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A Theory of Fractal Fluids

E. G. Schlechtendahl
Institut für Reaktorentwicklung

Kernforschungszentrum Karlsruhe

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E. G. Schlechtendahl

Kernforschungszentrum Karlsruhe GmbH, Karlsruhe

Kernforschungszentrum Karlsruhe GmbH
Postfach 3640, 7500 Karlsruhe 1

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ABSTRACT

Practical fluid mechanics problems may be characterized by a length scale ε, below which we are not able or not willing to resolve details of density or velocity distributions. Conventional fluid mechanics assumes that below the ε-scale all essential quantities (in particular the density and velocity) can be considered as continuous and are accurately enough described by their average values taken over control volumes of size ε. This assumption, however, is not valid for many problems of interest. The category of these problems includes both turbulent flow and two-phase flow. The fundamental property of these fluids is that local flow characteristics (such as density or velocity) at any given point must be considered to be different from any (possibly weighted) average of this quantity in a small control volume surrounding this point, even for the smallest control volumes we can practically choose.

This paper presents a new method to deal with such problems of fluid mechanics. The method is based on a rigorous treatment of the transport theorem and uses Taylor series expansions (with third order accuracy in the spatial variables) of the essential functions whenever exact solutions could not be obtained. A closed system of partial differential equations is derived. It contains conventional fluid dynamics as a special case as well as results which can be derived from the kinetic theory of gases.

The new set of equations contains balance equations for the conventional quantities mass (identical to the conventional form) and momentum (with an essential difference as compared to the conventional formulation), as well as some quantities which are not treated by the conventional theories, such as the first spatial moment of the density distribution (its divergence gives a quantitative measure of the density pattern, bubble flow versus droplet flow), the first spatial moment of the velocity distribution, and the velocity correlation (which is a symmetrical tensor representing the deviation of the local velocity from its mass weighted average).

Zusammenfassung

Eine Theorie fraktaler Fluide

Fluidmechanik-Probleme sind in der Praxis oft durch einen Längenmaßstab ε gekennzeichnet, unterhalb dessen wir Details der Dichte- oder Geschwindigkeitsverteilung nicht mehr auflösen können oder nicht mehr wollen. Die konventionelle Fluidmechanik nimmt an, daß unterhalb von ε alle wesentlichen Funktionen, speziell Dichte und Geschwindigkeit, als stetig angesehen werden und durch ihre über ein Kontrollvolumen gemittelten Werte genau genug beschrieben werden können. Diese Annnahme trifft jedoch für viele interessierende Probleme nicht zu. Dieser Problembereich umfaßt turbulente Strömungen ebenso wie Zweiphasen-Strömungen. Die fundamentale Eigenschaft solcher Fluide ist, daß die lokalen Verteilungsfunktionen der wichtigen Groessen an jedem Punkt sich wesentlich von jedem (auch gemittelten) Mittelwert in der Umgebung dieses Punktes unterscheiden koennen, so klein wir auch diese Umgebung waehlen.

Hier wird nun eine neue Methode vorgeschlagen, Fluidmechanik-Probleme dieser Art zu behandeln. Die Methode baut auf einer strengen Auswertung des Transport-Theorems auf und verwendet Taylorreihen-Entwicklungen der wesentlichen Funktionen bis zur Genauigkeit dritter Ordnung in den räumlichen Variablen, soweit nicht exakte Gleichungen abgeleitet werden können. Ein in sich geschlossenes System von Differentialgleichungen wird hergeleitet. Es beinhaltet die übliche Strömungsmechamik als Sonderfall ebenso wie Ergebnisse der kinetischen Theorie der Gase.

Der neue Satz von Gleichungen enthält Bilanzgleichungen für die üblichen Größen wie Masse (wie in der gewöhnlichen Formulierung) und Impuls (mit einem wesentlichen Unterschied im Vergleich zu der üblichen Form), aber auch einige Größen, die in den gewöhnlichen Theorien nicht auftauchen, wie das erste Moment der räumlichen Dichteverteilung (dessen Divergenz gibt ein Maß für die Dichteverteilung, Blasenströmung bzw. Tröpfchenströmung), das erste Moment der räumlichen Geschwindigkeitsverteilung und die Geschwindigkeitskorrelation (dies ist ein symmetrischer Tensor, der etwas über die Abweichung der lokalen Geschwindigkeit von dem mit der Dichte gewichteten lokalen Mittelwert aussagt).

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LIST OF SYMBOLS

symbol	first used in or before equation	definition of symbols (symbols which are used locally only are defined in the appropriate place)
a	3.06	average value of a_o in 3D-space
a^{Pi}	3.18	average value over rectangular control surface
a_m^{Pi}	3.19	mean value over opposite rectangular control surface
a_m^{Pi}	3.19	mean value over opposite rectangular control surface
a^{ε}	2.07	average value of a_o in ε-interval
a_m^{ε}	2.04	mean value taken over end points of the ε-interval
a_o	1.01	arbitrary function along x-axis
a_o	3.01	arbitrary function in 3D-space
c	15.1.06	speed of sound
e_{ni}	6.12	specific velocity correlation
g	4.07	specific force
g_o	4.06	local specific force
i	3.01	subscript indicating i-direction in 3D-space the summation convention does not apply to i
i_{ni}	11.01	moment of inertia
i^{ε}	2.17	second moment of a_o in ε-interval
j	3.01	subscript indicating j-direction in 3D-space the summation convention does not apply to j
k	3.01	subscript indicating k-direction in 3D-space the summation convention does not apply to k
ℓ	14.06	subscript, summation convention applied
m	5.02	subscript, summation convention applied
M	3.03	mass
n	2.02	exponent of Taylor series expansion
n	3.15	subscript, summation convention applied

symbol	first used in or before equation	definition of symbols (symbols which are used locally only are defined in the appropriate place)
o	3.08	center of mass offset from volumetric center
p	10.03	hydrodynamic pressure
Pi	3.17	rectangular control surface in 3D-space
Q	3.01	control volume (quadrilateral)
r	2.13	subscript, enumeration of ε-intervals along y-axis
s	6.04	specific first moment of velocity
t	4.01	time
u	4.01	velocity
v	5.05	specific momentum
V	3.02	volume
x	1.01	one-dimensional coordinate
x	3.01	coordinate in 3D-space
y	1.01	center of an ε-interval along x-axis
y	3.01	center of an ε-interval along x-axis in 3D-space
y_1	1.01	left side of a large scale interval along x-axis
y_2	1.01	right side of a large scale interval along x-axis
z	3.05	coordinate of center of mass
α^{ε}	2.08	average first moment of a_o in ε-interval
α	3.07	average first moment of a_o in 3D-space
ϒ	14.06	body force couple
δ	4.14	Kronecker delta
ε	1.01	small length scale, also length of a small interval along x-axis
ε	3.01	length of a small interval along x-axis in 3D-space
Θ	3.09	volumetric moment of inertia of 3D control volume
Θ^{ε}	2.06	$\varepsilon^2/12$
λ	1.01	large length scale
μ^*	14.06	stress couple
ν	12.04	subscript indicating direction normal to boundary
ν	14.4.21	kinematic viscosity

symbol	first used in or before equation	definition of symbols (symbols which are used locally only are defined in the appropriate place)
ρ	3.04	average density
ρ_l	15.4.28	liquid density
ρ_o	3.01	density distribution function in 3D-space
τ	4.08	average surface force density (stress)
τ^*	15.04	micro-stress average
ψ_o	4.01	arbitrary three-dimensional field function
ω	12.01	subscript indicating direction parallel to boundary
Ω	4.01	surface of control volume
$\dot{}$	4.02	superscript, total derivative with respect to time
^{o}	12.01	superscript, boundary value
$,_n$	3.09	partial derivative with respect to x_n if applied to local values
$,_n$	3.13	partial derivative with respect to y_n if applied to average values or first moments

A THEORY OF FRACTAL FLUIDS

1. Introduction

Practical fluid mechanics problems may be characterized by two different length scales: λ and ε. λ is the typical length of the geometrical space in which we are interested to obtain a solution. Anything beyond the λ-scale may influence the inside of the interesting domain, but is considered to be imposed externally without any significant interaction. ε characterizes the minimum size which we are able or willing to resolve. Conventional fluid mechanics assumes that whatever happens below the ε-scale is sufficiently described by average values of density and velocity taken over control volumes of size ε. This assumption, however, is not valid for many problems of interest. The category of these problems includes both turbulent flow and two-phase flow. The fundamental property of these fluids is that

- local flow characteristics (such as density or velocity) at any given point must be considered to be different from the average of this quantity in a small control volume surrounding this point, even for the smallest control volumes we can practically choose.

Following Mandelbrot [1], we will call functions which exhibit this property "fractals". More formally (and for the sake of simplicity for the one-dimensional case only), let $a_o(x)$ be an arbitrary function along some axis x within the interval $y-\frac{1}{2}\varepsilon \le x \le y+\frac{1}{2}\varepsilon$ in a control volume defined by the length scale ε around the point y, then we consider the function to be fractal if for at least a certain range $y_1 \le x \le y_2$:

$$(1.01)\quad \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\int_{y-\frac{1}{2}\varepsilon}^{y+\frac{1}{2}\varepsilon} a_o(x)dx - a_o(y)\right) \neq 0$$

As we will see later, the preferred formulation is:

$$(1.02)\quad \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\int_{y-\frac{1}{2}\varepsilon}^{y+\frac{1}{2}\varepsilon} a_o(x)dx - \tfrac{1}{2}(a_o(y+\tfrac{1}{2}\varepsilon)+a_o(y-\tfrac{1}{2}\varepsilon))\right) \neq 0$$

Conventional continuum mechanics assumes that the essential properties of a fluid (density and velocity) may be described as continuous fields. This implies that we can always choose the size ε of a control volume small enough as to reduce the difference between the local values and the average values below any arbitrary value. We emphasize that this is merely an approximation, a mathematical fiction, so to say. We know that this approximation fails on a molecular scale. But even on the much larger scale of Brownian motion the fluid behaviour is chaotic or fractal in nature, as pointed out by Mandelbrot in [1]. Mandelbrot even claims that on the scale of turbulent flow phenomena the fluid has this fractal property. From this point of view, the continuity assumption for the velocity field on a small scale should be considered as a mathematical fiction.

Similar criticism applies to the usual treatment of multi-phase flow. Steam-water mixtures, e.g., are usually considered to consist of two properly separated geometric domains filled either with vapour or with liquid. However, bubbles floating in a sea of liquid may contain smaller droplets which themselves contain even smaller bubbles, and so forth. Hence serious doubts are justified regarding the application of the classical "continuum" concept to such situations.

A number of theories have been developed to treat continua which, on a small scale, cannot be described adequately by a density and a velocity value alone. These continua exhibit , e.g., velocity discontinuities in the form of small scale rotations and deformations which are superimposed to the (continuous) global velocity field. Early work in this field was done by the Cosserat brothers [2]. The common approach of these theories is to associate with the smallest control volumes of the "continuum" not only mass and momentum, but such properties as angular momentum or rotational inertia, as well. A survey of these theories is given by Cowin [3]. Most advanced are the theories of microfluids and micro-polar fluids as developed by Eringen [4, 5]. The underlying argument of all these theories is that no matter how small we make a control volume, there is still much to be said about its interior; the kinetic

theory of gases may be used as an example; it tells more about the behaviour on small scale than any field theory can achieve.(1)

In this paper, we develop a new theory which aims at a rigorous treatment of heterogeneous flow problems. Our approach differs from earlier work in that it is not based on an intuitive concept for the physical behaviour of the material below the ε scale. In our new theory, mainly mathematical reasoning backed up by the conventional continuum mechanics laws is used:

1. We consider functions which are continuous on a small scale, and we analyse some of their basic properties: average values and first moments over small control volumes.
2. We study the behaviour of these averages and first moments for a whole class of functions as we decrease the control volume size ε below any arbitrary finite value.
3. We show that in the limit case $\varepsilon \to 0$ this class of functions converges to a fractal function.
4. We apply the same procedure to the density and velocity field of a fluid, making rigorous use of Reynold's transport theorem.
5. We obtain a set of differential equations and boundary conditions which in special cases becomes identical or at least similar to familiar formulations of fluid flow.
6. We propose this set of equations as generalization of fluid flow theory, in fact, as a generalization of the Navier-Stokes equations.

2. One-dimensional functions

Let

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1 With classical thermodynamics, one can calculate almost everything crudely; with kinetic theory, one can calculate fewer things, but more accurately; and with statistical mechanics, one can calculate almost nothing exactly!
(E. Wigner)

$$(2.01)\quad a_o(x) \equiv a_o(0) + a_o^{(1)}(0)x + a_o^{(2)}(0)x^2/2 + a_o^{(3)}(0)x^3/6 + \ldots.$$

be an arbitrary function which is sufficiently smooth to be expanded into a Taylor series in the neighbourhood of $x = 0$ up to any desired order. If we choose a point with $x = y$ not too far from the origin $x = 0$, we can approximate $a_o(y)$ up to any desired accuracy by a truncated Taylor series: (2)

$$(2.02)\quad a_o(y) =^n a_o(0) + a_o^{(1)}(0)y + \ldots.. + a_o^{(n)}(0)y^n/n!$$

where the symbol $=^n$ means that the two sides of the equation are equal up to order n in the independent variable, and the $a_o^{(n)}$ are the coefficients of the Taylor series expansion. For the two points $x = y \pm \varepsilon/2$ we find

$$(2.03)^*\quad a_o(y\pm\tfrac{1}{2}\varepsilon) =^3 a_o(y) \pm ({}^{da_o(y)}/_{dy})\tfrac{1}{2}\varepsilon + ({}^{d^2a_o(y)}/_{dy^2})\varepsilon^2/8 \pm ({}^{d^3a_o(y)}/_{dy^3})\varepsilon^3/48$$

provided that ε is small enough (at most of the same order of magnitude as y). The mean value is

$$(2.04)^*\quad a_m^{\varepsilon}(y) \equiv \tfrac{1}{2}(a_o(y+\tfrac{1}{2}\varepsilon) + a_o(y-\tfrac{1}{2}\varepsilon)) =^3 a_o(y) + ({}^{d^2a_o(y)}/_{dy^2})\varepsilon^2/8$$

Equation (2.04) determines the difference between the local value a_o in the middle of the interval and the mean value. For a step-type discontinuity, it is common practice to use the mean of the values on the left-side and the right-side discontinuity boundaries as the local value in the discontinuity:

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2 Equation numbers with superscript * indicate, that the formal proof is performed with a REDUCE [5] program listed in the appendix.

$$(2.05)\quad a_o(y) =^1 a_m^\varepsilon(y)$$

Note, that this approach neglects the term with ε^2 in (2.04) and, hence, (except when the second derivates vanishes) is first-order accurate only! In contrast to equation (2.05) we intend to derive an equation for the local value of $a_o(y)$ which is better than just first-order accurate. For this purpose, let us determine the average value of $a_o(x)$ and of the first central moment $(x-y)*a_o(x)$ over the interval $y-\varepsilon/2 \le x \le y+\varepsilon/2$. With

$$(2.06)^*\quad \theta^\varepsilon \equiv \frac{1}{\varepsilon}\int_{y-\frac{1}{2}\varepsilon}^{y+\frac{1}{2}\varepsilon}(x-y)^2dx = \varepsilon^2/12$$

we find for the average value

$$(2.07)^*\quad a^\varepsilon(y) \equiv \frac{1}{\varepsilon}\int_{y-\frac{1}{2}\varepsilon}^{y+\frac{1}{2}\varepsilon}a_o(x)dx$$

$$=^5 \frac{1}{\varepsilon}\int_{y-\frac{1}{2}\varepsilon}^{y+\frac{1}{2}\varepsilon}\left(\Sigma_{n=1\ldots4}a_o^{(n)}(0)x^n/n!\right)dx$$

$$=^5 a_o(y) + \tfrac{1}{2}\theta^\varepsilon\left(d^2a_o(y)/dy^2\right) + {}^3/_{40}(\theta^\varepsilon)^2\left(d^4a_o(y)/dy^4\right)$$

$$=^3 a_o(y) + \tfrac{1}{2}\theta^\varepsilon\left(d^2a_o(y)/dy^2\right)$$

and for the first spatial moment

$$(2.08)^*\quad \alpha^\varepsilon(y) \equiv \frac{1}{\varepsilon}\int_{y-\frac{1}{2}\varepsilon}^{y+\frac{1}{2}\varepsilon}(x-y)a_o(x)dx$$

$$=^5 \theta^\varepsilon\left(da_o(y)/dy\right) + \frac{3}{10}(\theta^\varepsilon)^2\left(d^3a_o(y)/dy^3\right)$$

$$=^3 \theta^\varepsilon\left(da_o(y)/dy\right)$$

Subsequently in this paper, we will often have to prove the validity of certain approximations formulated in terms of local quantities, average quantities, first-order moments and their derivatives. We will generally perform the formal proof of such approximations up to second-order accuracy. However,

since no terms of order ε^3 appear in equations (2.07) and (2.08) we conclude that such approximations -if they are formally proven to be valid up to order two- will also be valid up to order three. Hence, we will use the $=^3$ symbol even in cases where the proof by Taylor series expansion is performed only up to second order. As a general rule, we will avoid the explicit use of 0 and of second derivatives in these approximations and will attempt to formulate all approximations in terms of average values and first-order moments. Thus, if we satisfy ourselves with third-order approximations, we find:

$$(2.09)^* \quad a^\varepsilon(y) =^3 a_o(y) + \tfrac{1}{2}\alpha^\varepsilon(y)_{,y}$$

where the subscript "$_{,y}$" indicates differentiation with respect to y. Thus, we can determine the <u>local</u> value $a_o(y)$ of a function $a_o(x)$ with third-order accuracy, provided that we know the <u>average value</u> $a^\varepsilon(y)$ and the first moment function $\alpha^\varepsilon(y)$ in the neighbourhood of $x = y$.

So far, we have found that the first spatial moment α^ε is useful in providing a highly accurate approximation for the difference between the average value of a function and its local value in the middle of an interval. We will now show that α^ε can provide an <u>exact</u> expression for the difference between the average value and the mean value claculated from the local values at the two end points of the interval.

Instead of starting from a Taylor series expansion of function $a_o(x)$, we may differentiate the quantity $\alpha^\varepsilon(y)$ with with respect to y by applying partial differentiation:

$$(2.10) \quad \frac{d}{dy}\alpha^\varepsilon(y) = \frac{1}{\varepsilon}\frac{d}{dy}\int_{y-\frac{1}{2}\varepsilon}^{y+\frac{1}{2}\varepsilon}(x-y)a_o(x)dx$$

$$= \tfrac{1}{2}\left(a_o(y+\tfrac{1}{2}\varepsilon) + a_o(y-\tfrac{1}{2}\varepsilon) \right) + \frac{1}{\varepsilon}\int_{y-\frac{1}{2}\varepsilon}^{y+\frac{1}{2}\varepsilon}\frac{d}{dy}((x-y)a_o(x))dx$$

$$= \tfrac{1}{2}\left(a_o(y+\tfrac{1}{2}\varepsilon) + a_o(y-\tfrac{1}{2}\varepsilon) \right) - \frac{1}{\varepsilon}\int_{y-\frac{1}{2}\varepsilon}^{y+\frac{1}{2}\varepsilon}a_o(x)dx$$

$$= \tfrac{1}{2}\left(a_o(y+\tfrac{1}{2}\varepsilon) + a_o(y-\tfrac{1}{2}\varepsilon) \right) - a^\varepsilon(y)$$

$$= a_m^{\varepsilon}(y) - a^{\varepsilon}(y)$$

This equation relates the mean value of a function in an interval to the average value. Note, that (2.10) is exact, while (2.09) is merely an approximation. For this reason, we prefer the formulation (1.02) over (1.01) for the definition of fractal functions.

Let us consider the case of $\varepsilon \to 0$. Obviously, if we maintain the underlying continuous function a_o, both the mean value and the average value will converge to the local value and, at the same time, $\alpha^{\varepsilon}(y)_{,y}$ will vanish. This can be shown by estimating the magnitude of α^{ε} for $\varepsilon \to 0$. It is obvious, that this function must converge towards 0 everywhere provided that the integrand does not contain any singularities. Proof:

$$(2.11)\quad \alpha^{\varepsilon}(y) \equiv \frac{1}{\varepsilon}\int_{y-\frac{1}{2}\varepsilon}^{y+\frac{1}{2}\varepsilon}(x-y)a_o(x)dx \le \frac{1}{\varepsilon}\int \tfrac{1}{2}\varepsilon\,|a_o|_{max}dx \le \tfrac{1}{2}\varepsilon\,|a_o|_{max}$$

where $|a_o|_{max}$ is the maximum absolute value of a_o within the integration interval. This means that the postulated existence of a function α^{ε} with non-vanishing derivative

$$(2.12)\quad \lim_{\varepsilon\to 0}\alpha^{\varepsilon}(y)_{,y} \neq 0$$

contradicts with the continuity assumption. Vice versa, (2.12) implies discontinuity of the underlying function not only at individual points but <u>everywhere, where (2.12) applies</u>. Hence, the question arises whether (2.12) is at all a meaningful postulate for fractal functions. Obiously, we have to extend the concept of taking the derivative of a functions in a suitable way from the class of continuous functions into the class of fractal functions for this purpose. We will show that such an extended definition of derivatives is indeed possible, and we will demonstrate that there is at least one type of function for which the postulate (2.12) is meaningful.

For this purpose, let us divide the x-axis into intervals of equal length ε as shown in Figure 1 on page 10. The intervals are centered around points $x = y_r$. The function a_o is assumed to consist of segments which, within each interval, might appear as indicated in the figure. Continuity at the boundaries between the intervals is assumed. We assume further that the function is almost periodic. These assumptions may be combined into the following requirements for the class of functions which we want to consider:

$$(2.13)\quad \left(\frac{da_o(x)}{dx}\right)\Big|_{x=y_r} =^3 \tfrac{1}{2}\left(a^\varepsilon(y_{r+1}) - a^\varepsilon(y_{r-1}) \right)/\varepsilon$$

$$(2.14)\quad 0^\varepsilon\left(\frac{d^2a_o(x)}{dx^2}\right)\Big|_{x=y_r} =^3 \tfrac{1}{2}\left(\alpha^\varepsilon(y_{r+1}) - \alpha^\varepsilon(y_{r-1}) \right)/\varepsilon$$

The relations (2.09) and (2.10) still hold for this class of almost periodic function, if we define

$$(2.15)\quad a^\varepsilon(y),_y\Big|_{x=y_r} \equiv \tfrac{1}{2}\left(a^\varepsilon(y_{r+1}) - a^\varepsilon(y_{r-1}) \right)/\varepsilon$$

$$(2.16)\quad \alpha^\varepsilon(y),_y\Big|_{x=y_r} \equiv \tfrac{1}{2}\left(\alpha^\varepsilon(y_{r+1}) - \alpha^\varepsilon(y_{r-1}) \right)/\varepsilon$$

For this particular class of functions the knowledge of the average value and the first moment at the discrete points y_{r+1}, y_r, and y_{r-1} allows us evaluate the difference between average value and mean value (or local value) in the interval around y_r.

We now consider not just a single function a_o, but rather a family of such functions which are characterized by the fact that the period of the (almost) parabolic section of the function decreases as well. Thus, while the average value over a period ε remains constant with decreasing ε so does the local value in the middle of such a period. Hence, for $\varepsilon \to 0$ this family of functions will lead to a function which is highly discontinuous everywhere and for which (2.12) holds. Of course, such functions are not common-place in continuum mechanics. In general, continuity of the local fields of density and velocity is assumed except for isolated jump conditions. It was not until the author had read Mandelbrot's book on "The Fractal Geometry of Nature" [1] that he was en-

couraged to deal with density and velocity distributions which have to be considered as everywhere discontinuous.

The reader who is familiar with the theory of homogenization (see [7], e. g.) may recognize a similarity between this theory and our approach.

The definitions (2.15) and (2.16) give us the possibility to treat fractal functions formally as if they were continuous. Hence, in the subsequent chapters, we will not repeat the approach which we have followed here. We will formally operate on continuous functions, evaluate their averages and first moments, calulate derivatives with respect to the center of control volumes, but always remember that such derivatives should be interpreted in the sense of (2.15) and (2.16).

At the end of this chapter, we will point out the limitations of the approximation up to third order which we have chosen. For this purpose let us investigate an approximation that is accurate up to fifth order. We first determine the moment of inertia i^ε:

$$(\text{chap..17})^* \quad i^\varepsilon(y) \equiv \frac{1}{\varepsilon}\int_{y-\frac{1}{2}\varepsilon}^{y+\frac{1}{2}\varepsilon}(x-y)^2 a_o(x)dx$$

$$=^5 0^\varepsilon a_o(y) + \frac{9}{10}(0^\varepsilon)^2\left(d^2a_o(y)/dy^2\right)$$

We now find for arbitrary values of y with fifth-order accuracy

$$(2.18)^* \quad a^\varepsilon(y) =^5 a_o(y) + \tfrac{1}{2}0^\varepsilon\left(d^2a_o(y)/dy^2\right) + {}^3/_{40}(0^\varepsilon)^2\left(d^4a_o(y)/dy^4\right)$$

and for arbitrary values of y with fourth-order accuracy

$$(2.19)^* \quad a^\varepsilon(y) =^4 a_o(y) + \frac{5}{8}\alpha^\varepsilon(y)_{,y} - \frac{1}{8}i^\varepsilon(y)_{,yy}$$

which becomes fifth-order accurate at y = 0.

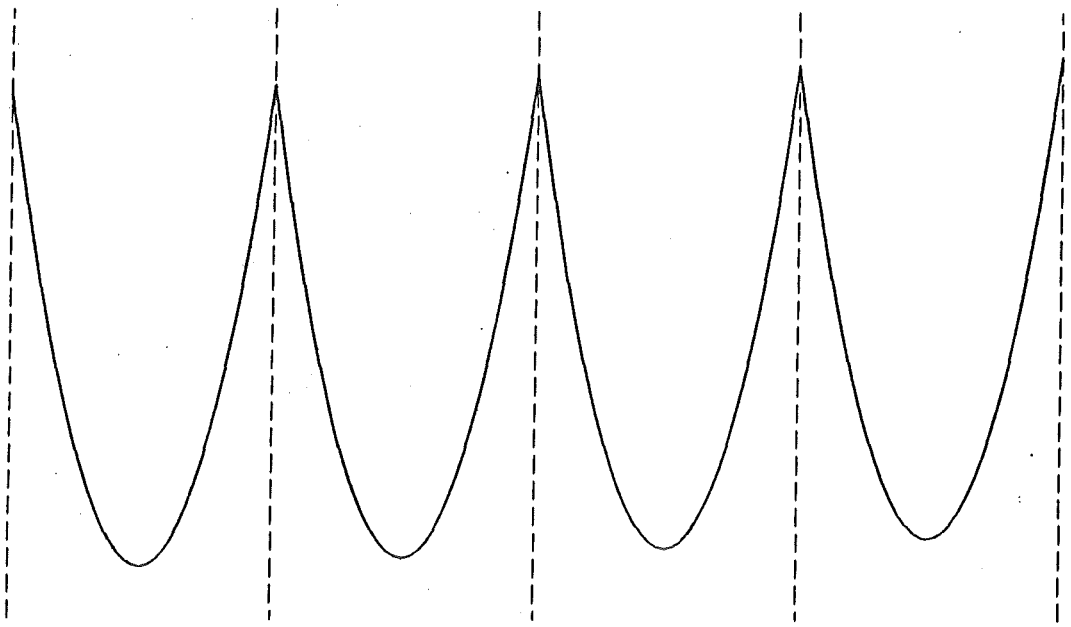

Figure 1. A piecewise parabolic function: A function of this kind represents a possible approximation of a heterogeneous function as described by (2.12).

Nevertheless, when we approach the three-dimensional problems in the next chapters, we will make only one step, so to say, beyond conventional continuum mechanics. We will consider terms up to order three only.

3. Three-dimensional functions

The same approach may be taken for three-dimensional functions of the type $\rho_o(x_i,x_j,x_k)a_o(x_i,x_j,x_k)$, where $a_o(x_i,x_j,x_k)$ is an arbitrary function and $\rho_o(x_i,x_j,x_k)$ is the "density" taken as a weighting function. For both a_o and ρ_o we assume that they can be developed into Taylor series in the neighbourhood of point (y_i,y_j,y_k). We define the quadrilateral

$$(3.01)\quad \{Q\} \equiv \{y_i-\varepsilon_i/2 \le x_i \le y_i+\varepsilon_i/2,\ y_j-\varepsilon_j/2 \le x_j \le y_j+\varepsilon_j/2,\ y_k-\varepsilon_k/2 \le x_k \le y_k+\varepsilon_k/2\}$$

We define the following quantities (using the abbreviation $dM \equiv \rho_o dV$)

$$(3.02)\quad V \equiv \iiint dV \qquad \text{volume}$$

$$(3.03)\quad M \equiv \iiint dM \qquad \text{mass}$$

$$(3.04)\quad \rho \equiv M/V \qquad \text{density}$$

$$(3.05)\quad z_i \equiv \iiint x_i\, dM/M \qquad \text{center of mass coordinate}$$

$$(3.06)\quad a \equiv \iiint a_o\, dM/M \qquad \text{average value of } a_o$$

$$(3.07)\quad \alpha_i \equiv \iiint (x_i - z_i)\, a_o\, dM/M \qquad \text{first moment of } a_o$$

where the integration is performed always over the quadrilateral $\{Q\}$.

Using Taylor series, we can show that the offset of the center of mass from the geometric midpoint of the control volume, which is defined by

$$(3.08)\quad o_i \equiv z_i - y_i$$

can be determined with third-order accuracy from(3)

$$(3.09)^*\quad \rho o_i =^3 \theta_i \rho_{o,i}$$

with

$$(3.10)^*\quad \theta_i \equiv \iiint (x_i - y_i)^2 dV/V$$

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3 Note on subscripts: The summation convention applies to all subscripts except subscripts i, j, k. Any equation formulated for subscript i holds as well for a cycled permutation of i, j, k.

Furthermore, we find:

$$(3.11)^{*}\quad \rho_o a_o =^{3} \rho a - \Sigma_{i=1..3}\ \tfrac{1}{2}\theta_i(\rho_o a_o)_{,ii}$$

$$(3.12)^{*}\quad \alpha_i =^{3} \theta_i a_{o,i}$$

Using a combination of (3.07), (3.08) and (3.09), we can write

$$(3.13)^{*}\quad \rho_o a_o =^{3} \rho a - \tfrac{1}{2}(\rho\alpha_n + \rho o_n a)_{,n}$$

where (as we conclude for the special case $a_o = 1$) the local density ρ_o is approximated by:

$$(3.14)^{*}\quad \rho_o =^{3} \rho - \Sigma_{i=1..3}\ \tfrac{1}{2}\theta_i\rho_{o,ii} =^{3} \rho - \tfrac{1}{2}(\rho o_n)_{,n}$$

We can interprete equation (3.13) and (3.14) as follows: Provided that we know the average values ρa and ρ in a quadrilateral and the behaviour of the first-order moment functions $\rho\alpha_i$ and ρo_i in the immediate neighbourhood, then we can evaluate the corresponding local values at the volumetric center with third-order accuracy. We can also determine the local value $a_o(z_i,z_j,z_k)$ at the center of mass according to:

$$(3.15)^{*}\quad a_o(z_i,z_j,z_k) =^{3} a - \tfrac{1}{2}\alpha_{n,n}$$

Here and subsequently in this paper, we often have to choose among several equally valid approximations. We always make this choice according to the following strategy:

- o avoid the explicit use of θ_n^{ε} and of second derivatives;
- o whenever a_o is a product of two functions, we choose the approximation such that the two functions are treated symmetrically unless we have specific reasons to proceed otherwise. Thus, in (3.13) we prefer
  - $\tfrac{1}{2}(\rho\alpha_i + \rho o_i a)$ over
  - $\rho\alpha_i$ or
  - $\rho o_i a$ or

- any otherwise weighted average of these approximations

which all are valid expressions for $\Theta_i \rho_{,i} a_{o,i}$.

We now like to derive an equation which will correspond to (2.10).

$$(3.16)\quad \rho_o(y_i,y_j,y_k) = 1$$

We introduce the average values over the small rectangular plate

$$(3.17)\quad \{Pi\} \equiv \{x_i = y_i,\ y_j-\varepsilon_j/2 \le x_j \le y_j+\varepsilon_j/2,\ y_k-\varepsilon_k/2 \le x_k \le y_k+\varepsilon_k/2\}$$

$$(3.18)\quad a^{Pi} \equiv \iint a_o dx_j dx_k / \iint dx_j dx_k$$

and the mean value

$$(3.19)\quad a_m^{Pi}(y_i,y_j,y_k) \equiv \tfrac{1}{2}\left(a^{Pi}(y_i+\tfrac{1}{2}\varepsilon_i,y_j,y_k) + a^{Pi}(y_i-\tfrac{1}{2}\varepsilon_i,y_j,y_k) \right)$$

Using partial differentiation with respect to y_i, we find exactly:

$$(3.20)\quad {}^{\partial}/_{\partial y_i} \iiint (x_i-y_i) a_o(x_i,x_j,x_k) dV/V$$

$$= \tfrac{1}{2}\varepsilon_i \iint a_o(y_i+\tfrac{1}{2}\varepsilon_i,x_j,x_k) dx_j dx_k / V$$

$$+ \tfrac{1}{2}\varepsilon_i \iint a_o(y_i-\tfrac{1}{2}\varepsilon_i,x_j,x_k) dx_j dx_k / V$$

$$+ \iiint {}^{\partial}/_{\partial y_i} ((x_i-y_i) a_o(x_i,x_j,x_k)) dV/V$$

$$= \tfrac{1}{2}\left(a^{Pi}(y_i+\tfrac{1}{2}\varepsilon_i,y_j,y_k) + a^{Pi}(y_i-\tfrac{1}{2}\varepsilon_i,y_j,y_k) \right) - a(y_i,y_j,y_k)$$

$$= a_m^{Pi}(y_i,y_j,y_k) - a(y_i,y_j,y_k)$$

Thus, for constant density ρ_o, this equation gives a meaningfull interpretation to the function α_i, which represents the first moment of a_o: its derivative is equal to the difference between the mean value of the average taken over the two bounding rectangles and the volume average value of a function in a quadrilateral. Equation (3.20) is the three-dimensional generalization of equation (2.10). It is exact, not an approximation.

Before we close this chapter let us state a few rules for future reference:

$$(3.21)\quad {}^{\partial}/_{\partial y_i} \iiint a_o(x_i,x_j,x_k)dV/V$$

$$= \iiint {}^{\partial}/_{\partial x_i} a_o(x_i,x_j,x_k)dV/V$$

$$= \iint a_o(y_i+\tfrac{1}{2}\varepsilon_i,x_j,x_k)dx_j dx_k/V - \iint a_o(y_i-\tfrac{1}{2}\varepsilon_i,x_j,x_k)dx_j dx_k/V$$

$$(3.22)\quad {}^{\partial}/_{\partial y_i} \iiint (x_n-y_n)a_o(x_i,x_j,x_k)dV/V$$

$$= \iiint (x_n-y_n)\,{}^{\partial}/_{\partial x_i} a_o(x_i,x_j,x_k)dV/V$$

$$= \tfrac{1}{2}\varepsilon_i \iint (a_o(y_i+\tfrac{1}{2}\varepsilon_i,x_j,x_k)+a_o(y_i-\tfrac{1}{2}\varepsilon_i,x_j,x_k)dx_j dx_k/V - \delta_{in}\iiint a_o(x_i,x_j,x_k)dV/V$$

where δ_{ni} is the Kronecker delta.

Both equations are to be read as follows: Partial differentiation with respect to y_i of the first line or partial integration of the second line with respect to x_i always results in the expression of the last line. Hence, the two first lines are equal.

4. The fundamental equations

4.1. The Transport Theorem

We will make extensive use of the transport theorem (see, e. g., [8, p.347]). According to this theorem the total derivative of an integral is

$$(4.01)\quad \frac{d}{dt}\iiint \psi_o dV = {}^{\partial}/_{\partial t}\iiint \psi_{oo} dV + \iint \psi_{oo} u_n d\Omega_n$$

where the integral on the left-hand side is taken over a control volume whose boundaries are moving with the velocity field u_i while the first integral on the right-hand side assumes the boundaries of the (same) control volume to remain fixed in space (this is indicated by the partial derivative symbol). The second integral on the right-hand side is over the whole surface of the control volume, u_n is the n-component of the surface velocity, and $d\Omega_n$ is the projection of a surface element in the direction of coordinate x_n and is positive (negative) when the inner product of the outward normal vector on this surface element with the positive direction of x_n is positive (negative). We will now replace the arbitrary function ψ_o by the product $a_o\rho_o$. Assuming continuity, (4.01) may then be reformulated as:

$$(4.02)\quad {}^{d}/_{dt}\iiint a_o dM = \iiint \left({}^{\partial}/_{\partial t}(a_o\rho_o) + (a_o\rho_o u_n)_{,n} \right) dV$$

$$= \iiint a_o \left(\rho_{o,t} + u_n \rho_{o,n} + \rho_o u_{n,n} \right) dV + \iiint \rho_o \left(a_{o,t} + u_n a_{o,n} \right) dV$$

$$= \iiint \dot{a_o} dM + \iiint a_o \left(\dot{\rho_o} + \rho_o u_{n,n} \right) dV$$

using the abbreviation

$$(4.03)\quad \dot{a_o} \equiv {}^{\partial a_o}/_{\partial t} + u_n a_{o,n}$$

At this point, it is worth mentioning that the local velocity u_i is the velocity of an arbitrary boundary in space. There is no need to assume that it is related to the motion of material.

4.2. The local mass balance

We postulate the validity of the "local mass balance":

$$(4.04)\quad \rho_{o,t} + (\rho_o u_n)_{,n} = 0$$

It is at this point where we associate a special physical meaning with the velocity field u_i Here and subsequently u_i will indicate the material velocity. Note that if (4.04) is satisfied then $^{d}/_{dt}\iiint dM$ vanishes for arbitrary control volumes as can immediately be shown from (4.02). Furthermore, the second integral in the last line of (4.02) will vanish identically. We now may write (4.02) as

(4.05) $^{d}/_{dt}\iiint a_o dM = \iiint \overset{\bullet}{a_o} dM$

4.3. The local momentum balance

We also postulate the "local momentum balance"

(4.06) $\rho_o \overset{\bullet}{u_i} = \rho_o g_{o_i}$

where g_{o_i} is the i-component of the local specific force vector. We introduce

(4.07) $g_i \equiv \iiint g_{o_i} dM/M$
(4.08) $\tau_{ni} \equiv \iiint (x_n - z_n) g_{o_i} dM/V$

We will see (in "15.3. Conventional fluid dynamics" on page 34) that the first moment of the force can be interpreted as a stress (or surface forces). We use the same notational convention as in [4]: a positive value of τ_{ni} indicates tension; the first subscript indicates the normal direction of the sides upon which surface forces are acting; the second subscript indicates the direction of surface forces.

4.4. The significance of the local balance equations

The question may arise as to the meaning of these "local" balance equations (4.04) and (4.06) in a theory which emphasizes that balance equations have a physical meaning only for finite control volumes. We can associate three different interpretations with the locality property:

1. The quantities and equations are considered to be defined locally in the strict sense as is general practice in classical continuum mechanics. This appears to be the least attractive interpretation.

2. The equations 4.04 and 4.06 might be interpreted as definition equations instead of representations of physical laws. In particular, 4.04 might be considered as the defining equation for a weighting factor ρ_o (except for a constant factor) which will satisfy

   (4.09) $\iiint a_o(\dot{\rho_o} + \rho_o u_{n,n})dV \equiv 0$

   in (4.02) identically for a given velocity field u_i and arbitrary control volumes as well as arbitrary functions a_o. Similarly, (4.06) might be interpreted simply as a substitution of g_{o_i} for $\dot{u}_i$.

3. We prefer the following interpretation: The term "local" should be interpreted as an average over control volumes which are much smaller than the dimensions ε of the control volumes we actually concern ourselves with. This interpretation fits best into the concept of fractals [1]. One dominant property of fractals is the self-similarity which means that essential features are independent of the sample size under investigation. In our case, we consider the mass and momentum balance equations together with the quantitities density, velocity, and specific force as the essential features. We require that the mass and momentum balance equations can be formulated independently of the sample size chosen. This will be in fact demonstrated in the "5. The Mass Balance" on page 18 and "8. The Momentum Balance" on page 22.

4.5. Constitutive assumptions

In order to close the system of equations which we are presently developing we would have to add constitutive assumptions for g_i and τ_{ni}. Here, we will leave these quantities undefined. Let us note, however, that we do not require the usual assumption of symmetry for τ_{ni}.

5. The Mass Balance

The balance of mass for a body is

$$(5.01)\quad \frac{d}{dt}\iiint dM = 0$$

Note, that this equation applies to a body of momentary quadrilateral shape, but with moving boundaries. We can derive the corresponding balance equation for the fixed quadrilateral control volume (this means: $V_{,t} = 0$) by applying the transport theorem (4.01) to (5.01):

$$(5.02)\quad \frac{d}{dt}\iiint \rho_o dV = {}^{\partial}/_{\partial t}\iiint dM + \iiint (\rho_o u_m)_{,m} dV$$

$$= (V\rho)_{,t} + \iiint (\rho_o u_m)_{,m} dV = 0$$

where "$_{,t}$" indicates the partial derivative with respect to t. Taking into account

$$(5.03)\quad V_{,t} = 0$$

produces

$$(5.04)\quad \rho_{,t} + \iiint (\rho_o u_m)_{,m} dV/V = 0$$

We now introduce the specific momentum v_i according to

$$(5.05)\quad v_i = \iiint u_i dM/M$$

Using (3.21), we find exactly that

$$(5.06)\quad \iiint (\rho_o u_m)_{,m} dV/V = (\rho v_m)_{,m}$$

where the subscript "$_{,m}$" indicates partial differentiation with respect to the center coordinate y_m of the quadrilateral. Thus we obtain

(5.07) $\rho,_t + (\rho v_m),_m = 0$

Note, that equation (5.07) is exact, not an approximation. Note also, that it is formally identical to the local mass balance (4.04) and, hence, preserves the self-similarity property of fractals.

6. The Balance Equation for Arbitrary Quantities

Let us assume that for some arbitrary quantity "a" we have the balance equation

(6.01) $\frac{d}{dt}\iiint a_o dM = \text{rhs}(a)$

for a control volume whose boundaries move with the material velocity u_i. The term "rhs(a)" stands for "right-hand side of the balance equation for quantity a". We can derive the balance equation for the fixed quadrilateral by means of the transport theorem (4.01):

(6.02) $\frac{d}{dt}\iiint a_o dM = {}^{\partial}/_{\partial t}\iiint a_o dM + \iiint (a_o \rho_o u_n),_n dV$

Hence,

(6.03) ${}^{\partial}/_{\partial t}\iiint a_o dM + \iiint (a_o \rho_o u_n),_n dV = \text{rhs}(a)$

For n=1...3 and m=1...3, we introduce

(6.04) $s_{nm} = \iiint (x_n - z_n)(u_m - v_m)dM/M$

Using (3.21) and (5.05), and applying third-order approximation, we find:

(6.05)* $\iiint (a_o \rho_o u_n),_n dV/V - (a\rho v_n),_n = (\iiint a_o \rho_o u_n dV/V - a\rho v_n),_n$

$$= (\iiint a_o \rho_o (u_n - v_n)dV/V),_n =^3 \tfrac{1}{2}(\rho s_{mn} a,_m + \rho \alpha_m v_n,_m),_n$$

Note, that because of

$$(6.06)^{*}\quad \rho s_{in} a_{,i} =^{3} \rho \alpha_i v_{n,i} =^{3} \theta_i a_{o,i} u_{n,i}$$

we could formulate an infinite number of approximations by choosing an arbitrarily weighted average of the two terms $\rho s_{in} a_{,i}$ and $\rho \alpha_i v_{n,i}$ above. However, as the expression on the left-hand side of the equation (6.06) are symmetrical in a_o and u_n as well as in a and v_n, we prefer to preserve this symmetry on the right-hand side also. See remarks following (3.15). Equation (6.05) allows an approximate substitution of the second term on the right-hand side of (6.02).

With (3.02) through (3.06), and (5.07) we can substitute the first term on the right-hand side of the equation (6.02):

$$(6.07)\quad \frac{1}{V}\frac{\partial}{\partial t}\iiint a_o dM = \rho a_{,t} + a\rho_{,t} = \rho a_{,t} - a(\rho v_n)_{,n}$$

Thus, we obtain finally from (6.03), (6.05), and (6.07):

$$(6.08)\quad \rho a_{,t} + \rho v_m a_{,m} + \tfrac{1}{2}(\rho s_{mn} a_{,m} + \rho \alpha_m v_{n,m})_{,n} =^{3} rhs(a)/V$$

We introduce the total derivative $a^{\bullet}$ by rewriting (6.01) as:

$$(6.09)\quad a^{\bullet} M \equiv \frac{d}{dt}\iiint a_o dM = rhs(a)$$

Thus, we now have a formal rule for replacing the total derivative $a^{\bullet}$ of a specific quantity "a" in terms of partial derivatives with respect to time and space:

$$(6.10)\quad a^{\bullet} =^{3} a_{,t} + v_m a_{,m} + \tfrac{1}{2}(\rho s_{mn} a_{,m} + \rho \alpha_m v_{n,m})_{,n}/\rho$$

We note that this rule is more complicated than the corresponding rule for <u>local</u> quantities (see 4.03) which we repeat here:

$$(6.11)\quad a_o^{\bullet} \equiv a_{o,t} + u_m a_{o,m}$$

In the case of velocity (that is: $a_o = u_i$), however, we will not use the approximation (6.08) but will instead derive an exact formulation by introducing the velocity correlation:

$$(6.12)\quad \rho e_{ni} \equiv \iiint (u_n - v_n)(u_i - v_i)dM/V$$

In terms of the velocity correlation we may write

$$(6.13)\quad \rho \overset{\bullet}{v_i} = \rho v_{i,t} + \rho v_m v_{i,m} + (\rho e_{ni})_{,n}$$

While the general form (6.10) is an approximation, equation (6.13) is exact.

7. The Balance Equation for the Center of Mass

The equation, which determines the change of the center of mass coordinates z_i may be written in two forms. The first form is derived from the transport theorem as formulated in (4.05) substituting x_i for a_o, and u_i for $\overset{\bullet}{a_o}$: (4)

$$(7.01)\quad \frac{d}{dt}\iiint x_i dM = \iiint u_i dM$$

A second form may be derived from the transport theorem as formulated in (4.01):

$$(7.02)\quad \frac{d}{dt}\iiint x_i dM = {}^{\partial}/_{\partial t}\iiint x_i dM + \iiint (x_i \rho_o u_m)_{,m} dV$$

$$= {}^{\partial}/_{\partial t}(o_i \rho V) + {}^{\partial}/_{\partial t}(y_i \rho V) + \iiint x_i (\rho_o u_m)_{,m} dV + \iiint \rho_o u_i dV$$

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4 The substitution for u_i results from (6.11) according to

$$\overset{\bullet}{x_i} = x_{i,t} + u_m x_{i,m} = 0 + u_m \delta_{im} = u_i$$

where we have made use of (3.08). Comparison with (7.01) produces

$$(7.03)\quad {}^{\partial}/_{\partial t}(o_i \rho V) + {}^{\partial}/_{\partial t}(y_i \rho V) + \iiint x_i (\rho_o u_m)_{,m} dV = 0$$

For the spatially fixed control volume {Q}, we know that

$$(7.04)\quad y_{i,t} = 0$$
$$(7.05)\quad V_{,t} = 0$$

If we apply the continuity equation in the form (5.04), we obtain:

$$(7.06)\quad \rho o_{i,t} + \iiint (x_i - z_i)(\rho_o u_m)_{,m} dV/V = 0$$

Using (3.21) and (3.22), we obtain exactly

$$(7.07)\quad \rho o_{i,t} + \rho v_m o_{i,m} + (\rho s_{im})_{,m} = 0$$

where we have used the relationship

$$(7.08)\quad \iiint (x_i - z_i) u_m dM = \iiint (x_i - y_i) u_m dM - o_i v_m M$$

8. The Momentum Balance

For arbitrary bodies, the balance of momentum may be written as:

$$(8.01)\quad {}^{d}/_{dt} \iiint u_i \rho_o dV = \iiint g o_i dM$$

Using previously the introduced definitions (4.08), (4.10), (5.05), and the mass balance (5.07), we obtain

$$(8.03)\quad \rho v_i^{\bullet} = \rho g_i$$

which is formally identical to the "local" momentum balance (4.06) and, thus, preserves the self-similarity of the momentum balance equation. With (6.13), we obtain finally

$$(8.04)\quad \rho v_{i,t} + \rho v_m v_{i,m} + (\rho e_{im})_{,m} = \rho g_i$$

Note, that this equation is exact, not an approximation.

9. The Balance of the First Spatial Moment of Momentum

Using the transport theorem, the balance of first spatial moment of momentum with respect to the center of mass may be written as:

$$(9.01)\quad {}^{d}/_{dt} \iiint (x_n - z_n)(u_i - v_i)\rho_o dV$$

$$= \iiint (x_n - z_n)(\overset{\bullet}{u}_i - \overset{\bullet}{v}_i)dM + \iiint (u_n - v_n)(u_i - v_i)dM$$

$$= \iiint (x_n - z_n) g_{o_i} dM + \iiint (u_n - v_n)(u_i - v_i)dM$$

$$= (\tau_{ni} + \rho e_{ni})V$$

Here we have used previously defined substitutions and rules. In short, we can write

$$(9.02)\quad \rho \overset{\bullet}{s}_{ni} = \tau_{ni} + \rho e_{ni}$$

Using (6.10) and the correspondence

$$a_o \rightarrow x_n u_i$$
$$a \rightarrow \iiint (x_n - z_n)(u_i - v_i)dM/M = s_{ni}$$
$$\alpha_n \rightarrow 0$$

Neglecting terms that are less accurate than of order three in ε, the final formulation is:

$$(9.03)\quad \rho s_{ni,t} + \rho v_m s_{mi,n} \simeq \tau_{ni} + \rho e_{ni}$$

10. Velocity Correlation, Kinetic Energy, and Hydrodynamic Pressure

For the velocity correlation which we introduced in (6.12)

$$(10.01)\quad \rho e_{ni} = \iiint (u_n - v_n)(u_i - v_i)\, dM/V$$

Third order approximation produces the following result:

$$(10.02)^{*}\quad \rho e_{ni} =^{3} \rho\theta_{l} u_{n,l} u_{i,l}$$

$$=^{3} \tfrac{1}{2}\rho(v_{n,l} s_{li} + s_{ln} v_{i,l})$$

However, we will modify this approximation to some extent. The reason is that the expression

$$(10.03)\quad p \equiv {}^{1}/_{3}\rho e_{mm}$$

(which represents two thirds of the kinetic energy of the control volume relative to the moving center of mass) must be non-negative. This requirement is not necessarily satisfied by the approximation (10.01). We propose the following formulation:

$$(10.04)\quad \rho e_{ni} =^{3} (\rho e_{ni})_{approximate} - ({}^{1}/_{3}\delta_{ni}(\rho e_{mm})_{approximate} - p)$$

Thus, (10.02) becomes

$$(10.05)\quad \rho e_{ni} =^{3} \tfrac{1}{2}\rho(v_{n,l} s_{li} + s_{ln} v_{i,l} - {}^{2}/_{3}\delta_{ni} s_{ll} v_{m,m}) - p\delta_{ni}$$

As we will see later in "15.2. Kinetic theory of gases" on page 33, the expression defined in (10.03) is properly interpreted as the pressure in the fluid, or to be more precise: the hydrodynamic pressure if we want to allow for distinguishing it from a thermodynamic pressure.

With this result, the momentum balance (10.04) now reads

$$(10.06)\quad \rho v_{i,t} + \rho v_m v_{i,m} = \rho g_i - p_{,i} - \tfrac{1}{2}\rho(v_{m,l}s_{li} + s_{lm}v_{i,l} - {}^2/_3\delta_{mi}s_{ll}v_{n,n})_{,m}$$

and the balance for the moment of momentum (9.03) becomes

$$(10.07)\quad \rho s_{ni,t} + \rho v_m s_{mi,n} \simeq \tau_{ni} - \delta_{ni} + \tfrac{1}{2}\rho(v_{n,l}s_{li} + s_{ln}v_{i,l} - {}^2/_3\delta_{ni}s_{ll}v_{m,m})$$

11. The Balance of Micro-Inertia

The theory of micro-fluids [4] has introduced the concept of micro-inertia. This property represents the local moment of inertia of arbitrarily small control volumes. Using our notation, we can define the micro-inertia i_{ni} as:

$$(11.01)\quad \rho i_{ni} \equiv \iiint (x_i - z_i)(x_n - z_n)\,dM/V$$

The values ρi_{ni} represent the terms of the so-called Euler tensor (see [8]). Note that $i_{ni} = i_{in}$. An approximation of ρ_o by Taylor series gives

$$(11.02)^*\quad \rho i_{ii} =^3 \rho\theta_i$$

$$(11.03)^*\quad \rho i_{ij} =^3 0$$

We note that in the context of a consistent third-order approximation the balance of micro-inertia is trivial. The values of i_{in} would remain constant if we neglect terms of higher than third order. However, we can easily add the balance equation for the micro-inertia to the equations derived previously as we do not need any approximations. In fact, the various versions of micro-fluid theories in the literature include a balance equation for the micro-inertia. The transport theorem applied to the definition of the micro-inertia (11.01) results in:

$$(11.04)\quad \rho \overset{\bullet}{i}_{ni} = \iiint (x_i - z_i)(u_n - v_n)\,dM/V + \iiint (u_i - v_i)(x_n - z_n)\,dM/V$$

or (without any approximation)

$$(11.05)\quad \overset{\bullet}{i}_{ni} = s_{ni} + s_{in}$$

However, (11.05) is of little help unless we express it in terms of partial derivatives with respect to space and time. But we know that the formulation (6.10) can provide third order accuracy only. Hence, in order to maintain consistency in the order of approximation we have no other choice but to consider the balance of micro-inertia as irrelevant in our theory.

12. The Boundary Conditions

We will deal with boundary conditions of the type "rigid wall" only. The problem is that the average quantities which appear in our differential equations are not defined on the boundary. The points nearest to the boundary where these quantities are still defined are at a location with

$$(12.01)\quad x_\omega = \tfrac{1}{2}\varepsilon_\omega$$

where we assume subscript ω to indicate the direction normal to the boundary and pointing into the fluid domain, subscript ν to indicate the directions parallel to the boundary, and (without loss of generality) the boundary to be located at $x_\omega = 0$.

In order to include points in the range $0 \le x_\omega < \tfrac{1}{2}\varepsilon_o$ properly in our analysis we have to provide suitable definitions for volume averages and first moments of the arbitrary functions a_o in this domain. We achieve this by extending the function a_o into the range $-\tfrac{1}{2}\varepsilon_o \le x_\omega < 0$. We will then define the volume averages and the first moments on the basis of such extended functions. With respect to the volume averages we extend the functions by their mirror images relative to the boundary, using equal signs for corresponding coordinates on both sides $x_\omega = 0$:

$$(12.02)\quad \rho_o(-x_\omega) = \rho_o(x_\omega)$$
$$(12.03)\quad u_\omega(-x_\omega) = u_\omega(x_\omega)$$

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$$(12.04)\quad u_\nu(-x_\omega) = u_\nu(x_\omega)$$
$$(12.05)\quad a_o(-x_\omega) = a_o(x_\omega)$$

With respect to the first moments, however, we extend the functions by their mirror image applying the oposite sign on the other side of the boundary. The purpose of this different treatment is to guarantee that a volume integral taken over half the range $0 \leq x_\omega < \varepsilon_\omega$ would be identical to the corresponding volume integral on the other side of the boundary; and the same argument applies to the first order moments.

We can now apply (2.09) directly to correlate average values with local values on the boundary via derivatives of the first order moments, by introducing surface average values on the boundaries according to:

$$(12.06)\quad \rho^\circ = \lim_{\varepsilon_\omega \to 0} \int\int \rho_\circ d\Omega_\omega / \int\int d\Omega_\omega$$

$$(12.07)\quad a^\circ \rho^\circ = \lim_{\varepsilon_\omega \to 0} \int\int a_\circ \rho_\circ d\Omega_\omega / \int\int d\Omega_\omega$$

where $d\Omega_\omega$ is a surface element of this surface. Furthermore, the symmetry assumptions (12.02) through (12.05) provide suitable boundary conditions for the average values. Thus, the complete set of boundary conditions at rigid walls becomes:

$$(12.08)\quad \rho_{,\omega} = 0$$
$$(12.09)\quad \rho o_{\omega,\omega} = \tfrac{1}{2}(\rho - \rho^\circ)$$
$$(12.10)\quad \rho o_{\nu,\omega} = 0$$
$$(12.11)\quad v_{i,\omega} = 0$$
$$(12.12)\quad s_{\omega i,\omega} = \tfrac{1}{2}(v_i - v_i^\circ)$$
$$(12.13)\quad s_{\nu i,\omega} = 0$$
$$(12.14)\quad e_{\nu i,\omega} = 0$$
$$(12.15)\quad a_{,\omega} = 0$$
$$(12.16)\quad \alpha_{\omega,\omega} = \tfrac{1}{2}(a - a^\circ)$$
$$(12.17)\quad \alpha_{\nu,\omega} = 0$$

where ρ°, a°, and v_i° are prescribed on the boundary.

Note, that from a formal point of view we could prescribe non zero values on the right-hand side of all above boundary conditions. It is a matter of phys-

ical realisation possibilities whether such non-zero values are meaningful or not.

13. Summary of Equations

Let us now summarize all equations obtained so far. We will note that the system of equations is closed provided that we add

- o boundary conditions for boundaries which are not of the type wall but which are submerged in the fluid. These boundary conditions will be of Dirichlet type depending on the characteristics of the system of equations in the neighbourhood of the boundary (elliptic or hyperbolic with inflowing characteristics).
- o constitutive equations for the specific body force vector g_i and the stress tensor τ_{ni}. In this paper, we do not investigate the restrictions imposed upon these tensors from energetic and thermodynamic considerations. We refer to the literature on microfluids in this respect, e.g. [4].

It is important to remember that all these equations apply to control volume of a predefined shape and size (a quadrilateral with side lengths ε_i, ε_j, ε_k) which has to be identical everywhere throughout the domain considered and for all times. Furthermore, the initial conditions from which an analysis can start have to be determined for these control volumes.

We now list the differential equations for all quantities together with their boundary conditions on walls. The subscripts ω and ν indicate the directions normal and parallel to the wall, respectively. The the superscript $^\circ$ indicates the surface average of the local value distribution on the boundary.

Note: Here and subsequently we will not use the $\stackrel{3}{=}$-symbol any longer to indicate that an equation is of approximate nature only. We will rather use the simple =-symbol with the implicit understanding that it means an approximation.

Density:

$$\rho_{,t} + (\rho v_m)_{,m} = 0$$
$$\rho_{,\omega} = 0$$

Offset of the centre of mass inside the control volumes:

$$\rho o_{i,t} + \rho v_m o_{i,m} = -(\rho s_{mi})_{,m}$$
$$\rho o_{\omega,\omega} = \rho^{\circ} - \rho$$
$$o_{\nu,\omega} = 0$$

Specific momentum:

$$\rho v_{i,t} + \rho v_m v_{i,m} = \rho g_i - p_{,i} - \tfrac{1}{2}\rho(v_{m,l}s_{li} + s_{lm}v_{i,l} - {}^{2}/_{3}\delta_{mi}s_{ll}v_{n,n})_{,m}$$
$$\rho v_{i,t} + \rho v_m v_{i,m} + (\rho e_{im})_{,m} = \rho g_i$$
$$v_{i,\omega} = 0$$

Specific first moment of momentum:

$$\rho s_{ni,t} + \rho v_m s_{mi,n} = \tau_{ni} - \delta_{ni} + \tfrac{1}{2}\rho(v_{n,l}s_{li} + s_{ln}v_{i,l} - {}^{2}/_{3}\delta_{ni}s_{ll}v_{m,m})$$
$$s_{\omega i,\omega} = v_i^{\circ} - v_i$$
$$s_{\nu i,\omega} = 0$$

Specific value of an arbitrary quantity:

$$\rho a_{,t} + \rho v_m a_{,m} + \tfrac{1}{2}(\rho s_{mn}a_{,m} + \rho\alpha_m v_{n,m})_{,n} = \mathrm{rhs}(a)\rho$$
$$a_{,\omega} = 0$$

Specific value of the first moment of an arbitrary quantity:

$$\rho\alpha_{i,t} + \rho v_m \alpha_{i,m} = \mathrm{rhs}(\alpha_i)\rho$$
$$\alpha_{\omega,\omega} = a^{\circ} - a$$
$$\alpha_{\nu,\omega} = 0$$

14. Discussion of Results

Let us now discuss the results of this theory in comparison with conventional continuum mechanics and the theory of microfluids. The difference is best illustrated by the balance equations for the momentum and the first moment of momentum.

14.1. Comparison with Conventional Continuum Mechanics

Our momentum equation (10.04) reads

$$(14.01)\quad \rho v_{i,t} + \rho v_m v_{i,m} + (\rho e_{im})_{,m} = \rho g_i$$

This formulation coincides with the conventional momentum balance equation

$$(14.02)\quad \rho v_{i,t} + \rho v_m v_{i,m} = \rho g_i + \tau_{mi,m}$$

if we assume

$$(14.03)\quad \tau_{ni} = -\rho e_{ni}$$

Later, in "15.3. Conventional fluid dynamics" on page 34, we will see that this assumption is equivalent to assuming

$$(14.04)\quad s_{ni} = 0$$

Remembering that ρe_{ni} is a measure of the local velocity correlation and, hence, an indication of the amount of turbulence, we expect that the formulation (14.01) will allow to describe the effects of turbulence more properly than the conventional approaches.

14.2. Comparison with microfluid theory

We use [4] as a reference and represent the equations in a notation that is adjusted to this paper. The balance equations for momentum and the first moment of momentum for the microfluid read

$$(14.05)\quad \rho v_{i,t} + \rho v_m v_{i,m} = \rho g_i + \tau_{ni,n}$$

$$(14.06)\quad \rho s_{ni,t} + \rho v_m s_{ni,m} = \rho \gamma_{ni} + \tau_{ni} - \mu^{*}_{\ell ni,\ell} - \tau^{*}_{ni}$$

where ρ, v_i, s_{ni}, g_i, and τ_{ni} have the same meaning as in our theory and

- γ_{ni} is the body force couple
- $\overset{*}{\tau}_{ni}$ is the micro-stress average;
- $\overset{*}{\mu}_{\ell ni}$ is the stress couple.

We compare these equations with our balance equations for the momentum (8.04) and its first moment (9.03), and with the conventional fluid mechanics (14.02). We note that the momentum balance equation for the microfluid is identical to the conventional formulation. Any exchange of momentum can occur only due to transport with velocity v_n. The exchange term $(\rho e_{im})_{,m}$ on the left-hand side of equation (14.01) would have to be represented in the microfluid formulation in terms of constitutive equations included in the stress gradient on the right-hand side. This approach, however, pays little credit to the fact that the exchange terms result from mere kinematics and do not have the physical background of a force. The difference between the two formulations of the balance equation for the first moment of momentum can be explained as follows:

- Agreement can then be achieved if we postulate, e.g.:

$$\gamma_{ni} - \mu_{\ell ni,\ell} - \overset{*}{\tau}_{ni} = \rho e_{ni}$$

The three quantities on the left-hand side are assumed to be governed by independent constitutive relations in the micro-fluid theory, while our theory of fractal fluides only those constitutive relations (specific body force g_i and stress tensor τ_{ni}) which are also known in the conventional theories.

A mayor difference between Eringen's work and our theory appears if one studies Eringen's work in more detail: He emphasizes the micro-inertia while in our paper emphasis is laid upon the offset of the center of mass from the volumetric center of the control volume. As the center of mass offset is a quantity which is proportional to ε^2 while the inertia tensor is of the order ε^4 only, we consider the center of mass treatment to be more important. A signif-

icant deficiency is certainly that the microfluid theory does not offer a way to treat the effect of the velocity correlation upon the momentum balance.

14.3. Boundary conditions

A significant difference between this theory and both the microfluids and conventional fluid theories is the formulation of the boundary conditions. The new theory allows to prescribe local densities (a liquid film or a gaseous film for two-phase mixtures, e. g.) to be prescribed on a wall without imposing a Dirichlet boundary condition for the average density. The same argument applies to velocity.

15. Special Aspects

So far, we have not made any assumptions regarding the constitutive relations for the specific force g_i and its first moment τ_{ni}. Any special selection for these quantities will result in a special behaviour of the fluid. We will now make some special choices such as to emphasize particular aspects of this theory.

15.1. One-dimensional pressure waves through stagnant fluids

We assume

(15.1.01) $\rho \simeq \rho_o \simeq \text{const}$

(15.1.02) $v_i \simeq 0$

(15.1.03) $e_{ni} \simeq 0$

(15.1.04) $g_i \simeq 0$

(15.1.05) $\tau_{12} = \tau_{13} = \tau_{2n} = \tau_{3n} \simeq 0$

We evaluate τ_{11} from the usual acoustic approximation for pressure waves travelling in the direction of coordinate axis 1 by assuming for the local force field:

(15.1.06) $g_{o_1} = -c^2 \rho_{o,1}/\rho_o$

Thus,

(15.1.07) $\tau_{11} = -c^2 \iiint (x_1 - z_1)\rho_o{}_{,1} dV/V = -c^2 \rho o_{1,1}$

Then, the remaining essential equations are

(15.1.08) $o_{1,t} + s_{11,1} = 0$
(15.1.09) $s_{11,t} = -c^2 o_{1,1}$

which may be combined into the single equation:

(15.1.10) $o_{1,t} + c^2 o_{1,11} = 0$

This partial differential equation describes the propagation of a wave through the fluid with wave velocity c. We note that the equations derived from our theory allow the phenomenon of pressure wave propagation to be treated separately from gross fluid motion.

15.2. Kinetic theory of gases

We assume

(15.2.11) $\rho = \rho_o = \text{const}$
(15.2.12) $v_i = 0$
(15.2.13) $s_{in} = 0$
(15.2.14) $e_{ij} = 0$
(15.2.15) $e_{ii} = e_{nn}/3$

With these assumptions we obtain from the balance of the first moment of momentum

(15.2.16) $\tau_{ii} = 2/3 * (e_{nn}/2)$

which indicates that all diagonal terms of the stress tensor are that the pressure effect on the walls of a small control volume equals the same and their value is equal to two thirds of the internal kinetic energy. This result

agrees with the one obtained from the kinetic theory of simple gases [9]. This result justifies the substitution of the hydrodynamic pressure introduced in "10. Velocity Correlation, Kinetic Energy, and Hydrodynamic Pressure" on page 24.

15.3. Conventional fluid dynamics

We now assume

$$(15.3.17)\quad s_{ni} = 0$$

From the balance of the first moment of momentum we obtain immediately

$$(15.3.18)\quad \tau_{ni} = -\rho e_{ni}$$

which we can insert into the momentum balance resulting in

$$(15.3.19)\quad v_{i,t} + v_m v_{i,m} = \rho g_i + \tau_{ni,n}$$

which is the classical formulation provided that

1. the specific force field g_i comprises the body forces only,
2. the first force moment τ_{ni} is interpreted as the classical stress component.

Note, that due to (15.3.18) and because of the inherent symmetry of the velocity correlation tensor e_{ni}, the stress tensor has to be symmetrical which is consistent with conventional continuum mechanics (Cauchy's second theorem).

15.4. Fractal Newton-Stokes fluids

Let us now consider the special case

$$(15.4.20)\quad g_i = 0$$

$$(15.4.21)\quad \tau_{ni} = -\delta_{ni}p + \rho\nu(v_{i,n}+v_{n,i}) - {}^{2}/_{3}\rho\nu\&\#,d_{ni}v_{m,m}$$

(15.4.21) is the usual formulation for Newton-Stokes fluids, see, for example, [9, page ???].

(15.4.22) $\rho_{,t} + (\rho v_m)_{,m} = 0$

(15.4.23) $\rho o_{i,t} + \rho v_m o_{i,m} - (\rho s_{mi})_{,m}$

(15.4.24) $\rho v_{i,t} + \rho v_m v_{i,m} = - p_{,i} - \frac{1}{2}\rho(v_{m,1}(2\nu+s_{1i}) + (2\nu+s_{lm})v_{i,1} - {}^2/_3\delta_{mi}(6\nu+s_{11})v_{n,n})_{,m}$

prüfen, ob z.B. 2ν

(15.4.25) $\rho s_{ni,t} + \rho v_m s_{mi,n} = \frac{1}{2}\rho(v_{n,1}(2\nu+s_{1i}) + (2\nu+s_{ln})v_{i,1} - {}^2/_3\delta_{ni}(6\nu+s_{11})v_{m,m})$

For o_i and s_{ni}, these equations degenerate into the balance equations of conventional fluid dynamics.

$H. = -pv_{n,n} + \rho\nu v_{n,m}v_{n,m}$

15.5. Void pattern description

An essential characteristic of two-phase flow is the so-called void pattern. The void pattern describes how the gaseous phase and the liquid phase (subscript 1) are mixed. Using (3.20), and with ρ_o instead of a_o, we can characterize such void patterns by the first moment of the density distribution, e.g., as:

- bubble flow

  The density at the boundaries of the control volumes (and, hence, the mean density) in all directions is equal to the liquid density:

  $(\rho o_i)_{,i} = (\rho o_j)_{,j} = (\rho o_k)_{,k} = \rho_1 - \rho$

- elongated bubbles in direction i

  The density at the boundaries of the control volumes (and, hence, the mean density) in directions j and k is equal to the liquid density. In direction i, the mean density equals the average density:

  $(\rho o_i)_{,i} = 0$

$$(\rho o_j)_{,j} = (\rho o_k)_{,k} = \rho_l - \rho$$

- o slug flow with slugs normal to direction i

  The density at the boundaries of the control volumes (and, hence, the mean density) in direction i is equal to the liquid density. In direction j and k, the mean density equals the average density:

$$(\rho o_j)_{,j} = (\rho o_k)_{,k} = 0$$
$$(\rho o_i)_{,i} = \rho_l - \rho$$

- o homogeneous flow

  The density at the boundaries of the control volumes (and, hence, the mean density) in all directions equals the average density:

$$(\rho o_i)_{,i} = (\rho o_j)_{,j} = (\rho o_k)_{,k} = 0$$

16. Concluding remarks

16.1. Work to be done

The author is fully aware of the still preliminary nature of his investigations. Particular areas requiring more intense studies are

- o the energy balance, in particular the conduction and production of heat;
- o thermodynamics aspects, in particular the second law of thermodynamics and its influence on the allowable formulations of the constitutive equations;
- o the question of objectivity (independence from the observer coordinate system);
- o the existence and uniqueness of solutions of this new set of equations in combination with various boundary conditions;
- o the influence of different ways of expressing approximations where the truncation of Taylor series does not provide for a unique formulation;
- o different constitutive assumptions, e.g., elastic stresses;
- o relaxation phenomena;

- last not least, practical application to simple situations (such as the flow past a backward facing step).

It is fair to state that in the past years fluid mechanics theoreticians have concentrated their efforts on various numerical treatments of the conventional fluid mechanics equations. On the other hand, the fundamental set of equations has been given too little attention. The variety of new formulations based on Eringen's micro-fluid theory or the approach taken in this paper should be taken as an initiative to pay more attention to the mathematical and physical foundations of fluid mechanics.

16.2. Acknowledgements

The author wishes to acknowledge the support and advice he received when developing this theory. First of all -in a chronological sense- it is due to a presentation of Prof. Dr. H. J. Wirz, Bruxelles, on Nov. 11, 1977, "Über die Probleme der physikalischen und numerischen Stabilität sowie der Sachgemäßheit bei einem aktuellen System von Differentialgleichungen der Mehrphasendynamik" that the author's interest in the subject was raised. Prof. Eringen's work on micro-fluids (and various derivations from this theory) was an indispensible source of information and encouragement. Dr. Malmberg's expertise in continuum mechanics and his continuous interest in this work contribute significantly to the present achievement. The author also wishes to acknowledge the discussions with his colleagues Dr. Benner, Dr. Grötzbach, Dr. Kleiser, and Dr. Schumann. Much credit is to be given to Prof. Smidt for providing the opportunity to pursue this work in his institute. He also formulated the guideline: What is required is not just another version of numerical treatment of turbulence or two-phase flow but a really new approach. We hope to have followed this guideline with some success.

17. References

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Proofs related to Chapter 2.

Time required for proof on a Siemens 7890 computer: 5 sec.

```
OFF ECHO;
ERRORCOUNT:=0$
OPERATOR A0;
FOR ALL X LET A0(X)=
 A00+A01*X+A02*X**X/2+A03*X*X*X/6+A04*X*X*X*X/24+A05*X*X*X*X*X/120;

COMMENT PROOF OF EQUATION (02.06);

THETA:=SUB(H=EPS,
         (SUB(X=Y+H/2,INT((X-Y)**2,X)) -
                SUB(X=Y-H/2,INT((X-Y)**2,X)))/H)$
ERROR:=THETA-EPS*EPS/12$
IF ERROR EQUAL 0 THEN WRITE "EQUATION (02.06) IS CORRECT"
ELSE ERRORCOUNT:=ERRORCOUNT+1;

AVERAGE:=SUB(H=EPS,(SUB(X=Y+H/2,INT(A0(X),X))-
                    SUB(X=Y-H/2,INT(A0(X),X)))/H)$
ALPHA  :=SUB(H=EPS,(SUB(X=Y+H/2,INT((X-Y)*A0(X),X))-
                    SUB(X=Y-H/2,INT((X-Y)*A0(X),X)))/H)$
INERT  :=SUB(H=EPS,(SUB(X=Y+H/2,INT((X-Y)**2*A0(X),X))-
                    SUB(X=Y-H/2,INT((X-Y)**2*A0(X),X)))/H)$

COMMENT PROOF OF EQUATION (02.03);

LET EPS*EPS*EPS*EPS=0;
LET EPS*EPS*EPS*  Y=0;
LET EPS*EPS*  Y*  Y=0;
LET EPS*  Y*  Y*  Y=0;
LET   Y*  Y*  Y*  Y=0;
ERROR:=  A0(Y+EPS/2)-A0(Y)
              -DF(A0(Y),Y)*EPS/2
                   -DF(A0(Y),Y,2)*EPS*EPS/8
                        -DF(A0(Y),Y,3)*EPS*EPS*EPS/48$
IF ERROR EQUAL 0 THEN
WRITE "EQUATION (02.03) IS CORRECT (PART 1)"
ELSE ERRORCOUNT:=ERRORCOUNT+1;

ERROR:=  A0(Y-EPS/2)-A0(Y)
              +DF(A0(Y),Y)*EPS/2
                   -DF(A0(Y),Y,2)*EPS*EPS/8
                        +DF(A0(Y),Y,3)*EPS*EPS*EPS/48$
IF ERROR EQUAL 0 THEN
WRITE "EQUATION (02.03) IS CORRECT (PART 2)"
ELSE ERRORCOUNT:=ERRORCOUNT+1;
COMMENT
```

CONTINUED ON NEXT PAGE

```
:FEE.

CONTINUATION                              END;
 COMMENT PROOF OF EQUATION (02.04);

 ERROR:=(A0(Y+EPS/2)+A0(Y-EPS/2))/2-A0(Y)-DF(A0(Y),Y,2)*EPS*EPS/8$
 IF ERROR EQUAL 0 THEN WRITE "EQUATION (02.04) IS CORRECT"
 ELSE ERRORCOUNT:=ERRORCOUNT+1;

 COMMENT PROOF OF EQUATION (02.07) (THIRD ORDER ACCURACY);

 AVERAGE:=SUB(H=EPS,(SUB(X=Y+H/2,INT(A0(X),X))-
                     SUB(X=Y-H/2,INT(A0(X),X)))/H)$
 ERROR:=AVERAGE-A0(Y)-THETA*DF(A0(Y),Y,2)/2$
 IF ERROR EQUAL 0 THEN
 WRITE "EQUATION (02.07) IS CORRECT (THIRD ORDER)"
 ELSE ERRORCOUNT:=ERRORCOUNT+1;

 COMMENT PROOF OF EQUATION (02.08) (THIRD ORDER ACCURACY);

 ALPHA  :=SUB(H=EPS,(SUB(X=Y+H/2,INT((X-Y)*A0(X),X))-
                     SUB(X=Y-H/2,INT((X-Y)*A0(X),X)))/H)$
 ERROR:=ALPHA-THETA*DF(A0(Y),Y)$
 IF ERROR EQUAL 0 THEN
 WRITE "EQUATION (02.08) IS CORRECT (THIRD ORDER)"
 ELSE ERRORCOUNT:=ERRORCOUNT+1;

 COMMENT PROOF OF EQUATION (02.09) ;
 COMMENT      NOTE THAT WE PROVE THIS ONLY FOR Y=0;
 COMMENT      WHICH MEANS SECOND ORDER ACCURACY;
 COMMENT      THE ARGUMENTS DISCUSSED IN THE PAPER;
 COMMENT      JUSTIFY THAT THIS REPRESENTS THIRD ORDERACCURACY;
 ERROR:=SUB(Y=0,AVERAGE-A0(Y)-DF(ALPHA,Y)/2)$
 IF ERROR EQUAL 0 THEN WRITE "EQUATION (02.09) IS CORRECT"
 ELSE ERRORCOUNT:=ERRORCOUNT+1;

 COMMENT PROOF OF EQUATION (02.07) (FIFTH ORDER ACCURACY);

 CLEAR EPS*EPS*EPS*EPS;   LET EPS*EPS*EPS*EPS*EPS*EPS=0;
 CLEAR EPS*EPS*EPS* Y ;   LET EPS*EPS*EPS*EPS*EPS* Y =0;
 CLEAR EPS*EPS* Y * Y ;   LET EPS*EPS*EPS*EPS* Y * Y =0;
 CLEAR EPS* Y * Y * Y ;   LET EPS*EPS*EPS* Y * Y * Y =0;
 CLEAR  Y * Y * Y * Y ;   LET EPS*EPS* Y * Y * Y * Y =0;
                          LET EPS* Y * Y * Y * Y * Y =0;
                          LET Y * Y * Y * Y * Y * Y =0;
 AVERAGE:=SUB(H=EPS,(SUB(X=Y+H/2,INT(A0(X),X))-
                     SUB(X=Y-H/2,INT(A0(X),X)))/H)$
 ERROR:=     AVERAGE-A0(Y)-THETA*DF(A0(Y),Y,2)/2
                    -(3/40)*THETA**2*DF(A0(Y),Y,4)$
 IF ERROR EQUAL 0 THEN
 WRITE "EQUATION (02.07) IS CORRECT (FIFTH ORDER)"
 ELSE ERRORCOUNT:=ERRORCOUNT+1;

 COMMENT
```

CONTINUED ON NEXT PAGE

```
CONTINUATION                                END;
 COMMENT PROOF OF EQUATION (02.08) (FIFTH ORDER ACCURACY);

 ALPHA  :=SUB(H=EPS,(SUB(X=Y+H/2,INT((X-Y)*A0(X),X))-
                     SUB(X=Y-H/2,INT((X-Y)*A0(X),X)))/H)$
 ERROR:=
 ALPHA-THETA*DF(A0(Y),Y)-3/10*THETA*THETA*DF(A0(Y),Y,3)$
 IF ERROR EQUAL 0 THEN
 WRITE "EQUATION (02.08) IS CORRECT (FIFTH ORDER)"
 ELSE ERRORCOUNT:=ERRORCOUNT+1;

 COMMENT PROOF OF EQUATION (02.17) ;

 CLEAR EPS*EPS*EPS*EPS*EPS*EPS;
 CLEAR EPS*EPS*EPS*EPS*EPS* Y ;
 CLEAR EPS*EPS*EPS*EPS* Y * Y ;
 CLEAR EPS*EPS*EPS* Y * Y * Y ;
 CLEAR EPS*EPS* Y * Y * Y * Y ;
 CLEAR EPS* Y * Y * Y * Y * Y ;
 CLEAR  Y * Y * Y * Y * Y * Y ;
 CLEAR EPS*EPS*EPS*EPS;
 LET   EPS*EPS*EPS*EPS*EPS*EPS=0;
 ERROR:=INERT-THETA*A0(Y)-9/10*THETA*THETA*DF(A0(Y),Y,2)$
 IF ERROR EQUAL 0 THEN WRITE "EQUATION (02.17) IS CORRECT"
 ELSE ERRORCOUNT:=ERRORCOUNT+1;

 COMMENT PROOF OF EQUATION (02.18) ;

 ERROR:=AVERAGE-A0(Y)
 -THETA*DF(A0(Y),Y,2)/2-THETA*THETA*DF(A0(Y),Y,4)*3/40$
 IF ERROR EQUAL 0 THEN
 WRITE "EQUATION (02.18) IS CORRECT"
 ELSE ERRORCOUNT:=ERRORCOUNT+1;

 COMMENT PROOF OF EQUATION (02.19) ;

 COMMENT CLEAR Y*EPS*EPS*Y*Y;
 COMMENT CLEAR Y*EPS*EPS*EPS*EPS;
 ERROR:=AVERAGE-A0(Y)-DF(ALPHA,Y)*5/8+DF(INERT,Y,2)/8$
 ERROR:=SUB(Y=0,ERROR)$
 IF ERROR EQUAL 0 THEN WRITE "EQUATION (02.19) IS CORRECT"
 ELSE ERRORCOUNT:=ERRORCOUNT+1;

 IF ERRORCOUNT EQUAL 0 THEN WRITE "ALL PROOFS WERE SUCCESSFULL"
 ELSE WRITE ERRORCOUNT," PROOFS FAILED";
 ;END;
```

Macros used for all subsequent proofs

In all subsequent proofs a set of macros called STANDARD is included. These macros are listed in this paragraph.

```
OFF ECHO;
COMMENT DECLARATION AND ASSIGMENT OF BASIC VALUES;

ARRAY Y(3),EPS(3),THETA(3),X(3),WWWW(3),WWW(3);
 X(1):=X1$ X(2):=X2$ X(3):=X3$
 Y(1):=Y1$ Y(2):=Y2$ Y(3):=Y3$
 EPS(1):=EPS1$ EPS(2):=EPS2$ EPS(3):=EPS3$
 WWWW(1):=WWWW1$ WWWW(2):=WWWW2$ WWWW(3):=WWWW3$
 WWW(1):=WWW1$ WWW(2):=WWW2$ WWW(3):=WWW3$
THETA(1):=THETA1$ THETA(2):=THETA2$ THETA(3):=THETA3$
 FOR N:=1:3 DO LET EPS(N)**2=12*THETA(N);

COMMENT THE FOLLOWING STATEMENTS TRUNCATE ANY POLYNOMIAL TO
COMMENT    THIRD ORDER ACCURACY WHEN EXPRESSED IN WWWW1,WWWW2,WWWW3;
COMMENT   SECOND ORDER ACCURACY WHEN EXPRESSED IN  WWW1, WWW2, WWW3;

 FOR N1:=0:4 DO
  FOR N2:=0:(4-N1) DO
     LET WWWW1**N1*WWWW2**N2*WWWW3**(4-N1-N2)=0;
 FOR L:=1:3 DO
  FOR N1:=0:2 DO
   FOR N2:=0:(2-N1) DO
      LET THETA(L)*WWWW1**N1*WWWW2**N2*WWWW3**(2-N1-N2)=0;
 FOR N1:=0:2 DO
  FOR N2:=0:(2-N1) DO
     LET THETA1**N1*THETA2**N2*THETA3**(2-N1-N2)=0;
 FOR N1:=0:3 DO
  FOR N2:=0:(3-N1) DO
     LET WWW1**N1*WWW2**N2*WWW3**(3-N1-N2)=0;
 FOR L:=1:3 DO
  FOR N1:=0:1 DO
   FOR N2:=0:(1-N1) DO
      LET THETA(L)*WWW1**N1*WWW2**N2*WWW3**(1-N1-N2)=0;

COMMENT TAYLOR SERIES EXPANSION OF THE ARBITRARY LOCAL FUNCTION A0;

  OPERATOR A0;
  PROCEDURE A0(X1,X2,X3);
                                        A00+A01*X1+A02*X2+A03*X3
                           +(A011*X1*X1+A022*X2*X2+A033*X3*X3)/2
                            +A012*X1*X2+A023*X2*X3+A031*X3*X1
         +(A0111*X1*X1*X1+A0222*X2*X2*X2+A0333*X3*X3*X3)/6
         +(A0122*X1*X2*X2+A0133*X1*X3*X3+A0233*X2*X3*X3
          +A0112*X1*X1*X2+A0113*X1*X1*X3+A0223*X2*X2*X3)/2
         + A0123*X1*X2*X3
  + (A01111*X1*X1*X1*X1+A02222*X2*X2*X2*X2+A03333*X3*X3*X3*X3)/24
   +(A01112*X1*X1*X1*X2+A01113*X1*X1*X1*X3+A02223*X2*X2*X2*X3
    +A01222*X1*X2*X2*X2+A01333*X1*X3*X3*X3+A02333*X2*X3*X3*X3)/6
   +(A01122*X1*X1*X2*X2+A01133*X1*X1*X3*X3+A02233*X2*X2*X3*X3)/4
   +(A01233*X1*X2*X3*X3+A01223*X1*X2*X2*X3+A01123*X1*X1*X2*X3)/2
   $
```

```
COMMENT TAYLOR SERIES EXPANSION OF THE LOCAL DENSITY RHO0;

  OPERATOR RHO0;
  PROCEDURE RHO0(X1,X2,X3);
                                    RHO00+RHO01*X1+RHO02*X2+RHO03*X3
                           +(RHO011*X1*X1+RHO022*X2*X2+RHO033*X3*X3)/2
                            +RHO012*X1*X2+RHO023*X2*X3+RHO031*X3*X1
          +(RHO0111*X1*X1*X1+RHO0222*X2*X2*X2+RHO0333*X3*X3*X3)/6
          +(RHO0122*X1*X2*X2+RHO0133*X1*X3*X3+RHO0233*X2*X3*X3
           +RHO0112*X1*X1*X2+RHO0113*X1*X1*X3+RHO0223*X2*X2*X3)/2
          + RHO0123*X1*X2*X3
  + (RHO01111*X1*X1*X1*X1+RHO02222*X2*X2*X2*X2+RHO03333*X3*X3*X3*X3)/24
   +(RHO01112*X1*X1*X1*X2+RHO01113*X1*X1*X1*X3+RHO02223*X2*X2*X2*X3
    +RHO01222*X1*X2*X2*X2+RHO01333*X1*X3*X3*X3+RHO02333*X2*X3*X3*X3)/6
   +(RHO01122*X1*X1*X2*X2+RHO01133*X1*X1*X3*X3+RHO02233*X2*X2*X3*X3)/4
   +(RHO01233*X1*X2*X3*X3+RHO01223*X1*X2*X2*X3+RHO01123*X1*X1*X2*X3)/2
   $

COMMENT TAYLOR SERIES EXPANSION OF THE LOCAL VELOCITY U;

  OPERATOR U;
  PROCEDURE U(N,X1,X2,X3);
       IF N=1 THEN
                      U10+U11*X1+U12*X2+U13*X3
                           +(U111*X1*X1+U122*X2*X2+U133*X3*X3)/2
                            +U112*X1*X2+U123*X2*X3+U131*X3*X1
          +(U1111*X1*X1*X1+U1222*X2*X2*X2+U1333*X3*X3*X3)/6
          +(U1122*X1*X2*X2+U1133*X1*X3*X3+U1233*X2*X3*X3
           +U1112*X1*X1*X2+U1113*X1*X1*X3+U1223*X2*X2*X3)/2
          + U1123*X1*X2*X3
  ELSE IF N=2 THEN
                      U20+U21*X1+U22*X2+U23*X3
                           +(U211*X1*X1+U222*X2*X2+U233*X3*X3)/2
                            +U212*X1*X2+U223*X2*X3+U231*X3*X1
          +(U2111*X1*X1*X1+U2222*X2*X2*X2+U2333*X3*X3*X3)/6
          +(U2122*X1*X2*X2+U2133*X1*X3*X3+U2233*X2*X3*X3
           +U2112*X1*X1*X2+U2113*X1*X1*X3+U2223*X2*X2*X3)/2
          + U2123*X1*X2*X3
  ELSE IF N=3 THEN
                      U30+U31*X1+U32*X2+U33*X3
                           +(U311*X1*X1+U322*X2*X2+U333*X3*X3)/2
                            +U312*X1*X2+U323*X2*X3+U331*X3*X1
          +(U3111*X1*X1*X1+U3222*X2*X2*X2+U3333*X3*X3*X3)/6
          +(U3122*X1*X2*X2+U3133*X1*X3*X3+U3233*X2*X3*X3
           +U3112*X1*X1*X2+U3113*X1*X1*X3+U3223*X2*X2*X3)/2
          + U3123*X1*X2*X3
  ELSE ERRORINCALLOFU $


  COMMENT                                  CONTINUED ON NEXT PAGE
```

```
CONTINUATION                                    END;

COMMENT TAYLOR SERIES EXPANSION OF THE LOCAL SPECIFIC FORCE G0;

  OPERATOR G0;
  PROCEDURE G0(M,X1,X2,X3);
  IF M=1 THEN
                                          G10+G11*X1+G12*X2+G13*X3
                              +(G111*X1*X1+G122*X2*X2+G133*X3*X3)/2
                               +G112*X1*X2+G123*X2*X3+G131*X3*X1
         +(G1111*X1*X1*X1+G1222*X2*X2*X2+G1333*X3*X3*X3)/6
         +(G1122*X1*X2*X2+G1133*X1*X3*X3+G1233*X2*X3*X3
          +G1112*X1*X1*X2+G1113*X1*X1*X3+G1223*X2*X2*X3)/2
         + G1123*X1*X2*X3
+ (G11111*X1*X1*X1*X1+G12222*X2*X2*X2*X2+G13333*X3*X3*X3*X3)/24
 +(G11112*X1*X1*X1*X2+G11113*X1*X1*X1*X3+G12223*X2*X2*X2*X3
  +G11222*X1*X2*X2*X2+G11333*X1*X3*X3*X3+G12333*X2*X3*X3*X3)/6
 +(G11122*X1*X1*X2*X2+G11133*X1*X1*X3*X3+G12233*X2*X2*X3*X3)/4
 +(G11233*X1*X2*X3*X3+G11223*X1*X2*X2*X3+G11123*X1*X1*X2*X3)/2
  ELSE IF M=2 THEN
                                          G20+G21*X1+G22*X2+G23*X3
                              +(G211*X1*X1+G222*X2*X2+G233*X3*X3)/2
                               +G212*X1*X2+G223*X2*X3+G231*X3*X1
         +(G2111*X1*X1*X1+G2222*X2*X2*X2+G2333*X3*X3*X3)/6
         +(G2122*X1*X2*X2+G2133*X1*X3*X3+G2233*X2*X3*X3
          +G2112*X1*X1*X2+G2113*X1*X1*X3+G2223*X2*X2*X3)/2
         + G2123*X1*X2*X3
+ (G21111*X1*X1*X1*X1+G22222*X2*X2*X2*X2+G23333*X3*X3*X3*X3)/24
 +(G21112*X1*X1*X1*X2+G21113*X1*X1*X1*X3+G22223*X2*X2*X2*X3
  +G21222*X1*X2*X2*X2+G21333*X1*X3*X3*X3+G22333*X2*X3*X3*X3)/6
 +(G21122*X1*X1*X2*X2+G21133*X1*X1*X3*X3+G22233*X2*X2*X3*X3)/4
 +(G21233*X1*X2*X3*X3+G21223*X1*X2*X2*X3+G21123*X1*X1*X2*X3)/2
  ELSE IF M=3 THEN
                                          G30+G31*X1+G32*X2+G33*X3
                              +(G311*X1*X1+G322*X2*X2+G333*X3*X3)/2
                               +G312*X1*X2+G323*X2*X3+G331*X3*X1
         +(G3111*X1*X1*X1+G3222*X2*X2*X2+G3333*X3*X3*X3)/6
         +(G3122*X1*X2*X2+G3133*X1*X3*X3+G3233*X2*X3*X3
          +G3112*X1*X1*X2+G3113*X1*X1*X3+G3223*X2*X2*X3)/2
         + G3123*X1*X2*X3
+ (G31111*X1*X1*X1*X1+G32222*X2*X2*X2*X2+G33333*X3*X3*X3*X3)/24
 +(G31112*X1*X1*X1*X2+G31113*X1*X1*X1*X3+G32223*X2*X2*X2*X3
  +G31222*X1*X2*X2*X2+G31333*X1*X3*X3*X3+G32333*X2*X3*X3*X3)/6
 +(G31122*X1*X1*X2*X2+G31133*X1*X1*X3*X3+G32233*X2*X2*X3*X3)/4
 +(G31233*X1*X2*X3*X3+G31223*X1*X2*X2*X3+G31123*X1*X1*X2*X3)/2
  ELSE ERRORINCALLTOG0$
```

COMMENT CONTINUED ON NEXT PAGE

```
CONTINUATION                              END;

COMMENT TAYLOR SERIES EXPANSION OF THE SPECIFIC VOLUME VSPEC;

   OPERATOR VSPEC;
   PROCEDURE VSPEC(XX1,XX2,XX3,H1,H2,H3);
           ( - 6*RHO0123*RHO00**2*XX3*XX2*XX1 -
            3*RHO0223*RHO00**2*H2**2/12*XX3 -
            3*RHO0223*RHO00**2*XX3*XX2**2 - 3*RHO0113*RHO00**2
            *H1**2/12*XX3 - 3*RHO0113*RHO00**2*XX3*XX1**2 -
            3*RHO0112*RHO00**2*H1**2/12*XX2 -
            3*RHO0112*RHO00**2*XX2*XX1**2 - 3*
            RHO0233*RHO00**2*H3**2/12*XX2 -
            3*RHO0233*RHO00**2*XX3**2*XX2 -
            3*RHO0133*RHO00**2*H3**2/12*XX1 -
            3*RHO0133*RHO00**2*XX3**2 *XX1 -
            3*RHO0122*RHO00**2*H2**2/12*XX1 -
            3*RHO0122*RHO00**2*XX2**2*XX1 -
            3*RHO0333*RHO00**2*H3**2/12*XX3 - RHO0333*
            RHO00**2*XX3**3 - 3*RHO0222*RHO00**2*H2**2/12*XX2 -
            RHO0222*RHO00**2*XX2**3 -
            3*RHO0111*RHO00**2*H1**2/12*XX1 - RHO0111*
            RHO00**2*XX1**3 + 12*RHO031*RHO03*RHO00*XX3**2*XX1
          + 12*RHO031*RHO02*RHO00*XX3*XX2*XX1
          + 12*RHO031*RHO01*RHO00*XX3* XX1**2 -
            6*RHO031*RHO00**2*XX3*XX1
          + 12*RHO023*RHO03*RHO00*XX3**2*XX2
          + 12*RHO023*RHO02*RHO00*XX3*XX2**2 + 12*RHO023*
            RHO01*RHO00*XX3*XX2*XX1 - 6*RHO023*RHO00**2*XX3*XX2
          + 12*RHO012*RHO03*RHO00*XX3*XX2*XX1
          + 12*RHO012*RHO02*RHO00* XX2**2*XX1
          + 12*RHO012*RHO01*RHO00*XX2*XX1**2 -
            6*RHO012*RHO00**2*XX2*XX1
          + 6*RHO033*RHO03*RHO00*H3**2/12*XX3 + 6*
            RHO033*RHO03*RHO00*XX3**3
          + 6*RHO033*RHO02*RHO00*H3**2/12*XX2
          + 6*RHO033*RHO02*RHO00*XX3**2*XX2 + 6*RHO033*RHO01
            *RHO00*H3**2/12*XX1 + 6*RHO033*RHO01*RHO00*XX3**2*XX1
            - 3*RHO033*RHO00**2*H3**2/12 -
            3*RHO033*RHO00**2*XX3**2 + 6*
            RHO022*RHO03*RHO00*H2**2/12*XX3
          + 6*RHO022*RHO03*RHO00*XX3*XX2**2
          + 6*RHO022*RHO02*RHO00*H2**2/12*XX2 + 6*RHO022
            *RHO02*RHO00*XX2**3
          + 6*RHO022*RHO01*RHO00*H2**2/12*XX1
          + 6*RHO022*RHO01*RHO00*XX2**2*XX1 - 3*RHO022*RHO00**2*
            H2**2/12 - 3*RHO022*RHO00**2*XX2**2
          + 6*RHO011*RHO03*RHO00*H1**2/12*XX3
          + 6*RHO011*RHO03*RHO00*XX3*XX1**2 + 6*RHO011
            *RHO02*RHO00*H1**2/12*XX2
          + 6*RHO011*RHO02*RHO00*XX2*XX1**2
          + 6*RHO011*RHO01*RHO00*H1**2/12*XX1 + 6*RHO011*RHO01
            *RHO00*XX1**3 - 3*RHO011*RHO00**2*H1**2/12 -
            3*RHO011*RHO00**2*XX1**2 - 6*RHO03**3*XX3**3 -
            18*RHO03**2*RHO02*XX3**2*XX2 - 18
            *RHO03**2*RHO01*XX3**2*XX1 + 6*RHO03**2*RHO00*XX3**2
            - 18*RHO03*RHO02**2*XX3*XX2**2 -
            36*RHO03*RHO02*RHO01*XX3*XX2*XX1
```

```
          + 12*RHO03*RHO02*RHO00*XX3*XX2 -
            18*RHO03*RHO01**2*XX3*XX1**2
          + 12*RHO03*RHO01*RHO00*XX3*XX1 - 6*RHO03*RHO00**2*XX3
             - 6*RHO02**3*XX2**3 - 18*RHO02**2*RHO01*XX2**2*XX1
          + 6*RHO02**2*RHO00*XX2**2 - 18*RHO02*RHO01**2*XX2*XX1**2
          + 12*RHO02* RHO01*RHO00*XX2*XX1 - 6*RHO02*RHO00**2*XX2 -
            6*RHO01**3*XX1**3 + 6*RHO01**2*RHO00*XX1**2 -
            6*RHO01*RHO00**2*XX1 + 6* RHO00**3)/(6*RHO00**4)$



COMMENT TAYLOR SERIES EXPANSION OF THE APPROXIMATE EVALUATION
        OF THE AVERAGE OVER A QUADRILATERAL;

OPERATOR AVERAGE;
PROCEDURE AVERAGE(POL,XX1,XX2,XX3,H1,H2,H3);
BEGIN SCALAR A;
A:=
  +SUB(X1=XX1,X2=XX2,X3=XX3,
       POL
   +DF(POL,X1,2)*H1**2/24
   +DF(POL,X2,2)*H2**2/24
   +DF(POL,X3,2)*H3**2/24      );
RETURN(A);
END$

COMMENT TAYLOR SERIES EXPANSION OF THE APPROXIMATE EVALUATION
        OF THE OFFSET OF THE CENTER OF MASS IN A QUADRILATERAL;

 OPERATOR O;
 PROCEDURE O(NAL,XX1,XX2,XX3,H1,H2,H3);
 BEGIN SCALAR ERG,VS,XRHO;
 XRHO:=AVERAGE((X(NAL)-Y(NAL))*
      RHOO(X1,X2,X3),Y1,Y2,Y3,H1,H2,H3)$
 XRHO:=SUB(Y1=WWWW1,Y2=WWWW2,Y3=WWWW3,XRHO)$
 VS:=VSPEC(WWWW1,WWWW2,WWWW3,H1,H2,H3)$
 XRHO:=VS*XRHO;
 ERG:=SUB(WWWW1=XX1,WWWW2=XX2,WWWW3=XX3,XRHO)$
 RETURN ERG;
 END$
```

COMMENT

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```
CONTINUATION                              END;

COMMENT ONEDIMENSIONAL INTEGRATION OF A POLYNOMIAL;

  OPERATOR INT1D;
    PROCEDURE INT1D(POL,X1,LB1,HB1);
       BEGIN SCALAR ERG;
       ERG:=INT(POL,X1);
       ERG:=SUB(X1=HB1,ERG)-SUB(X1=LB1,ERG);
       RETURN ERG;
       END$

COMMENT INTEGRATION OF A POLYNOMIAL OVER A QUADRILATERAL;

  OPERATOR INT3D;
    PROCEDURE INT3D(POL,X1,X2,X3,LB1,HB1,LB2,HB2,LB3,HB3);
       BEGIN SCALAR ERG;
       ERG:=INT(INT(INT(POL,X1),X2),X3);
       ERG:=SUB(X1=HB1,ERG)-SUB(X1=LB1,ERG);
       ERG:=SUB(X2=HB2,ERG)-SUB(X2=LB2,ERG);
       ERG:=SUB(X3=HB3,ERG)-SUB(X3=LB3,ERG);
       RETURN ERG;
       END$

COMMENT VOLUME OF A QUADRILATERAL;

   OPERATOR VOLUME;
   PROCEDURE VOLUME(X1,X2,X3,H1,H2,H3);
   H1*H2*H3$

COMMENT TAYLOR SERIES EXPANSION OF THE RECIPROCAL OF A POLYNOMIAL;
COMMENT THIS WAS USED TO DETERMINE THE APPROXIMATION FOR VSPEC;

  OPERATOR RECIPROC;
  PROCEDURE RECIPROC(POL,BASIS,XX1,XX2,XX3);
  BEGIN SCALAR REST,RECI0,RECI1,RECI2,RECI3;
  COMMENT 1/(1+REST)=1-REST+REST**2-REST**3;
  REST:=SUB(XX1=WWWW1,XX2=WWWW2,XX3=WWWW3,POL)/BASIS-1;
  RECI1:=REST;
  RECI2:=RECI1*REST;
  RECI3:=RECI2*REST;
  RECI0:=1-RECI1;
  RECI0:=RECI0+RECI2;
  RECI0:=RECI0-RECI3;
  RECI0:=SUB(WWWW1=XX1,WWWW2=XX2,WWWW3=XX3,RECI0/BASIS);
  RETURN RECI0;
  END$

   ;END;
```

Proofs related to Chapter 3.

Time required for proof on a Siemens 7890 computer: 7.5 min.

```
  IN STANDARD;
  OFF ECHO;
  ARRAY Z(3),RHOO(3),RHOALPHA(3),ALPHA(3);
  ERRORCOUNT:=0$
  RHOAVER:=AVERAGE(RHOO(X1,X2,X3),WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3)$
  VS:=VSPEC(WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3)$

  COMMENT PROOF OF CORRECTNESS OF VSPEC;

  IF RHOAVER*VS-1 EQUAL 0 THEN WRITE "FUNCTION VSPEC IS CORRECT"
  ELSE ERRORCOUNT:=ERRORCOUNT+1;

  COMMENT PROOF OF EQUATION (03.09);

  FOR J:=1:3 DO BEGIN
  RHOO(J):=AVERAGE((X(J)-WWWW(J))
    *RHOO(X1,X2,X3),WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3)$
  ERROR:=RHOO(J)-THETA(J)*DF(RHOO(WWWW1,WWWW2,WWWW3),WWWW(J))$
  IF ERROR EQUAL 0 THEN
  WRITE "EQUATION (03.09) IS CORRECT FOR I=",J
  ELSE ERRORCOUNT:=ERRORCOUNT+1;
  END;

  COMMENT PROOF OF CORRECTNESS OF THETA ACCORDING TO (3.10);

  FOR J:=1:3 DO IF
  THETA(J)-AVERAGE((X(J)-WWWW(J))**2,WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3)
           EQUAL 0 THEN WRITE "FUNCTION THETA IS CORRECT FOR I=",J
           ELSE ERRORCOUNT:=ERRORCOUNT+1; END$

  COMMENT PROOF OF CORRECTNESS OF FUNCTION O;
  FOR J:=1:3 DO BEGIN
  Z(J):=WWWW(J)+O(J,WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3)$
  ERROR:=RHOO(J)-RHOAVER*O(J,WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3);
  IF ERROR EQUAL 0 THEN
  WRITE "FUNCTION O(....) IS CORRECT FOR I=",J
  ELSE ERRORCOUNT:=ERRORCOUNT+1;
  END;

  COMMENT PROOF OF EQUATION (03.11);

  RHOA:=RHOO(WWWW1,WWWW2,WWWW3)*AO(WWWW1,WWWW2,WWWW3)$
  RHOAX:=SUB(WWWW1=X1,WWWW2=X2,WWWW3=X3,RHOA)$
  RHOAAVER:=AVERAGE(RHOAX,WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3)$
  ERROR := RHOAAVER-RHOA -
      FOR N:=1:3 SUM(THETA(N)*DF(RHOA,WWWW(N),2)/2)$
  IF ERROR EQUAL 0 THEN WRITE "EQUATION (03.11) IS CORRECT"
  ELSE ERRORCOUNT:=ERRORCOUNT+1;

  COMMENT
```

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```
CONTINUATION                              END;
 COMMENT PROOF OF EQUATION (03.12);

 FOR J:=1:3 DO BEGIN
 RHOALPHA(J):=
 AVERAGE((X(J)-Z(J))*RHOAX,WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3);
 ALPHA(J):=VS*RHOALPHA(J);
 ERROR:=ALPHA(J)-THETA(J)*DF(AO(WWWW1,WWWW2,WWWW3),WWWW(J))$
 IF ERROR EQUAL 0 THEN
 WRITE "EQUATION (03.12) IS CORRECT FOR I=",J
 ELSE ERRORCOUNT:=ERRORCOUNT+1;
 END;

 COMMENT PROOF OF EQUATION (03.13);

 ERROR:=
 RHOAAVER-RHOA-FOR N:=1:3 SUM( DF(RHOALPHA(N),WWWW(N))
             + DF(RHOAAVER*(Z(N)-WWWW(N)),WWWW(N)) )/2$
 ERROR:=SUB(WWWW1=WWW1,WWWW2=WWW2,WWWW3=WWW3,ERROR)$
 IF ERROR EQUAL 0 THEN WRITE "EQUATION (03.13) IS CORRECT"
 ELSE ERRORCOUNT:=ERRORCOUNT+1;

 COMMENT PROOF OF EQUATION (03.14);

 ERROR:=
 RHOAVER-RHOO(WWWW1,WWWW2,WWWW3)-
 FOR N:=1:3 SUM(THETA(N)*DF(RHOO(WWWW1,WWWW2,WWWW3),WWWW(N),2)/2)$
 IF ERROR EQUAL 0 THEN
 WRITE "EQUATION (03.14) IS CORRECT (PART 1)"
 ELSE ERRORCOUNT:=ERRORCOUNT+1;

 ERROR:=
 RHOAVER-RHOO(WWWW1,WWWW2,WWWW3) -
    FOR N:=1:3 SUM(DF(RHOO(N),WWWW(N))/2)$
 ERROR:=SUB(WWWW1=WWW1,WWWW2=WWW2,WWWW3=WWW3,ERROR)$
 IF ERROR EQUAL 0 THEN
 WRITE "EQUATION (03.14) IS CORRECT (PART 2)"
 ELSE ERRORCOUNT:=ERRORCOUNT+1;

 COMMENT PROOF OF EQUATION (03.15);

 ASPEC:=VS*RHOAAVER$
 ERROR:=ASPEC-AO(Z(1),Z(2),Z(3)) -
      FOR N:=1:3 SUM(DF(ALPHA(N),WWWW(N))/2)$
 ERROR:=SUB(WWWW1=WWW1,WWWW2=WWW2,WWWW3=WWW3,ERROR)$
 IF ERROR EQUAL 0 THEN WRITE "EQUATION (03.15) IS CORRECT"
 ELSE ERRORCOUNT:=ERRORCOUNT+1;

 IF ERRORCOUNT EQUAL 0 THEN WRITE "ALL PROOFS WERE SUCCESSFULL"
 ELSE WRITE ERRORCOUNT," PROOFS FAILED";
```

Proofs related to Chapter 6.

Time required for proof on a Siemens 7890 computer: 52.5 min.

```
  OFF ECHO;
  IN STANDARD;
  ARRAY RHOAU(3),RHOU(3),Z(3),
  RHOSPEED(3),SPEED(3),RHOALFA(3),ALFA(3),RHOS(3,3),S(3,3);
  ERRORCOUNT:=0$
  RHOAVER:=AVERAGE(RHOO(X1,X2,X3),WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3)$
  VS:=VSPEC(WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3)$
  RHOA:=RHOO(WWWW1,WWWW2,WWWW3)*AO(WWWW1,WWWW2,WWWW3)$
  RHOAX:=SUB(WWWW1=X1,WWWW2=X2,WWWW3=X3,RHOA)$
  RHOAAVER:=AVERAGE(RHOAX,WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3)$
  ASPEC:=VS*RHOAAVER$
  FOR M:=1:3 DO
    Z(M):=WWWW(M)+O(M,WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3)$
  FOR N:=1:3 DO BEGIN
    RHOAU(N):=RHOA*U(N,WWWW1,WWWW2,WWWW3)$
    RHOAU(N):=SUB(WWWW1=X1,WWWW2=X2,WWWW3=X3,RHOAU(N));
    RHOU(N):=RHOO(WWWW1,WWWW2,WWWW3)*U(N,WWWW1,WWWW2,WWWW3)$
    RHOU(N):=SUB(WWWW1=X1,WWWW2=X2,WWWW3=X3,RHOU(N))$
    RHOSPEED(N):=AVERAGE(RHOU(N),WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3)$
    SPEED(N):=VS*RHOSPEED(N)$
       FOR M:=1:3 DO RHOS(M,N):=AVERAGE((X(M)-Z(M))
            *RHOU(N),WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3)$
    RHOALFA(N):=AVERAGE((X(N)-Z(N))
         *RHOAX,WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3)$
    END;

  ERROR :=0$

  COMMENT PROOF OF EQUATION (06.05);

  FOR N:=1:3 DO BEGIN;
    RESULT:=0$
    RESULT:=RESULT+SUB(WWWW1=WWW1,WWWW2=WWW2,WWWW3=WWW3,
            AVERAGE(RHOAU(N),WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3))$
    APPROX:=0$
    APPROX:=APPROX
            +SUB(WWWW1=WWW1,WWWW2=WWW2,WWWW3=WWW3,SPEED(N))
            *SUB(WWWW1=WWW1,WWWW2=WWW2,WWWW3=WWW3,RHOAAVER);
    OPTION1:=0$
    FOR M:=1:3 DO
      OPTION1:=OPTION1
            +SUB(WWWW1=WWW1,WWWW2=WWW2,WWWW3=WWW3,
                  RHOS(M,N)*DF(ASPEC,WWWW(M)));
    OPTION2:=0$
    FOR M:=1:3 DO
      OPTION2:=OPTION2
            +SUB(WWWW1=WWW1,WWWW2=WWW2,WWWW3=WWW3,
                  RHOALFA(M)*DF(SPEED(N),WWWW(M)));
  COMMENT                                    CONTINUED ON NEXT PAGE
```

```
CONTINUATION                              END;

 COMMENT PROOF OF EQUATION (06.06);

   IF (OPTION1-OPTION2) EQUAL 0 THEN
      WRITE "EQUATION (06.06) IS CORRECT FOR N=",N
      ELSE ERRORCOUNT:=ERRORCOUNT+1;
   ERROR:=ERROR+RESULT-APPROX-(OPTION1/2+OPTION2/2);

  END;
 IF ERROR EQUAL 0 THEN
 WRITE "EQUATION (06.05) IS CORRECT"
 ELSE ERRORCOUNT:=ERRORCOUNT+1;
 IF ERRORCOUNT EQUAL 0 THEN WRITE "ALL PROOFS WERE SUCCESSFULL"
 ELSE WRITE ERRORCOUNT," PROOFS FAILED";
```

Proofs related to Chapter 10.

Time required for proof on a Siemens 7890 computer: 74.3 min.

```
  ON  ECHO;
  IN STANDARD;
  ARRAY RHOU(3),RHOS(3,3),Z(3);
  ARRAY RHOUU(3,3),RHOSPEED(3),SPEED(3);
  ERRORCOUNT:=0$

  COMMENT PROOF OF CORRECTNESS EQUATION (10.06);

  RHOAVER:=AVERAGE(RHOO(X1,X2,X3),WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3)$
  VS      :=VSPEC(WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3)$
  FOR M:=1:3 DO Z(M):=WWWW(M)+O(M,WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3);
  FOR J:=1:3 DO BEGIN
       RHOU(N):=RHOO(WWWW1,WWWW2,WWWW3)*U(N,WWWW1,WWWW2,WWWW3)$
       FOR N:=1:3 DO
           RHOUU(N,J):=RHOU(J)*U(N,WWWW1,WWWW2,WWWW3);
       RHOU(J):=SUB(WWWW1=X1,WWWW2=X2,WWWW3=X3,RHOU(J))$
       FOR M:=1:3 DO
         RHOS(M,J):=AVERAGE((X(M)-Z(M))*RHOU(J),WWWW1,WWWW2,WWWW3,
                    EPS1,EPS2,EPS3);
       RHOSPEED(J):=AVERAGE(RHOU(J),WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3);
       SPEED(J):=VS*RHOSPEED(J);
  END;

  FOR J:=1:3 DO BEGIN
  FOR N:=1:3 DO BEGIN
  RESULT:=0$
  RHOUUX:=SUB(WWWW1=X1,WWWW2=X2,WWWW3=X3,RHOUU(N,J));
  RESULT:=RESULT+
        AVERAGE(RHOUUX      ,WWWW1,WWWW2,WWWW3,EPS1,EPS2,EPS3);
  RESULT:=RESULT- RHOSPEED(N)*SPEED(J);
  RESULT:=SUB(WWWW1=WWW1,WWWW2=WWW2,WWWW3=WWW3,RESULT);


  COMMENT                                    CONTINUED ON NEXT PAGE
```

```
CONTINUATION                              END;
 OPTION1:=0;
 OPTION2:=0;
 FOR M:=1:3 DO BEGIN
 OPTION1:=OPTION1+RHOS(M,N)*DF(SPEED(J),WWWW(M));
 OPTION1:=SUB(WWWW1=WWW1,WWWW2=WWW2,WWWW3=WWW3,OPTION1);
 OPTION2:=OPTION2+DF(SPEED(N),WWWW(M))*RHOS(M,J);
 OPTION2:=SUB(WWWW1=WWW1,WWWW2=WWW2,WWWW3=WWW3,OPTION2);
 END;
 IF OPTION1-OPTION2 EQUAL 0 THEN
 WRITE "THE TWO APPROXIMATIONS ARE IDENTICAL FOR I=",J," AND N=",N
 ELSE ERRORCOUNT:=ERRORCOUNT+1;
 ERROR:=RESULT-(OPTION1+OPTION2)/2;
 IF ERROR EQUAL 0 THEN
 WRITE "EQUATION (10.06) IS CORRECT FOR I=",J," AND N=",N
 ELSE ERRORCOUNT:=ERRORCOUNT+1;
 END;
 END;
 IF ERRORCOUNT EQUAL 0 THEN WRITE "ALL PROOFS WERE SUCCESSFULL"
 ELSE WRITE ERRORCOUNT," PROOFS FAILED";
```