\leq k<n) = \\ &= \int_{-\infty}^{\infty} \text{prob}(S_k\leq h\wedge S_n>h|e\leq E\leq e+de, S_0+I_0=z \text{ for } 1\leq k<n)\cdot f_E(e)de = \\ &= \int_{-\infty}^{\infty} p_n(z;k-e)\cdot f_E(e)de \ . \end{aligned}$$

The last part of the Theorem follows immediately from the definition of $L_E(z;k)$. □

6.5 References

E.L. Lehmann (1959), Testing Statistical Hypotheses. John Wiley and Sons, New York

D.C. Montgomery and L.A. Johnson (1976), Forecasting and Time Series Analysis, McGraw Hill Book Company

E.S. Page (1954), Continuous Inspection Schemes. Biometrika 41, pp. 100-115

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7. Reference Facility, Measurement Model and Losspatterns

We consider the process of the chemical extraction of plutonium in a reference reprocessing plant with a throughput of 1000 tons heavy metal per year which has been described by Kluth at al. (1981), and where for the sake of simplicity the chemical extraction is separated into five process areas: Head-end, first, second and third plutonium cycle and plutonium concentration.

7.1 Reference Facility and Measurement Model

The following analysis is based on a very simple process and measurement model. The process inventory is collected in five process units which correspond to the five areas mentioned above. Transfers are made in form of transfer units: There are three input batches, two Pu product batches and one waste batch per working day. The process is stationary which means that the inventory in the five areas is constant, and that there are no unmeasured losses. The working year consists of 200 working days, one inventory period consists of five working days. In section 8.2.2 a reference time interval of 60 inventory periods i.e. 300 working days is considered. All these source data are collected in Table 7.1 .

The measurement model may be described as follows: In case of the inventory we assume that the measurements of the different inventory units are mutually independent, and that the systematic errors cancel in the balance statistics since it is assumed that the measurement procedures are not recalibrated during the reference time. In case of the transfer measurements we take into account random and systematic errors. Again we assume that the systematic errors are constant during the whole inventory period. The relative standard deviations of all types of measurements are collected in Table 7.2.

The variance of one inventory determination then is the sum of the variances of the measurements of the five inventory units. The variance of the sum of all transfer measurements for one inventory period generally is

$$\text{var(input)} + \text{var(product)} + \text{var(waste)} =$$

$$= \sum_{k=1}^{3} (n_k \cdot \sigma_{rk}^2 + n_k^2 \cdot \sigma_{sk}^2),$$

Table 7.1: Data of the Reference Reprocessing Plant (after Kluth et al (1981))

Heavy metal throughput (t/yr)	1000
Pu-throughput (t/yr)	10
Working days per year	200
Length of an inventory period (working days)	5
Reference time (working days)	300
Input	
Number of input batches per working day	3
Pu content of one batch (Kg)	16.73
Product	
Number of product batches per working day	2
Pu content of one batch (kg)	25
Waste	
Number of waste batches per working day	1
Pu content of one batch (kg)	0.2
Inventory (kg Pu)	
Headend	196.5
Ist Pu-cycle	7.6
2nd Pu-cycle	50.
3rd Pu-cycle	134.
Pu-concentration	62.5

where n_k is the number of transfers of the k-class (input, product, waste) during one inventory period, and σ^2_{rk} and σ^2_{sk} are the (absolute) variances of the random and of the systematic errors of the single measurement of the k-th class. Therefore, the variance of the material balance test statistic for the i-th inventory period is

$$\text{var}(MUF_i) = 2 \cdot \text{var}(\text{inventory}) + \sum_{k=1}^{3} (n_k \cdot \sigma^2_{rk} + n_k^2 \cdot \sigma^2_{sk}), \; i=1,2,\ldots \; .$$

The covariance between two subsequent material balance test statistics is equal to the negative variance of the intermediate inventory plus the variance of the systematic errors; the covariance between two material balance test statistics, which do not follow each other immediately, is equal to the variance of the systematic errors alone.

All these variances and covariances are collected in Table 7.3, in addition the variance of the material balance test statistic is given for the total reference time of 300 working days. The large value of this variance is caused by the persistent systematic errors. If one assumed, e.g., that all measurements would be recalibrated after each inventory period, and that the systematic errors of the inventory measurements could be neglected, then one would obtain a standard deviation of only 20.8 kg Pu for the whole reference time.

7.2 Loss Pattern

Since as mentioned earlier, for an arbitrary number of inventory periods there exists no uniformly accepted optimization criterion, and since therefore one cannot determine pessimistic loss or diversion scenarios, so-called loss patterns were agreed upon which are collected in Table 7.4. In the first group A, losses occur in 40 periods, namely without interruption starting with the first (A1) eleventh (A2) and twentyfirst (A3) period. In the second group (B) the losses occur on two series à six periods namely from the 10th to the 15th and from the 35th to the 4oth (B1) and 10 respectively 20 periods later (B2 and B3). In the last group (C) the losses occur in eigth periods, namely every fifth period, starting with the first (C1), 11th (C2) and 21st (C3) period.

Table 7.2a: Plutonium Inventory and Relative Standard Deviation of the Random Measurement Errors

Process unit	Pu-inventory (kg)	Relative standard deviation of random error
Headend	196.5	0.01
Ist Pu-cycle	7.6	0.01
2nd Pu-cycle	50.	0.005
3rd Pu-cycle	134.	0.005
Pu-concentration	62.5	0.005

Table 7.2b: Transfer Measurements and Relative Standard Deviations of Random and Systematic Errors

Transfer	Number of batches per working day	Concent of one batch (kg Pu)	Rel. std. dev. of random errors	Rel. std. dev. of systematic errors
Input	3	16.73	0.01	0.01
Product	2	25	0.002	0.002
Waste	1	0.2	0.25	0.25

Table 7.3: Variances of Inventory, Throughput, Single Material Balance and Total Material Balance Test Statistic

Variance of the inventory

$$var(I) = (196.5x0.01)^2+(7.6x0.01)^2+(50x0.005)^2+(134x0.005)^2+(62.5x0.005)^2$$
$$= 4.476 = (2.116)^2 \ (kg^2 Pu)$$

Variance of transfers for one inventory period ($\hat{=}$ 5 working days)

$$var(input) = var(input, random) + var(input, syst) =$$
$$= 15x(16.73x0.01)^2+15^2x(16.73x0.01)^2 = 6.72$$

$$var(product) = var(prod, random) + var(prod, syst) =$$
$$= 10x(25x0.002)^2+10^2x(25x0.002)^2 = 0.275$$

$$var(waste) = var(waste, random) + var(waste, syst) =$$
$$= 5x(0.2x0.25)^2+5^2x(0.2x0.25)^2 = 0.074$$

Variance of the material balance test statistic for one inventory period

$$var(MUF) = 2x4.76+6.72+0.275+0.074 = 16.02 = 4.002^2 \ (kg^2 \ Pu)$$

Covariance between two material balance test statistics

$$cov(MUF_i, MUF_j) = \begin{cases} -4.476+6.3+0.25+0.0625 = \\ \quad = 2.135 & \text{for } |i-j| = 1 \\ 6.3+0.25+0.0625 = 6.613 & \text{for } |i-j|>1 \end{cases}$$

$$\frac{cov(MUF_i, MUF_j)}{var(MUF_i)} = \begin{cases} 0.1332 & \text{for } |i-j| = 1 \\ 0.413 & \text{for } |i-j|>1. \end{cases}$$

Variance of the balance for the reference time

$$var(\sum_{i=1}^{60} MUF_i) = 2x4.476+60x(0.42+0.025+0.0125) +$$
$$+60^2x(6.30+0.25+0.0625) = (154.38)^2 \quad kg^2 \ Pu$$

Table 7.4: Loss Patterns for the Analysis of Sequential Test Procedures for 60 Inventory periods, Total Loss M (kg Pu) .

Loss Pattern	Loss of Amount	in Periods
A1		1,2,3,.......,39,40
A2	$\frac{M}{40}$	11,12,13,.....,49,50
A3		21,22,23,.....,59,60
B1		10-15 and 35-40
B2	$\frac{M}{12}$	20-25 and 45-50
B3		30-35 and 55-60
C1		1,6,11,......,31,36
C2	$\frac{M}{8}$	11,16,21,.....,41,46
C3		21,26,31,.....,51,56

7.3 Reference

M. Kluth, H. Haug, M. Schmieder (1981)
Konzept zur verfahrenstechnischen Auslegung einer 1000 Jahrestonnen PUREX-Referenzanlage mit Basisdaten für eine Spaltstoffflußkontrolle, Report of the Nuclear Research Center Karlsruhe, KfK 3204.

8. Numerical Results

In the following we present some numerical calculations, based on the reference plant and on the loss patterns described in the foregoing chapter, in order to illustrate the efficiencies of various decision procedures discussed in the foregoing chapters. Before, we collect the quantitative rules of those procedures which will be used subsequently.

8.1 Test procedures

Six test procedures have been selected out of the many ones which have been described before. The first three procedures are based on the original MUF variables, given by (2-1); the remaining ones are based on the transformed variables, given by (5-16). In Table 9.1 these six test procedures are collected, and some information is given in which way the resulting probabilities of detection for various loss patterns can be evaluated.

We will denote the observations of random variables by corresponding small letters, i.e., the observations of MUF, MUFR, T by muf, mufr, t etc.

8.1.1 Neyman-Pearson test

For a given loss pattern $\underline{M}' = (M_1,...,M_n)$ the Neyman-Pearson test statistic is given by, see (3-7),

$$T_1 := \underline{M}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{MUF}, \tag{8-1}$$

and the test procedure is

$$\underline{M}' \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{muf} \begin{cases} > k_\alpha & : \text{reject } H_0 \\ \leq k_\alpha & : \text{reject } H_1, \end{cases} \tag{8-2}$$

where $\underline{\underline{\Sigma}}$ is the covariance matrix of the random vector

$$\underline{MUF}' = (MUF_1, ..., MUF_n)$$

and where the significance threshold k_α is determined with the help of the false alarm probability α.

The worst case loss pattern, i.e., that pattern which leads to the lowest possible probability of detection has in our case the form

$$M_1 = M_n \tag{8-3}$$

$$M_2 = M_3 = \ldots = M_{n-1};$$

the Neyman-Pearson test statistic for this worst case pattern is

$$T_{1w} := \sum_{i=1}^{n} MUF_i = I_1 + T_1 + \ldots + T_n - I_{n+1}. \tag{8-4}$$

8.1.2 <u>Truncated sequential test</u>

The test statistic of the truncated sequential test after the i-th inventory period is, see (4-1),

$$T_{2,i} := CUMUF_i := \sum_{j=1}^{i} MUF_i = I_1 + T_1 + \ldots + T_i - I_{i+1}, \quad i=1..n \tag{8-5}$$

and the test procedure is

$$cumuf_i \begin{cases} > s_i & : \text{reject } H_0 \\ \le s_i & : \text{no decision} \end{cases} \quad \text{for } i<n \tag{8-6a}$$

$$cumuf_n \begin{cases} > s_n & : \text{reject } H_0 \\ \le s_n & : \text{reject } H_1 \end{cases} \tag{8-6b}$$

The significance thresholds s_i, i=1,...,n, are determined in such a way that for each single test the same single false alarm probability is given, and furthermore, that the overall false alarm probability does not exceed a given value.

8.1.3 Onesided CUSUM test for MUF_i, $i=1,...,n$

The test statistic of the truncated sequential test after the i-th inventory period is, see (6-12),

$$T_{3,i} = \sum_{j=1}^{i}(MUF_j-k) - \min_{0\leq l\leq i} \sum_{j=1}^{l}(MUF_j-k), \tag{8-7}$$

and the test procedure is

$$t_{3,i} \begin{cases} >h : & \text{reject } H_0 \\ \leq h : & \text{no decision} \end{cases} \quad \text{for } i<n \tag{8-8a}$$

$$t_{3,n} \begin{cases} >h : & \text{reject } H_0 \\ \leq h : & \text{reject } H_1 \end{cases} \tag{8-8b}$$

The parameters h and k are determined in such a way that the overall false alarm probability for n inventory periods does not exceed a given value and furthermore, that a constant loss is detected with as high a probability as possible.
In our case, we have chosen $k = 0$.

8.1.4 Twosided test for $MUFR_i$, $i=1,...,n$

Here, the test statistics are, see (5-16),

$$T_{4,1} := MUFR_i = \sum_{j=1}^{i} a_{ij} \cdot MUF_j, \quad a_{ii} = 1, \tag{8-9}$$

where the a_{ij} are determined by the recursive relations (5-36), and the test procedure is

$$|mufr_i| \begin{cases} >c_i & : \text{reject } H_o \\ \leq c_i & : \text{no decision} \end{cases} \qquad \text{for } i<n \qquad (8\text{-}10a)$$

$$|mufr_n| \begin{cases} >c_n & : \text{reject } H_o \\ \leq c_n & : \text{reject } H_1 \end{cases} \qquad (8\text{-}10b)$$

where the significance thresholds $c_i, i=1,\ldots,n$, are determined in the same way as for the truncated sequential test.

8.1.5 Twosided CUSUM test for $MUFR_i$, $i=1,\ldots,n$

The test statistic of the twosided CUSUM test after the i-th period is, see, e.g., Nadler and Robbins (1971),

$$T^+_{5,i} = \sum_{j=1}^{i}(MUFR_j-k) - \min_{0\leq l\leq i} \sum_{j=1}^{l}(MUFR_j-k) \qquad (8\text{-}11)$$

$$T^-_{5,i} = \max_{0\leq l\leq i} \sum_{j=1}^{l}(MUFR_j+k) - \sum_{j=1}^{i}(MUFR_j+k),$$

and the test procedure is

$$t^+_{5,i} \text{ or } t^-_{5,i} \begin{cases} >h & : \text{reject } H_o \\ \leq h & : \text{no decision;} \end{cases} \qquad \text{for } i<n \qquad (8\text{-}12a)$$

$$t^+_{5,n} \text{ or } t^-_{5,n} \begin{cases} >h & : \text{reject } H_1 \\ \leq h & : \text{reject } H_1 \end{cases} \qquad (8\text{-}12b)$$

the parameters h and $k\geq 0$ are determined analoguously to the onesided CUSUM test for MUF_i, $i=1,\ldots,n$.

8.1.6 Power One Test

This test uses a test statistic which is similar to that of the truncated sequential test; it differs insofar as it uses the transformed and standardized material balance test statistics

$$T_{6,i} := \sum_{j=1}^{i} \frac{MUFR_j}{\sqrt{var(MUFR_j)}} = \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} a_{jk} \cdot MUF_k}{\sqrt{var(\sum_{k=1}^{j} a_{jk} MUF_k)}} , \tag{8-13}$$

where the a_{ij} are determined by the recursive relations (5-36). The test procedure is

$$|t_{6,i}| \begin{cases} >b_i & : \text{reject } H_0 \\ \leq b_i & : \text{no decision,} \end{cases} \tag{8-14a}$$

$$|t_{6,n}| \begin{cases} >b_n & : \text{reject } H_0 \\ \leq b_n & : \text{reject } H_1 \end{cases} \tag{8-14b}$$

where the significance thresholds b_i are given by

$$b_i = ((i+m) \cdot (-2 \cdot \ln a + \ln(\frac{i}{m} + 1)))^{\frac{1}{2}}, \quad m>0, \tag{8-15}$$

and where the parameter a is the total false alarm probability for the infinite sequence of tests and m controls the distribution of false alarms over time.

In fact, this test has the property that even for an infinite sequence of observations the false alarm probability may be smaller than one whereas for the alternative hypothesis of a constant loss the null hypothesis is rejected with probability one (therefore the name of the test).

Observation / Test	MUF_i, i=1,...,n	$MUFR_i$, i=1,...,n
Neyman-Pearson Test	Guaranteed probability of detection and pessimistic loss pattern can be determined analytically (Avenhaus and Jaech 1981). Test statistic is overall balance for the whole time period. Test procedure (9-2).	(Gives no new information compared to the Neyman-Pearson test based on the MUF_i.)
Single Tests	Probability of detection can be determined only with simulation methods. So far, this procedure has not been used by anybody.	Guaranteed probability of detection and pessimistic loss pattern can be determined (Avenhaus and Frick 1975). Probability of detection for special loss patterns can be determined numerically (Laude 1983). Test procedure (9-10).
CUMUF Test	Guaranteed probability of detection and pessimistic loss pattern can be determined analytically (Beedgen 1983). Test procedure (9-6).	Use of normalized variables leads to Power One test (Robbins and Siegmund 1969). Probability of detection for finite time horizon can be determined only with simulation methods. (Sellinschegg) Test procedure (9-14).
CUSUM Test	For the no loss and the constant loss case integral equations for the probabilities of detection after the i-th period and for the average run length can be formulated and solved numerically. Test procedure (9-8).	For the no loss case integral equations for the probability of detection after the i-th period and for the average run length can be formulated and solved numerically. Test procedure (9-12).

Table 8.1: Selected test procedures for the Near Real Time Accountancy, based on the material balance test statistics MUF_i, (2-1), and on the transformed statistics $MUFR_i$, (5-16), and possibilities for determining their efficiencies - probability of detection or average run length - for various loss patterns.

8.2 Examples

According to Table 7.3 the standard deviation for the material balance test statistic for one inventory period is

$$\sqrt{var(MUF)} = \sigma = 4 \text{ (kg Pu)} ;$$

the major part of this is caused by the errors of the inventory determination and of the systematic errors of the input measurements.

The probability of detection 1-ß for one inventory period is given by the formula (see, e.g. Avenhaus 1977)

$$1-\beta = \phi\left(\frac{M}{\sqrt{var(MUF)}} - U_{1-\alpha}\right),$$

where M is the amount to be diverted, α the false alarm probability, ϕ the normal distribution function and U its inverse.

By use of appropriate tables of the normal distribution function (see, e.g., Abramovitz and Stegun 1972) one finds immediately that 4·3.3 = 13.2 kg Pu have to be diverted in order that for a false alarm probability $\alpha = 0.05$ one gets a detection probability $1-\beta = 0.95$. If a detection probability $1-\beta = 0.50$ is satisfying, then only 1.8 kg Pu have to be diverted.

8.2.1 Two Balance Periods

Let us consider first the Neyman-Pearson test for the worst loss pattern from the safeguards authority's point of view. According to (3.12) it is given by $(M_1,M_2) = (M/2,M/2)$, and according to (3.11) the guaranteed probability of detection is

$$1-\beta_{NP}^{*} = \phi\left(\frac{M}{\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2} - U_{1-\alpha}\right) .$$

With $\sigma_1 = \sigma_2 = \sigma$ we get for $\alpha = 0.05$

$$1-\beta_{NP}^{*} = \phi\left(\frac{M}{6.03} - 1.65\right) ,$$

i.e. we get a probability of detection 1-ß = 0.95 for a loss

$$M = 3.3\sigma = 19.89 \quad (\text{kg Pu})$$

a comparatively smaller amount than for one balance period.

In the following we consider only two procedures out of the remaining five we listed in section 8.1; the reason for this is the fact that those procedures which are not considered now display their special properties only for many periods.

In case of the independently transformed test statistics $MUFR_i$ the test procedure described in section 5.4 leads with $\sigma_1 = \sigma_2 = \sigma_1\ \rho_{12} = \rho$, according to page 5-17 to the following expression for the probability of detection

$$1-\beta = 1-\phi\left(U_{1-\alpha_1} - \frac{M_1}{\sigma}\right)\cdot\phi\left(U_{1-\alpha_2} - \frac{M_2-\rho M_1}{\sigma\cdot\sqrt{1-\rho^2}}\right) ,$$

the overall false alarm probability α is given by

$$1-\alpha = (1-\alpha_1)\cdot(1-\alpha_2).$$

In Table 8.2 numerical values of 1-ß are given for

$$1-\alpha_1 = 1-\alpha_2 = \sqrt{1-\alpha} = \sqrt{0.95} = 0.975$$

and for the two cases

$$M_1 = M_2 = M_2 \text{ and } M_1 = 0,\ M_2 = M.$$

Note:

In section 8.1 we described a two-sided test procedure, whereas here we used a one-sided procedure. The reason is that M_1 is greater or equal to zero and $M_2-\rho\cdot M_1$ is also greater or equal to zero for all interesting cases thus, the two-sided procedure is not reasonable here. For many periods the situation is different since there the transformation frequently leads to $E(MUFR_i) < 0$ even if $E(MUF_i) > 0$.

For the CUMUF procedure we get the following formular for the false alarm probability α and the detection probability 1-ß:

$$1-\alpha = L(-U_{1-\alpha_1}, -U_{1-\alpha_2}, \rho)$$

$$\beta = L\left(\frac{M_1}{\sigma_1} - U_{1-\alpha_1}, \frac{M_2}{\sigma_{12}} - U_{1-\alpha_2}, \rho\right) ;$$

here, $L(h,k,\rho)$ is according to Abramovitz and Stegun (1972) defined by

$$L(h,k,\rho) = \frac{1}{2\pi} \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot \int_h^\infty dt_1 \int_k^\infty dt_2 \, \mathrm{expl}\left[-\frac{1}{2} \cdot \frac{1}{1-\rho^2} \cdot (t_1^2 - 2_\rho t_1 t_2 + t_2^2)\right]$$

and σ_1^2 and σ_{12}^2 are given as

$$\sigma_1^2 = \mathrm{var}(MUF_1), \; \sigma_{12}^2 = \mathrm{var}(MUF_1 + MUF_2).$$

In order to determine the correletion ρ between MUF_1 and MUF_1+MUF_2, we have to go back to the error model. According to (2.3) we have

$$MUF_1 = I_1 + \sum_{i=1}^{15} (ZE_i + SE) - \sum_{i=1}^{20} (ZP_i + SP) - \sum_{i=1}^{5} (ZW_i + SW) - I_2$$

$$MUF_2 = I_2 + \sum_{i=1}^{30} (ZE_i + SE) - \sum_{i=1}^{20} (ZP_i + SP) - \sum_{i=1}^{10} (ZW_i + SW) - I_3,$$

where E,P and W refer to input, product and waste respectively. Therefore, the covariance of MUF_1 and MUF_1+MUF_2 is

$$\mathrm{cov}(MUF_1, MUF_1+MUF_2) =$$
$$= \mathrm{var}(I_1) + 15\cdot\mathrm{var}(ZE) + 15\cdot 30\cdot\mathrm{var}(SE) + 10\cdot\mathrm{var}(ZP) +$$
$$+ 10\cdot 20\cdot\mathrm{var}(SP) + 5\cdot\mathrm{var}(ZW) + 5\cdot 10\cdot\mathrm{var}(SW).$$

With the data of Table 7.3 we get

$$\mathrm{cov}(MUF_1, MUF_1+MUF_2) = 18.07$$

and with

$$\sqrt{\mathrm{var}(MUF_1)} = 4, \quad \sqrt{\mathrm{var}(MUF_1+MUF_2)} = 6.03$$

finally the correlation coefficient

$$\rho = \frac{\mathrm{cov}(MUF_1, MUF_1 + MUF_2)}{\sqrt{\mathrm{var}(MUF_1)} \cdot \sqrt{\mathrm{var}(MUF_1 + MUF_2)}} = 0.751 .$$

From the table for the bivariate normal distribution we get for $\alpha = 0.05$

$$1-\alpha_1 = 1-\alpha_2 = 0.968$$

In Table 8.2 numerical values of $1-\beta$ are given for the two cases

$$M_1 = M_2 = \frac{M}{2} \text{ , and } \quad M_1 = 0 \text{ , } \quad M_2 = M .$$

Some observations should be mentioned:

- Naturally for the loss pattern (0,M) the probability of detection for the Neyman-Pearson test is larger than 0.999.
- In case of the independently transformed test statistics one has a comparativily strong dependence on the loss pattern.
- The CUMUF test is "robust" against various loss patterns. According to chapter 4 the probability of detection for the loss pattern (0,M) is the guaranteed probability of detection, i.e., for $M = 19.89$ it cannot be smaller than the value given in the Table.

Table 8.2: Probability of detection for two balance periods and various test procedures; total false alarm probability $\alpha = 0.05$, total loss $M = 19.89$

Test procedure	Probability of detection
Neyman-Pearson Test for pessimistic loss pattern $(\frac{M}{2}, \frac{M}{2})$	0.95
Independently transformed test statistic; loss pattern $(\frac{M}{2}, \frac{M}{2})$	0.88
Independently transformed test statistics; loss pattern (0,M)	0.999
CUMUF test loss pattern $(\frac{M}{2}, \frac{M}{2})$	0.936
CUMUF test loss pattern (0,M)	0.926

8.2.2 60 Balance Periods

In the following we consider the case of 60 balance periods of 5 operating days each which corresponds to a operation time of 1,5 years. For the statistical tests a total false alarm probability of 5% is assumed and we consider only the case that no measurement device is recalibrated, i.e. constant systematic errors.

Taking the criteria detection probability and timeliness of detection into account we do not have yet an optimal loss or detection strategy. Therefore, we assume a set of plausible loss patterns, which are described in Tab. 7.4. We assume a loss during 40 balance periods, but this loss may occur somehow during 60 periods.

In group A of our loss patterns we have a constant loss during 40 balance periods beginning with the first (A1), eleventh (A2) and twentyfirst (A3) period. In group B the loss occurs in two 6 period blocks ranging from 10 to 15 and 35 to 40 (B1), so to 25 and 45 to 50 (B2) and 30 to 35 and 55 to 60 (B3).

In group C we consider a discrete loss in 8 balance periods every fifth period beginning with period 1(L1), 11(12) and 21(C3).

In Tables 8.3-8.3 the probabilities of detecting a loss not later than after the 60th balance are presented for different total amounts of losses.

The comparison of there three tables shows that the total amount of loss has no influence on the structure of the detection probabilities for the different test procedures and loss patterns. Therefore, it is sufficient to discuss Tab. 8.5.

In the first row the optimal achievable detection probabilities (for the Neyman-Pearson-test) are presented under the assumption that we know the actual loss pattern. It is obvious that those tests (T_4,T_5,T_6) which use the transformed values of the MUF series ($MUFR_i$) lead to considerably higher detection probabilities than those tests (T_2,T_3) which use the original values. The test T_3 leads to higher detection probabilities for all loss patterns except A_1.

The test T_5 is the best one for all loss patterns and the detection probabilities are close to the maximum achievable ones. The test T_4 gives better results than T_6 only for the B patterns since T_4 is based on the individual MUFR's and thus more sensitive for abrupt losses.

Additionally to the described 9 loss pattern the very important loss pattern of a continuous constant loss in each balance period was considered.

Table 8.3: Probability of detection for various test procedures and the loss pattern given in Table 8.1, M=30 kg Pu

	T_1	T_2	T_3	T_4	T_5	T_6
A1	.973	.084	.075	.134	.764	.324
A2	.921	.053	.075	.104	.547	.457
A3	.973	.051	.075	.122	.889	.708
B1	.999	.063	.075	.586	.913	.521
B2	.998	.055	.075	.746	.963	.716
B3	.999	.053	.075	.814	.978	.704
C1	.999	.063	.075	.194	.831	.424
C2	.999	.054	.075	.198	.487	.380
C3	.999	.052	.075	.245	.875	.699

T_1: Neyman-Pearson test for specific loss pattern

T_2: Truncated Sequential CUMUF

T_3: Onesided CUSUM Test of MUF_i; i=1,2,...

T_4: Twosided test of $MUFR_i$, i=1,2,...

T_5: Twosided CUSUM Test of $MUFR_i$, i=1,2,...

T_6: Sequential twosided test of Power One for $MUFR_i$, i=1,2,...

Table 8.4: Probability of detection for various test procedures and the loss pattern given in Table 8.1, M=40 kg Pu

	T_1	T_2	T_3	T_4	T_5	T_6
A1	.999	.096	.085	.233	.962	.569
A2	.992	.060	.085	.159	.812	.710
A3	.999	.057	.085	.197	.992	.931
B1	.999	.078	.085	.912	.999	.834
B2	.999	.062	.085	.982	.999	.935
B3	.999	.059	.085	.994	.999	.934
C1	.999	.078	.085	.346	.982	.699
C2	.999	.061	.085	.376	.775	.614
C3	.999	.056	.085	.487	.988	.926

T_1: Neyman-Pearson test for specific loss pattern

T_2: Truncated Sequential CUMUF

T_3: Onesided CUSUM Test of MUF_i; i=1,2,...

T_4: Twosided test of $MUFR_i$, i=1,2,...

T_5: Twosided CUSUM Test of $MUFR_i$, i=1,2,...

T_6: Sequential twosided test of Power One for $MUFR_i$, i=1,2,...

Table 8.5: Probability of detection for various test procedures and the loss pattern given in Table 8.1, M=50 kg Pu

	T_1	T_2	T_3	T_4	T_5	T_6
A1	.999	.117	.094	.390	.998	.792
A2	.999	.066	.094	.245	.957	.892
A3	.999	.060	.094	.311	.999	.994
B1	.999	.094	.094	.996	.999	.979
B2	.999	.065	.094	.999	.999	.995
B3	.999	.059	.094	.999	.999	.994
C1	.999	.094	.094	.554	.999	.895
C2	.999	.064	.094	.620	.945	.813
C3	.999	.056	.094	.769	.999	.991

T_1: Neyman-Pearson test for specific loss pattern

T_2: Truncated Sequential CUMUF

T_3: Onesided CUSUM Test of MUF_i; i=1,2,...

T_4: Twosided test of $MUFR_i$, i=1,2,...

T_5: Twosided CUSUM Test of $MUFR_i$, i=1,2,...

T_6: Sequential twosided test of Power One for $MUFR_i$, i=1,2,...

The constant loss case is very similar to the loss pattern with the lowest guaranted detection probability (worst case loss pattern). In Table 8.6 the detection probabilities for two values of total loss are presented. In this case we recognize that tests T_2 and T_3 are superior to T_4 to T_6. T_3 is almost as good as the Neyman-Pearson-test. In Figure 8.1 the detection probabilities of the truncated sequential CUMUF test are compared with optimal values of the Neyman-Pearson test where a loss pattern with minimal guaranteed detection probability is assumed, the difference in the values is always less than 10%; this recommends the truncated sequantial CUMUF as a possible test procedure.

Table 8.6: Probability of detection for various test procedures and saddlepoint loss pattern according to Eq. (3.12)

M Kg Pu	T_1	T_2	T_3	T_4	T_5	T_6
50	.093	.085	.093	.049	.053	.052
500	.945	.903	.944	.219	.264	.496

In the following we discuss the question of timeliness of detection of losses. Until now we have not considered this question at all. However, it was necessary to discuss the overall probability of detection since this criteria nevertheless will be very important.

A reasonable and in quality control frequently used criterion is the average run length of the test until the rejection of the null hypothesis. For a finite sequence of periods it used only if the overall Probability of detection is nearby one. Table 8.7 gives some examples. It indicates that in those cases where a comparison of different tests is possible, no major differences can be observed.

The test T_5 is better than T_6 and T_4 is better than T_5 and T_6 for pattern B just as the overall probability of detection. For the loss pattern B it can be concluded than on the overage 10-14 balance periods after the first loss occured it will be detected.

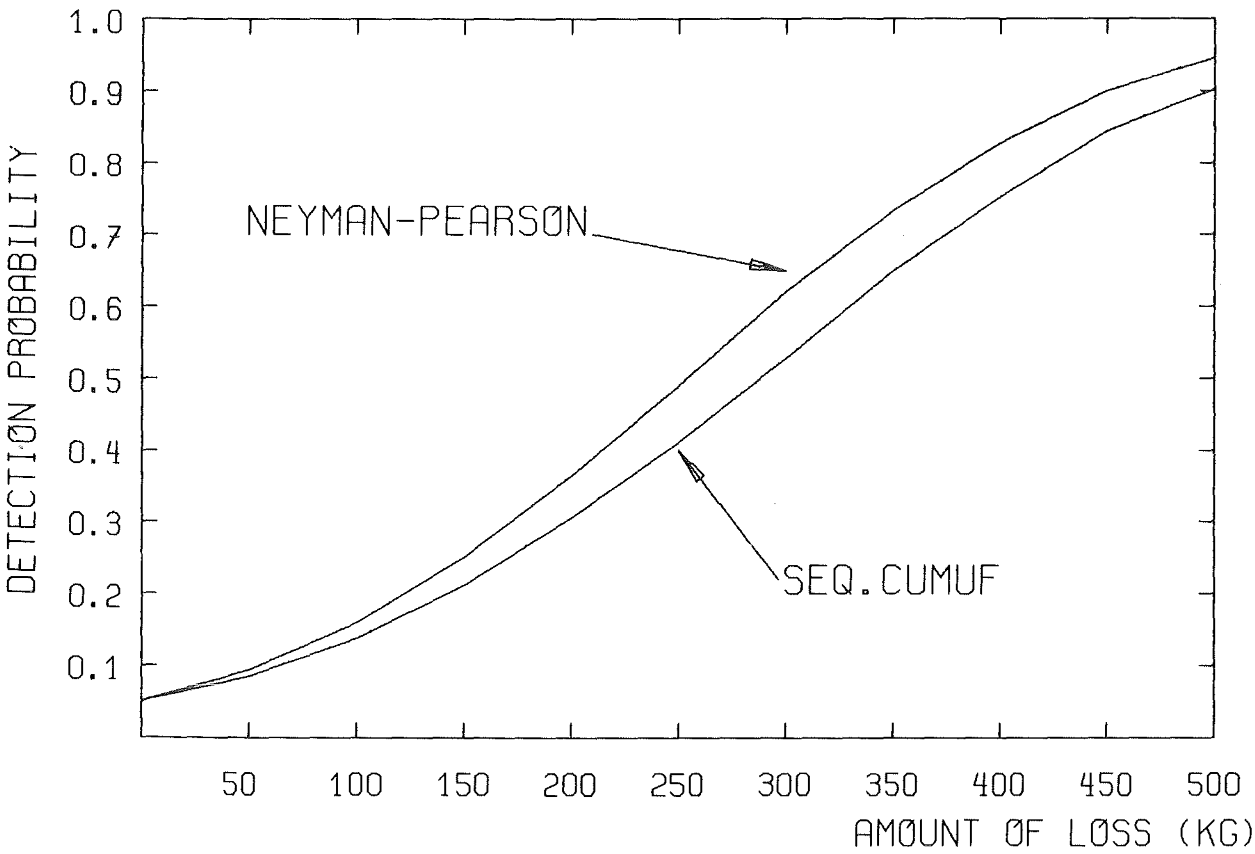

Figure 8.1 : Detection probabilities for the Neyman-Pearson test and truncated sequential CUMUF test where the loss pattern with minimal guaranteed detection probability is assumed.

Table 8.7: Average Run Length for test procedures T_4, T_5, T_6 and loss patterns given in Table 8.1, M=50 kg Pu . Probability of detection greater than .99 is given.

	T_4	T_5	T_6
A1	-	49.54	-
A2	-	-	-
A3	-	31.50	34.41
B1	14.46	14.16	-
B2	22.03	23.32	24.01
B3	31.62	33.08	34.17
C1	-	47.15	-
C2	-	-	-
C3	-	29.48	32.36

The best information about the timeliness of the detection of any loss is given by the run length distribution, i.e. the development of the probabilities of detection with time. Figures 8.2 - 8.4 present some examples for tests T_5 and T_6.

Figure 8.2 shows that for loss pattern A1, in which the loss occurs right at the beginning of the evaluation period both tests are not able to indicate the loss up that period in which the loss is terminated. But afterwards both tests are indicating the loss with a high detection probability. This indicates that an extension of the loss over all the considered balance periods would not be detected with a reasonable probability. When the loss does not occur at the beginning of the evaluation period, the it is indicated immediately by both test procedures, see loss patterns A2 and A3. Furthermore, we find that T_6 does detect the beginning of a loss pattern with a slightly better detection probability than T_5. But after a few periods T_5 becomes more sensitive.

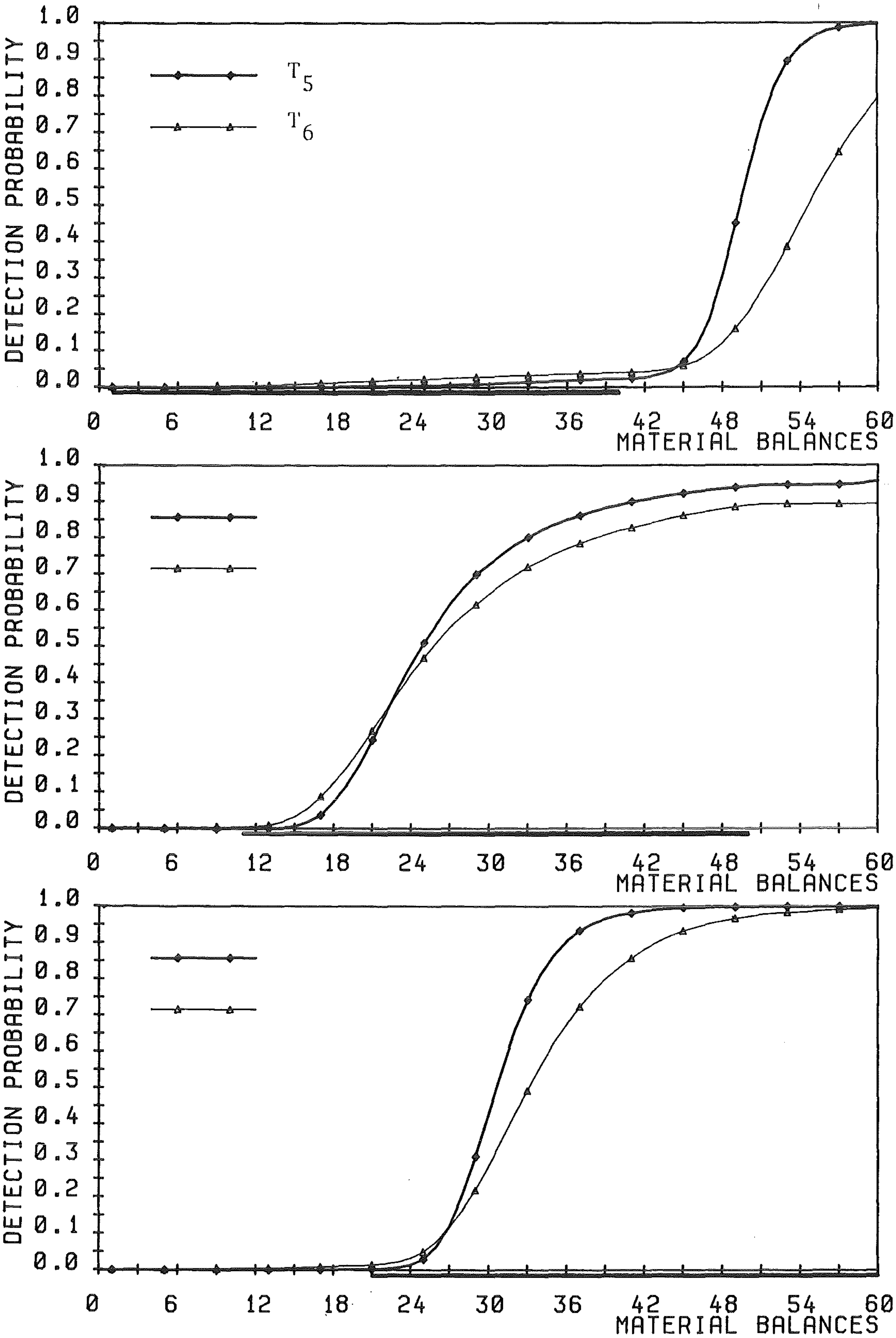

Figure 8.2 : Development of detection probabilities (run length distribution) for tests T_5 and T_6 and loss pattern A; M = 50 kg

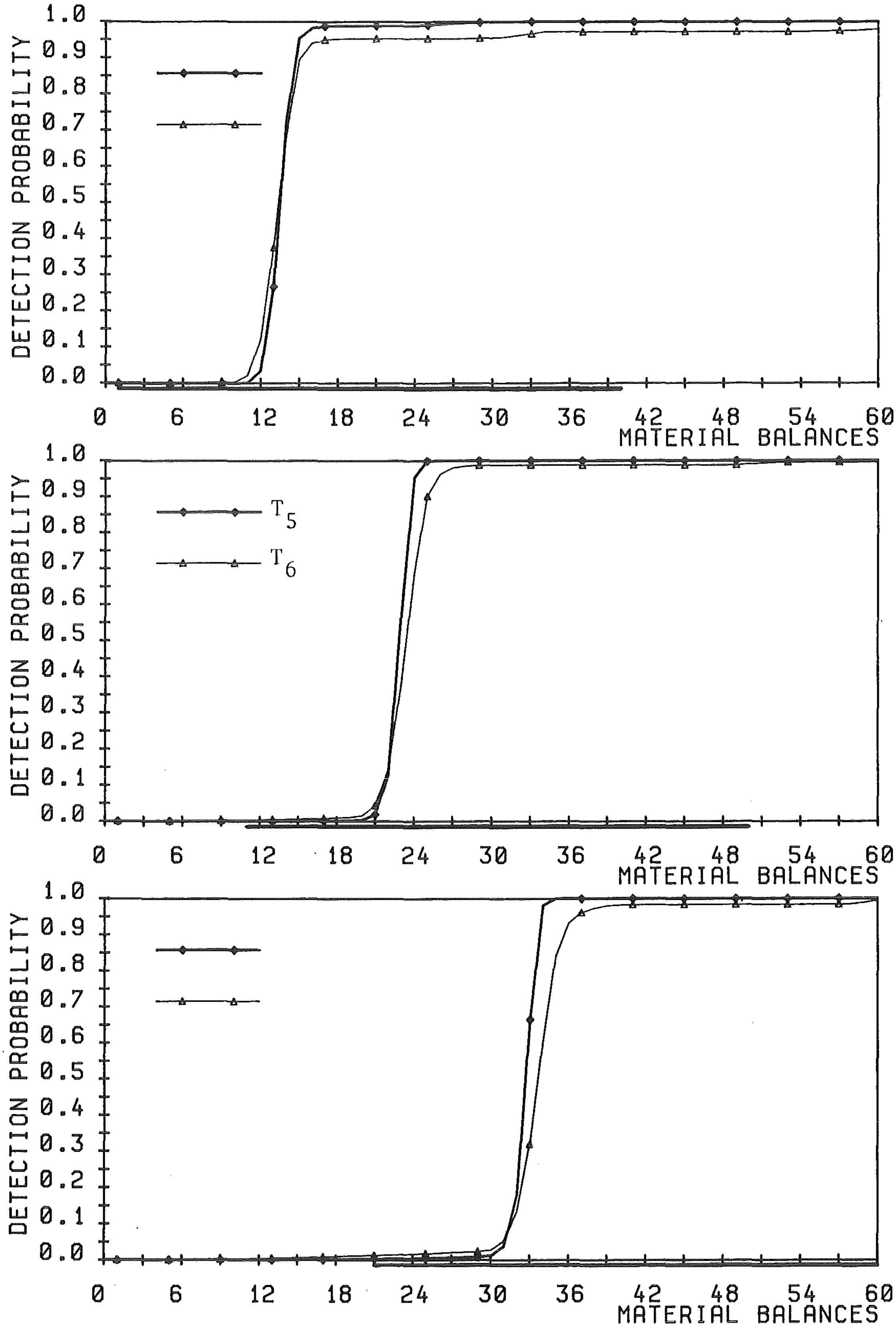

Figure 8.3 : Development of detection probabilities (run length distribution) for tests T_5 and T_6 and loss pattern B; M = 50 kg

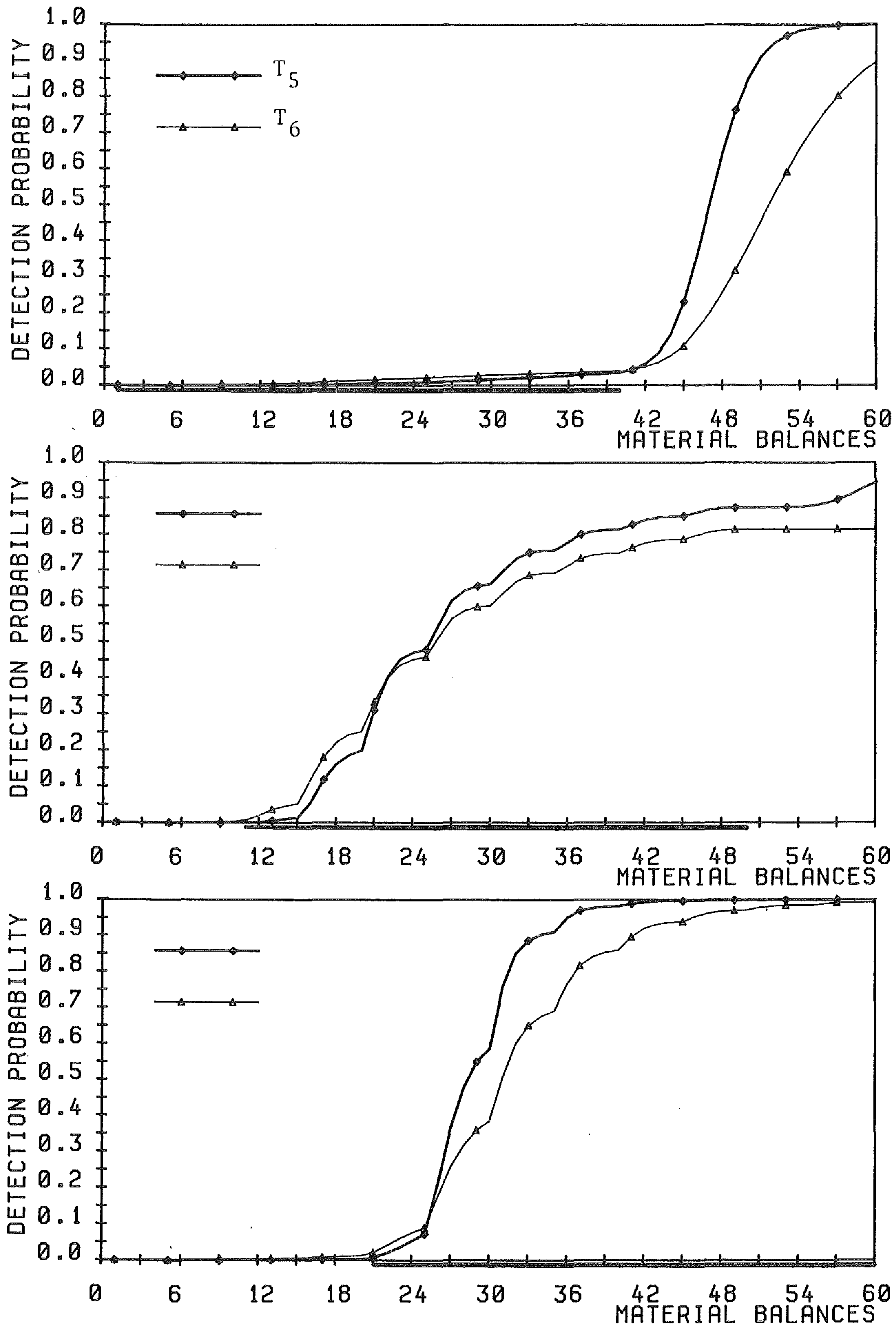

Figure 8.4 : Development of detection probabilities (run length distribution) for tests T_5 and T_6 and loss pattern C; M = 50 kg

Figure 8.3 shows that loss patterns B represent an abrupt loss which is detected immediately after its occurence. The translation of the time axis does not give any new information. Finally, the comparison of Figures 8.2 and 8.4 indicates that loss patterns C are only a special form of continuous loss.

Summarizing the analysis for the considered test procedures it can be shown, that an optimum test procedure with a maximum achievable detection probability for all possible loss patterns does not exist. It is evident that at least two different types of loss patterns have to be considered. Firstly, patterns in which a constant loss occurs in each balance period and secondly, such patterns in which a loss occurs in some but not in all balance periods under consideration. For the first type of patterns the test procedures based on the cumulative sum of the MUF values show the best detection probabilities. The statistics for these tests can be calculated without knowing the variance - covariance structure. This information is necessary if we want to make any statistical inference. For the truncated sequential CUMUF test it has to be mentionend that a guaranteed detection probability for all possible loss patterns can be calculated.

For the second type of loss patterns test procedures based on the transformed MUF values show the best detection probabilities. It has to be mentioned that for calculating these statistics the exact variance - covariance structure of the MUF values has to be known. The question of "robustness" of the test procedures based on the transformed MUF's against changes in the variance co-variance structure has to be further investigated.

Thus, our analysis shows that with a combination of two test, one based on cumulative sum of MUF's and another one based on transformed MUF's, most of the considered loss patterns can be covered with sufficient high detection probability and in short detection time. For two tests, the development of false alarms can be kept under control; e.g. with the help of Bonferoni's inequality.

8.3 References

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