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Thermodynamics of a Visco-Plastic Material Model with Internal Variables

Summary

The thermodynamic restrictions for the constitutive relations of an elastic-viscoplastic material model with internal variables are derived by requiring compatibility with the Clausius-Duhem entropy inequality (second law). The material model is a thermodynamic extension of a simplified version of the mechanical Robinson model, which does not account for isotropic hardening and its thermomechanical history dependence.

The usual evaluation of the second law implies sufficiently smooth processes. However, the evolution equations of the Robinson model are discontinuous at certain states. To account for the discontinuities in the rates of the state variables, the analysis is based on the time-integral form of the basic equations. Within the frame of the assumptions, it is shown that the time-integral entropy inequality does not impose additional constitutive restrictions beyond those obtained when smooth processes apply.

The evaluation of the residual entropy inequality is done first for the uniaxial state of stress and then for the general threedimensional state. For the uniaxial state a limited analysis was performed which resulted in conditions on the material parameters which are sufficient but which were not proved to be necessary. On the other hand, a more detailed analysis for the general multiaxial state yielded explicit conditions which are necessary and sufficient to ensure a non-negative entropy production. These restrictions do not depend anymore on the state variables, except the temperature. This result was obtained since the values of some material parameters were assumed to be restricted to certain ranges which is, however, consistent with present applications of the material model. Further, simplified but only sufficient conditions were derived which encompass those for the uniaxial case. They have the handy form of an upper bound criterion for one of the thermodynamic parameters.

These results are supplemented by constitutive restrictions deduced from the stability requirement of equilibrium states.

With respect to the simplified purely mechanical Robinson model it is shown that it is formally consistent with the second law in the following sense: A thermodynamically extended version of the mechanical model can always be set up in such a way that the entropy production is ensured to be non-negative, whatever the parameters of the simplified mechanical model are. However, the extended

model may still fail when confronted with caloric measurements even if it is compatible with mechanical measurements of isothermal experiments.

Thermodynamik eines visko-plastischen Materialmodells mit inneren Variablen

Zusammenfassung

Die thermodynamischen Einschränkungen für die konstitutiven Gleichungen eines elastisch visko-plastischen Materialmodells mit inneren Variablen werden aus der Forderung nach Verträglichkeit mit der Clausius-Duhem Entropieungleichung (Zweiter Hauptsatz) abgeleitet. Das Materialmodell ist eine thermodynamische Erweiterung einer vereinfachten Version des mechanischen Robinson-Modells, die die isotrope Verfestigung und deren Abhängigkeit von thermomechanischen Prozeßgeschichten nicht erfaßt.

Die übliche Auswertung des Zweiten Hauptsatzes setzt hinreichend glatte Prozesse voraus. Für gewisse Zustände sind die Evolutionsgleichungen des Robinson-Modells nun allerdings diskontinuierlich. Um auch Diskontinuitäten in den Raten der Zustandsgrößen berücksichtigen zu können, geht die Analyse von zeitlich integralen Grundgleichungen aus. Im Rahmen der sonstigen Voraussetzungen wird gezeigt, daß die zeitlich integrale Entropieungleichung zu keinen zusätzlichen konstitutiven Einschränkungen führt, die über die Restriktionen hinausgehen, wie man sie erhält, wenn nur glatte Prozesse betrachtet werden.

Die Auswertung der Restentropieungleichung wird zunächst für den einachsigen und dann für den allgemeinen dreiachsigen Spannungszustand durchgeführt. Für den einachsigen Fall wurde eine begrenzte Analyse durchgeführt, die für die Materialparameter Bedingungen ergab, die hinreichend sind, deren Notwendigkeit aber nicht nachgewiesen wurde. Andererseits ergab eine detaillierte Analyse für den allgemeinen mehrachsigen Spannungszustand explizite Bedingungen, die hinreichend und notwendig sind, um eine nicht-negative Entropieproduktion zu gewährleisten. Diese Einschränkungen hängen von den Zustandsvariablen, mit Ausnahme der Temperatur, nicht mehr ab. Dieses Ergebnis wurde erhalten, da angenommen worden war, daß einige Materialparameter in ihrem Wertebereich beschränkt sind; allerdings ist dieser verträglich mit den gegenwärtigen Anwendungen des Materialmodells. Weiterhin wurden vereinfachte, aber nur hinreichende Bedingungen abgeleitet, die jene für den einachsigen Fall mit einschließen. Sie haben die nützliche Form einer oberen Schranke für einen der thermodynamischen Parameter.

Diese Resultate werden ergänzt durch konstitutive Restriktionen, die aus der Forderung der Stabilität von Gleichgewichtszuständen folgen.

Für das vereinfachte, rein mechanische Robinson-Modell wird gezeigt, daß es in folgendem Sinne mit dem Zweiten Hauptsatz formal verträglich ist: Es kann immer eine thermodynamisch erweiterte Version des mechanischen Modells in der Weise aufgestellt werden, daß die Nicht-Negativität der Entropieproduktion sichergestellt ist, wie auch die Parameter des vereinfachten mechanischen Modells gewählt werden. Dies schließt allerdings nicht aus, daß das erweiterte Modell beim Vergleich mit kalorischen Meßergebnissen versagt, selbst wenn es mit mechanischen Meßergebnissen isothermer Versuche in Übereinstimmung ist.

Contents

1. Introduction

For the description of the visco-plastic high temperature deformation behaviour of metals various constitutive models have been proposed. Many of these models imply that the state of the material is not only described by the instant values of the stress, temperature, total strain, and plastic strain but also by an additional set of internal variables.

The concept of internal variables* has been introduced in thermodynamics and continuum thermomechanics in such references as [1 - 8] and has been considerably extended and applied more recently [9 - 36]**. This list of publications encloses linear as well as nonlinear constitutive models and rather different thermodynamic approaches, e.g., the classical Theory of Irreversible Processes (TIP) and Rational Thermodynamics. However, an appreciation of these various contributions is beyond the scope of this study.

Some general remarks should be made. It is assumed that the material response can be characterized by the evolution of two sets of variables, namely, the *independent variables* and the *dependent variables*. Of course, the choice of these variables depends on the kind of material to be modeled. For example, if a classical thermoelastic material is considered, the independent variables are chosen to be the instant values of the strain and temperature and possibly its spatial gradient, and the dependent variables are the specific internal energy (or specific free energy), the stress, specific entropy, and the heat flux if heat conduction is accounted for.

Strain and temperature are in principle easily measured and directly controlled at a specimen; therefore, these quantities and their spatial and temporal gradients belong to the group of *external (or observable) variables*. But the instant values of external variables may not be sufficient to determine the dependent variables. Here additional variables are required which reflect internal processes in the material. They cannot be easily determined and no direct control is possible since these *internal (hidden or conceiled) variables* are not directly connected to any

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* Early writers did not use this name: Bridgman [4] used the term "conceiled variables".

** The cited references neither represents a complete list of all contributors up to 1975 nor does this list contain all relevant publications of the cited authors. Since 1975 the concept of internal variables in continuum thermomechanics sees an expanding development and application up to present days (e.g. see CISM course "Internal variables in thermodynamics and continuum mechanics", 1988).

additional external force variables. Consequently, the form of the balance equations for linear momentum, moment of momentum, and energy are the same whether or not internal variables are accounted for.

Examples of interpretations of internal variables with different character are

- degree of advancement of chemical reactions or phase transformations (concentrations of constituents)
- quantities related to the density, motion, and arrangement of dislocations
- residual stresses between microelements contained in the macroelement of polycrystals, concrete, or sand
- internal stresses or displacements in rheological models.

The evolution of the internal variables is usually* assumed to be governed by a set of ordinary differential equations such that the rates of the internal variables are uniquely determined by the external and internal independent variables (constitutive equations for internal variables).

The constitutive equations for the dependent variables such as stress, free energy, and entropy are usually assumed to be algebraic relations between these variables and the independent external and internal variables. Thus, the dependent variables depend on the history of the independent external variables only through the instant values of the internal variables. Of course, the constitutive relations for the dependent variables and the evolution equations for the internal variables cannot be chosen independently because the second law imposes certain thermodynamic restrictions.**

For the description of visco-plastic material behaviour models have been set up taking into account thermodynamic concepts [e.g. 19 - 28, 30, 35]. However, the references [37 - 41, 43 - 57] demonstrate that various internal variable models have been proposed which are not embedded in a truly thermodynamic frame although they involve the temperature: The temperature appears either simply as a parameter affecting materials "constants" or it allows for thermomechanical history dependence in the evolution equations [41, 43, 56]. The missing thermodynamic frame is seen in the fact that quantities like internal energy, free energy

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* More complex cases are conceivable: for example, the evolution equations may involve also the rates of the external variables or they may involve divergence terms of the internal variables (spatial differential equations)

** Other constitutive principles have to be observed, too.

and entropy are not present in the theory, and thermodynamic restrictions on the various material parameters are not considered. Therefore, these models are actually purely mechanical models. However, it appears to be desirable that even a mechanical model should be interpretable as a special case of a more embracing thermomechanical model, and in general thermodynamical requirements are expected to impose restrictions even on the structure of the purely mechanical models.

The Robinson model [54 - 57] is such a mechanical model not embedded in a thermodynamic frame. It has been implemented into a Finite Element Code and extensive numerical studies have been performed; also parameter identification is underway at the Nuclear Research Center Karlsruhe [58 - 61]. Therefore, it appeared appropriate to complement those studies by investigations of the thermodynamic consistency of the model: Here the question should be answered whether a thermodynamic analysis yields restrictions on the material parameters of the Robinson model.

Such an analysis requires a thermodynamic extension of the model, i.e., an assumption about the structure of the free energy. But it is conceiveable that restrictions are obtained which involve only the aspects of the original mechanical model and thus are not subject to the additional assumptions.

The above questions are analysed in this report under a set of simplifications made because of the apparent mathematical complexities:

- Isotropic hardening was ignored and therefore also the thermomechanical history dependence of the drag stress. This simplification reduces the number of evolution equations for the internal variables by one.

- The static recovery term in the evolution equation of the back stress, which controls the kinematic hardening, is modified in the same way as was done by Hornberger and Stamm [58 - 60]. This not only eliminates a discontinuity in the evolution equation but also allows for the existence of equilibrium states.

- The second law of thermodynamics is taken to be the *classical Clausius-Duhem entropy inequality*. Further, its evaluation is based on the Coleman-Noll argument. More advanced principles and concepts, e.g., the approach of Müller [62] (see also [63]), require a more elaborate analysis.

The organization of the present study is as follows. In section 2 a general discussion of thermodynamic restrictions for a class of internal variable models is given.

An essential starting point is the thermodynamics with internal variables formulated by Coleman and Gurtin [13] within the frame of Rational Thermodynamics. Results obtained for smooth processes are summarized (section 2.1). This is supplemented (section 2.2) by observations on the residual entropy inequality at equilibrium states and stability considerations of these states.

The derivitions of thermodynamic restrictions is usually based on the assumption that processes are smooth functions of time; thus, the local instant form of the Clausius-Duhem entropy inequality is an adequate starting point. In fact, this is an assumption contained in the work [13, 63]. However, a peculiar property of the Robinson model is the fact that the evolution equations involve discontinuities in the rates of the internal variables. Therefore, these discontinuities require a more general starting point, i.e., a time integral formulation of the entropy inequality and the balance equations. A general internal variable model, allowing for discontinuities in the evolution functions for the internal variables, is set up (section 2.3) such that the Robinson model (without thermomechanical history dependence) is a special case. For this model the time integral form of the entropy inequality is evaluated to study whether or not additional constitutive restrictions are obtained for the discontinuities in the evolution functions.

The thermodynamics with internal state variables formulated by Coleman and Gurtin [13] has been used by Perzyna et al. [19 - 22] and Kratochvil and Dillon [27, 28] to analyse the thermodynamic restrictions imposed on some elastic-viscoplastic or elastic-plastic material models. In these and other studies the inelastic strains are interpreted as internal state variables such that the results of Coleman and Gurtin could be directly applied. This interpretation is also used in the present analysis. With this interpretation, and observing some rather general constitutive assumptions compatible with the simplified Robinson model, the residual entropy inequality is formulated (section 2.4). This inequality is to be satisfied for all states* and it requires further evaluation when specific forms of the evolution functions are given.

In section 3 a formal description of the Robinson model for the multiaxial and uniaxial state of stress is given, and section 4 describes a relatively simple extension of the Robinson model within a thermodynamic frame.

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* Admissible according to the assumed constitutive equations.

The residual entropy inequality is evaluated in section 5, first for the uniaxial state of stress, then for the all embracing multiaxial state of stress, and this is concluded with a comparative analysis.

A discussion of the stability of equilibrium states of the thermodynamically extended Robinson model is enclosed in section 6; a further constitutive restriction is obtained from these considerations.

In the final section 7 the results obtained for the thermodynamically extended version as well as for the mechanical version of the simplified Robinson model are discussed. Future work is indicated.

2. On Thermodynamic Restrictions for Constitutive Equations of Internal Variable Models

2.1 Constitutive Restrictions for an Internal Variable Model based on the Evaluation of the Clausius-Duhem Entropy Inequality for Smooth Processes (A Summary)

Starting from the set of assumed constitutive equations for a three-dimensional deformable heat conductor with (N) scalar internal variables α_γ*, $\gamma = 1,\ldots,N$,

$$\left.\begin{array}{lll}
\sigma_{kl} = \hat{\sigma}_{kl}(\varepsilon_{mn}, T, g_k, \alpha_\gamma) = \hat{\sigma}_{lk} & & \text{: stress} \\
q_k = \hat{q}_k(\quad '' \quad) & & \text{: heat flux} \\
\epsilon = \hat{\epsilon}(\quad '' \quad) & & \text{: specific internal energy} \\
\eta = \hat{\eta}(\quad '' \quad) & & \text{: specific entropy}
\end{array}\right\} \quad (2.1)$$

and with the evolution equations for the internal variables

$$\dot{\alpha}_\gamma = f_\gamma(\varepsilon_{mn}, T, g_k, \alpha_\nu) \quad , \quad \gamma, \nu = 1, 2, \ldots, N \qquad (2.2)$$

the "dissipation postulate", in connection with the local instant Clausius-Duhem entropy inequality

$$\sigma := \rho\dot{\eta} + \left(\frac{q_k}{T}\right)_{,k} - \frac{\rho r}{T} \geq 0$$

or the equivalent dissipation inequality** (2.3)

$$\gamma := T\sigma = -\rho\dot{\psi} - \rho\eta\dot{T} + \sigma_{kl}\dot{\varepsilon}_{lk} - \frac{q_k T_{,k}}{T^2} \geq 0,$$

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[*] Latin subscripts are associated with cartesian tensors and Greek subscripts with scalar tuples. The summation convention applies to repeated indices. A comma denotes partial differentiation.

[**] The dissipation inequality is derived from the entropy inequality by eliminating the heat sources from $(2.3)_1$ via the local energy balance equation, introducting the free energy, and multiplying the result with the absolute temperature $T > 0$.

was evaluated in [13, 63]. Here the free energy (Helmholtz function) $\psi = \epsilon - T\eta$ is introduced.

Two methods were applied [63] which yield the same restrictions on the constitutive functions, i.e.,

$$\left.\begin{aligned} \hat{\sigma}_{mn} &= \rho \, \frac{1}{2} \left(\frac{\partial \hat{\psi}}{\partial \varepsilon_{mn}} + \frac{\partial \hat{\psi}}{\partial \varepsilon_{nm}} \right) \\ \hat{\eta} &= - \frac{\partial \hat{\psi}}{\partial T} \\ \hat{\epsilon} &= \hat{\psi} - T \frac{\partial \hat{\psi}}{\partial T} \\ \frac{\partial \hat{\psi}}{\partial g_m} &= 0 \end{aligned} \right\} \qquad (2.4)$$

and the residual dissipation inequality

$$\gamma := - \rho \frac{\partial \hat{\psi}}{\partial \alpha_\gamma} f_\gamma - \frac{1}{T} \hat{q}_k g_k \geq 0. \qquad (2.5)$$

Condition $(2.4)_4$ implies that $\hat{\psi}$, $\hat{\varepsilon}$, $\hat{\eta}$, and $\hat{\sigma}_{mn}$ are independent of the temperature gradient g_k. It is also evident that the prescription of the free energy function $\hat{\psi}$ completely determines the constitutive functions $\hat{\sigma}_{mn}$, $\hat{\eta}$ and $\hat{\varepsilon}$.

A few remarks on the logic how the restrictions (2.4) and (2.5) are derived from (2.3) within the frame of Rational Thermodynamics should be made; a more detailed analysis is given in [63].

The "dissipation postulate" requires that (2.3) holds for all smooth admissible processes, i.e., smooth solutions of the balance equations (momentum and energy) and the constitutive equations (2.1) and (2.2).

Two approaches can be used to realize an arbitrary thermodynamic process in the body. From a physical standpoint these processes are controlled by prescribing the distribution and history of the body forces and the heat supply as well as initial and boundary conditions. In the approach of Coleman and Noll [117] the *body force* and the *heat supply* are required to be assignable in any way but initial and boundary conditions need not to be considered explicitely. This assures that *locally* and *instantly* the observable variables strain ε_{kl} and temperature T and it gradients $g_k = T_{,k}$ as well as their rates $\dot{\varepsilon}_{kl}$, $\dot{T}$, $\dot{g}_k$ may take arbitrary values. For example, the energy balance equation

$$\rho\dot{\epsilon} = \rho\left[\frac{\partial\hat{\epsilon}}{\partial\varepsilon_{mn}}\dot{\varepsilon}_{mn} + \frac{\partial\hat{\epsilon}}{\partial T}\dot{T} + \frac{\partial\hat{\epsilon}}{\partial g_k}\dot{g}_k + \frac{\partial\hat{\epsilon}}{\partial\alpha_\gamma}\dot{\alpha}_\gamma\right] = \sigma_{kl}\dot{\varepsilon}_{lk} - q_{k,k} + \rho r \tag{2.6}$$

does not impose a coupling on the rates $\dot{\varepsilon}_{kl}$, $\dot{T}$, $\dot{g}_k$ since r is arbitrary assignable by assumption. Therefore, the balance equations can always be satisfied for an arbitrary set of values ε_{kl}, T, g_k and $\dot{\varepsilon}_{kl}$, $\dot{T}$, $\dot{g}_k$.

In the approach of Liu [118] and Müller [62] extended by the author [63] the body force and heat supply are not assignable any more but only the initial and boundary conditions. Thus, the balance equations represent auxiliary conditions which have to be considered explicitely to account for the coupling; details may be found in [63].

To elucidate the results (2.4) and (2.5) somewhat further the more simple approach of Coleman and Noll is followed. The local energy balance allows to eliminate the heat supply ρr from $(2.3)_1$. Insertion of the assumed constitutive functions (2.1) and (2.2) into (2.3) yields an expression for the entropy production σ which involves only the independent variables ε_{mn}, T, g_k, α_ρ and the rates of the observable variables $\dot{\varepsilon}_{mn}$, $\dot{T}$, $\dot{g}_k$; thus

$$\sigma = \hat{\sigma}(\varepsilon_{mn}, T, g_k, \alpha_\gamma, \dot{\varepsilon}_{mn}, \dot{T}, \dot{g}_k) \geq 0.$$

This expression, when written down explicitly, contains the rates only *linearly*. According to the Coleman-Noll approach the arguments of the $\hat{\sigma}$-function are arbitrary and independent quantities at any material point and at any instant and the inequality must be satisfied for any arbitrary list of arguments, i.e. arbitrary rates and arbitrary state variables. Thus, one obtaines two groups of necessary and sufficient conditions on the constitutive functions:

(1) The satisfaction of the inequality for *arbitrary rates* can be easily realized since the rates are only linearly involved. One obtains (2.4) and this represents restrictions on the constitutive functions and not on the state variables

ε_{kl}, T, g_k, α_ρ since they can take locally and instantly arbitrary values in a process.

(2) The above result allows to reduce the dissipation inequality to the residual inequality (2.5). Here the left hand side is only depending on the state variables and a set of material parameters if the general form of the involved constitutive functions are assumed to be given. Thus, inequality (2.5) represents an algebraic condition. Necessary and sufficient conditions on the material parameters have to be derived such that inequality (2.5) is satisfied for arbitrary and independent state variables ε_{kl}, T, g_k, α_ρ. In general the realization of this requirement may be a formidable mathematical task.

In the following the dissipation (entropy production rate times absolute temperature) due to a change in the internal variables, i.e., the first term in (2.5), will be called *intrinsic dissipation*; the contribution due to heat conduction, that is the second term in (2.5), is the *thermal dissipation*.

For zero temperature gradient $g_k = T_{,k} = 0$ one obtains from (2.5) for the intrinsic dissipation

$$-\varrho \frac{\partial \hat{\psi}(\varepsilon_{mn}, T, \alpha_\xi)}{\partial \alpha_\gamma} f_\gamma(\varepsilon_{mn}, T, 0, \alpha_\nu) \geq 0 .$$

However, in general a separation of the two dissipation terms in (2.5) into two inequalities is not possible. A separation is obtained when the evolution equations of the internal variables are assumed to be independent of the temperature gradient, i.e.,

$$\dot{\alpha}_\gamma = f^*_\gamma(\varepsilon_{mn}, T, \alpha_\nu) \tag{2.7}$$

Then, from (2.5) two separate inequalities are derived when it is observed that locally the temperature gradient may assume arbitrary values:

intrinsic dissipation

$$-\varrho \frac{\partial \hat{\psi}(\varepsilon_{mn}, T, \alpha_\xi)}{\partial \alpha_\gamma} f^*_\gamma(\varepsilon_{mn}, T, \alpha_\nu) \geq 0 \tag{2.8}$$

thermal dissipation

$$-\frac{1}{T}\,\hat{q}_k(\varepsilon_{mn}, T, g_l, \alpha_\nu)\, g_k \qquad \geq 0 \qquad \Bigg\} \quad (2.8)$$

These inequalities have to be satisfied whatever the values of the variables $\varepsilon_{mn} = \varepsilon_{nm}$, T, g_k, and α_ν.

If we consider an *isothermal relaxation process at uniform temperature*, i.e.,

$$\varepsilon_{mn} = \text{const.}, \quad T = \text{const.}, \quad g_k = 0, \tag{2.9}$$

then

$$\frac{d\hat{\psi}}{dt} = \frac{\partial\hat{\psi}}{\partial\alpha_\gamma}\,\dot{\alpha}_\gamma$$

and from (2.5) it follows

$$\left.\frac{d\hat{\psi}}{dt}\right|_{\substack{\varepsilon_{kl},\, T \,\hat{=}\, \text{const.}\\ g_k = 0}} \leq 0; \tag{2.10}$$

thus, during such a process the free energy is a decreasing function of time.* The same conclusion applies when equ. (2.7) is assumed to be valid, but the temperature field needs not to be uniform.

Under the assumption (2.7) the inequality $(2.8)_1$ is valid and it requires that the free energy $\hat{\psi}$ and the evolution function f^*_γ are intimately related. This relation may be given a geometrical interpretation. For constant strain and temperature, i.e.,

$$\varepsilon_{kl} = \varepsilon^*_{kl} = \text{const.}, \qquad T = T^* = \text{const.}$$

a constant free energy

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* The case $d\psi/dt \equiv 0$ is excluded.

$$\hat{\psi}(\varepsilon^*_{mn}, T^*, \alpha_\gamma) = const.$$

represents a surface in the subspace of the internal variables α_γ. The n-tuple $\{\partial\hat{\psi}/\partial\alpha_\gamma\}_{\varepsilon,T = const}$ is a vector which is normal to this surface. The changes (rates) in the internal variables and thus the evolution functions f^*_γ represent an other vector in this subspace, and the left hand side of $(2.8)_1$ is simply the scalar product of these two vectors. Consequently, the *inequality $(2.8)_1$ requires that the two vectors form a blunt angle* as shown in fig. 1 for the case of two internal variables, and the choice of the evolution functions f^*_γ and the free energy $\hat{\psi}$ must be such that this condition is satisfied for all variables ε^*_{mn}, T^*, and α_γ. Note that we exclude the case where the two vectors are orthogonal such that the intrinsic dissipation $(2.8)_1$ vanishes identically for all variables.

2.2 Some General Observations Concerning the Evaluation of the Residual Entropy Inequality at Equilibrium States and Stability Considerations

We focus our attention on an arbitrary material point x_k in the body. Equally well we may consider a finite body in a homogeneous state.

An equilibrium state at a material point is defined by the condition that the independent variables

$$\Pi := \{\varepsilon_{mn}, T, g_k, \alpha_\gamma\} \tag{2.11}$$

do not change with time,

$$\dot{\Pi} = \{\dot{\varepsilon}_{mn}, \dot{T}, \dot{g}_k, \dot{\alpha}_\gamma\} = \{0\} \tag{2.12}$$

and that the temperature gradient g_k vanishes:

$$g_k = 0. \tag{2.13}$$

Consequently, the dependent variables σ_{kl}, q_k, ε, η are independent of time. It should be noted that this definition does not necessarily include the vanishing of the heat flux and the stress.

Since the rates of the internal variables α_γ vanish at equilibrium, it follows for the evolution function f_γ, equ. (2.2),

$$\dot{\alpha}_\gamma = f_\gamma(\varepsilon_{mn}, T, 0, \alpha_\nu) = 0 \quad , \quad \gamma = 1, \ldots N. \tag{2.14}$$

It will be required that the functions f_γ satisfy these conditions at least at one point ($\varepsilon_{mn}{}^E$, T^E, 0, $\alpha_\gamma{}^E$) in the space of the independent variables. Thus, every equilibrium state

$$\Pi^E := \{\varepsilon^E_{mn}, T^E, 0, \alpha^E_\gamma\} \tag{2.15}$$

is a solution of (2.14) and therefore the equilibrium values $\varepsilon_{mn}{}^E$, T^E, $\alpha_\nu{}^E$ are not independent of each other.

Inspection of the residual dissipation inequality (2.5) and the constitutive functions $\hat{\psi}$, f_ν, and $\hat{q}_k$ shows that the entropy production rate σ or the dissipation γ can be represented as a function of the independent variables ε_{mn}, T, g_k, and α_ν, i.e.,

$$T\sigma = \gamma = \hat{\gamma}(\varepsilon_{mn}, T, g_k, \alpha_\nu). \tag{2.16}$$

The functions $\hat{\psi}$ and $\hat{q}_k$, up to and including higher derivatives, are continuous by assumption. The evolution functions f_ν are presently assumed to be of class C^2 but in the later course of the development discontinuous functions will be allowed. Consequently, for the moment being, the dissipation γ is assumed to be at least a function of class C^2 of its arguments.

Generally, the residual dissipation inequality is required to hold for all sets $\mathcal{T}$ where it is defined. The evaluation of this requirement may be rather cumbersome. However, restricted but important conclusions with respect to the constitutive functions can be obtained if the residual dissipation inequality is evaluated at an equilibrium state. In the following some general observations are made and discussed.

Taking due account of (2.12) and (2.13), it is evident that the dissipation γ vanishes at an equilibrium state

$$\hat{\gamma}(\varepsilon_{mn}{}^E, T^E, 0, \alpha_\nu{}^E) = 0. \tag{2.17}$$

Generally, it is assumed that the materials under consideration are such that the dissipation is positive except at an equilibrium state. If this assumption applies, then the dissipation has a relative minimum at an equilibrium state. The assumed continuity and differentiability of γ then implies that the first variation of γ vanishes at this state and the second variation is positive definite*, i.e.,

$$(\delta\hat{\gamma})_E = 0 \quad , \quad (\delta^2\hat{\gamma})_E > 0 \tag{2.18}$$

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* If the second variation vanishes the third variation must be positive etc.

for all variations of the state variables ε_{mn}, T, g_m, and α_γ. If γ is not strictly positive in the neighborhood of an equilibrium state but possibly vanishing, then $(2.18)_2$ is relaxed to read

$$(\delta^2 \gamma)_E \geq 0 . \tag{2.19}$$

The first variation yields the following conditions at an equilibrium state

$$\left.\begin{aligned}
\left[\frac{\partial\hat{\gamma}}{\partial\varepsilon_{mn}}\right]_E &= \left[-\rho\frac{\partial^2\hat{\psi}}{\partial\alpha_\gamma\partial\varepsilon_{mn}} f_\gamma - \rho\frac{\partial\hat{\psi}}{\partial\alpha_\gamma}\frac{\partial f_\gamma}{\partial\varepsilon_{mn}} - \frac{1}{T}\frac{\partial\hat{q}_k}{\partial\varepsilon_{mn}} g_k\right]_E = 0 \\
\left[\frac{\partial\hat{\gamma}}{\partial T}\right]_E &= \left[-\rho\frac{\partial^2\hat{\psi}}{\partial\alpha_\gamma\partial T} f_\gamma - \rho\frac{\partial\hat{\psi}}{\partial\alpha_\gamma}\frac{\partial f_\gamma}{\partial T} - \right. \\
&\qquad\qquad \left.\left(\frac{1}{T}\frac{\partial\hat{q}_k}{\partial T} - \frac{1}{T^2}\hat{q}_k\right) g_k\right]_E = 0 \\
\left[\frac{\partial\hat{\gamma}}{\partial g_m}\right]_E &= \left[-\rho\frac{\partial\hat{\psi}}{\partial\alpha_\gamma}\frac{\partial f_\gamma}{\partial g_m} - \frac{1}{T}\frac{\partial\hat{q}_k}{\partial g_m} g_k - \frac{1}{T}\hat{q}_k\right]_E = 0 \\
\left[\frac{\partial\hat{\gamma}}{\partial\alpha_\rho}\right]_E &= \left[-\rho\frac{\partial^2\hat{\psi}}{\partial\alpha_\gamma\partial\alpha_\rho} f_\gamma - \rho\frac{\partial\hat{\psi}}{\partial\alpha_\gamma}\frac{\partial f_\gamma}{\partial\alpha_\rho} - \frac{1}{T}\frac{\partial\hat{q}_k}{\partial\alpha_\rho} g_k\right]_E = 0
\end{aligned}\right\} \quad (2.20)$$

and this reduces to

$$\left.\begin{aligned}
(2.20)_1 \;&\Longrightarrow\; \left[\frac{\partial\hat{\psi}}{\partial\alpha_\gamma}\frac{\partial f_\gamma}{\partial\varepsilon_{mn}}\right]_E &&= 0 \\
(2.20)_2 \;&\Longrightarrow\; \left[\frac{\partial\hat{\psi}}{\partial\alpha_\gamma}\frac{\partial f_\gamma}{\partial T}\right]_E &&= 0 \\
(2.20)_3 \;&\Longrightarrow\; \left[\frac{\partial\hat{\psi}}{\partial\alpha_\gamma}\frac{\partial f_\gamma}{\partial g_m} + \frac{1}{\rho T}\hat{q}_k\right]_E &&= 0 \\
(2.20)_4 \;&\Longrightarrow\; \left[\frac{\partial\hat{\psi}}{\partial\alpha_\gamma}\frac{\partial f_\gamma}{\partial\alpha_\rho}\right]_E &&= 0
\end{aligned}\right\} \quad (2.21)$$

If the variables ε_{mn}, T, g_k, and α_γ are renamed and collected in the array

$$\Pi_A = \{\varepsilon_{mn}, T, g_k, \alpha_\gamma\}_A \quad , \quad A = 1, 2, \ldots, 2\times3+1+3+N, \tag{2.22}$$

then the condition on the second variation (2.19) can be represented as

$$\left(\delta^2 \hat{\gamma}\right)_E = \left(\frac{\partial^2 \hat{\gamma}}{\partial \Pi_A \partial \Pi_B}\right) \delta\Pi_A \delta\Pi_B \geq 0 \qquad \forall\, \delta\Pi_A \wedge \delta\Pi_B, \tag{2.23}$$

It is obvious that this implies that the diagonal members of the matrix $\left(\partial^2 \hat{\gamma} / \partial \Pi_A \, \partial \Pi_B\right)_E$ must be non-negative, that is,

$$\left(\frac{\partial^2 \hat{\gamma}}{\partial \Pi_A \partial \Pi_A}\right)_E \geq 0 \quad , \qquad \text{no summation;} \tag{2.24}$$

these are necessary conditions but they are not sufficient. In fact condition (2.23) implies that the matrix $\left(\partial^2 \hat{\gamma} / \partial \Pi_A \, \partial \Pi_B\right)_E$ is positive semi-definite. A necessary and sufficient conditon is that the principal minor determinants of the matrix are positive semi-definite [64]. We will not go into the details of this analysis at this level of generality.

However, some further discussion of (2.21) is in place. It is fairly obvious that conditions (2.21) are satisfied if it is assumed that

$$\left[\frac{\partial \hat{\psi}}{\partial \alpha_\gamma}\right]_E = 0 \tag{2.25}$$

and then condition $(2.21)_3$ implies that

$$\left[\hat{q}_k\right]_E = 0 \, , \tag{2.26}$$

However, the assumption (2.25) appears to be rather arbitrary. On the other hand, if the n x n matrix $\left[\partial f_\gamma / \partial \alpha_\gamma\right]$ is not singular,

$$\det\left[\frac{\partial f_\gamma}{\partial \alpha_\varsigma}\right]_E \neq 0 \, , \tag{2.27}$$

then equ. (2.25) and (2.26) *follow* from $(2.21)_4$. Enquiring which mathematical or physical requirement may yield the condition (2.27), the following observations are made.

Assume that the set $\mathcal{T}^E$, definition (2.15), satisfies the equilibrium condition (2.14). The (n) conditions (2.14) represent (n) nonlinear algebraic equations for the (n) equilibrium values α_γ^E. According to the implicit function theorem [65, 66], there exists a unique continuous set of functions

$$\alpha_\gamma = g_\gamma(\varepsilon_{mn}, T) \quad , \qquad \gamma = 1, \dots, N \tag{2.28}$$

in the neighborhood of the equilibrium state $\mathcal{T}^E$ which are continuous differentiable if the Jacobian matrix $[\partial f_\gamma / \partial \alpha_\rho]$ at the equilibrium point is non-singular, i.e. (2.27) applies. Thus, if (2.27) is valid, then, a unique solution α_γ^E of equ. (2.14) in terms of ε^E_{mn} and T^E exists. However, it should be noted that the implicit function theorem is only a sufficient condition. One can construct examples* which do not satisfy (2.27) but still allow a unique solution (2.28).

Another interpretation of the requirement (2.27) is obtained from *stability conditions of an equilibrium state.* We consider an equilibrium state $\mathcal{T}^E$ and assume that a perturbation has changed the variables $\mathcal{T}$ to

$$\mathcal{T}^0 := \left\{ \varepsilon^E_{mn} \, , \, T^E \, , \, 0 \, , \, \alpha^0_\nu \right\} . \tag{2.29}$$

Note that the observable variables of the disturbed state have the same values as in the equilibrium state $\mathcal{T}^E$, but the internal variables have changed from α_ρ^E to α_ρ^0. In the following we restrict ourselves to infinitesimal perturbations such that

$$\alpha_\varrho^* = \alpha_\varrho^E + d\alpha_\varrho .$$

---

*

$$f_1\left(A(\varepsilon,T), \alpha_1, \alpha_2\right) = \left(\alpha_1 - A(\varepsilon,T)\right)^2 \alpha_2^2$$

$$f_2\left(A(\varepsilon,T), \alpha_1, \alpha_2\right) = \left(\alpha_2 - A(\varepsilon,T)\right)^2 \alpha_1^2$$

where A is some continuous function; here all derivatives of f_i with respect to A, α_1, α_2 vanish at the equilibrium point $\alpha_1 = \alpha_2 = A(\varepsilon, T)$

A *static stability criterion* for the equilibrium state $\mathcal{T}^E$ would require that the equilibrium state $\mathcal{T}^E$ does not admit an other neighboring equilibrium state $\mathcal{T}^o$, i.e., the equilibrium state $\mathcal{T}^E$ is uniquely determined by the choice of $\varepsilon_{mn}{}^E$ and T^E.

If a neighboring equilibrium state exists, then the equilibrium condition for the perturbed state

$$
\begin{aligned}
&f_\gamma(\varepsilon_{mn}{}^E, T^E, 0, \alpha_\rho{}^E + d\alpha_\rho{}^E) = \\
&\left(\frac{\partial f_\gamma}{\partial \alpha_\rho}\right)_E d\alpha_\rho + \left(\frac{\partial^2 f_\gamma}{\partial \alpha_\rho \, \partial \alpha_\mu}\right)_E d\alpha_\rho \, d\alpha_\mu + \ldots \quad = 0 \qquad (2.30) \\
&\qquad\qquad\qquad\qquad \gamma = 1, \ldots, N
\end{aligned}
$$

must admit a *nontrivial* solution for $d\alpha_\rho$. However, when the matrix $(\partial f_\gamma / \partial \alpha_\rho)_E$ can be shown to be nonsingular, then, except for higher order terms in $d\alpha_\rho$, only a trivial solution for $d\alpha_\rho$ is obtained; thus, within the frame of this linearized analysis, a neighboring equilibrium state does not exist. Therefore, the condition

$$
\det\left(\frac{\partial f_\gamma}{\partial \alpha_\rho}\right)_E \neq 0
$$

assures stability of the equilibrium state $\mathcal{T}^E$ under constant strain and temperature.

The characterization of stability of an equilibrium state may also be done from a kinetic point of view. In fact, this seems to be the most natural approach since the transition from the perturbed to the equilibrium state is a process involving the rates of the internal variables. On the other hand, it is well known from the stability theory of elastostatics that a kinetic stability analysis is the most general and safe method [67, 68].

We consider again a perturbation of the equilibrium state $\mathcal{T}^E$ such that at time $t = 0$ the state is at $\mathcal{T}^o$, def. (2.29). The evolution of the internal variables is studied under the constraint of constant strain and temperature, ε_{mn}^E and T^E with $g_k = 0$. This evolution is governed by the following initial value problem

$$\left.\begin{array}{l} \dot{\alpha}_\gamma = f_\gamma(\varepsilon_{mn}^E, T^E, 0, \alpha_\xi) \\ \alpha_\gamma = \alpha_\gamma^0 \quad , \quad \text{for } t = 0 . \end{array}\right\} \quad (2.31)$$

Assume that the *equilibrium state* $\mathcal{T}^E$ *is locally asymptotically stable* under constant strain and temperature. Roughly speaking, this means that the evolution functions f_γ have properties at $\mathcal{T}^E$ such that the solution $\alpha_\gamma(t)$ of (2.31) tends to α_γ^E with increasing time, i.e.,

$$\alpha_\gamma(t) \longrightarrow \alpha_\gamma^E \qquad \text{for} \quad t \longrightarrow \infty , \qquad (2.32)$$

whatever the choice of the initial values α_γ^0 in the neighborhood of the equilibrium values α_γ^E.

A more precise definition is as follows [13]. We assume that the evolution functions f_γ admit at least one equilibrium point $\mathcal{T}^E$ in the $\mathcal{T}$ -space. The *domain of attraction D at constant strain and temperature of an equilibrium state* is the set of all initial values α_γ^0 such that the solution $\alpha_\gamma(t)$ of the initial value problem (2.31) exists for all $t \geqq 0$ and tends to α_γ^E. An equilibrium state $\mathcal{T}^E$ is said to be *locally asymptotically stable at constant strain and temperature* if D contains a neighborhood of α_γ^E, i.e., if there exists a $\varepsilon > 0$ such that every set of initial values α_γ^0 with $|\alpha_\gamma^0 - \alpha_\gamma^E| < \varepsilon$, is in D. Note that D is, of course, depending on $\mathcal{T}^E$. This definition implies that the equilibrium point α_γ^E is entirely within D but not on its boundary as illustrated in fig. 2 for the case of two internal variables.

Assume that the deviation from the equilibrium state

$$\xi_\gamma(t) = \alpha_\gamma(t) - \alpha_\gamma^E \qquad (2.33)$$

is small then the right hand side of (2.31) may possibly be developed in a convergent Taylor series

$$\begin{aligned} f_\gamma(\varepsilon_{mn}{}^E, T^E, 0, \alpha_\gamma) = {} & f_\gamma(\varepsilon_{mn}{}^E, T^E, 0, \alpha_\gamma{}^E) + \\ & + \left(\frac{\partial f_\gamma}{\partial \alpha_\varrho}\right)_E \xi_\varrho + \frac{1}{2}\left(\frac{\partial^2 f_\gamma}{\partial \alpha_\varrho \partial \alpha_\nu}\right)_E \xi_\varrho \xi_\nu + \dots \end{aligned} \tag{2.34}$$

such that (2.31) reads

$$\left.\begin{aligned} \dot{\xi}_\gamma &= \left(\frac{\partial f_\gamma}{\partial \alpha_\gamma}\right)_E \xi_\varrho + n_\gamma(\varepsilon_{mn}{}^E, T^E, \alpha_\varrho{}^E, \xi_\nu) \\ \xi_\gamma &= \alpha_\gamma{}^0 - \alpha_\gamma{}^E \qquad \text{for } t=0 \end{aligned}\right\} \tag{2.35}$$

where n_γ collects the terms nonlinear in ξ_ν. Instead of analyzing the stability properties of the nonlinear initial value problem (2.35), one may question whether the analysis of the linearized problem

$$\left.\begin{aligned} \dot{\xi}_\gamma &= \left(\frac{\partial f_\gamma}{\partial \alpha_\varrho}\right)_E \xi_\varrho \\ \xi_\gamma &= \alpha_\gamma{}^0 - \alpha_\gamma{}^E \qquad \text{for } t=0 \end{aligned}\right\} \tag{2.36}$$

gives sufficient information. Here the following theorems apply (e.g. [69]). Consider a nonlinear system of ordinary differential equations of first order

$$\dot{z}_i = A_{ij} z_j + n_i(z_n) \qquad , \ i=1,\dots,N$$

with initial conditions (2.37)

$$z_i(0) = e_i$$

where A_{ij} is a constant matrix and $n_i(z_n)$ satisfies the condition

$$n_i(z_n = 0) = 0 . \tag{2.38}$$

Then theorem (1) is valid [69]:

Theorem (1): If

(a) every solution of

$$\dot{z}_i = A_{ij} z_j \tag{2.39}$$

approaches zero as t → ∞

(b) $n_i(z_n)$ is continuous in some region about $z_i = 0$, $i = 1, \ldots, n$

(c) $\| n_i \| / \| z_i \| \rightarrow 0$ as $\| z_i \| \rightarrow 0$
(nonlinearity condition)*,
then $z_i = 0$ is a stable solution of (2.37). Furthermore, every solution of (2.37) for which $\| z_i(0) \|$ is sufficiently small approaches zero as $t \rightarrow 0$ (asymptotic stability).

Note that condition (a) implies asymptotic stability of the equilibrium state $z_i = 0$ of the linear system (2.39). However, from the theory of linear systems of ordinary differential equations of first order with constant coefficients it is known (e.g. [67]) that the asymptotic stability is assured if and only if all eigenvalues λ_ζ of the matrix A_{ij} have negative real parts, i.e.,

$$R_e(\lambda_\xi) < 0$$

irrespective of whether the eigenvalues are simple or not. A necessary and sufficient algebraic criterion for this requirement is due to Hurwitz [67, 70].

Theorem (2): If

(a) the matrix A_{ij} of the linear reduced system (2.39) possesses at least one characteristic root with positive real part

(b) $\| n_i(z_i) \| / \| z_i \| \rightarrow 0$ as $\| z_i \| \rightarrow 0$,

then the equilibrium point $z_i = 0$ is unstable not only for the linear reduced system (2.39) but also for the nonlinear system (2.37).

---

* $\| z_i \|$ is a norm of the n-tuple z_i, e.g. $\| a_i \| = \Sigma | a_i |$

From Theorem (1) und (2) and the above remarks it follows that the stability properties of the equilibrium state of the nonlinear system (2.37) can be judged in some cases on the basis of the matrix

$$A_{ij} = \left(\frac{\partial f_i}{\partial \alpha_j} \right)_E \quad , \quad i,j = 1,2,\dots,N$$

alone, ignoring the nonlinear terms $n_i\,(z_i)$. These criteria are related to the Jacobian $(\partial\, f_i/\partial\, a_j)_E$ as follows:

(a) If for *all* eigenvalues λ_ζ of the Jacobian $(\partial\, f_i/\partial\, a_j)_E$

$$R_e\,(\lambda_\zeta) < 0$$

then the equilibrium state Π^E of the nonlinear system (2.37) is locally asymptotically stable,

(b) if for only one eigenvalue λ_J

$$R_e\,(\lambda_J) > 0 ,$$

then the equilibrium state Π^E is unstable.

A peculiar situation (critical case) arises if one or more eigenvalues have a vanishing real part. Then, according to the linearized analysis, the equilibrium state Π^E may be stable or unstable* but certainly not asymptotically stable. However, most important, such a linearized analysis is not sufficient anymore: The stability properties of Π^E cannot be judged on the basis of the properties of the Jacobian $(\partial f_i/\partial\, a_j)_E$ alone but the nonlinear terms $n_\gamma(z_i)$ have to be accounted for [67]. In the present context it is important to note that asymptotic stability may possibly be assured even in this case. The following theorem may illustrate this [71]:

<u>Theorem (3):</u> If

the Jacobian matrix $(\partial f_i/\partial a_j)_E$ has a simple eigenvalue $\lambda_K = 0$ at the equilibrium state Π^E but only eigenvalues with negative real parts in the neigh-

---

* In the strictly linear case further properties of the Jacobian $(\partial f_i/\partial\, a_j)_E$ must be analysed.

borhood of $\mathcal{T}^{E}$, then the equilibrium state $\mathcal{T}^{E}$ is asymptotically stable. However, if at every state in the neighborhood of $\mathcal{T}^{E}$, excluding $\mathcal{T}^{E}$, the Jacobian has an eigenvalue with a positive real part, then $\mathcal{T}^{E}$ is unstable.

Assume that the Jacobian $(\partial f_i / \partial \alpha_j)_E$ has a simple eigenvalue $\lambda_K = 0$. Accounting for the fact that the determinant of the Jacobian can be represented by the product of the eigenvalues

$$\det(\partial f_i / \partial \alpha_j)_E = \lambda_1 \ \lambda_2 \ \ldots \ \lambda_K \ \ldots \ \lambda_N ,$$

it then follows that

$$\det(\partial f_i / \partial \alpha_j)_E = 0.$$

Thus, it is shown that, even for the case of a vanishing determinant of the Jacobian matrix at the equilibrium state, asymptotic stability may possibly be assured.

The aim of the above discussion was to develop rigorous arguments on which the non-singularity of the Jacobian matrix $(\partial f_i / \partial \alpha_j)_E$ at an equilibrium state $\mathcal{T}^{E}$ could be based. The following conclusions were obtained. Neither the requirement that the equilibrium conditions (2.14) should admit a unique solution for the internal variables in terms of the observable variables, nor the condition of asymptotic stability is necessary for the non-singularity of the Jacobian. There exist cases where either the first or the second condition is satisfied but still the Jacobian is singular. It is certainly true that the requirement of asymptotic stability of the equilibrium state puts restrictions on the evolution functions f_γ at $\mathcal{T}^{E}$ but there is presently no necessary and simple theorem available. The theorems available express only sufficient conditions.

However, Coleman and Gurtin [13], assuming locally asymptotic stability under constant strain and temperature, have proved the condition (2.25), i.e.,

$$\left(\frac{\partial \hat{\psi}}{\partial \alpha_\gamma} \right)_E = 0$$

in a direct way without enforcing (2.27).* If strain and temperature are constant

---

* Following the previous discussion on the validity of (2.27), the argument of Bowen [14] that asymptotic stability at constant strain and temperature implies (2.25) appears to be erroneous.

and the temperature gradient vanishes, then (2.10) applies, i.e., the free energy is a decreasing function of time. Thus

$$\hat{\psi}(\varepsilon_{mn}^E, T^E, \alpha_\gamma^0) \geq \hat{\psi}(\varepsilon_{mn}^E, T^E, \alpha_\gamma(t)) \geq \hat{\psi}(\varepsilon_{mn}^E, T^E, \alpha_\gamma^E) \tag{2.40}$$

where α_γ^0 characterizes the perturbed state in the neighborhood of α_γ^E, and $\alpha_\gamma(t)$ represents the solution of (2.31) which tends to the equilibrium value α_γ^E. Since the equilibrium state is locally asymptotically stable, every set of internal state variables α_γ in the neighborhood of α_γ^E may be a point along same relaxation trajectory which ultimately leads to the equilibrium values α_γ^E. Thus, it is not permitted that the equilibrium state is on the boundary of the domain of attraction (fig. 2). Consequently,

$$\hat{\psi}(\varepsilon_{mn}^E, T^E, \alpha_\gamma) \geq \hat{\psi}(\varepsilon_{mn}^E, T^E, \alpha_\gamma^E) \tag{2.41}$$

where α_γ is any arbitrary set of internal state parameters close to α_γ^E. Since the free energy is assumed to be continuous and differentiable, inequality (2.41) yields

$$\left(\frac{\partial\hat{\psi}}{\partial\alpha_\gamma}\right)_E = \left(\frac{\partial\hat{\psi}}{\partial\alpha_\gamma}(\varepsilon_{mn}^E, T^E, \alpha_\gamma)\right)_{\alpha_\gamma=\alpha_\gamma^E} = 0 . \tag{2.42}$$

Assume that the material under consideration is strictly dissipative, i.e., the intrinsic dissipation is positive everywhere in the neighborhood of an equilibrium state. Then the equality sign in (2.41) applies only at the equilibrium state. Consequently, the free energy has a relative minimum at the equilibrium state with respect to all variations in the internal variables

$$(\delta_\alpha \hat{\psi})_E = 0 , \quad (\delta_\alpha^2 \hat{\psi})_E > 0 . \tag{2.43}$$

Condition $(2.43)_1$ is equivalent to (2.42) and $(2.43)_2$ reads*

---

* If the second variation of $\hat{\psi}$ should vanish at the equilibrium state then higher variations should be considered.

$$\left(\delta_\alpha^2 \hat{\psi}\right)_E = \left(\frac{\partial^2 \hat{\psi}(\varepsilon_{mn}{}^E, T^E, \alpha_\mu)}{\partial \alpha_\gamma \, \partial \alpha_\xi}\right)_{\alpha_\mu = \alpha_\mu{}^E} \delta\alpha_\gamma \, \delta\alpha_\xi > 0 \qquad \forall \, \delta\alpha_\nu \tag{2.44}$$

which means that the matrix of the second derivatives is positive definite. It is obvious that equs. (2.43) represent important constitutive restrictions on the structure of the free energy, and we come back to this when the Robinson model is analysed.

Conditions (2.43) are consequences of three independent requirements:

(I) The entropy production or dissipation is non-negative.

(II) The considered equilibrium state is locally asymptotically stable.

(III) The free energy is of class C^2.

Note that requirement (I) involves both the free energy $\hat{\psi}$ and the evolution functions f_γ, that requirement (II) is a condition on the evolution functions alone, and that requirement (III) applies only to the free energy. The second requirement has been discussed to some extend but a necessary and sufficient criterion for the functions f_γ is not available. A sufficient condition is given by (2.27).

2.3 Internal Variable Models with Constraints and Discontinuities

2.3.1 Stating the Questions

It should be recalled that the derivation of the constitutive restrictions (2.4) and (2.5) is based on the dissipation postulate requiring that the Clausius-Duhem entropy inequality is satisfied identically for *all smooth* thermomechanical processes. Thermomechanical processes are represented by solutions to the nonlinear differential equations generated by the balance laws of mass, momentum, and energy as well as the assumed constitutive relations, prescribed initial and boundary conditions, and given body forces and heat supply. However, smoothness of these solutions cannot be taken for granted. Discontinuities in space and time of spatial gradients, temporal rates, or even the amplitudes of the solutions may be induced by discontinuities in the initial values, external constraints, forces or energy input, or by discontinuities in the constitutive relations. But even for smooth initial data, etc., material nonlinearities may increase the solution amplitude rapidly such that shock waves are generated. Of course, dissipation is a mechanism which may smoothen out the discontinuities.

In the present context only discontinuities in the constitutive relations are of interest. When modeling elastic-plastic or elastic-viscoplastic material behavior, smoothness of the constitutive functions is not necessarily acceptable. Therefore, the validity of the arguments and continuity requirements leading to (2.4) and (2.5) have to be reassessed when the evolution equations (2.2) change according to certain constraint conditions. However, the constitutive equations (2.1) are still assumed to be continuous, at least up to and including their first derivatives with respect to all their independent variables.

Thermomechanical processes with discontinuities and in connection with an evaluation of an entropy inequality have been primarily considered (eg. [72-78]) when the discontinuity is represented by a spatial singular surface (e.g. a shock, i.e., a mathematical model of a transition zone with large spatial gradients of field quantities). In addition material properties (like a surface density of internal energy or entropy) may be attached to a singular surface* to represent immanent physical properties of interfaces or thin layers [72, 73, 76, 78]. Therefore, constitutive relations are required to describe these material properties.

---

* Daher and Maugin [78] call this simply an "interface" or a "thermodynamical singular surface" in contrast to a "free singular surface" described above.

The jump conditions related to the balance of mass, impulse, and energy play the role of transition conditions between the adjacend regimes separated by the discontinuity surface. On the other hand, it has been pointed out by Dafermos [79, 81] that the jump condition related to the entropy inequality at an ordinary singular surface ("free singular surface") rules out certain processes* that are otherwise compatible with the balance laws of mass, momentum, and energy (Entropy Admissibility Criterion). Thus, for a spatial singular surface without material properties attached to it, the entropy inequality represents a restriction on processes and not on the constitutive behavior of the bulk material.

The presence of a discontinuity in the evolution equations for the internal variables may produce spatial discontinuities (surfaces, lines). However, even all material points of a threedimensional domain of the body may experience a discontinuity at the same time. This situation is realized especially under homogeneous conditions. Under these conditions only discontinuities in the time domain have to be accounted for. Eringen [83] has discussed thermodynamic processes which are discontinuous at discrete times in somewhat general terms without including spatial discontinuities. He assumed that the class of thermomechanical processes to be considered does include "local linear continuations" at the instant of discontinuity. However, this approach is somewhat artificial and does not clearly show the physical implications involved. Here another argumentation is developed to derive consequences from the entropy inequality when a discontinuity of the thermomechanical process is induced in the time domain by properties of the evolution functions.

The primary question to be answered is, whether the general constitutive restrictions (2.4) and (2.5), derived for smooth processes, are affected or must be supplemented by additional restrictions if the evolution functions f_γ are subject to constraint conditions yielding a discontinuous response. It is clear that an analysis of this question in full generality is beyond the range of this study. Therefore, we will restrict the analysis to a limited class of constitutive models which contains the Robinson model as a special case.

Before we proceed along these lines the constitutive class is described in more detail. It is assumed that there exist two constraint functions C_i $(\varepsilon_{mn}, T, g_k, \alpha_\nu)$,

---

* For example, in classical gas dynamics rarefaction shocks are compatible with the balance laws of mass, momentum, and energy but are inadmissible because they do not obey the entropy inequality.

$i = 1,2$ such that the following constraint conditions control the evolution equations as indicated:

$$\dot{\alpha}_\gamma = f_\gamma = \begin{cases} h_\gamma(\varepsilon_{mn}, T, g_k, \alpha_\nu)\,; & C_1 > 0 \text{ and } C_2 > 0 \\ l_\gamma(\varepsilon_{mn}, T, g_k, \alpha_\nu)\,; & C_1 \leq 0 \text{ or } C_2 \leq 0 \end{cases} \tag{2.45}$$

Thus, if the conjunction of the two conditions $C_1 > 0$ and $C_2 > 0$ applies, then the function h_γ governs the evolution of the internal variable α_γ; in all the other cases the function l_γ determines the rate $\dot{\alpha}_\gamma$.

Here two different cases are considered:

(1) Continuous transition, i.e., f_γ is at least of class C^0,

$$h_\gamma = l_\gamma \qquad \text{on } C_1 = 0\,, \; C_2 = 0$$

(2) Discontinuous transition*

$$h_\gamma \neq l_\gamma \tag{2.46}$$

on a surface defined by the constraints in equ. (2.45), i.e., composed of parts of the surfaces $C_1 = 0$ and $C_2 = 0$.

It should be noted that case (1) or (2) still permits the derivatives of the evolution functions to be discontinuous.

In the $(10+N)$-dimensional space of the independent variables

$$\mathcal{T} = \{\varepsilon_{mn} = \varepsilon_{nm}\,, \; T\,, \; g_k\,, \; \alpha_\gamma\} \tag{2.47}$$

in the following denoted by „$\mathcal{T}$-space", the conditions $C_1 = 0$ and $C_2 = 0$ represent two surfaces which may intersect. Locally, i.e., at a material point, the thermomechanical history is completely defined if ε_{mn}, T, g_k are given as functions of time and if the initial values of the internal variables are prescribed. Assuming

---

* Physical intuition may suggest that the discontinuous case is nonrealistic or even in contrast to some general principle; nevertheless, constitutive models are proposed which have this property.

that (2.45) allows a unique solution, then the α_γ are obtained by integration and the variables σ_{kl}, ε, η, and q_k may be calculated. The history of the independent variables $\mathcal{T}$ up to time t_e may be represented by a trajectory (process path) in the $\mathcal{T}$-space and the time t $(0 \leqq t \leqq t_e)$ is the parameter of the trajectory. The material rates

$$\frac{D}{Dt}\mathcal{T} = \dot{\mathcal{T}} = \{\dot{\varepsilon}_{mn} = \dot{\varepsilon}_{nm}, \dot{T}, \dot{g}_k, \dot{\alpha}_\gamma\} \tag{2.48}$$

are collected in a vector in the $\mathcal{T}$-space which is tangential to the trajectory. Thus, a segment of the trajectory is defined by ($\dot{\mathcal{T}}$ dt). At all points in the $\mathcal{T}$-space where these rates are continuous functions of time the trajectory is a continuous *smooth* curve. An exception is when a trajectory, originally in the domain $(C_1 > 0 \cap C_2 > 0)$, traverses either the surface $C_1 = 0$ or $C_2 = 0$, i.e., the boundary of the domain $(C_1 \leqq 0 \cup C_2 \leqq 0)$, and vice versa; then the *continuous trajectory suffers a jump (kink) in its slope* on the surface $C_1 = 0$ or $C_2 = 0$ (fig. 3)*.

Consider a *smooth trajectory*. The segment ($\dot{\mathcal{T}}$ dt) at a point is not completely arbitrary even though the rates $\mathbb{R} = \{\dot{\varepsilon}_{mn}, \dot{T}, \dot{g}_k\}$ may take locally arbitrary values by a suitable choice of the thermomechanical process. This is so since the components ($\dot{\alpha}_\gamma$dt) of the segment depend only on the position in the $\mathcal{T}$-space and are fixed. Consequently, a given trajectory can be traversed only in the direction compatible with the evolution functions. Thus, to every trajectory a direction is attached except where the rates $\dot{\alpha}_\gamma$ vanish (α_γ = const. during the process).

Consider a process following a trajectory which intersects the surface $C_1 = 0$ or $C_2 = 0$ with a discontinuous transition (case (2), equ. (2.46)). Although most trajectories on either side of the surface of discontinuity can be passed through only in one direction, the surface can be intersected from both sides if a suitable choice for the rates $\dot{\varepsilon}_{ij}$, $\dot{T}$, $\dot{g}_k$ is made. This is illustrated by the following observation. The normal vector of the surface* $C = 0$ pointing from the negative domain $C < 0$ to the positive domain $C > 0$ is denoted by $\partial C/\partial \mathcal{T}_i$ where $\mathcal{T}_i$ is a component of the ordered set (2.47). The limiting values of the segment vectors tangential to a trajectory on either side of the surface $C = 0$ are denoted by $\dot{\mathcal{T}}^-_i$ dt and $\dot{\mathcal{T}}^+_i$ dt. A trajectory intersecting the surface from the negative to the positive side satisfies the conditions

---

* C stands for either C_1 or C_2 also in fig. 3.

$$\frac{\partial C}{\partial \Pi_i}\,\dot{\Pi}_i^- > 0 \quad \text{and} \quad \frac{\partial C}{\partial \Pi_i}\,\dot{\Pi}_i^+ > 0 . \tag{2.49}$$

If

$$\frac{\partial C}{\partial \Pi_i}\,\dot{\Pi}_i^- = \frac{\partial C}{\partial \varepsilon_{kl}}\,\dot{\varepsilon}_{kl}^- + \frac{\partial C}{\partial T}\,\dot{T}^- - \frac{\partial C}{\partial g_m}\,\dot{g}_m^- + \frac{\partial C}{\partial \alpha_g}\,\dot{\alpha}_g^- = 0$$

then the segment is tangential to the surface. Since the rates $\dot{\varepsilon}_{kl}^-$, $\dot{T}^-$, $\dot{g}_m^-$ are locally independent and arbitrary, they may be chosen such that the segment $\dot{\Pi}_i^-$ dt is pointing away from the surface C = 0, into the domain C < 0, i.e.,

$$\frac{\partial C}{\partial \Pi_i}\,\dot{\Pi}_i^- < 0 . \tag{2.50}$$

However, it should be observed that the segment Π_i^- dt built with the rates $\dot{\Pi}_i^-$ in (2.50) is part of a trajectory which is different from that trajectory which contains the a segment $\dot{\Pi}_i^-$ dt constructed with the rates of $(2.49)_1$.

Similar arguments apply to the segment ($\dot{\Pi}_i^+$ dt). Therefore, the surfaces can be intersected from both sides. An exceptional case is envisaged if the normal of the surface C = 0 has no components along the coordinates ε_{kl}, T, g_m, i.e., if

$$\frac{\partial C}{\partial \varepsilon_{kl}} = 0 \;, \quad \frac{\partial C}{\partial T} = 0 \;, \quad \frac{\partial C}{\partial g_m} = 0 \;, \quad \frac{\partial C}{\partial \alpha_g} \neq 0 . \tag{2.51}$$

Here the segment ($\dot{\Pi}^-$ dt) is always either pointing towards or away from the surface C = 0 whatever the choice of ($\dot{\varepsilon}_{mn}^-$, $\dot{T}^-$, $\dot{g}_k^-$).

The more readily treatable case (1) of (2.46) is considered first.

2.3.2 Continuous Evolution Functions

First, it is noted that the constraint condition do depend only on the variables but not on the rates

$$\mathbb{R} = \left\{ \dot{\varepsilon}_{mn} = \dot{\varepsilon}_{nm} , \dot{T} , \dot{g}_m \right\} \tag{2.52}$$

of the observable variables $\{\varepsilon_{mn}, T, g_m\}$. Second, it is observed that the constraint conditions do not restrict processes to a certain region of the $\mathbb{T}$-space. A simple example with this property may illustrate this: Assume that

$$\dot{\alpha}_\gamma = f_\gamma = \begin{cases} h_\gamma(\varepsilon_{mn}, T, g_m, \alpha_\nu) ; & C(\varepsilon_{mn}, T, g_m, \alpha_\nu) = 0 \\ \\ l_\gamma(\varepsilon_{mn}, T, g_m, \alpha_\nu) ; & C(\varepsilon_{mn}, T, g_m, \alpha_\nu) < 0 \end{cases} \tag{2.53}$$

and that the states in the region where $C > 0$ are inadmissible. Thus, when a process is such that the first condition applies along the trajectory, then

$$\dot{C} = \frac{\partial C}{\partial \varepsilon_{mn}} \dot{\varepsilon}_{mn} + \frac{\partial C}{\partial T} \dot{T} + \frac{\partial C}{\partial g_m} \dot{g}_m + \frac{\partial C}{\partial \alpha_\gamma} h_\gamma = 0. \tag{2.54}$$

Evidently, for a trajectory on the surface $C = 0$ the rates $\mathbb{R}$ of the observable variables are not independent any more. If the actual state is on the surface $C = 0$ but the process is such that it leaves the surfaces $C = 0$, then the rates $\mathbb{R}$ are subject to the constraint condition

$$\dot{C} = \frac{\partial C}{\partial \varepsilon_{mn}} \dot{\varepsilon}_{mn} + \frac{\partial C}{\partial T} \dot{T} + \frac{\partial C}{\partial g_m} \dot{g}_m + \frac{\partial C}{\partial \alpha_\gamma} h_\gamma < 0, \tag{2.55}$$

i.e., positive rates $\dot{C}$ are inadmissible.

For the states on the surface $C = 0$ the two conditions (2.54) and (2.55) on the rates $\mathbb{R}$ represent the inadmissibility condition of states in the domain $C > 0$. Consequently, instead of (2.53) and the verbal inadmissibility requirement we may write

$$\text{If}\begin{Bmatrix} C(\dots)=0 \\ \\ C(\dots)<0 \end{Bmatrix}\text{then}\quad \dot{\alpha}_\gamma=\begin{Bmatrix} h_\gamma(\dots) \\ \\ l_\gamma(\dots) \end{Bmatrix}\text{and } \mathbb{R}\begin{cases} \text{constrained by } \dot{C}\leq 0 \\ \\ \text{unconstrained} \end{cases} \tag{2.56}$$

It is seen that the inadmissibility of a domain in the $\mathbb{T}$-space implies constraint conditions for the rates of the observable variables. Obviously, this requires special notice when the local entropy inequality is evaluated for arbitrary admissible processes.

It is important to note that the constraint conditions in the assumed model (2.45) do involve only the actual state $\mathbb{T}$ but not details of the process, e.g., rates of the observable variables.

Finally, it is noted that the other constitutive functions $\hat{\sigma}_{mn}$, $\hat{q}_k$, $\hat{\varepsilon}$ and $\hat{\eta}$ are assumed to be at least of class* C^1 with respect to all their arguments and thus are not subject to any "switch" conditions.

The Clausius-Duhem entropy inequality takes now the same form as in ref. [63], equ. (2.15),

$$\begin{aligned}
&\left[\varrho\frac{\partial\hat{\eta}}{\partial\varepsilon_{mn}}-\frac{1}{T}\varrho\frac{\partial\hat{\epsilon}}{\partial\varepsilon_{mn}}+\frac{1}{T}\hat{\sigma}_{mn}\right]\dot{\varepsilon}_{mn}\\
&+\left[\varrho\frac{\partial\hat{\eta}}{\partial T}-\frac{1}{T}\varrho\frac{\partial\hat{\epsilon}}{\partial T}\right]\dot{T}\\
&+\left[\varrho\frac{\partial\hat{\eta}}{\partial g_m}-\frac{1}{T}\varrho\frac{\partial\hat{\epsilon}}{\partial g_m}\right]\dot{g}_m\\
&+\left[\varrho\frac{\partial\hat{\eta}}{\partial\alpha_\gamma}-\frac{1}{T}\varrho\frac{\partial\hat{\epsilon}}{\partial\alpha_\gamma}\right]f_\gamma-\frac{\hat{q}_k g_k}{T^2}\geq 0
\end{aligned} \tag{2.57}$$

---

* They are continuous functions up to and including their first derivatives.

where, depending on the set $\mathbb{T}$, the functions f_γ either take the form h_γ or l_γ.

In the following the Coleman-Noll approach [13], interpreted and described in [63], is used. With the same arguments as given in [63] the functions

$u_k(X_m,t)$: displacement vector

$T(X_m,t)$: absolute temperature

$\underset{0}{\alpha}_\gamma(X_m)$: initial values of internal variables

are sufficient to describe an admissible thermodynamic process; of course, it is implied that the new evolution equations (2.45) admit locally an unique solution.

For an arbitrary admissible thermodynamic process, characterized by an arbitrary choice of smooth functions $U_k(X_m, t)$, $T(X_m, t)$, and $\underset{0}{\alpha}_\gamma(X_m, t)$, the elements of the two sets

$$\begin{aligned} \mathbb{T} &= \{\varepsilon_{mn} = \varepsilon_{nm},\ T,\ g_m = T_{,m},\ \alpha_\gamma\} \\ \mathbb{R} &= \{\dot\varepsilon_{mn} = \dot\varepsilon_{nm},\ \dot T,\ \dot g_m = \dot T_{,m}\} \end{aligned} \qquad (2.58)$$

may take *locally* (i.e., for a material point) arbitrary values independent of each other at any time.

Consider a state $\mathbb{T}$ such that $C_1 > 0$ and $C_2 > 0$ are satisfied simultaneously. Since the rates $\mathbb{R}$ are independent of the state $\mathbb{T}$ and arbitrary as well as linearly involved in the entropy inequality, the following restrictions are obtained

$$\left.\begin{aligned} &\rho\frac{1}{2}\left(\frac{\partial\hat\eta}{\partial\varepsilon_{mn}} + \frac{\partial\hat\eta}{\partial\varepsilon_{nm}}\right) - \frac{1}{T}\rho\frac{1}{2}\left(\frac{\partial\hat e}{\partial\varepsilon_{mn}} + \frac{\partial\hat e}{\partial\varepsilon_{nm}}\right) + \frac{1}{T}\hat\sigma_{mn} = 0 \\ &\frac{\partial\hat\eta}{\partial T} - \frac{1}{T}\frac{\partial\hat e}{\partial T} = 0 \\ &\frac{\partial\hat\eta}{\partial g_m} - \frac{1}{T}\frac{\partial\hat e}{\partial g_m} = 0 \end{aligned}\right\} \qquad (2.59)$$

and the residual entropy inequality becomes

$$\rho\left(\frac{\partial\hat{\eta}}{\partial\alpha_\gamma} - \frac{1}{T}\frac{\partial\hat{\varepsilon}}{\partial\alpha_\gamma}\right)h_\gamma - \frac{\hat{q}_k\, g_k}{T^2} \geq 0. \tag{2.60}$$

If the state $\mathbb{T}$ is such that $C_1 < 0$ or $C_2 < 0$, the standard argumentation yields the same restrictions as (2.59) and the entropy inequality is identical to (2.60) except the functions h_γ have to be interchanged by l_γ:

$$\rho\left(\frac{\partial\hat{\eta}}{\partial\alpha_\gamma} - \frac{1}{T}\frac{\partial\hat{\varepsilon}}{\partial\alpha_\gamma}\right)l_\gamma - \frac{\hat{q}_k\, g_k}{T^2} \geq 0. \tag{2.61}$$

The same conclusions are obtained if the state $\mathbb{T}$ is on the surface $C_1 = 0$ (or $C_2 = 0$) since the rates $\mathbb{R}$ are not interrelated and may take arbitrary values on $C_1 = 0$. Naturally, there are *special trajectories* which stay on the surface $C_1 = 0$ such that

$$\dot{C}_1 = \frac{\partial C_1}{\partial\varepsilon_{mn}}\dot{\varepsilon}_{mn} + \frac{\partial C_1}{\partial T}\dot{T} + \frac{\partial C_1}{\partial g_m}\dot{g}_m + \frac{\partial C_1}{\partial\alpha_\nu} l_\nu = 0. \tag{2.62}$$

This represents a constraint on the rates $\mathbb{R}$. Thus, if the strain rates and the rates of the temperature gradient are taken as arbitrary, the temperature rate $\dot{T}$ must obey the following relation

$$\dot{T} = \frac{-1}{\partial C_1/\partial T}\left[\frac{\partial C_1}{\partial\varepsilon_{mn}}\dot{\varepsilon}_{mn} + \frac{\partial C_1}{\partial g_m}\dot{g}_m + \frac{\partial C_1}{\partial\alpha_\nu} l_\nu\right].$$

Inserting this relation into the Clausius-Duhem entropy inequality (2.57) and collecting terms with the same rates, one obtains for states $\mathbb{T}$ on $C_1 = 0$ and arbitrary rates $\dot{\varepsilon}_{mn} = \dot{\varepsilon}_{nm}$ and $\dot{g}_m$

$$
\begin{aligned}
&\Big[\varrho\,\frac{\partial\hat{\eta}}{\partial\varepsilon_{mn}} - \frac{1}{T}\varrho\,\frac{\partial\hat{\epsilon}}{\partial\varepsilon_{mn}} + \frac{1}{T}\hat{\sigma}_{mn} \\
&\qquad - \Big(\varrho\,\frac{\partial\hat{\eta}}{\partial T} - \frac{1}{T}\varrho\,\frac{\partial\hat{\epsilon}}{\partial T}\Big)\frac{\partial C_1/\partial\varepsilon_{mn}}{\partial C_1/\partial T}\Big]_{sym}\overset{\circ}{\varepsilon}_{mn} \\
&+\Big[\varrho\,\frac{\partial\hat{\eta}}{\partial g_m} - \frac{1}{T}\varrho\,\frac{\partial\hat{\epsilon}}{\partial g_m} \\
&\qquad - \Big(\varrho\,\frac{\partial\hat{\eta}}{\partial T} - \frac{1}{T}\varrho\,\frac{\partial\hat{\epsilon}}{\partial T}\Big)\frac{\partial C_1/\partial g_m}{\partial C_1/\partial T}\Big]\overset{\circ}{g}_m \\
&+\Big[\varrho\,\frac{\partial\hat{\eta}}{\partial\alpha_\gamma} - \frac{1}{T}\varrho\,\frac{\partial\hat{\epsilon}}{\partial\alpha_\gamma} \\
&\qquad - \Big(\varrho\,\frac{\partial\hat{\eta}}{\partial T} - \frac{1}{T}\varrho\,\frac{\partial\hat{\epsilon}}{\partial T}\Big)\frac{\partial C_1/\partial\alpha_\gamma}{\partial C_1/\partial T}\Big]l_\gamma - \frac{\hat{q}_k g_k}{T^2} \geq 0.
\end{aligned}
\tag{2.63}
$$

Since $\overset{\circ}{\varepsilon}_{mn}$ and $\overset{\circ}{g}_m$ are arbitrary on the surface $C_1 = 0$ and linearly involved in (2.64), the following conditions on $C_1 = 0$, i.e.,

$$
\left.
\begin{aligned}
&\Big[\varrho\,\frac{\partial\hat{\eta}}{\partial\varepsilon_{mn}} - \frac{1}{T}\varrho\,\frac{\partial\hat{\epsilon}}{\partial\varepsilon_{mn}} + \frac{1}{T}\hat{\sigma}_{mn} \\
&\qquad - \Big(\varrho\,\frac{\partial\hat{\eta}}{\partial T} - \frac{1}{T}\varrho\,\frac{\partial\hat{\epsilon}}{\partial T}\Big)\frac{\partial C_1/\partial\varepsilon_{mn}}{\partial C_1/\partial T}\Big]_{sym} = 0 \\
&\Big[\varrho\,\frac{\partial\hat{\eta}}{\partial g_m} - \frac{1}{T}\varrho\,\frac{\partial\hat{\epsilon}}{\partial g_m} \\
&\qquad - \Big(\varrho\,\frac{\partial\hat{\eta}}{\partial T} - \frac{1}{T}\varrho\,\frac{\partial\hat{\epsilon}}{\partial T}\Big)\frac{\partial C_1/\partial g_m}{\partial C_1/\partial T}\Big] = 0
\end{aligned}
\right\}
\tag{2.64}
$$

and a residual entropy inequality

$$\Big[\Big(\varrho\frac{\partial\hat{\eta}}{\partial\alpha_\gamma} - \frac{1}{T}\varrho\frac{\partial\hat{\epsilon}}{\partial\alpha_\gamma}\Big) - \Big(\varrho\frac{\partial\hat{\eta}}{\partial T} - \frac{1}{T}\varrho\frac{\partial\hat{\epsilon}}{\partial T}\Big)\frac{\partial C_1/\partial\alpha_\gamma}{\partial C_1/\partial T}\Big]l_\gamma - \frac{\hat{q}_k g_k}{T^2} \geq 0, \tag{2.65}$$

are obtained.

However, the relations (2.64) and (2.65) do not represent new restrictions on the constitutive equations since they are automatically satisfied if the constitutive relations obey the restrictions (2.59) and (2.60). This is simply due to the fact that on $C_1 = 0$ actually all combinations of rates $\mathcal{R}$ are admissible. Of course, the same argumentation applies to states on the surface $C_2 = 0$.

Finally, it should be noted that the introduction of the free energy $\hat{\psi} = \hat{\varepsilon} - \hat{\eta}T$ allows to write the restrictions (2.59) and (2.60) or (2.61), respectively in the simple form (2.4) and (2.5).

Summarizing, we observe that the constitutive restrictions (2.4) and the residual entropy inequality (2.5) retain their validity for every point in the $\mathcal{T}'$-space of independent variables, including points on the surfaces $C_1 = 0$ and $C_2 = 0$. Of course, the evolution functions f_γ in (2.5) are to be interpreted according to (2.45) with (2.46).

2.3.3 Discontinuous Evolution Functions

It is important to note that the analysis of the internal variable model in [13, 63] and in section 2.1 is based on the assumption that the forcing functions (e.g., volume force, heat supply), the initial and boundary conditions, and *all* constitutive functions are sufficiently smooth so that spatial and temporal derivatives of the independent and dependent variables exist for all material points and all states $\mathcal{T}'$. The presence of a discontinuity in the rates of the internal variables for states on the surfaces $C_1 = 0$ or $C_2 = 0$ in the $\mathcal{T}'$-space invalidates this prerequisite for trajectories which intersect these surfaces. However, the previous argumentation and results still apply to smooth processes which do not intersect the surfaces of discontinuity in the $\mathcal{T}'$-space. Therefore, and considering the discussion in section 2.3.1, the question arises whether the Clausius-Duhem entropy inequality yields additional restrictions on the processes passing through states on the surfaces of discontinuity in the $\mathcal{T}'$-space or on the constitutive relations at these states. Thus, observing the remarks in section 2.3.1, it appears sufficient to treat discontinuities in the time domain only.

The internal energy per unit mass is assumed to be a function of the independent variables collected in the set $\mathcal{T}$,

$$\epsilon = \hat{\epsilon}(\varepsilon_{kl}, T, g_m, \alpha_\gamma)$$

and again it is assumed to be continuous up to and including at least the first derivatives with respect to its arguments. Then its material time derivative is

$$\frac{D}{Dt}\epsilon = \dot{\epsilon} = \frac{\partial\hat{\epsilon}}{\partial\varepsilon_{kl}}\dot{\varepsilon}_{kl} + \frac{\partial\hat{\epsilon}}{\partial T}\dot{T} + \frac{\partial\hat{\epsilon}}{\partial g_m}\dot{g}_m + \frac{\partial\hat{\epsilon}}{\partial\alpha_\gamma}\dot{\alpha}_\gamma \qquad (2.66)$$

whenever these time derivatives are well defined. For states on the surfaces of discontinuity in the $\mathcal{T}$-space, i.e., $C_1=0$ or $C_2=0$, the rates of the internal variables are discontinuous, and this may induce discontinuities in the rates of the other variables. But it is not immediately obvious which quantities are affected. Therefore, an analysis is required which accounts for temporal discontinuities in the basic principles.

The energy balance equation is usually formulated as an *integral* statement with respect to some finite part of the body but *local* (*instant*) in time, i.e.,

$$\left.\begin{array}{l}
\frac{D}{Dt}(\underset{\text{internal energy}}{E} + \underset{\text{kinetic energy}}{K}) = \underset{\text{mechanical power}}{P} + \underset{\text{heating}}{H} \\[2ex]
\frac{D}{Dt}\int_V \rho(\epsilon + \frac{1}{2}\bar{v}\cdot\bar{v})\,dV = \underbrace{\int_V \rho\,\bar{b}\cdot\bar{v}\,dV + \oint_O \bar{t}\cdot\bar{v}\,dO}_{\text{total mechanical power}} \\[2ex]
\qquad\qquad + \underbrace{\int_V \rho r\,dV - \oint_O \bar{q}\cdot\bar{n}\,dO}_{\text{total heating}}
\end{array}\right\} \qquad (2.67)$$

where the time derivative refers to a material volume V contained in the surface O. The integration is performed in the present deformed configuration of the body. The quantitites are as follows

ϵ : internal energy per unit mass

ρ : density in the present configuration

$\bar{v}$: velocity vector

$\bar{b}$: body force per unit mass

r : heat supply per unit mass

$\bar{q}$: heat flux vector

$\bar{n}$: external unit vector on the deformed boundary surface O.

The present formulation does allow for the presence of strong *spatial* discontinuities in the body, well known in continuum mechanics (e.g. a shock). However, interfaces or layers, which can also be modeled by a strong discontinuity with different material properties (e.g. surface density of internal energy or entropy), are not contained in the above formulation. Nevertheless, in the following spatial discontinuities are excluded by assumption.

More important is the implicit assumption in equ. (2.67) that there exists a time derivative of the total energy. Allowing for a discontinuity in the rates of the internal variables, this is by no means obvious. Thus, a more general energy balance equation than (2.67) is required.

The total energy (E + K) of a material body is assumed to be discontinuous at a finite number of discrete times t*. Further, (E + K) is required to be bounded in the time domain and the existence of the limits

$$\lim_{\tau \to 0} (E+K)_{t^*-\tau} \;, \qquad \lim_{\tau \to 0} (E+K)_{t^*+\tau}$$

is implied. These properties are viewed as the result of a mathematical idealization (passage to the limit) of an energy history with a large time derivative in a small time interval $(t^*-\tau) \leqq t \leqq (t^*+\tau)$.

Taking the total energy as a continuous and smooth function of time, the derivative D(E + K)/Dt is defined and continuous and its integration over the arbitrary interval $(t_1 \leqq t^* \leqq t_2)$ gives

$$(E+K)_{t_2} - (E+K)_{t_1} \;.$$

This is valid also for a derivative D(E + K)/Dt discontinuous but bounded at a finite number of instants.

Time integration of the *power and heat terms* in (2.67) is well defined as long as these contributions are continuous or even discontinuous but *bounded functions of time* (finite jumps at a finite number of discrete times are permitted). Thus, the time-space integral energy balance reads

$$\left[\int_V \varrho\left(\epsilon + \tfrac{1}{2}\bar{v}\cdot\bar{v}\right)dV\right]_{t_2} - \left[\int_V \varrho\left(\epsilon + \tfrac{1}{2}\bar{v}\cdot\bar{v}\right)dV\right]_{t_1} = \int_{t_1}^{t_2}\left\{P + H\right\}dt. \tag{2.68}$$

which is valid for all time intervals $t_1 \leqq t \leqq t_2$ and material volumes V. This is the desired general formulation which contains the usual instant formulation as a special case: Assuming the power and heat (P + H) and the total energy (E + K) to be continuous and smooth functions of time, the localization in the time domain yields the standard formulation (2.67). But in addition, for more general functions P, H, E, and K, additional conditions (jump conditions in the time domain) are obtained.

In a similar way the instant formulation of the Clausius-Duhem entropy inequality (e.g., ref. [63], equ. (1.1)) is generalized to read

$$\left[\int_V \varrho\eta\, dV\right]_{t_2} - \left[\int_V \varrho\eta\, dV\right]_{t_1} \geq \int_{t_1}^{t_2}\left\{\int_V \frac{\varrho r}{T}dV - \oint_O \frac{\bar{q}\cdot\bar{n}}{T}dO\right\}dt. \tag{2.69}$$

The time-space integral energy balance (2.68) and the entropy inequality (2.69) may also be formulated in the undeformed reference configuration R of the body [84]

$$\left.\begin{aligned} &\left[\int_{V_R} \rho_R \left(\epsilon + \tfrac{1}{2}\bar{v}\cdot\bar{v}\right) dV_R\right]_{t_1}^{t_2} = \\ &\int_{t_1}^{t_2}\left\{\int_{V_R} \rho_R\, \bar{b}\cdot\bar{v}\, dV_R + \oint_{O_R} \bar{t}_R\cdot\bar{v}\, dO_R \right. \\ &\qquad \left. + \int_{V_R} \rho_R\, r\, dV_R - \oint \bar{q}_R\cdot\bar{n}_R\, dO_R\right\} dt\,, \end{aligned}\right\} \quad (2.70)$$

$$\left.\begin{aligned} &\left[\int_{V_R} \rho_R\, \eta\, dV_R\right]_{t_1}^{t_2} \geq \\ &\int_{t_1}^{t_2}\left\{\int_{V_R} \frac{\rho_R\, r}{T}\, dV_R - \oint_{O_R} \frac{\bar{q}_R\cdot\bar{n}_R}{T}\, dO_R\right\} dt \end{aligned}\right\} \quad (2.71)$$

where V_R and O_R are volume and surface in the reference configuration and

$\rho_R = \rho\, dV/dV_R$: density in the initial undeformed reference configuration

$\bar{t}_R = \bar{t}\, dO/dO_R$: stress vector per unit undeformed surface

$\bar{q}_R\cdot\bar{n}_R = \bar{q}\cdot\bar{n}\, dO/dO_R$: heat flux through the surface in the undeformed configuration

$\bar{n}_R$: external unit vector on the undeformed reference surface

The requirement of invariance of the energy balance equation (2.70) under Galilean transformations yields the integral balance of momentum

$$\left[\int_{V_R} \varrho_R \bar{v}\, dV_R\right]_{t_1}^{t_2} = \int_{t_1}^{t_2} \left\{ \int_{V_R} \varrho_R \bar{b}\, dV_R + \oint_{O_R} \bar{t}_R\, dO_R \right\} dt. \qquad (2.72)$$

We now assume that the resultant force

$$\bar{F}(t) = \int_{V_R} \varrho_R \bar{b}\, dV_R + \oint_{O_R} \bar{t}_R\, dO_R \qquad (2.73)$$

is a vector function of *bounded variation* in the time domain, piecewise continuous but with discontinuities* at a finite number of time instants t_i. Therefore, the vector function $\bar{F}(t)$ is (Riemann) integrable [65]. The integral

$$\bar{J}(t) := \int_{t_0}^{t} \bar{F}(\tau)\, d\tau \qquad (2.74)$$

is uniquely defined at all instants $t_0 \leqq t$ in the time domain. Further, since $\bar{F}(t)$ is integrable, $\bar{J}(t)$ may be proved to be a continuous vector function [85]. Since $\bar{F}(t)$ is integrable everywhere in the time domain, it is also integrable in any interval $t_0 \leqq t_1 \leqq t \leqq t_2$ [85]. Since equ. (2.72) is required to hold for all intervals (t_1, t_2), one obtains from (2.72)

$$\left[\int_{V_R} \varrho_R \bar{v}\, dV_R\right]_{t_0}^{t} = \bar{J}(t) \qquad (2.75)$$

and thus the impulse

$$\int_{V_R} \varrho_R \bar{v}\, dV_R$$

is a continuous vector function of time. Since this applies to any volume V_R, the velocity field is a continuous function of time.

---

* It is assumed that the limiting values $\bar{F}(t_i + 0)$ and $\bar{F}(t_i - 0)$ exist; the value $\bar{F}(t_i)$ may be different from the limiting values but is assumed to be bounded.

With the even *stronger assumption of continuity of the resultant force $\bar{F}(t)$* for all volumes V_R and all instants t_i it may be proven that the material time derivative of the total impulse and also of the velocity field $\bar{v}(X_m, t)$ is continuous in the time domain. Consequently, the time-integral formulation of the balance of linear momentum (2.72) can be replaced by the instant formulation

$$\frac{D}{Dt}\int_{V_R} \rho_R\, \bar{v}\, dV_R = \int_{V_R} \rho_R\, \bar{b}\, dV_R + \oint_{O_R} \bar{t}_R\, dO_R\,. \tag{2.76}$$

The continuity and smoothness of the velocity field in the time domain implies continuity and smoothness of the kinetic energy, and thus the integral energy balance may be written as

$$\Big[\int_{V_R} \rho_R\, \varepsilon\, dV_R\Big]_{t_1}^{t_2} = \int_{t_1}^{t_2}\Big\{\int_{V_R} \rho_R\, (\bar{b} - \bar{v})\cdot\bar{v}\, dV_R + \oint_{O_R} \bar{t}_R\cdot\bar{v}\, dO_R + \int_{V_R} \rho_R\, r\, dV_R - \oint_{O_R} \bar{q}_R\cdot\bar{n}_R\, dO_R\Big\}\, dt \tag{2.77}$$

having used

$$\frac{D}{Dt}\int_{V_R} \rho_R\, \frac{1}{2}\bar{v}\cdot\bar{v}\, dV_R = \int_{V_R} \rho_R\, \dot{\bar{v}}\cdot\bar{v}\, dV_R\,.$$

Assuming that the *total heat input H* (per unit time) transmitted to the material volume V_R is a *bounded function of time* and using the same mathematical arguments as applied to the integral balance of linear momentum, it is shown that the total internal energy E and the specific internal energy ε are continuous functions of time.

If the *stronger assumption of continuity of the transmitted heat* (for an arbitrary volume V_R) is made, then the continuity of the material time derivative of E and ε follows as a consequence and an instant formulation of (2.77) is permitted.

Thus, under the above continuity assumption for the forces and the heat input, one obtains

$$\left.\begin{aligned}
&\lim_{\tau\to 0}\left[\epsilon(X_M, t_c+\tau) - \epsilon(X_M, t_c)\right] = 0\\
&\lim_{\tau\to 0}\left[\epsilon(X_M, t_c) - \epsilon(X_M, t_c-\tau)\right] = 0\\
&\text{and}\\
&\lim_{\tau\to 0}\frac{\epsilon(X_M, t_c+\tau)-\epsilon(X_M, t_c)}{\tau} = \lim_{\tau\to 0}\frac{\epsilon(X_M, t_c)-\epsilon(X_M, t_c-\tau)}{\tau}
\end{aligned}\right\} \quad (2.78)$$

where t_c is any instant during the thermomechanical process including those where the process path reaches a surface of discontinuity $C_1=0$ or $C_2=0$ in the $\mathcal{T}$ -space. Equ. (2.78) states that the internal energy and its material time derivative are continuous functions of time.

One of the basic constitutive assumptions is that the internal energy is taken as a function of the independent variables $\mathcal{T}$, i.e.,

$$\epsilon = \hat{\epsilon}\underbrace{(\varepsilon_{kl}, T, g_m, \alpha_\gamma)}_{\mathcal{T}}$$

which is assumed to be at least of class C^1 in all its arguments. If the variables $\mathcal{T}$ are assumed to be continuous functions of time, then, obviously, the internal energy is also a continuous function of time, and this is compatible with condition $(2.78)_1$.

In an analogous way the integral entropy inequality (2.71) is analysed to derive some consequences for the entropy η. We assume that the entropy supply

$$\int_{V_R}\frac{\rho_R\, r}{T}\, dV_R$$

and the entropy flux

$$- \oint_{O_R} \frac{\bar{q}_R \cdot \bar{n}_R}{T} \, dO_R$$

transferred to a material volume V_R are continuous functions of time (class C^0). This is based on the previous assumption that the local quantities r and $\bar{q}_R \cdot \bar{n}_R$, i.e., the heat supply and the heat flux as well as the temperature history $T(X_m, t)$ are continuous.

Consider a trajectory in the $\mathcal{T}$ -space which intersects one of the surfaces of discontinuity $C_1 = 0$ or $C_2 = 0$. Let t_c be the instant of intersection. We consider a time interval

$$t_c - \tau = t_1 \leq t \leq t_2 = t_c + \tau .$$

Observing the continuity of the integrand of the r.h.s. of (2.71) and passing to the limit $\tau \rightarrow 0$, we obtain from (2.71)

$$\lim_{\tau \rightarrow 0} \left[\int_{V_R} \rho_R \, \eta \, dV_R \right]_{t_c-\tau}^{t_c+\tau} \geq 0$$

or

$$\lim_{\tau \rightarrow 0} \int_{V_R} \rho_R \left[\eta(X_M, t_c+\tau) - \eta(X_M, t_c-\tau) \right] dV_R \geq 0. \tag{2.79}$$

Requiring validity for all material volumes V_R, equ. (2.79) yields

$$\lim_{\tau \rightarrow 0} \left[\eta(X_M, t_c+\tau) - \eta(X_M, t_c-\tau) \right] \geq 0. \tag{2.80}$$

It appears that this condition allows the entropy to be discontinuous in time. However, the entropy η is a function of the variables $\mathcal{T}$ by assumption, i.e.,

$$\eta = \hat{\eta} \underbrace{(\varepsilon_{KL}, T, g_m, \alpha_\gamma)}_{\mathcal{T}}$$

and according to the basic constitutive assumption, $\hat{\eta}$ is at least of class C^1 in all its arguments.

Since the set $\mathcal{T}$ is continuous in time, so is the entropy $\eta(X_m, t)$; consequently, the equality condition in equ. (2.80) is satisfied. Thus, the basic continuity assumption on $\hat{\eta}$ is in accordance with the requirement (2.80).

We consider now two time intervals just before and after the instant of intersection t_c, i.e.,

$$t_c - \tau = t_1 \leq t \leq t_2 = t_c - 0$$

$$t_c + 0 = t_1 \leq t \leq t_2 = t_c + \tau .$$

Application of the integral entropy balance equation (2.71) to these intervals, observing the continuity of the integrand on the r.h.s. of (2.71), and using the mean value theorem gives*

$$\left.\begin{aligned} \left[\int_{V_R} \varrho_R\, \eta \, dV_R\right]_{t_c-\tau}^{t_c-0} &\geq \tau \left\{ \int_{V_R} \frac{\varrho_R\, r}{T}\, dV_R - \oint_{O_R} \frac{\bar{q}_R \cdot \bar{n}_R}{T}\, dO_R \right\}_{MV^-} \\ \left[\int_{V_R} \varrho_R\, \eta \, dV_R\right]_{t_c+0}^{t_c+\tau} &\geq \tau \left\{ \int_{V_R} \frac{\varrho_R\, r}{T}\, dV_R - \oint_{O_R} \frac{\bar{q}_R \cdot \bar{n}_R}{T}\, dO_R \right\}_{MV^+} \end{aligned}\right\} \quad (2.81)$$

Passing to the limit $\tau \to 0$, one obtains

$$\left.\begin{aligned} \int_{V_R} \varrho_R \left(\frac{D}{Dt}\eta\right)_{t_c-0} dV_R &\geq H(t_c-0) \\ \int_{V_R} \varrho_R \left(\frac{D}{Dt}\eta\right)_{t_c+0} dV_R &\geq H(t_c+0) \end{aligned}\right\} \quad (2.82)$$

---

* The subcript MV± refers to the mean values of the two internals.

where

$$H_{(t)} := \int_{V_R} \frac{\varrho_R r}{T} \, dV_R - \oint_{O_R} \frac{\bar{q}_R \cdot \bar{n}_R}{T} \, dO_R \, . \tag{2.83}$$

Since H is continuous in time, the right hand sides of $(2.82)_1$ and $2.82)_2$ are equal. However, equs. (2.82) show that the entropy rate just before and after the instant of intersection are not required to be equal. Thus, according to the entropy inequality, the entropy rate may well be discontinuous.

The continuity of the rates of internal energy and kinetic energy allows to write the time-space integral energy balance equation (2.70) in the instant form (2.67). Further, the assumed absense of spatial discontinuities allows to formulate the *local version* of the energy balance valid at all material points. The same arguments are applicable to the integral balance of linear momentum (2.72). These local forms, not given here explicitely, allow to derive the *local reduced form of the energy balance* (2.6), i.e., a differential equation where the power of the body force is eliminated.

An instant form of the integral entropy inequality is valid only at those instants when the entropy rate $\dot{\eta}$ is continuous. The variables Π are continuous functions of time with continuous rates, except for states $\Pi = \Pi_c$ on the surfaces of discontinuity. Consequently, the rate

$$\dot{\eta} = \frac{\partial \hat{\eta}}{\partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} + \frac{\partial \hat{\eta}}{\partial T} \dot{T} + \frac{\partial \hat{\eta}}{\partial g_m} \dot{g}_m + \frac{\partial \hat{\eta}}{\partial \alpha_\gamma} \dot{\alpha}_\gamma \tag{2.84}$$

is continuous at all states except Π_c and condition (2.82) clearly shows that continuity is not required when the trajectory intersects the surfaces $C_1 = 0$ or $C_2 = 0$.

Consequently, the validity of an instant local energy balance equation and an instant local entropy inequality for all states $\Pi \neq \Pi_c$ allows to write the Clausius-Duhem entropy inequality in the form (2.3). For all states $\Pi \neq \Pi_c$ the usual argument yielding the results (2.4) and (2.5) applies [13, 63]; thus, the dependence of the functions $\hat{\varepsilon}$, $\hat{\eta}$ and $\hat{\sigma}_{ij}$ on the temperature gradient g_m is dropped for $\Pi \neq \Pi_c$. However, since *these functions are assumed to be of class C^1*, this applies also to states $\Pi = \Pi_c$. Therefore, in the following the dependence on g_m can be dropped for all states in the Π-space.

The material time derivative of the specific internal energy just before and after the intersection of the surfaces of discontinuity $C_1 = 0$ or $C_2 = 0$ are given by (2.66) at the appropriate instants. According to the continuity of this rate, one obtains

$$\begin{aligned}
&\left(\frac{\partial \hat{\epsilon}}{\partial \varepsilon_{kl}}\right)_c \left[\dot{\varepsilon}_{kl}(X_M, t_c + 0) - \dot{\varepsilon}_{kl}(X_M, t_c - 0)\right] + \\
&\left(\frac{\partial \hat{\epsilon}}{\partial T}\right)_c \left[\dot{T}(X_M, t_c + 0) - \dot{T}(X_M, t_c - 0)\right] + \\
&\left(\frac{\partial \hat{\epsilon}}{\partial \alpha_\gamma}\right)_c \left[\dot{\alpha}_\gamma(X_M, t_c + 0) - \dot{\alpha}_\gamma(X_M, t_c - 0)\right] = 0.
\end{aligned} \tag{2.85}$$

Since the velocity (and displacement) field are continuous in time, the strain rates are continuous and thus the first term of (2.85) vanishes. With the notation

$$\dot{T}(X_M, t_c \pm 0) = \dot{T}^{(\pm)} \tag{2.86}$$

equ. (2.85) reduces to

$$\left(\frac{\partial \hat{\epsilon}}{\partial T}\right)_c \left(\dot{T}^{(+)} - \dot{T}^{(-)}\right) + \left(\frac{\partial \hat{\epsilon}}{\partial \alpha_\gamma}\right)_c \left(\dot{\alpha}_\gamma^{(+)} - \dot{\alpha}_\gamma^{(-)}\right) = 0. \tag{2.87}$$

Consequently, the discontinuity in the evolution equations (2.45) induces a discontinuity in the temperature rate

$$\left(\dot{T}^{(+)} - \dot{T}^{(-)}\right) = \frac{-\left(\frac{\partial \hat{\epsilon}}{\partial \alpha_\gamma}\right)_c}{\left(\frac{\partial \hat{\epsilon}}{\partial T}\right)_c} \left(\dot{\alpha}_\gamma^{(+)} - \dot{\alpha}_\gamma^{(-)}\right). \tag{2.88}$$

The relation between the two discontinuities is linear and it is obvious from (2.88) that an increase in the rates of the internal variable yields a decrease in the temperature rate if

$$\left(\frac{\partial \hat{e}}{\partial \alpha_\gamma}\right)_c \Big/ \left(\frac{\partial \hat{e}}{\partial T}\right)_c > 0 \,, \quad \gamma = 1,\ldots,N.$$

A segment of a trajectory in the $\mathcal{T}$-space is defined by the rates of the variables $\mathcal{T}$, i.e., ($\dot{\mathcal{T}}$ dt). According to the above assumptions and the result (2.88), the following statement holds:

> If the limiting segment on one side of the surface of discontinuity in the $\mathcal{T}$-space - e.g. ($\dot{\mathcal{T}}^{-}$dt) - is given, then the limiting segment on the other side of the surface - e.g., ($\dot{\mathcal{T}}^{+}$dt) - is completely determined.

The jump in the entropy rate at $t = t_c$ is defined as

$$\dot{\eta}^{(+)} - \dot{\eta}^{(-)} = \lim_{\tau \to 0} \left(\dot{\eta}(X_M, t_c + \tau) - \dot{\eta}(X_M, t_c - \tau) \right); \tag{2.89}$$

observing (2.84) one obtains

$$\begin{aligned} \dot{\eta}^{(+)} - \dot{\eta}^{(-)} = &\left(\frac{\partial \hat{\eta}}{\partial T}\right)_c \left(\dot{T}^{(+)} - \dot{T}^{(-)}\right) + \\ &+ \left(\frac{\partial \hat{\eta}}{\partial \alpha_\gamma}\right)_c \left(\dot{\alpha}_\gamma^{(+)} - \dot{\alpha}_\gamma^{(-)}\right); \end{aligned} \tag{2.90}$$

here the continuity of the strain rates is taken into account. With (2.88) and the definition of the free energy ψ (page 7) the r.h.s. of (2.90) simplifies to

$$\dot{\eta}^{(+)} - \dot{\eta}^{(-)} = -\left(\frac{1}{T} \frac{\partial \hat{\psi}}{\partial \alpha_\gamma}\right)_c \left(\dot{\alpha}^{(+)} - \dot{\alpha}^{(-)}\right) \tag{2.91}$$

and it is obvious that the jump in the entropy rate is linearly related to the jumps in the rates of the internal variables. A comparison with (2.5) clearly shows that the discontinuity in the entropy rate is proportional to the discontinuity of the entropy production rate γ, i.e.*,

$$\dot{\eta}^{(+)} - \dot{\eta}^{(-)} \sim \sigma^{(+)} - \sigma^{(-)} \sim \gamma^{(+)} - \gamma^{(-)}. \tag{2.92}$$

---

* Note that the entropy production due to heat condition is continuous.

It should be remarked that the discontinuity in the evolution functions will also induce a jump in the stress rate

$$\dot{\sigma}_{ij}^{(+)} - \dot{\sigma}_{ij}^{(-)} = \left(\frac{\partial\hat{\sigma}_{ij}}{\partial T}\right)_c \left(\dot{T}^{(+)} - \dot{T}^{(-)}\right) + \left(\frac{\partial\hat{\sigma}_{ij}}{\partial\alpha_\gamma}\right)_c \left(\dot{\alpha}_\gamma^{(+)} - \dot{\alpha}_\gamma^{(-)}\right)$$

$$= \left[\left(\frac{\partial\hat{\sigma}_{ij}}{\partial\alpha_\gamma}\right)_c - \left(\frac{\partial\hat{\sigma}_{ij}}{\partial T}\right)_c \left(\frac{\partial\hat{\varepsilon}/\partial\alpha_\gamma}{\partial\hat{\varepsilon}/\partial T}\right)_c\right]\left(\dot{\alpha}_\gamma^{(+)} - \dot{\alpha}_\gamma^{(-)}\right) \qquad (2.93)$$

where $\hat{\sigma}_{ij}$ and $\hat{\varepsilon}$ are related to the free energy $\hat{\psi}$ by $(2.4)_1$ and $(2.4)_3$ which are valid also on the surface of discontinuity in the $\mathcal{T}$-space.

At this point it is appropriate to summarize the main results. The *basic assumptions* are

(1) The constitutive functions $\hat{\sigma}_{ij}$, $\hat{\varepsilon}$, $\hat{\eta}$, and $\hat{q}_i$ are continuous and have at least continuous first derivatives with respect to the variables $\mathcal{T} = \{\varepsilon_{ij}, T, g_m, \alpha_\gamma\}$.

(2) The evolution functions f_γ for the internal variables are allowed to be discontinuous along a surface composed of parts of the surfaces $C_1 = 0$ and $C_2 = 0$ in the space of the independent variables ($\mathcal{T}$-space).

(3) Spatial discontinuities are excluded from the discussion. Thus, it is understood that all material points of some material volume $V_{\mathfrak{R}}$ experience a discontinuity in the rates of the internal variables at the same time. This situation is realized especially under homogeneous conditions in the material volume $V_{\mathfrak{R}}$.

(4) The thermomechanical process is such that the rate of the energy input $(P + H)$ to a material volume $V_{\mathfrak{R}}$ is a continuous function of time. In accordance with this assumption it is also assumed that the resultant force on a material volume is continuous. It should be noted that energy and force transfer to a material volume may also be assumed to be discontinuous; for example, an isothermal process would require such a discontinuity. This will yield other jump conditions but no restrictions on the constitutive relations.

When trajectories in the $\mathcal{T}$-space are considered which do not intersect the surfaces of discontinuity, the standard Coleman-Noll argument may be applied yielding the constitutive restrictions (2.4) and (2.5) for all points in the $\mathcal{T}$-space except the surfaces of discontinuity. However, since the constitutive functions (2.1) are *assumed* to be continuous up to and including their first derivatives with respect to all arguments, the restrictions (2.4) apply also to states on the surface $C_1 = 0$ or $C_2 = 0$. Therefore, the independence of the functions (2.1) on the temperature gradient is valid for all states in the $\mathcal{T}$-space.

A special analysis, representing the main part of this section, is required for trajectories intersecting the surfaces of discontinuity. Under the above assumptions (1) to (4) and using *a time-space integral version of the energy balance* equation, the velocity field, kinetic energy, and internal energy are shown to be continuous with first material time derivatives continuous for all instants including those when a jump in the rate of the internal variables occurs. With these continuity properties it is then proved that the discontinuity in the evolution functions produces a discontinuity in the rate of the absolute temperature and stress.

The *time-space integral entropy inequality (2.69)* applied to a time interval enclosing the instant t_c of passing through the discontinuity imposes a restriction on the jump in the entropy at $t = t_c$: Only non-negative jumps are permitted. However, since the entropy is assumed to be at least a C^1-function of the independent variables $\mathcal{T}$ and these variables are continuous in time, the entropy must be a continuous function of time.

On the other hand, the entropy rates just before and after $t = t_c$ are not required to be equal. Indeed, it is shown that there is a jump in the entropy rate proportional to the jump in the rate of the internal variables.

It is concluded that the above constitutive assumptions are in accordance with the time-integral Clausius-Duhem entropy inequality, i.e., no further constitutive restrictions are obtained when a discontinuity in the evolution equations for the internal variables is present as described. However, this discontinuity naturally induces discontinuities in process variables.

2.4 The Residual Entropy Inequality for Constitutive Models with Internal Variables of Different Character

In the following it is assumed that the (n) internal variables a_γ may be separable in two sets*:

(1) a first set representing two 3x3 symmetric tensors** $a^{(i)}_{kl}$, i = 1, 2, in the threedimensional Euclidian (x_1, x_2, x_3)-space, and

(2) a single scalar κ.

Further, instead of the evolution equations (2.45) having the same constraint conditions, we consider evolution equations somewhat more general than (2.45):

$$\dot{\alpha}^{(i)}_{kl} = f^{(i)}_{kl} = \begin{cases} h^{(i)}_{kl}(\varepsilon_{mn}, T, g_m, \alpha^{(i)}_{rs}, \varkappa); & C^{(i)}_1 > 0 \cap C^{(i)}_2 > 0 \\ \\ e^{(i)}_{kl}(\varepsilon_{mn}, T, g_m, \alpha^{(i)}_{rs}, \varkappa); & C^{(i)}_1 \leq 0 \cup C^{(i)}_2 \leq 0 \end{cases}$$

$$i = 1, 2 \tag{2.94}$$

$$\dot{\varkappa} = f(\varepsilon_{mn}, T, g_m, \alpha^{(i)}_{rs}, \varkappa); \qquad \text{unconstrained} . \tag{2.95}$$

Again the Coleman-Noll argument in connection with the Clausius-Duhem entropy inequality is applied. The difference in the constraint functions does not affect the general structure of the constitutive restrictions. One obtains (2.4) and the residual dissipation inequality is

$$-\rho \frac{\partial \hat{\psi}}{\partial \alpha^{(1)}_{mn}} f^{(1)}_{mn} - \rho \frac{\partial \hat{\psi}}{\partial \alpha^{(2)}_{mn}} f^{(2)}_{mn} - \rho \frac{\partial \hat{\psi}}{\partial \varkappa} f - \frac{1}{T} \hat{q}_k g_k \geq 0 \tag{2.96}$$

---

* It is evident that this is only a special case in a more general setting with (k) tensors and (n) scalars. Requiring material frame indifference, Coleman and Gurtin [13] have proved that internal variables of *vectorial* character are not allowed in the evolution equations.

** It appears that the introduction of internal variables with a tensorial character is due to Perzyna and Wojno [19] where the form of the functions $f^{(1)}_{mn}$ and $f^{(2)}_{mn}$ depends on the variables

$$\mathbb{T} = \{ \mathcal{E}_{mn} = \mathcal{E}_{nm} , T , g_m , \alpha^{(1)}_{rs} , \alpha^{(2)}_{rs} , \varkappa \} \tag{2.97}$$

according to the constraint conditions.

Up to now the physical interpretation of the internal variables was left open. However, in section 1 it was already mentioned that these quantities may represent rather different physical entities, depending on the various physical phenomena to be studied. In microscopic level theories of crystalline materials the plastic deformation is governed by the movement, multiplication, interaction, and arrangement of crystal defects known as dislocations. Obviously, the parameters describing their evolution are not subjected to direct external control and therefore it appears reasonable to consider those parameters as internal (state) variables. However, it is still rather difficult to characterize these variables in a reasonable finite form for a particular material.

In dislocation theories of materials, the movement of dislocations cause permanent geometrical changes in the shape of the body. Therefore, an average of these dislocation movements in a volume element is a measure of the local shape changes which may be described by the macroscopic inelastic strain tensor ε^p_{kl}.* According to Kratochvil and Dillon [27], a detailed discussion of the relation between the inelastic strain and the motion of dislocations was given by Kröner and Rieder [86].

Following Kratochvil and Dillon [27], the first set of internal variables $\alpha^{(1)}_{kl}$ is identified as the inelastic strains ε^p_{kl}, and the following notations are introduced

$$\left.\begin{aligned} \alpha^{(1)}_{kl} &= \varepsilon^p_{kl} \\ \alpha^{(2)}_{kl} &= \alpha_{kl} \; . \end{aligned}\right\} \tag{2.98}$$

---

* The introduction of an "inelastic" strain tensor obviously deserves a much more elaborated analysis.

The evolution equations of the internal variables are denoted as follows

$$\left.\begin{aligned}
\dot{\varepsilon}^p_{kl} &= f^{(1)}_{kl} = f^p_{kl} = \begin{cases} f^{p+}_{kl}(\mathbb{T}) \;; & C^p_1(\mathbb{T}) > 0 \;\cap\; C^p_2(\mathbb{T}) > 0 \\[2ex] f^{p-}_{kl}(\mathbb{T}) \;; & C^p_1(\mathbb{T}) \le 0 \;\cup\; C^p_2(\mathbb{T}) \le 0 \end{cases} \\[4ex]
\dot{\alpha}_{kl} &= f^{(2)}_{kl} = f^\alpha_{kl} = \begin{cases} f^{\alpha+}_{kl}(\mathbb{T}) \;; & C^\alpha_1(\mathbb{T}) > 0 \;\cap\; C^\alpha_2(\mathbb{T}) > 0 \\[2ex] f^{\alpha-}_{kl}(\mathbb{T}) \;; & C^\alpha_1(\mathbb{T}) \le 0 \;\cup\; C^\alpha_2(\mathbb{T}) \le 0 \;. \end{cases}
\end{aligned}\right\} \quad (2.99)$$

With these notations the residual dissipation inequality (2.68) takes the following form

$$-\varrho \frac{\partial \hat{\psi}}{\partial \varepsilon^p_{mn}} f^p_{mn} - \varrho \frac{\partial \hat{\psi}}{\partial \alpha_{mn}} f^\alpha_{mn} - \varrho \frac{\partial \hat{\psi}}{\partial \varkappa} f - \frac{\hat{q}_k g_k}{T} \ge 0 . \qquad (2.100)$$

To circumvent superfluous generalities, the following specific assumptions are made

(1) The evolution functions f^p_{mn} , f^α_{mn} ,and f are independent of the temperature gradient. Consequently, the residual dissipation inequality (2.100) can be splitted into two separate inequalities analogous to (2.8).

(2) All constitutive functions ψ (and consequently η and ε), f^p_{mn} , f^α_{mn} ,and f do not depend explicitly on the plastic strain ε^p_{mn} and on the total strain ε_{mn} but on the thermoelastic strain, i.e.,

$$\varepsilon^e_{mn} := \varepsilon_{mn} - \varepsilon^p_{mn} - \varepsilon^{th}_{mn} \qquad (2.101)$$

where ε^{th}_{mn} is the thermal strain. Thus, instead of the list of independent variables $\mathbb{T}^t$, equ. (2.11), we have*

$$\mathbb{T}^* = \{ \varepsilon^e_{mn} , T , \alpha_{kl} , \varkappa \} . \tag{2.102}$$

(3) The evolution functions f^α_{mn} are assumed to consist of two parts

$$\dot{\alpha}_{mn} = f^\alpha_{mn}(\mathbb{T}^*) = \Omega(\mathbb{T}^*)\, \dot{\varepsilon}^p_{mn}(\mathbb{T}^*) + Y(\mathbb{T}^*)\, \alpha_{mn} \tag{2.103}$$

thus, the rate α_{mn} is affected by the plastic strain rate $\dot{\varepsilon}^p_{mn}$ since the first part depends linearly on it. Here the scalar functions Ω and Y are subject to the following constraint conditions

$$\left.\begin{aligned}
\Omega(\mathbb{T}^*) &= \begin{cases} \Omega^+(\mathbb{T}^*) \;; & C^\alpha_1(\mathbb{T}^*) > 0 \;\cap\; C^\alpha_2(\mathbb{T}^*) > 0 \\[2ex] \Omega^-(\mathbb{T}^*) \;; & C^\alpha_1(\mathbb{T}^*) \le 0 \;\cup\; C^\alpha_2(\mathbb{T}^*) \le 0 \end{cases} \\[3ex]
Y(\mathbb{T}^*) &= \begin{cases} Y^+(\mathbb{T}^*) \;; & C^\alpha_1(\mathbb{T}^*) > 0 \;\cap\; C^\alpha_2(\mathbb{T}^*) > 0 \\[2ex] Y^-(\mathbb{T}^*) \;; & C^\alpha_1(\mathbb{T}^*) \le 0 \;\cup\; C^\alpha_2(\mathbb{T}^*) \le 0 \end{cases}
\end{aligned}\right\} \tag{2.104}$$

where Ω^+ and Ω^- as well as Y^+ and Y^- are not necessarily continuous on $C_1^\alpha = 0$ and $C_2^\alpha = 0$.

The dissipation inequality (2.100) yields the following conditions**

$$\left.\begin{aligned}
-\Big[\rho \frac{\partial \hat{\psi}}{\partial \varepsilon^p_{mn}} + \rho \frac{\partial \hat{\psi}}{\partial \alpha_{mn}} \Omega\Big]_{sym} f^p_{mn} - \rho \Big[\frac{\partial \hat{\psi}}{\partial \alpha_{mn}}\Big]_{sym} \alpha_{mn} Y - & \\
- \rho \Big[\frac{\partial \hat{\psi}}{\partial \varkappa}\Big] f \;\; &\ge 0 \\
- \hat{q}_k\, g_k / T \;\; &\ge 0
\end{aligned}\right\} \tag{2.105}$$

---

* Note that the temperature gradient g_m is still an independent variable but only the heat flux depends on it.

** Since ε^p_{mn} and α_{mn} are symmetric, only the symmetric part of the bracketed terms are involved.

Observing

$$\psi = \hat{\psi}(\Pi') = \tilde{\psi}(\Pi^*) \tag{2.106}$$

we get

$$\left.\begin{aligned}
\frac{\partial\hat{\psi}}{\partial\varepsilon^p_{mn}} &= \frac{\partial\tilde{\psi}}{\partial\varepsilon^e_{kl}}\frac{\partial\varepsilon^e_{kl}}{\partial\varepsilon^p_{mn}} = -\frac{\partial\tilde{\psi}}{\partial\varepsilon^e_{kl}}\delta_{km}\delta_{ln} = -\frac{\partial\tilde{\psi}}{\partial\varepsilon^e_{mn}}\\
\frac{\partial\hat{\psi}}{\partial\alpha_{mn}} &= \frac{\partial\tilde{\psi}}{\partial\alpha_{mn}}\\
\frac{\partial\hat{\psi}}{\partial\varkappa} &= \frac{\partial\tilde{\psi}}{\partial\varkappa}\\
\frac{\partial\hat{\psi}}{\partial\varepsilon_{mn}} &= \frac{\partial\tilde{\psi}}{\partial\varepsilon^e_{mn}}
\end{aligned}\right\} \tag{2.107}$$

such that (2.105)$_1$ is transformed to read

$$\begin{aligned}
-\left[-\varrho\frac{\partial\tilde{\psi}}{\partial\varepsilon^e_{mn}} + \varrho\frac{\partial\tilde{\psi}}{\partial\alpha_{mn}}\Omega\right]_{sym} f^p_{mn} - \varrho\left[\frac{\partial\tilde{\psi}}{\partial\alpha_{mn}}\right]_{sym}\alpha_{mn}\, y\\
-\varrho\left[\frac{\partial\tilde{\psi}}{\partial\varkappa}\right] f \geq 0
\end{aligned} \tag{2.108}$$

or with (2.4)$_1$ and (2.107)$_4$

$$\begin{aligned}
\sigma_{mn}\,\dot{\varepsilon}^p_{mn} - \varrho\Big\{\Omega\left[\frac{\partial\tilde{\psi}}{\partial\alpha_{mn}}\right]_{sym} f^p_{mn} + y\left[\frac{\partial\tilde{\psi}}{\partial\alpha_{mn}}\right]_{sym}\alpha_{mn}\\
+\left[\frac{\partial\tilde{\psi}}{\partial\varkappa}\right] f\Big\} \geq 0 .
\end{aligned} \tag{2.109}$$

The first term represents the plastic stress power per unit volume. It is obvious from this expression that the plastic stress power may be positive or negative depending on the sign of the bracketed term. However, if the internal variables α_{mn} and κ are not present in the theory, then the bracketed term has to be deleted from the l.h.s. of (2.109) and restriction (2.109) states that the plastic power must be positive if the decoupling (2.105) is acceptable.

3. Formal Description of the Robinson Model

3.1 The Multiaxial Stress State

Here we state in a formal way the multiaxial elastic-viscoplastic constitutive equations developed by Robinson [54 - 56]. As pointed out in section 1 the reader is reminded that the Robinson model is actually a purely mechanical model although nonisothermal processes are accounted for. The model presupposes infinitesimal deformations and initially isotropic material response. The references [54 - 56] use a slightly different nomenclature and are of somewhat different generality but here the version given in [56, 57] is summarized.

The total strain tensor ε_{kl} is assumed to be separable into its elastic, viscoplastic, and thermal parts, i.e.,

$$\varepsilon_{kl} = \varepsilon^{e}_{kl} + \varepsilon^{p}_{kl} + \varepsilon^{th}_{kl} \tag{3.1}$$

The thermal part is given by

$$\varepsilon^{th}_{kl} = A^{th}(T-T_0)\,\delta_{kl} \qquad (3.2)_A$$

implying isotropic thermal expansion. The elastic strain is related to the stress by

$$\sigma_{kl} = C_{kl\,mn}\,\varepsilon^{e}_{mn}$$

where $C_{kl\,mn}$ is the fourth order elasticity tensor; in the isotropic case

$$C_{klmn} = \lambda\,\delta_{kl}\,\delta_{mn} + \mu\,(\delta_{km}\,\delta_{ln} + \delta_{kn}\,\delta_{lm})$$

and equ. (3.2) reads (Hooke's law)

$$\sigma_{kl} = \lambda\,\varepsilon^{e}_{mm}\,\delta_{kl} + \mu\,(\varepsilon^{e}_{kl} + \varepsilon^{e}_{lk}). \qquad (3.2)_B$$

In terms of Young's modulus E and Poisson's number ν, Lame's constants are

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}\,, \qquad \mu = \frac{E}{2(1+\nu)}\,. \tag{3.3}$$

The essential part of the model is contained in relations for the viscoplastic strains ε^p_{kl} and the internal variables a_{kl} and κ:

$$\dot{\varepsilon}^p_{kl} = \begin{cases} A\,F^n \dfrac{\Sigma_{kl}}{\sqrt{J_2}} \ ; & F>0 \ \cap \ S_{kl}\Sigma_{kl}>0 \\ \\ 0 \ ; & F\le 0 \ \cup \ S_{kl}\Sigma_{kl}\le 0 \end{cases} \tag{3.4}$$

$$\dot{a}_{kl} = \begin{cases} \dfrac{H}{G^\beta}\,\dot{\varepsilon}^p_{kl} - R\,G^{m-\beta}\dfrac{a_{kl}}{\sqrt{I_2}} \ ; & G>G_0 \ \cap \ S_{kl}a_{kl}>0 \\ \\ \dfrac{H}{G_0^\beta}\,\dot{\varepsilon}^p_{kl} - R\,G_0^{m-\beta}\dfrac{a_{kl}}{\sqrt{I_2}} \ ; & G\le G_0 \ \cup \ S_{kl}a_{kl}\le 0 \end{cases} \tag{3.5}$$

$$\dot{\kappa} = \Gamma\,S_{kl}\,\dot{\varepsilon}^p_{kl} + \Theta\,\dot{T} \tag{3.6}$$

in which

$S_{kl} = \sigma_{kl} - \frac{1}{3}\sigma_{mm}\delta_{kl}$: deviatoric stress

$a_{kl} = \alpha_{kl} - \frac{1}{3}\alpha_{mm}\delta_{kl}$: deviatoric "internal stress (back stress)"

$\Sigma_{kl} = S_{kl} - a_{kl}$: effective stress

$J_2 = \frac{1}{2}\Sigma_{kl}\Sigma_{kl}$: 2nd invariant of effective stress (3.7)

$I_2 = \frac{1}{2}a_{kl}a_{kl}$: 2nd invariant of back stress

$F = J_2/\kappa^2 - 1$: "yield" function

κ : drag stress

$G = I_2/\kappa_0^2$

The quantities A, n, m, β, H, R, and G_0 are material parameters* which may depend on temperature. κ_0 is the initial value of κ at the reference temperature T_0. The quantities Γ and Θ are functions of the plastic work W^p and the absolute temperature

$$\Gamma = \hat{\Gamma}(W^p, T) \quad , \quad \Theta = \hat{\Theta}(W^p, T) \tag{3.8}$$

where

$$W^p = \int S_{kl} \, \dot{\varepsilon}^p_{kl} \, dt \, . \tag{3.9}$$

From the structure of these relations and as pointed out in [54] it is evident that the actual internal variables α_{kl} (or their deviatoric part a_{kl}) and κ describe kinematic and isotropic hardening, respectively.

As indicated in [56] the form of the evolutionary equation for κ, equ. (3.6), does allow for thermomechanical history dependence of cyclic hardening response. Equ. (3.6) is a growth law for κ which is not a perfect differential. Thus the functions Γ and Θ are independent.

It is noteworthy that none of the evolution equations contain the rates of the external observable variables $\dot{\varepsilon}_{kl}$ and $\dot{T}$, except the growth law for κ; here the temperature rate $\dot{T}$ is involved.

The constitutive equations for the internal variable models discussed in [13, 27] as well as [63] did not involve the rates of the external observables. Thus, the constitutive restrictions, including the residual entropy inequality (e.g., equ. (2.4) and (2.5)), are not necessarily applicable for the case when the rates of external variables are involved. Therefore, a simpler version of Robinsons model is studied which does not involve isotropic hardening at all, i.e., the evolution equation for κ is ignored and κ is set equal to its initial value

$$\kappa = \kappa_0 = const. \tag{3.10}$$

This version was presented by Robinson in ref. [54, 55] and during a lecture [57]. The more general case described above will be treated in a future paper.

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* The quantity G_0 is actually chosen apriory as a small number; this prevents the first term in (3.5) to become singular.

For future purposes we require a representation of the functions f^p_{kl}, equ. (2.98), as well as Ω and Y appearing in the general evolution equation (2.103). With (3.4) and (3.5) one finds

$$f^p_{kl} = \begin{cases} f^{p+}_{kl} := A F^n \dfrac{\Sigma_{kl}}{\sqrt{J_2}} \; ; & C^p_1 := F > 0 \cap C^p_2 := s_{kl}\Sigma_{kl} > 0 \\ f^{p-}_{kl} := 0 \; ; & C^p_1 \le 0 \quad \cup \quad C^p_2 \le 0 \end{cases} \tag{3.11}$$

$$\Omega = \begin{cases} \Omega^+ := H/G^\beta \; ; & C^\alpha_1 := G - G_0 > 0 \cap C^\alpha_2 := s_{kl} a_{kl} > 0 \\ \Omega^- := H/G_0^\beta \; ; & C^\alpha_1 \le 0 \quad \cup \quad C^\alpha_2 \le 0 \end{cases} \tag{3.12}$$

$$Y = \begin{cases} Y^+ := -R \dfrac{G^{m-\beta}}{\sqrt{I_2}} \; ; & C^\alpha_1 > 0 \quad \cap \quad C^\alpha_2 > 0 \\ Y^- := -R \dfrac{G_0^{m-\beta}}{\sqrt{I_2}} \; ; & C^\alpha_1 \le 0 \quad \cup \quad C^\alpha_2 \le 0 \end{cases} \tag{3.13}$$

The following remarks are now appropriate:

(1) It is noted that in the evolution equations (3.4) and (3.5) only the deviatoric part of the internal variables does appear. A constitutive equation for the trace (α_{kk}) is not required, and the rate of the deviatoric trace ($\dot{a}_{kk}$) is vanishing. This peculiarity is not in conflict with the formalism described in section 2.

(2) The evolution equations for the plastic strain ε^p_{kl} and the deviatoric internal variables a_{kl} are discontinuous on certain surfaces in the $\mathbb{T}^*$-space determined by the constraint conditions. This observation will be illustrated to some detail for the uniaxial case (section 3.2). It should be recalled that such a discontinuity is explicitely accounted for in the derivations described in section 2.

(3) The right hand side of the evolution equations (3.4) and (3.5) depend on the deviatoric stress s_{kl}, the deviatoric internal variables a_{kl}, and the absolute temperature T. This set of variables differs from the assumed list of independent state variables $\mathcal{T}^*$, equ. (2.102). However, a dependence on the elastic strains ε^e_{kl} may easily be formulated if one uses Hooke's law, equ. (3.2).

3.2 The Uniaxial Stress State (Pure Tension or Pure Shear)

The uniaxial constitutive relations are obtained by ignoring the isotropic hardening, such that equ. (3.10) applies. Assuming a uniaxial state of tension or shear, one obtains:

Table 1:

	Tension	Shear
σ_{kl}	$\begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
s_{kl}	$\begin{bmatrix} \frac{2}{3}\sigma & 0 & 0 \\ 0 & -\frac{1}{3}\sigma & 0 \\ 0 & 0 & -\frac{1}{3}\sigma \end{bmatrix}$	$\begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
α_{kl}	$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
a_{kl}	$\begin{bmatrix} \frac{2}{3}\alpha & 0 & 0 \\ 0 & -\frac{1}{3}\alpha & 0 \\ 0 & 0 & -\frac{1}{3}\alpha \end{bmatrix}$	$\begin{bmatrix} 0 & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Σ_{kl}	$\begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} (\sigma-\alpha)$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (\tau-\beta)$
$I_2 = \frac{1}{2} a_{kl} a_{lk}$	$\frac{1}{3} \alpha^2$	β^2
$J_2 = \frac{1}{2} \Sigma_{kl} \Sigma_{lk}$	$\frac{1}{3} (\sigma-\alpha)^2$	$(\tau-\beta)^2$
$K_2 = \frac{1}{2} a_{kl} \Sigma_{lk}$	$\frac{1}{3} \alpha (\sigma-\alpha)$	$\beta (\tau-\beta)$
$s_{kl} \Sigma_{lk}$	$\frac{2}{3} \sigma (\sigma-\alpha)$	$2\tau (\tau-\beta)$
$s_{kl} a_{lk}$	$\frac{2}{3} \sigma\alpha$	$2\tau\beta$

In the *case of uniaxial tension* Hooke's law (3.2) reduces to

$$\sigma_{11} = \sigma = E \varepsilon^e_{11} = E \varepsilon^e ; \qquad (3.14)$$

time differentiation yields

$$\dot{\sigma} = E \dot{\varepsilon}^e ; \qquad (3.15)$$

here it is implied that E is a constant. It is convenient to introduce dimensionless quantities

$$\sigma^* = \frac{\sigma}{\sqrt{3}\, \varkappa_0} \quad , \quad \alpha^* = \frac{\alpha}{\sqrt{3}\, \varkappa_0} \qquad (3.16)$$

The constraint functions F and G then take the form

$$F = (\sigma^* - \alpha^*)^2 - 1 , \qquad G = (\alpha^*)^2 . \tag{3.17}$$

Instead of using the parameter G_0, we may define another parameter α_0 and α_0^* related to the parameter G_0 by

$$G_0 = \frac{(\alpha_0)^2}{3\varkappa_0^2} = (\alpha_0^*)^2 . \tag{3.18}$$

With these definitions the plastic strain rate function f^p_{kl} is given by

$$\dot{\varepsilon}^p_{kl} = f^p_{kl} = \begin{cases} \begin{pmatrix} \frac{2}{3}\sqrt{3} & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix} A F^n \operatorname{sgn}(\sigma^* - \alpha^*) ; & F > 0 \cap \\ & \sigma^*(\sigma^* - \alpha^*) > 0 \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; & F \leq 0 \cup \sigma^*(\sigma^* - \alpha^*) \leq 0 . \end{cases} \tag{3.19}$$

The rate of the deviatoric back stress tensor $\dot{a}_{kl}$ is determined by (3.5) which reduces to a single equation for the dimensionless back stress α^*

$$\dot{\alpha}^* = \begin{cases} \frac{H}{\varkappa_0 (\alpha^*)^{2\beta}} \frac{\dot{\varepsilon}^p}{\frac{2}{3}\sqrt{3}} - \frac{R(\alpha^*)^{2(m-\beta)}}{\varkappa_0} \operatorname{sgn} \alpha^* ; & |\alpha^*| > |\alpha_0^*| \cap \sigma^* \alpha^* > 0 \\ \frac{H}{\varkappa_0 (\alpha_0^*)^{2\beta}} \frac{\dot{\varepsilon}^p}{\frac{2}{3}\sqrt{3}} - \frac{R(\alpha_0^*)^{2(m-\beta)}}{\varkappa_0} \operatorname{sgn} \alpha^* ; & |\alpha^*| \leq |\alpha_0^*| \cup \\ & \sigma^* \alpha^* \leq 0 \end{cases} \tag{3.20}$$

where

$$\dot{\varepsilon}^{p} := \dot{\varepsilon}^{p}_{11} \, . \tag{3.21}$$

The functions $\Omega^{\pm}$ and $Y^{\pm}$, equ. (3.12) and (3.13), take the following form

$$\left.\begin{array}{l} \Omega = \begin{cases} \Omega^{+} := \dfrac{H}{(\alpha^{*})^{2/\beta}} \; ; & |\alpha^{*}| > |\alpha^{*}_{0}| \wedge \sigma^{*}\alpha^{*} > 0 \\[2ex] \Omega^{-} := \dfrac{H}{(\alpha^{*}_{0})^{2/\beta}} \; ; & |\alpha^{*}| \leq |\alpha^{*}_{0}| \vee \sigma^{*}\alpha^{*} \leq 0 \end{cases} \\[6ex] Y = \begin{cases} Y^{+} := -\dfrac{R}{\varkappa_{0}} \dfrac{(\alpha^{*})^{2(m-\beta)}}{|\alpha^{*}|} \; ; & |\alpha^{*}| > |\alpha^{*}_{0}| \wedge \sigma^{*}\alpha^{*} > 0 \\[2ex] Y^{-} := -\dfrac{R}{\varkappa_{0}} \dfrac{(\alpha^{*}_{0})^{2(m-\beta)}}{|\alpha^{*}|} \; ; & |\alpha^{*}| \leq |\alpha^{*}_{0}| \vee \sigma^{*}\alpha^{*} \leq 0 \, . \end{cases} \end{array}\right\} \tag{3.22}$$

For the *case of pure shear* the analogous equations are as follows. Hooke's law reduces to

$$\tau_{12} = \tau = \frac{E}{1+\nu} \, \varepsilon^{e}_{12} \, . \tag{3.23}$$

With the dimensionless variables*

$$\tau^{*} = \frac{\tau}{\varkappa_{0}} \quad , \quad \beta^{*} = \frac{\beta}{\varkappa_{0}} \tag{3.24}$$

one obtains

$$F = (\tau^{*} - \beta^{*})^{2} - 1 \quad , \qquad G = (\beta^{*})^{2} \tag{3.25}$$

and

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* It should be observed that the scaling factor is different from (3.16).

$$G_0 = (\beta_0^*)^2. \tag{3.26}$$

Then the plastic strain rate function is given by

$$\dot{\varepsilon}^p_{kl} = f^p_{kl} = \begin{cases} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} A F^n \operatorname{sgn}(\tau^* - \beta^*) \; ; & F > 0 \cap \\ & \tau^*(\tau^* - \beta^*) > 0 \\ \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; ; & F \leq 0 \; \cup \; \tau^*(\tau^* - \beta^*) \leq 0 \end{cases} \tag{3.27}$$

and the evolution equations (3.5) simplify to a single differential equation for the dimensionless back stress β*:

$$\dot{\beta}^* = \begin{cases} \dfrac{H}{\varkappa_0 G^\beta} \dot{\varepsilon}^p_{12} - \dfrac{R G^{m-\beta}}{\varkappa_0} \operatorname{sgn} \beta^* \; ; & |\beta^*| > |\beta_0^*| \cap \tau^* \beta^* > 0 \\ \\ \dfrac{H}{\varkappa_0 G_0^\beta} \dot{\varepsilon}^p_{12} - \dfrac{R G_0^{m-\beta}}{\varkappa_0} \operatorname{sgn} \beta^* \; ; & |\beta^*| \leq |\beta_0^*| \cup \tau^* \beta^* \leq 0 \end{cases} \tag{3.28}$$

The functions Ω± and Y± are given by

$$\left. \Omega = \begin{cases} \Omega^+ := \dfrac{H}{(\beta^*)^2} \; ; & |\beta^*| > |\beta_0^*| \cap \tau^* \beta^* > 0 \\ \\ \Omega^- := \dfrac{H}{(\beta_0^*)^2} \; ; & |\beta^*| \leq |\beta_0^*| \cup \tau^* \beta^* \leq 0 \end{cases} \right\} \tag{3.29}$$

$$y = \left\{ \begin{array}{ll} y^{+} := - \dfrac{R\,(\beta^{*})^{2(m-\beta)}}{\varkappa_0\,|\beta^{*}|} & ; \; |\beta^{*}| > |\beta_0^{*}| \wedge \tau^{*}\beta^{*} > 0 \\ y^{-} := - \dfrac{R\,(\beta_0^{*})^{2(m-\beta}}{\varkappa_0\,|\beta^{*}|} & ; \; |\beta^{*}| \leq |\beta_0^{*}| \vee \tau^{*}\beta^{*} \leq 0 \end{array} \right\} \tag{3.29}$$

The evolution equations for the plastic strain $\varepsilon^{P} = \varepsilon^{P}_{11}$ and the dimensionless back stress α^{*} change their structure at the boundary of certain regimes in the state space (σ^{*}, α^{*}) defined by the constraint conditions. For better perception these regimes and their properties are indicated in fig. 4 for ε^{P} and fig. 5 for α^{*}. Also the boundaries in the (σ^{*}, α^{*})-space, where a discontinuity in the rates is observed, are indicated.

4. Extension of the Robinson Model within a Thermodynamic Frame

4.1 The Free Energy*

Obviously, the Robinson model is a purely mechanical model although temperature dependence is included. However, the temperature only plays the role of a parameter and details of the functional dependence have not been described yet. Since the consistence of the model with thermodynamic principles is to be analysed, it is necessary to extend the model. The constitutive functions to be introduced are the specific internal energy $\hat{\epsilon}$ and the specific entropy $\hat{\eta}$. But the thermodynamic restrictions (2.4) show that it suffices to introduce the free energy

$$\psi = \hat{\psi}(\Pi) = \tilde{\psi}(\Pi^*); \quad \Pi^* = \{\varepsilon^e_{mn}, T, \alpha_{kl}, \varkappa\}. \tag{4.1}$$

With the above remarks and observing the peculiarities of the simplified Robinson model, it is necessary to introduce

$$\psi = \check{\psi}(\Pi^{**})$$

where (4.2)

$$\Pi^{**} = \{\varepsilon^e_{kl}, T, a_{kl}\}.$$

Since the restriction $(2.4)_1$ shows that the free energy is a potential for the stress

$$\sigma_{kl} = \frac{1}{2}\rho\left(\frac{\partial\hat{\psi}}{\partial\varepsilon_{kl}} + \frac{\partial\hat{\psi}}{\partial\varepsilon_{lk}}\right) = \frac{1}{2}\rho\left(\frac{\partial\check{\psi}}{\partial\varepsilon^e_{kl}} + \frac{\partial\check{\psi}}{\partial\varepsilon^e_{lk}}\right)$$

we have, observing Hooke's law,

$$\frac{1}{2}\rho\left(\frac{\partial\check{\psi}}{\partial\varepsilon^e_{kl}} + \frac{\partial\check{\psi}}{\partial\varepsilon^e_{lk}}\right) = \lambda\,\varepsilon^e_{mm}\,\delta_{kl} + 2\mu\,\varepsilon^e_{kl}. \tag{4.3}$$

Thus, except for integration constants, the free energy function ψ may be obtained

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* Further discussion of this subject is given in section 7.

from (4.3) by partial integration in the domain of the elastic strains. One finds

$$\check{\psi}(\Pi^{**}) = \frac{1}{\varrho}\frac{1}{2}\left(2\mu\, \varepsilon^e_{kl}\, \varepsilon^e_{lk} + \lambda\, \varepsilon^e_{mm}\, \varepsilon^e_{nn}\right) + \check{\psi}^{aT}(a_{mn}, T). \tag{4.4}$$

This almost trivial result should not make one believe that this completes the determination of the free energy function. Instead, the most important part is the characterization of the "integration constant" which depends on the deviatoric internal variables a_{kl} and the absolute temperature, and which is assumed not to depend on any other variable. This especially implies its independence on any material descriptor which is in accordance with the assumed isotropic material behavior. Further, it should be noted that the deviatoric internal variables a_{kl} do appear only in the "integration constant" $\check{\psi}^{aT}$, since Lame's constants are assumed independent of a_{kl} but still may be functions of T.

The free energy contribution $\check{\psi}^{aT}(a_{kl}, T)$ due to the internal variables a_{kl} is required to be invariant under coordinate transformations. The variables a_{kl} have the dimension of stress and combine with the Cauchy stress σ_{kl} to form the effective stress Σ_{kl}. Therefore, it is reasonable to assume that the back stress a_{kl} transforms in the same way as the Cauchy stress, i.e.,

$$a^*_{kl} = Q_{km}\, a_{mn}\, Q_{nl} \qquad \text{or} \qquad \underline{a}^* = \underline{Q}\,\underline{a}\,\underline{Q}^T$$

where $\underline{Q}$ is a proper orthogonal tensor. Thus, implying isotropic material behavior, it is required that

$$\check{\psi}^{aT}(\underline{a}, T) = \check{\psi}^{aT}(\underline{Q}\,\underline{a}\,\underline{Q}^T, T) \quad \text{for} \quad \forall\, \underline{Q}\,,\ \det\underline{Q} = +1.$$

This condition is satisfied if and only if $\check{\psi}^{aT}$ can be represented as a function of the temperature and the invariants

$$i_1 = \operatorname{tr}\underline{a} = a_{mm} = 0$$

$$i_2 = \frac{1}{2}(\operatorname{tr}\underline{a})^2 - \operatorname{tr}(\underline{a}\,\underline{a}) = -\frac{1}{2}a_{kl}a_{lk} = -I_2$$

$$i_3 = \det\underline{a}$$

In agreement with the continuity requirements (section 2.3.3) it is assumed that ψ^{aT} allows a polynomial representation in a_{kl}. Dropping terms of cubic order (i.e., the determinand i_3), one gets*

$$\check{\psi}^{aT} = \check{\psi}^{T}(T) + \frac{1}{\varrho} M(T) \underbrace{\frac{1}{2} a_{mn} a_{nm}}_{I_2} . \tag{4.5}$$

This simple form will be the basis for the further analysis.

The specific entropy and the internal energy are related to the free energy by the restrictions $(2.4)_2$ and $(2.4)_1$, i.e.,

$$\left.\begin{aligned} \eta &= - \frac{\partial \hat{\psi}}{\partial T} \\ \epsilon &= \hat{\psi} - T \frac{\partial \hat{\psi}}{\partial T} \end{aligned}\right\} \tag{4.6}$$

where

$$\psi = \hat{\psi}(\varepsilon_{kl}, T, \overbrace{\varepsilon^{p}_{mn}, a_{pq}}^{\hat{=}\, \alpha_\gamma}) . \tag{4.7}$$

It should be noted that the free energy, according to (4.4), involves a different set of arguments

$$\psi = \check{\psi}(\varepsilon^{e}_{kl}, T, a_{pq}) \tag{4.8}$$

where the elastic strain ε^e_{kl} is given by (3.1) which involves the temperature T. Consequently,

$$\begin{aligned} \frac{\partial \hat{\psi}}{\partial T} &= \frac{\partial \check{\psi}}{\partial T} + \frac{\partial \check{\psi}}{\partial \varepsilon^{e}_{mn}} \frac{\partial \varepsilon^{e}_{mn}}{\partial T} \\ &= \frac{\partial \check{\psi}}{\partial T} - \frac{\partial \check{\psi}}{\partial \varepsilon^{e}_{mn}} \frac{\partial \varepsilon^{th}_{mn}}{\partial T} \\ &= \frac{\partial \check{\psi}}{\partial T} - \frac{1}{\varrho}(2\mu + 3\lambda) \varepsilon^{e}_{kk} \frac{d}{dT}\left[A^{th} \cdot (T - T_0)\right] \end{aligned} \tag{4.9}$$

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* The contribution $\psi^T(T)$ need not to be specified within the context of this study.

and

$$\frac{\partial \check{\psi}}{\partial T} = \frac{1}{\rho}\frac{1}{2}\left(2\frac{d\mu}{dT}\varepsilon^e_{kl}\varepsilon^e_{lk} + \frac{d\lambda}{dT}\varepsilon^e_{mm}\varepsilon^e_{nn}\right) + \frac{1}{\rho}\frac{dM}{dT}\frac{1}{2}a_{mn}a_{nm} + \frac{d\check{\psi}^T}{dT}. \tag{4.10}$$

Thus

$$\begin{aligned}\eta &= -\left(\frac{\partial \check{\psi}}{\partial T} - \frac{1}{\rho}(2\mu+3\lambda)\frac{d}{dT}\left[A^{th}\colon(T-T_0)\right]\varepsilon^e_{kk}\right)\\ &= -\left\{\frac{1}{\rho}\frac{1}{2}\left(2\frac{d\mu}{dT}\varepsilon^e_{kl}\varepsilon^e_{lk} + \frac{d\lambda}{dT}\varepsilon^e_{mm}\varepsilon^e_{nn}\right)\right.\\ &\quad -\frac{1}{\rho}(2\mu+3\lambda)\frac{d}{dT}\left[A^{th}\colon(T-T_0)\right]\varepsilon^e_{kk}\\ &\quad \left. +\frac{1}{\rho}\frac{dM}{dT}\frac{1}{2}a_{mn}a_{nm} + \frac{d\check{\psi}^T}{dT}\right\}\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\epsilon &= \check{\psi} - T\left(\frac{\partial \check{\psi}}{\partial T} - \frac{1}{\rho}(2\mu+3\lambda)\frac{d}{dT}\left[A^{th}\colon(T-T_0)\right]\varepsilon^e_{kk}\right)\\ &= \frac{1}{\rho}\frac{1}{2}\left[2\left(\mu - T\frac{d\mu}{dT}\right)\varepsilon^e_{kl}\varepsilon^e_{lk} + \left(\lambda - T\frac{d\lambda}{dT}\right)\varepsilon^e_{kl}\varepsilon^e_{lk}\right]\\ &\quad +\frac{1}{\rho}(2\mu+3\lambda)T\frac{d}{dT}\left[A^{th}\colon(T-T_0)\right]\varepsilon^e_{kk}\\ &\quad +\frac{1}{\rho}\left(M - T\frac{dM}{dT}\right)\frac{1}{2}a_{mn}a_{nm} + \left(\check{\psi}^T - T\frac{d\check{\psi}^T}{dT}\right).\end{aligned} \tag{4.12}$$

These equations have to be supplemented by an appropriate constitutive law for the heat flux. Here simply Fourier's law is assumed

$$q_k = -\Lambda(T)\frac{\partial T}{\partial x_k}, \quad \Lambda > 0. \tag{4.13}$$

It should be noted that the parameter $M_{(T)}$ and the purely thermal contribution to the free energy, i.e., $\psi^T{}_{(T)}$, do not appear in the stress-strain relation (3.8) and in the evolution equations for the plastic strain εp_{kl} and the back stress a_{kl}. Therefore, these quantities are not involved in the balance of linear momentum. However, these quantities play their part in the energy balance equation

$$\rho \dot{\epsilon} - \sigma_{kl} \dot{\varepsilon}_{lk} + q_{k,k} - \rho r = 0 \tag{4.14}$$

if the rate of the internal energy is expressed in terms of its arguments. Starting from

$$\epsilon = \hat{\psi} - T\frac{\partial \hat{\psi}}{\partial T} = \check{\psi} - T\left(\frac{\partial \check{\psi}}{\partial T} - \frac{1}{\rho}(2\mu+3\lambda)\frac{d}{dT}\left[A^{th}\cdot(T-T_0)\right]\varepsilon^e_{kk}\right) \tag{4.15}$$

one obtains for continuous rates of the back stress

$$\begin{aligned}
\dot{\epsilon} = {} & \frac{\partial \check{\psi}}{\partial \varepsilon^e_{kl}} \dot{\varepsilon}^e_{kl} + \frac{\partial \check{\psi}}{\partial a_{kl}} \dot{a}_{kl} + \frac{\partial \check{\psi}}{\partial T} \dot{T} \\
& - \dot{T}\left[\frac{\partial \check{\psi}}{\partial T} - \frac{1}{\rho}(2\mu+3\lambda)\frac{d}{dT}\left[A^{th}\cdot(T-T_0)\right]\varepsilon^e_{kk} \right] \\
& - T\Big[\frac{\partial^2 \check{\psi}}{\partial T \partial \varepsilon^e_{mn}} \dot{\varepsilon}^e_{mn} + \frac{\partial^2 \check{\psi}}{\partial T \partial a_{mn}} \dot{a}_{mn} + \frac{\partial^2 \check{\psi}}{\partial T^2} \dot{T} \\
& - \frac{1}{\rho}\left((2\mu+3\lambda)\frac{d}{dT}\left[A^{th}\cdot(T-T_0)\right]\right)^{\bullet}\varepsilon^e_{kk} \\
& - \frac{1}{\rho}(2\mu+3\lambda)\frac{d}{dT}\left[A^{th}\cdot(T-T_0)\right]\dot{\varepsilon}^e_{kk}\Big] \\
= {} & \frac{1}{\rho}\sigma_{kl}\dot{\varepsilon}_{kl} + \frac{1}{\rho} M a_{kl} \dot{a}_{kl} \\
& + \dot{T}\Big\{ \frac{1}{\rho}(2\mu+3\lambda)\frac{d}{dT}\left[A^{th}\cdot(T-T_0)\right]\varepsilon^e_{kk} - \frac{\partial^2 \check{\psi}}{\partial T^2} T \\
& \quad + \frac{1}{\rho} T \frac{d}{dT}\left[(2\mu+3\lambda)\frac{d}{dT}\left[A^{th}\cdot(T-T_0)\right]\right]\varepsilon^e_{kk} \Big\} \\
& + T\left\{ \frac{1}{\rho}(2\mu+3\lambda)\frac{d}{dT}\left[A^{th}\cdot(T-T_0)\right]\delta_{mn} - \frac{\partial^2 \check{\psi}}{\partial T \partial \varepsilon^e_{mn}} \right\}\dot{\varepsilon}^e_{mn} - T\frac{\partial^2 \check{\psi}}{\partial T \partial a_{mn}} \dot{a}_{mn} .
\end{aligned} \tag{4.16}$$

If the elastic constants λ, μ, the expansion coefficient A^{th}, and the parameter M are temperature independent, and if the specific heat at constant elastic strain (~stress) and constant back stress is introduced, i.e.

$$c_p := -T \frac{\partial^2 \check{\psi}}{\partial T^2} = -T \frac{d^2 \check{\psi}^T(T)}{dT^2}, \tag{4.17}$$

then equ. (4.16) simplifies to

$$\begin{aligned} \rho \dot{\epsilon} &= \sigma_{kl} \dot{\varepsilon}_{kl} + M a_{kl} \dot{a}_{kl} + \\ &+ \dot{T} \left\{ (2\mu + 3\lambda) A^{th} \varepsilon^e_{kk} + c_p \right\} + T (2\mu + 3\lambda) A^{th} \dot{\varepsilon}_{mm} \end{aligned}$$

and the energy balance reads

$$\begin{aligned} &\left\{ c_p + (2\mu + 3\lambda) A^{th} \varepsilon^e_{kk} \right\} \dot{T} - \sigma_{kl} \underbrace{(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^e_{kl})}_{\dot{\varepsilon}^p_{kl}} + \\ &+ M a_{kl} \dot{a}_{kl} + (2\mu + 3\lambda) A^{th} T \dot{\varepsilon}^e_{mm} = \rho r - q_{k,k} . \end{aligned} \tag{4.18}$$

If in addition the body is constrained against deformation at constant temperature

$$\varepsilon_{kl} = const. \quad , \quad T = const.$$

(relaxation process at constant temperature), and if this process takes place in a domain where the plastic deformation rate vanishes, i.e.,

$$\dot{\varepsilon}^p_{kl} = 0$$

then the elastic volume change vanishes

$$\dot{\varepsilon}^e_{kk} = 0$$

and (4.18) reduces finally to

$$M a_{kl} \dot{a}_{kl} = \rho r - q_{k,k} . \tag{4.19}$$

Note that the right hand side represents the heat the body exchanges with its surrounding. It is positive if heat is transferred to the body. It is now evident that under the simplifying assumptions and for this special process the heat exchanged is directly proportional to the parameter M. In principle this observation allows to determine the parameter M.

The term

$$-\varrho \frac{\partial \overset{\vee}{\psi}}{\partial a_{mn}} \dot{a}_{mn} = - M a_{mn} \dot{a}_{mn} \tag{4.20}$$

is the contribution to the intrinsic dissipation due to a change in the back stress. In fact, it is equal to the total intrinsic dissipation if the plastic strain rates vanish; consequently it is necessarily positive. Therefore, within the frame of the assumptions made, the left hand side of (4.19) must be negative which implies that heat must be extracted from the body if the temperature is to be held constant during such a relaxation process.

4.2 Existence of Equilibrium States

For every natural body we require the existence of at least one equilibrium state. This state is characterized by time independent and uniform fields. Typically, an equilibrium state is attained after a sufficiently long time when initial non-uniformities and time dependent processes in the body have smoothed out. Of course, during this phase the boundary conditions have to be such that this relaxation process can take place, e.g., adiabatic enclosure and no transfer of mechanical work. Whether or not this state can be attained can be checked by requiring that all balance equations, constitutive equation, and boundary conditions are satisfied in the body. However, the constitutive equations are the most crucial part.

Therefore, the existence of an equilibrium state requires that the evolution equations for the plastic strain rate and the rate of the internal variables admit a set of values (σ^E_{ij}, a^E_{ij}) such that these rates vanish, i.e.,

$$\dot{\varepsilon}^p_{kl} = 0 \quad , \qquad \dot{a}_{kl} = 0 \, . \tag{4.21}$$

Condition $(4.21)_1$ is satisfied if and only if

$$F \leq 0 \qquad \text{or} \qquad s_{kl} \Sigma_{kl} \leq 0 \, . \tag{4.22}$$

For the uniaxial case these conditions reduce to

$$F = (\sigma^* - \alpha^*)^2 - 1 \leq 0 \qquad \text{or} \qquad \sigma^* (\sigma^* - \alpha^*) \leq 0 \, ;$$

the appropriate regimes in the (σ^*, a^*)-space are indicated in fig. 4.

If $(4.21)_1$ is satisfied, then inspection of $(4.21)_2$ shows that the second term in $(3.5)_2$ should become zero. However, it is obvious that there exists no set of values $a_{kl} \neq 0$ which satisfies this requirement; further, performing the limiting process $a_{kl} \to 0$, we obtain a finite non-zero limit value, e.g.,

$$\lim_{a_{kl}\to 0} \frac{1}{2}\overset{\circ}{a}_{kl}\overset{\circ}{a}_{lk} = \left(R G_0^{m-\beta}\right)^2.$$

Therefore, vanishing of the rate of the back stress can be obtained only approximately as G_0 is made sufficiently small. A rigorous satisfaction of $(4.21)_2$ can be obtained if the original version of the evolution equation for the internal variables a_{kl} is modified. A convenient approach is to extend the recovery term in $(3.5)_1$ from the region $G > G_0$ into the region $G < G_0$, continuously. Consequently, instead of (3.5), we have

$$\overset{\circ}{a}_{kl} = \begin{cases} \dfrac{H}{G^{\beta}}\overset{\circ}{\varepsilon}{}^{p}_{kl} - R G^{m-\beta}\dfrac{a_{kl}}{\sqrt{I_2}}; & G > G_0 \cap S_{kl}a_{lk} > 0 \\[2ex] \dfrac{H}{G_0^{\beta}}\overset{\circ}{\varepsilon}{}^{p}_{kl} - R G^{m-\beta}\dfrac{a_{kl}}{\sqrt{I_2}}; & G \le G_0 \cup S_{kl}a_{lk} \le 0 \end{cases} \tag{4.23}$$

where it is understood that

$$m - \beta > 0. \tag{4.24}$$

This inequality is derived as follows. With $\overset{\circ}{\varepsilon}{}^{p}_{kl} = 0$ and $(3.7)_8$ one obtaines from $(4.23)_2$

$$\frac{1}{2}\overset{\circ}{a}_{kl}\overset{\circ}{a}_{lk} = R^2\left(\frac{I_2}{\mathcal{K}_0^2}\right)^{2(m-\beta)} = R^2\left(\frac{\frac{1}{2}a_{kl}a_{lk}}{\mathcal{K}_0^2}\right)^{2(m-\beta)}.$$

Thus, if all rates $\overset{\circ}{a}_{kl}$ vanish, as it should in an equilibrium state, then the r.h.s. must vanish, i.e.,

$$\left(I_2\right)^{2(m-\beta)} = 0.$$

If (m-β) > 0, then $I_2 = 0$ and consequently*

$$a_{kl} = 0 \quad , \quad k,l = 1,2,3 \; . \tag{4.25}$$

If (m-β) < 0, then $I_2 \rightarrow \infty$ to assure that all rates $\overset{\circ}{a}_{kl}$ vanish; however, this appears to be unreasonable.

With (4.23) and (4.24) an equilibrium state is possible when the deviatoric stress s_{kl} and back stress a_{kl} satisfy the conditions

$$\frac{\frac{1}{2} s_{kl}\, s_{lk}}{\varkappa_0^{\,2}} - 1 \leq 0 \qquad \text{and} \qquad a_{kl} = 0 \, . \tag{4.26}$$

Finally, it should be noted that the modification $(4.23)_2$ has been used by Hornberger and Stamm [58 - 60].

For completeness we indicate that the uniaxial evolution laws $(3.20)_2$ and $(3.28)_2$ should appropriately be changed. Thus, for example, $(3.20)_2$ reads

$$\overset{\circ}{\alpha}{}^{*} = \frac{H}{\varkappa_0 (\alpha_0^{*})^{4/3}} \frac{\overset{\circ}{\varepsilon}{}^{p}}{\frac{2}{3}\sqrt{3}} - \frac{R}{\varkappa_0} (\alpha^{*})^{2(m-\beta)} \operatorname{sgn} \alpha^{*} ; \quad |\alpha^{*}| \leq |\alpha_0^{*}| \;\cup\; \sigma^{*}\alpha^{*} \leq 0 \, . \tag{4.27}$$

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* Note that I_2 is a sum of quadratic terms.

5. Constitutive Restrictions for the Extended Robinson Model

5.1 Preliminaries

The extended and modified Robinson model is required to be consistent with the dissipation postulate where the entropy inequality is given in the Clausius-Duhem form. The requirement to satisfy the Clausius-Duhem entropy inequality for all processes, i.e., especially for *all rates* of the independent observable variables, was imposed on a whole class of constitutive relations (section 2) yielding a set of restrictions (e.g. on the stress, entropy etc.) and the residual entropy inequality in the form (2.109). Further, this inequality has to be satisfied for *all admissible variables* $\mathbb{T}$ *, equ. (2.102), whether the state is an equilibrium state or not.

The extended Robinson model (section 4) uses a free energy function depending on the variables $\mathbb{T}$ **

$$\psi = \check{\psi}(\mathbb{T}^{**}) \quad , \qquad \mathbb{T}^{**} = \{\varepsilon^e_{kl}, T, a_{mn}\}$$

and the scalar quantities Ω and Y (3.12) and (3.13) are given by*

$$\begin{aligned} \Omega &= \overset{\Delta}{\Omega}(\mathbb{T}^{\Delta}) \\ Y &= \overset{\Delta}{Y}(\mathbb{T}^{\Delta}) \end{aligned} \quad , \qquad \mathbb{T}^{\Delta} = \{\sigma_{kl}, T, a_{mn}\} . \qquad (5.1)$$

Thus, the dissipation inequality (2.109) takes the following alternative forms

$$\left.\begin{aligned} &\sigma_{mn}\,\dot{\varepsilon}^p_{nm} - \rho\Big\{\overset{\Delta}{\Omega}\Big[\frac{\partial\check{\psi}}{\partial a_{mn}}\Big]_{sym}\underbrace{f^p_{nm}}_{\dot{\varepsilon}^p_{nm}} + \overset{\Delta}{Y}\Big[\frac{\partial\check{\psi}}{\partial a_{mn}}\Big]_{sym} a_{nm}\Big\} \geq 0 \\ &\text{or} \\ &\Big[\sigma_{mn} - \rho\,\overset{\Delta}{\Omega}\Big[\frac{\partial\check{\psi}}{\partial a_{mn}}\Big]_{sym}\Big]\dot{\varepsilon}^p_{nm} - \rho\,\overset{\Delta}{Y}\Big[\frac{\partial\check{\psi}}{\partial a_{mn}}\Big]_{sym} a_{nm} \geq 0 \end{aligned}\right\} \qquad (5.2)$$

and with (4.4) and (4.5) this reduces to

---

* Note that the elastic strains ε^e_{kl} may be uniquely expressed in terms of the stresses σ_{kl}

$$[\sigma_{mn} - \overset{\Delta}{\Omega}(\Pi^{\Delta})\, M_{(T)}\, a_{mn}]\, \underbrace{\dot{\varepsilon}^{p}_{mn}}_{f^{p}_{mn}(\Pi^{\Delta})} - \overset{\Delta}{Y}(\Pi^{\Delta})\, M_{(T)}\, a_{mn}\, a_{nm} \geq 0. \tag{5.3}$$

In an equilibrium state, which is partly characterized by (4.21), the left hand side of (5.3) vanishes.

The inequality (5.3) has to be satisfied for every set of variables (σ_{mn}, a_{mn}, T), or more precisely, (s_{mn}, a_{mn}, T) since the hydrostatic stress* is not involved in (5.3). It appears that the general threedimensional case requires a cumbersome analysis although the material reponse is assumed to be isotropic. Therefore, the onedimensional stress state is analysed first. Obviously, conditions to be obtained here cannot be as strong as in the multiaxial case.

5.2 Thermodynamic Restrictions for the Uniaxial State of Stress

A uniaxial state of tension is considered. According to section (3.2), one has

$$\sigma_{mn}\, \dot{\varepsilon}^{p}_{nm} = \sigma_{11}\, \dot{\varepsilon}^{p}_{11} = \sigma\, \dot{\varepsilon}^{p}$$

$$a_{mn}\, \dot{\varepsilon}^{p}_{nm} = a_{11}\, \dot{\varepsilon}^{p}_{11} + a_{22}\, \dot{\varepsilon}^{p}_{22} + a_{33}\, \dot{\varepsilon}^{p}_{33} = \alpha\, \dot{\varepsilon}^{p}$$

$$a_{mn}\, a_{nm} = 2\, J_{2} = \frac{2}{3}\alpha^{2}$$

and

$$\dot{\varepsilon}^{p} = \dot{\varepsilon}^{p}_{11} = \begin{cases} \frac{2}{3}\sqrt{3}\, A F^{n} \operatorname{sgn}(\sigma^{*} - \alpha^{*}); & F > 0 \cap \sigma^{*}(\sigma^{*} - \alpha^{*}) > 0 \\ 0 & ; F \leq 0 \cup \sigma^{*}(\sigma^{*} - \alpha^{*}) \leq 0 \end{cases} \tag{5.4}$$

---

* Note that $\sigma_{mn}\, \dot{\varepsilon}^{p}_{mn} = s_{mn}\, \dot{\varepsilon}^{p}_{mn}$

$$\hat{\Omega}(\mathbb{T}^\Delta) = \begin{cases} \dfrac{H}{(\alpha^*)^{2\beta}} & ; \; |\alpha^*| > |\alpha_0^*| \; \cap \; \sigma^*\alpha^* > 0 \\[2ex] \dfrac{H}{(\alpha_0^*)^{2\beta}} & ; \; |\alpha^*| \le |\alpha_0^*| \; \cup \; \sigma^*\alpha^* \le 0 \end{cases} \tag{5.5}$$

$$\hat{y}(\mathbb{T}^\Delta) = -\frac{R}{\varkappa_0}\,\frac{(\alpha^*)^{2(m-\beta)}}{|\alpha^*|} \, . \tag{5.6}$$

With dimensionless variables inequality (5.3) reduces to

$$[\sigma^* - \hat{\Omega} M \alpha^*]\,\dot{\varepsilon}^p - \tfrac{2}{3}\sqrt{3}\,\varkappa_0\,\hat{y}\,M\,(\alpha^*)^2 \ge 0 \tag{5.7}$$

or more explicitely using (5.4)

Case A: $F > 0 \; \cap \; \sigma^*(\sigma^* - \alpha^*) > 0$

$$[\sigma^* - \hat{\Omega} M \alpha^*]\,A F^n \operatorname{sgn}(\sigma^* - \alpha^*) - \varkappa_0\,\hat{y}\,M\,(\alpha^*)^2 \ge 0 \tag{5.8}$$

Case B: $F \le 0 \; \cup \; \sigma^*(\sigma^* - \alpha^*) \le 0$

$$-\varkappa_0\,\hat{y}\,M\,(\alpha^*)^2 \ge 0 \, . \tag{5.9}$$

In fig. 6 the (σ*, α*)-plane is separated in the regions 0 to V. According to fig. 4, in regions O, II, and V, which may or may not include part of their boundary, the plastic strain rate vanishes. Consequently, states corresponding to case B are in these regions. From (5.9) with (5.6) one obtains

$$RM\,\frac{(\alpha^*)^{2(m-\beta)}}{|\alpha^*|}\,(\alpha^*)^2 = RM\,(\alpha^{*2})^{m-\beta+1/2} \ge 0 \, , \; \forall \alpha^*, T.$$

With (4.17) it is assured that the dissipation vanishes at the equilibrium state $\alpha^* = 0$ if RM is bounded. Thus, non-negative intrinsic dissipation requires

$$R_{(T)} \, M_{(T)} \geq \quad , \qquad \forall T. \tag{5.10}$$

Excluding the case that either R or M are identically zero, they are required to have the same sign except for special values of the temperature where either R or M may vanish. If the original Robinson model is used, with Y given by $(3.22)_2$, distinctions according to the constraints in $(3.22)_2$ must be made when (5.9) is evaluated. However, the inequality (5.10) is obtained again.

The algebraic analysis for case A is more involved. This condition applies in regions I, III, IV, and VI of fig. 6.

Region III and VI

In *region III* the second term of (5.8) is explicitely

$$-\varkappa_0 \, \overset{\Delta}{y} \, M \, (\alpha^*)^2 = + \, RM \, \frac{(\alpha^{*2})^{m-\beta}}{|\alpha^*|} \, \alpha^{*2} \tag{5.11}$$

and with condition (5.10) this is clearly non-negative. The first term of (5.8) in region III is determined to be

$$\begin{aligned} &[\sigma^* - \overset{\Delta}{\Omega} M \alpha^*] A F^n \operatorname{sgn}(\sigma^* - \alpha^*) = \\ &= [-|\sigma^*| - \frac{HM}{(\alpha_0^*)^{2\beta}} \alpha^*] A F^n (-1) = [|\sigma^*| + HM \frac{\alpha^*}{(\alpha_0^*)^{2\beta}}] A F^n \end{aligned} \tag{5.12}$$

and this expression is clearly positive since $\alpha^* \geqq 0$ in this region and H and A are assumed positive. Consequently, in region III the intrinsic dissipation is positive.

In *region VI* a similar consideration yields the same conclusion.

Region I and IV

In *region I* inequality (5.8) simplifies to

$$\left.\begin{aligned} \gamma^* &= \underbrace{\left[\sigma^* - C^*(\alpha^*)\right]}_{\leq 0}\underbrace{\left[(\sigma^* - \alpha^*)^2 - 1\right]}_{\geq 0}^{m} + D^*(\alpha^*) \geq 0, \\ &\qquad\qquad\qquad\qquad\qquad \text{for } \alpha^* > \alpha_0^* > 0 \\ \gamma^* &= \underbrace{\left[\sigma^* - C_0^*(\alpha^*)\right]}_{\leq 0}\underbrace{\left[(\sigma^* - \alpha^*)^2 - 1\right]}_{\geq 0}^{m} + D^*(\alpha^*) \geq 0, \\ &\qquad\qquad\qquad\qquad\qquad \text{for } 0 \leq \alpha^* \leq \alpha_0^* \end{aligned}\right\} \tag{5.13}$$

where

$$\left.\begin{aligned} C^*(\alpha^*) &= HM \frac{\alpha^*}{(\alpha^*)^{2/\beta}}, \qquad C_0^*(\alpha^*) = HM \frac{\alpha^*}{(\alpha_0^*)^{2/\beta}} \\ D^*(\alpha^*) &= \frac{RM}{A} \frac{(\alpha^*)^{2(m-\beta)}(\alpha^*)^2}{|\alpha^*|} = \frac{RM}{A} (\alpha^*)^{2(m-\beta)} |\alpha^*|. \end{aligned}\right\} \tag{5.14}$$

The second term on the left side of (5.13) is non-negative for all values of α^*, i.e.,

$$D^*(\alpha^*) \geq 0 \quad , \quad \forall \alpha^* \geq 0 \tag{5.15}$$

since (5.10) and $A > 0$ implies

$$\frac{RM}{A} \geq 0 .$$

If RM/A is strictly positive, then $D^*(\alpha^*)$ vanishes only if $\alpha^* = 0$ (note the assumption (4.24)). The first term on the left side of (5.13) requires some discussion. The second factor

$$F = \left[(\sigma^* - \alpha^*)^2 - 1\right]^m \tag{5.16}$$

is definitely positive in region I except along the boundary between region I and 0 where F vanishes. The first factor

$$\left[\sigma^* - C^*(\alpha^*)\right] \quad \text{or} \quad \left[\sigma^* - C_0^*(\alpha^*)\right] \tag{5.17}$$

may become negative then and only then when the common factor HM of the functions $C^*(\alpha^*)$ and $C_0^*(\alpha^*)$ is positive. H is positive by assumption but M is required to be positive if R is assumed positive; then

$$HM > 0. \tag{5.18}$$

With (5.18) the terms (5.17) may be either positive or negative depending on the values of σ^* and α^*. Thus, the continuous function

$$\sigma^* = \begin{cases} C_0^*(\alpha^*) = HM\,\kappa^*\,(\alpha_0^*)^{-2\beta} & ; \quad 0 \le \alpha^* \le \alpha_0^* \\ C^*(\alpha^*) = HM\,(\alpha^*)^{1-2\beta} & ; \quad \alpha_0^* \le \alpha^* \end{cases} \tag{5.19}$$

represents a boundary which possibly separates region I into two subsections. For a sufficiently large value HM and

$$\beta > 1/2 \tag{5.20}$$

this is shown in fig. 7. Note that the function $C^*(\alpha^*)$ is monotonically decreasing or increasing with α^* according to whether $\beta > \frac{1}{2}$ or $\beta < \frac{1}{2}$. The further discussion is restricted to the assumption (5.20) which is in accordance with the data assumed by Robinson [55, 56].

Consequently, for *every* set of values (σ^*, α^*) represented by a point in region I, which is outside the shaded area, the factors (5.17) are positive and consequently the intrinsic dissipation is positive. The shaded area, in the following denoted as the "critical region", contains states (σ^*, α^*) where the factors (5.17) are negative and thus the intrinsic dissipation may possibly become negative. The boundary of the critical region is given by equ. (5.19) and

$$F = 0\,, \quad \text{i.e.,} \quad \sigma^* = \alpha^* + 1\,. \tag{5.21}$$

The analysis of region IV can be done along the same line of arguments. One obtains a critical region of the same shape but with a point symmetric orientation (fig. 7).

It is obvious that for a special set of material parameters the critical region in regions V and VI may vanish. This is the case when the maximum point C of the curve (5.21), with coordinates

$$\begin{aligned} \sigma^*_{max} &= C^*(\alpha_0^*) = C_0^*(\alpha_0^*) = HM\,(\alpha_0^*)^{1-2\beta}, \quad \beta > 1/2 \\ \alpha^*_{max} &= \alpha_0^* \end{aligned} \tag{5.22}$$

is just on the upper boundary of region 0 (point B'). Thus, the critical region vanishes if

$$C_0^*(\alpha_0^*) = HM\,(\alpha_0^*)^{1-2\beta} \leq \alpha_0^* + 1, \quad for\ \beta \geq 1/2$$

or

$$HM \leq (1+\alpha_0^*)\,(\alpha_0^*)^{2\beta-1}, \quad for\ \beta \geq 1/2 . \tag{5.23}$$

Of course, the same result is obtained for the critical region in region IV. Consequently, if the material parameters are such that

$$A, H, R, \varkappa_0 > 0$$

and, according to (5.10),

$$M > 0$$

and if in addition inequality (5.23) is satisfied, then the intrinsic dissipation is positive for *all* states in the (σ^*, α^*)-plane. However, it is evident that relation (5.23) is only a sufficient condition. Thus, by a suitable choice of the transition value α_0^*, equ. (3.18), it is possible to assure that the intrinsic dissipation is positive for all states in the (σ^*, α^*)-space. Note that this choice is only depending on the material parameters H, M, and β. But it is obvious from (5.23) that the transition value is bounded from below, i.e., it cannot be made arbitrarily small.

> *Remark*: The original version of the Robinson model required a distinction in the recovery term of the evolution equation for the internal variables (back stress α_{kl}) such that the model did not admit an exact equilibrium state. Nevertheless, *by choosing* α_0^* *as small as possible,* an approximate equilibrium state could be achieved. The constraint (5.23) would also apply to the original version and assures positive intrinsic entropy production in the (σ^*, α^*)-space. However, the constraint that α_0^* should be above

some lower bound and simultaneously as small as possible represents an inconsistency of the original version of the Robinson model.

To obtain necessary and sufficient conditions for the uniaxial case, the following approach appears to be straight forward. Generally, it is required to find those conditions on the material parameters which just prevent the intrinsic dissipation γ^* from becoming negative at any state (σ^*, α^*) in the critical region. Thus, the vanishing of the absolute minimum of the intrinsic dissipation in the critical region defines the marginal case. For the uniaxial Robinson model the intrinsic dissipation γ^* is a function of the variables (σ^*, α^*) and of the material parameters which may depend on the absolute temperature. Equ. (5.13) and (5.14) show that γ^* is continuous in the variables (σ^*, α^*) but discontinuous in its derivative $d\gamma^*/d\alpha^*$ at $\alpha^* = \alpha_0^*$ (along B' C in fig. 7). Therefore the two regions A B' C and B' B" C are discussed separately.

If the absolute minimum of the γ^*-function in the critical region is represented by the extremal properties

$$\frac{\partial \gamma^*}{\partial \sigma^*} = 0 \quad , \quad \frac{\partial \gamma^*}{\partial \alpha^*} = 0 \; , \tag{5.24}$$

then the search for the absolute minium of γ^* reduces to the solution $(\sigma^*, \alpha^*)_{\min}$ of the two nonlinear algebraic equations (5.24). Obtaining a closed form solution is by no means obvious, and even if it can be done, the result has to be checked whether the formal solution is in the domain of definition. On the other hand, the absolute minimum may well be on B' C where $\partial\gamma^*/\partial\alpha^*$ is discontinuous.

Therefore, an alternative approach will be followed by deriving a qualitative picture of the γ^*-function in the two regions A B' C and B' B" C.

On the boundary A B' B" C, given by the functions (5.19) and (5.21), the intrinsic dissipation is determined from (5.13):

$$\gamma^* = D^*(\alpha^*) = \frac{RH}{A} (\alpha^*)^{2(m-\beta)+1} > 0 \; , \quad \text{on boundary } AB'B''C \; . \tag{5.25}$$

Thus, γ^* is strictly positive and independent of σ^*. Consequently, the smallest value of γ^* on the boundary is at point A, the largest at B", and at B' and C the intrinsic dissipation is the same.

Critical region B' B" C:

According to $(5.13)_1$,

$$\gamma^* = [\sigma^* - C^*(\alpha^*)][(\sigma^* - \alpha^*)^2 - 1]^n + D^*(\alpha^*)$$

and thus

$$\left.\begin{aligned} \frac{\partial \gamma^*}{\partial \sigma^*} &= [(\sigma^* - \alpha^*)^2 - 1]^{n-1} \Big\{ [(\sigma^* - \alpha^*)^2 - 1] + \\ &\qquad + [\sigma^* - C^*(\alpha^*)] 2n (\sigma^* - \alpha^*) \Big\} \\ \frac{\partial \gamma^*}{\partial \alpha^*} &= - [(\sigma^* - \alpha^*)^2 - 1]^{n-1} \Big\{ \frac{dC^*}{d\alpha^*} [(\sigma^* - \alpha^*)^2 - 1] - \\ &\qquad - [C^*(\alpha^*) - \sigma^*] 2n (\sigma^* - \alpha^*) \Big\} + \frac{dD^*}{d\alpha^*}. \end{aligned}\right\} \quad (5.26)$$

These functions take the following values at the boundary of region B' B" C if the material parameters are not yet restricted by the condition $\gamma^* \geqq 0$:

Table 2:

	γ^*	$\frac{\partial \gamma^*}{\partial \sigma^*}$	$\frac{\partial \gamma^*}{\partial a^*}$
B' B"	$D^*(a^*) > 0$ progressive	0	$\frac{dD^*(a^*)}{da^*} > 0$
B" C	$D^*(a^*) > 0$ progressive	> 0 $(= 0$ at B")	> 0
C B' $(a^* = a_0^* + 0)$	$\lesseqgtr 0$	$\lesseqgtr 0$	> 0

The results in table 2 give a qualitative picture of the $\gamma^*(\sigma^*, a^*)$-distribution in the region (B'B"C) as shown in fig. 8. Thus, the absolute minimum of γ^* in this region is to be expected on the intersecting boundary B'C $(a^* = a_0^* + 0)$. Since the slope

$\partial \gamma^*/\partial \sigma^*$ vanishes at B' and is positive at C, at least one extremum (minimum) must exist along B'C. This is determined by the condition

$$\left(\frac{\partial \gamma^*}{\partial \sigma^*}\right)_{\alpha^* = \alpha_0^* + 0} = 0 . \tag{5.27}$$

From $(5.26)_2$ one observes that this is satisfied for

$$\left[(\sigma^* - \alpha^*)^2 - 1\right]^{n-1} = 0 \tag{5.28}$$

or

$$\left[(\sigma^* - \alpha_0^*)^2 - 1\right] - \left[C(\alpha_0^*) - \sigma^*\right] 2n \, (\sigma^* - \alpha^*) = 0 , \tag{5.29}$$

Equ. (5.28) identifies point B' where (5.27) is satisfied. Equ. (5.29) is simply a quadratic equation for σ^* with the following two roots

$$\begin{aligned} \sigma_{1,2}^* &= \frac{(1+n)\alpha_0^* + n\,\zeta_0^*}{1+2n} \pm \left[\left(\frac{(1+n)\alpha_0^* + n\,\zeta_0^*}{1+2n}\right)^2 + \frac{1 - \alpha_0^{*2} - 2n\,\zeta_0^*}{1+2n}\right]^{1/2} \\ &= \frac{(1+n)\alpha_0^* + n\,\zeta_0^*}{1+2n} \pm \frac{1}{1+2n}\left[1+2n + n^2(\zeta_0^* - \alpha_0^*)^2\right]^{1/2} \end{aligned} \tag{5.30}$$

with

$$\zeta_0^* = C^*(\alpha_0^*) = C_0^*(\alpha_0^*) = \frac{HM}{(\alpha_0^*)^{2\beta-1}} . \tag{5.31}$$

Since the radicand on the right hand side of (5.30) is positive for all parameter combinations, equ. (5.30) represents two real roots. Inspection of (5.26) shows that

$$\left.\begin{aligned} \gamma^* &> 0 \\ \frac{\partial \gamma^*}{\partial \sigma^*} &> 0 \end{aligned}\right\} \quad \text{for} \quad \forall\, \sigma^* \geq \zeta_0^* , \; \alpha^* = \alpha_0^* \tag{5.32}$$

and

$$\left.\begin{aligned} \gamma^* &< 0 \\ \frac{\partial \gamma^*}{\partial \sigma^*} &> 0 \end{aligned}\right\} \quad \text{for} \quad \sigma^* \longrightarrow -\infty \; , \; \alpha^* = \alpha_0^* . \tag{5.33}$$

The second derivative of γ* is

$$\begin{aligned} \left(\frac{\partial^2 \gamma^*}{\partial \sigma^{*2}}\right)_{\alpha^* = \alpha_0^*} = {} & (n-1)\left[(\sigma^* - \alpha_0^*)^2 - 1\right]^{n-2} 2\,(\sigma^* - \alpha_0^*) \cdot \\ & \cdot \left\{\left[(\sigma^* - \alpha_0^*)^2 - 1\right] - \left[\zeta_0^* - \sigma^*\right] 2n\,(\sigma^* - \alpha_0^*)\right\} \\ & + \left[(\sigma^* - \alpha_0^*)^2 - 1\right]^{n-1} \cdot \\ & \cdot \left\{(2 + 2n)(\sigma^* - \alpha_0^*) - \left[\zeta_0^* - \sigma^*\right] 2n\right\} . \end{aligned} \tag{5.34}$$

It vanishes at point B' ($\sigma^* = 1 + \alpha_0^*$). At the positions ($\alpha_0^*$, σ_1^*) and (α_0^*, σ_2^*) one obtains with (5.29) and (5.30)

$$\begin{aligned} \left(\frac{\partial^2 \gamma^*}{\partial \sigma^*}\right)_{\sigma^* = \sigma_{1,2}^*} = {} & \left[(\sigma_{1,2}^* - \alpha_0^*) - 1\right]^{n-1} \cdot \\ & \cdot \left\{2(1+n)(\sigma_{1,2}^* - \alpha_0^*) - 2n\left[\zeta_0^* - \sigma_{1,2}^*\right]\right\} \\ = {} & \left[(\sigma_{1,2}^* - \alpha_0^*)^2 - 1\right]^{n-1} \cdot \\ & \cdot 2(1+n)\left[\sigma_{1,2}^* - \frac{(1+n)\alpha_0^* + n\zeta_0^*}{1+2n}\right] \\ = {} & \left[(\sigma_{1,2}^* - \alpha_0^*) - 1\right]^{n-1} \cdot \\ & \cdot 2\left\{\pm\left[1 + 2n + n^2(\zeta_0^* - \alpha_0^*)^2\right]^{1/2}\right\} . \end{aligned} \tag{5.35}$$

Therefore

$$\left(\frac{\partial^2 \gamma^*}{\partial \sigma^{*2}}\right)_{\sigma^*=\sigma_1^*} > 0 \longrightarrow \text{Minimum at} \quad (\alpha_0^*, \sigma_1^*)$$

$$\left(\frac{\partial^2 \gamma^*}{\partial \sigma^{*2}}\right)_{\sigma^*=\sigma_2^*} < 0 \longrightarrow \text{Maximum at} \quad (\alpha_0^*, \sigma_2^*) \tag{5.36}$$

If (5.29) is observed, then it is evident that the two extrema are positioned in the range $\sigma_{1,2}^* < {}_0C^*$. It is clear that the minimum at σ_1^* is in the interval $1+\alpha_0^* < \sigma_1^* < {}_0C^*$ but the maximum at σ_2^* may be within or outside this interval as shown in fig. 9. However, a decision can be obtained whether the maximum is in the interval $1+\alpha_0^* < \sigma^* < {}_0C^*$ or not; if it is, then γ^* must be somewhat larger than $D^*(\alpha_0^*)$ for σ^*-values slightly larger than $1+\alpha_0^*$. Inspection of $(5.26)_1$ shows that

$$\gamma^*(\alpha_0^*,\sigma^*) = D^*(\alpha_0^*) - \underbrace{[{}_0C^*-\sigma^*]}_{>0}\underbrace{[(\sigma^*-\alpha_0^*)^2-1]^{n-1}}_{>0\ ,\ \text{if}\ \varepsilon>0} < D^*(\alpha_0^*) \tag{5.37}$$

with

$$\sigma^* = 1+\alpha_0^*+\varepsilon\ , \qquad 0<\varepsilon\ll 1$$

and this is true for all parameter combinations subject to the assumptions made so far. Thus, it is concluded that the maximum at (α_0^*, σ_2^*) is outside the interval B'C and is consequently irrelevant.

These results and the general observations collected in table 2 prove that the general shape of the function $\gamma^*(\alpha^*, \sigma^*)$ in the region B'B"C is as indicated in fig. 8. Therefore, in this region γ^* has its minimum values along B'C at

$$\sigma^* = \sigma_1^* = \frac{(1+n)\alpha_0^* + n\,{}_0C^*}{1+2n} + \frac{1}{1+2n}\left[1+2n+n^2({}_0C^*-\alpha_0^*)^2\right]^{1/2} \tag{5.38}$$

and it is required that

$$\gamma^*(\alpha_0^*, \sigma_1^*) = D^*(\alpha_0^*) - [\zeta_0^* - \sigma_1^*][(\sigma_1^* - \alpha_0^*)^2 - 1]^m \geq 0 \qquad (5.39)$$

with

$$\zeta_0^* = \frac{HM}{(\alpha_0^*)^{2\beta-1}}, \qquad D^*(\alpha_0^*) = \frac{RM}{A}(\alpha_0^*)^{2(m-\beta)-1}.$$

If this condition is satisfied, then the intrinsic dissipation γ*(α*, σ*) is positive for all other states in the region B'B"C. This follows from the general shape of the function γ*(α*, σ*) in this region (table 2, fig. 8). Condition (5.39) together with (5.38) represents a restriction on the constitutive parameters and the logical status of condition (5.39) is as follows. Whenever the parameters have been chosen such that

$$\gamma^*(\alpha_0^*, \sigma_1^*) < 0, \qquad (5.40)$$

then it is assured that the set of constitutive parameters is *not thermodynamically consistent*: Then there exists a finite region of states in the critical region B'B"C where the intrinsic dissipation is negative. However, if

$$\gamma^*(\alpha_0^*, \sigma_1^*) \leq 0,$$

thermodynamical consistency is not yet assured since γ*(α*, σ*) may possibly be negative in the critical region B'CA. The following analysis demonstrates this.

Critical region B'CA

We calculate the slope

$$\left(\frac{\partial \gamma^*}{\partial \alpha^*}\right)_{\alpha^* = \alpha_0^* - 0,\ \sigma^* = \sigma_1^*};$$

if this value is positive for the limiting case

$$\gamma^*(\alpha_0^*, \sigma_1^*) = 0, \qquad (5.41)$$

then there exist states with $\gamma^* < 0$ in region B'CA in the neighborhood of (α_0^*, σ_1^*). With $(5.13)_2$ we obtain

$$\left(\frac{\partial \gamma^*}{\partial \alpha^*}\right)_{\substack{\alpha^* = \alpha_0^* - 0 \\ \sigma^* = \sigma_1^*}} = \left(\frac{dD^*}{d\alpha^*}\right)_{\alpha_0^*} - \left(\frac{dC_0^*}{d\alpha^*}\right)_{\alpha_0^*} \left[(\sigma_1^* - \alpha_0^*)^2 - 1\right]^n + \left[C_0^*(\alpha_0^*) - \sigma_1^*\right] 2n (\sigma_1^* - \alpha_0^*) \left[(\sigma_1^* - \alpha_0^*)^2 - 1\right]^{n-1}. \tag{5.42}$$

But at the state (α_0^*, σ_1^*) condition (5.29), and thus (5.30), applies so that (5.42) reduces to

$$\left(\frac{\partial \gamma^*}{\partial \alpha^*}\right)_{\substack{\alpha^* = \alpha_0^* - 0 \\ \sigma^* = \sigma_1^*}} = \left(\frac{dD^*}{d\alpha^*}\right)_{\alpha_0^*} - \left[\left(\frac{dC_0^*}{d\alpha^*}\right)_{\alpha_0^*} - 1\right] \left[(\sigma_1^* - \alpha_0^*)^2 - 1\right]^n. \tag{5.43}$$

Observing (5.14), one deduces

$$\left.\begin{aligned} \left(\frac{dD^*}{d\alpha^*}\right)_{\alpha_0^*} &= \frac{RM}{A}\left(2(m-\beta)+1\right)(\alpha_0^*)^{2(m-\beta)} = D^*(\alpha_0^*)\,\frac{2(m-\beta)+1}{\alpha_0^*} \\ \left(\frac{dC_0^*}{d\alpha^*}\right)_{\alpha_0^*} &= \frac{HM}{(\alpha_0^*)^{2\beta}} = C_0^*(\alpha_0^*)\,\frac{1}{\alpha_0^*}\,. \end{aligned}\right\} \tag{5.44}$$

The limiting case equ. (5.41) gives

$$D^*(\alpha_0^*) = \left[C_0^*(\alpha_0^*) - \sigma_1^*\right]\left[(\sigma_1^* - \alpha_0^*)^2 - 1\right]^n. \tag{5.45}$$

Combining (5.44) and (5.45) with (5.43) yields

$$\begin{aligned} \left(\frac{\partial \gamma^*}{\partial \alpha^*}\right)_{\substack{\alpha^* = \alpha_0^* - 0 \\ \sigma^* = \sigma_1^*}} &= \frac{1}{\alpha_0^*}\left\{\left(C_0^*(\alpha_0^*) - \sigma_1^*\right)\left(2(m-\beta)+1\right) - \left(C_0^*(\alpha_0^*) - \alpha_0^*\right)\right\} \cdot \left[(\sigma_1^* - \alpha_0^*)^2 - 1\right]^n \\ &= \frac{1}{\alpha_0^*}\left\{\left(C_0^*(\alpha_0^*) - \sigma_1^*\right) 2(m-\beta) - (\sigma_1^* - \alpha_0^*)\right\} \cdot \left[(\sigma_1^* - \alpha_0^*)^2 - 1\right]^n. \end{aligned} \tag{5.46}$$

Since

$$\sigma_1^* - \alpha_0^* > 0$$

the r.h.s. of (5.46) may be written

$$\left(\frac{\partial \gamma^*}{\partial \alpha^*}\right)_{\substack{\alpha^* = \alpha_0^* - 0 \\ \sigma^* = \sigma_1^*}} = \frac{1}{\alpha_0^*}\left\{\left(C_0^*(\alpha_0^*) - \sigma_1^*\right)\left(\sigma_1^* - \alpha_0^*\right)2(m-\beta) - \left(\sigma_1^* - \alpha_0^*\right)^2\right\} \cdot \left[\left(\sigma_1^* - \alpha_0^*\right)^2 - 1\right]^n \Big/ \left(\sigma_1^* - \alpha_0^*\right) . \tag{5.47}$$

Accounting for (5.29) with σ* = σ₁*, again we finally obtain for (5.47)

$$\left(\frac{\partial \gamma^*}{\partial \alpha^*}\right)_{\substack{\alpha^* = \alpha_0^* - 0 \\ \sigma^* = \sigma_1^*}} = \left\{\left(\frac{m-\beta}{n} - 1\right)\left(\sigma_1^* - \alpha_0^*\right)^2 - \frac{m-\beta}{n}\right\} \cdot \frac{\left[\left(\sigma_1^* - \alpha_0^*\right)^2 - 1\right]^n}{\alpha_0^*\left(\sigma_1^* - \alpha_0^*\right)} . \tag{5.48}$$

The sign of the derivative is determined by the sign of the term within the {...}-brackets. It is evident that the derivative is negative for all values of (σ_1^* - α_0^*) > 0 whenever the exponents m, β, and n are conditioned by

$$\frac{m-\beta}{n} \leq 1 . \tag{5.49}$$

Thus, if the ratio (m-β)/n satisfies the upper bound condition, then it is assured that all states (α^*, σ^*) in the infinitesimal neighborhood of (α_0^*, σ_1^*) belonging to region BC'A have a positive intrinsic dissipation independently of the value (σ_1^*- α_0^*). If the above condition is not satisfied, then the same conclusion is obtained provided

$$\left(\frac{m-\beta}{n} - 1\right)\left(\sigma_1^* - \alpha_0^*\right)^2 - \frac{m-\beta}{n} < 0$$

or

$$\frac{m-\beta}{n} < \frac{\left(\sigma_1^* - \alpha_0^*\right)^2}{\left(\sigma_1^* - \alpha_0^*\right)^2 - 1} > 1 \tag{5.50}$$

On the other hand, if either (5.49) or (5.50) is not satisfied by the constitutive parameters, the intrinsic dissipation is negative at neighboring states of (α_0^*, σ_1^*) in region B'CA. Thus, satisfaction of condition (5.39) assures thermodynamical consistence for all states in the critical region B'B"C but not in region B'CA.

However, the satisfaction of the conditions (5.49) or (5.50) does not assure that the intrinsic dissipation does not possibly become negative at other states in region B'CA. Therefore, it is necessary to study the implications of a vanishing relative minimum in region B'CA.

From $(5.13)_2$ we obtain

$$\left.\begin{aligned}
\gamma^* &= D^*(\alpha^*) - [C_o^*(\alpha^*) - \sigma^*][(\sigma^* - \alpha^*)^2 - 1]^n \\
\frac{\partial \gamma^*}{\partial \sigma^*} &= [(\sigma^* - \alpha^*)^2 - 1]^{n-1}\Big\{[(\sigma^* - \alpha^*)^2 - 1] - \\
&\qquad - [C_o^*(\alpha^*) - \sigma^*]\,2n(\sigma^* - \alpha^*)\Big\} \\
\frac{\partial \gamma^*}{\partial \alpha^*} &= \frac{dD^*}{d\alpha^*} - [(\sigma^* - \alpha^*)^2 - 1]^{n-1}\Big\{\frac{dC_o^*(\alpha^*)}{d\alpha^*}[(\sigma^* - \alpha^*)^2 - 1] \\
&\qquad - [C_o^*(\alpha^*) - \sigma^*]\,2n(\sigma^* - \alpha^*)\Big\}.
\end{aligned}\right\} \tag{5.51}$$

The general trends of these functions at the boundary of region AB'C are collected in table 3.

Table 3:

	γ^*	$\frac{\partial \gamma^*}{\partial \sigma^*}$	$\frac{\partial \gamma^*}{\partial \alpha^*}$
B' A	$D^*(\alpha^*)>0$	0	$\frac{dD^*(\alpha^*)}{d\alpha^*}>0$
A C	$D^*(\alpha^*)>0$	>0 $(=0$ at A)	$\substack{\leqq \\ >} 0$
C B'	$\substack{\leqq \\ >} 0$ (see table 2)	$\substack{\leqq \\ >} 0$ (see table 2)	$\substack{\leqq \\ >} 0$

Following the variation of γ^* for a fixed value of α^* but variable σ^* (i.e., along a parallel to the σ^*-axes), one finds that the slope $\partial\gamma^*/\partial\sigma^*$ must vanish at a point in the interval $1+\alpha^*<\sigma^*<C_0^*(\alpha^*)$, and this corresponds to a minimum of the curve $(\gamma^*)_{\alpha^*=\text{const.}}$. Thus, the general shape of γ^* along parallels to the σ^*-axes is the same in the regions B'CA and B'B”C.

From

$$\frac{\partial \gamma^*}{\partial \sigma^*} = 0 \qquad \text{in region B'CA} \tag{5.52}$$

it follows (excluding the boundary B'C; see table 2)

$$[(\sigma_m^* - \alpha^*)^2 - 1] - [C_0^*(\alpha^*) - \sigma_m^*]\, 2n\, (\sigma_m^* - \alpha^*) = 0 \tag{5.53}$$

and consequently

$$\sigma_m^* = \frac{(1+n)\alpha^* + n\, C_0^*(\alpha^*)}{1+2n} + \frac{1}{1+2n}\left[1+2n+n^2\left(C_0^*(\alpha^*)-\alpha^*\right)^2\right]^{1/2}. \tag{5.54}$$

The requirement

$$\left(\frac{\partial \gamma^*}{\partial \alpha^*}\right)_{\substack{\sigma^* = \sigma_m^* \\ \alpha^* = \alpha_m^*}} = 0 \qquad \text{in region B'CA}$$

yields

$$\left(\frac{dD^*}{d\alpha^*}\right)_{\alpha_m^*} - \left[(\sigma_m^* - \alpha_m^*)^2 - 1\right]^u \left[\left(\frac{dC_0^*}{d\alpha^*}\right)_{\alpha_m^*} - 1\right] = 0$$

when (5.53) is observed and thus

$$\sigma_m^* = \alpha_m^* + \left[1 + \left[\frac{(dC_0^*/d\alpha^*)_{\alpha_m^*} - 1}{(dD^*/d\alpha^*)_{\alpha_m^*}}\right]^{1/u}\right]^{1/2}. \tag{5.55}$$

Combining (5.54) and (5.55) gives a nonlinear algebraic equation for α_m^*. If the choice of the material parameters is such that the solution α_m^* is outside the interval

$$\alpha_{min}^* := \frac{\alpha_0^*}{C_0^*(\alpha_0^*) - \alpha_0^*} < \alpha_m^* < \alpha_0^*,$$

then certainly a *relative* minimum of γ^* in region B'CA does not exist. In any case an analytical solution for α_m^* does not appear to be feasible. Therefore, the search for thermodynamic restrictions of the material parameters via the requirement of a non-negative relative minimum of the intrinsic dissipation is not tractable if restrictions of some generality are wanted. However, a lower bound for γ^* in region B'CA can be determined. Taking the minimum value of the first term on the r.h.s. of $(5.51)_1$ and the maximum value of the second term yields

$$\gamma_{lb}^* = D^*(\alpha_{min}^*) - \left[C_0^* - (1 + \alpha_0^*)\right]\left[(C_0^* - \alpha_0^*)^2 - 1\right]^u. \tag{5.56}$$

It is now evident that thermodynamical consistency of the material parameters is *assured* in case that

$$\gamma_{eb}^{*} > 0 . \tag{5.57}$$

The quality of this bound is difficult to estimate. An improved bound can possibly be found if the function (5.56) is studied in more detail. However, at this point a further extension of the analysis seems not to be justified since the results are restricted to the uniaxial case anyhow. It is more appropriate for the multiaxial state of stress which is tackled in the next section. In fact, for a limited but practically important range of parameters we will obtain a complete solution.

5.3 Thermodynamic Restrictions for the Multiaxial State of Stress

For the extended Robinson model without isotropic hardening and for a general state of stress the dissipation inequality is given by the inequality (5.3). Here $\hat{\Omega}$ is the positive discontinuous function

$$\Omega = \hat{\Omega} = \begin{cases} H/G^{\beta} \; ; & G > G_0 \quad \cap \; S_{kl}\, a_{kl} > 0 \\ H/G_0^{\beta} \; ; & G \leq G_0 \quad \cup \; S_{kl}\, a_{kl} \leq 0 \end{cases} \tag{5.58}$$

and, according to the assumed continuity of the recovery term (equ. (4.23)), the function $\hat{Y}$ is continuous and given by

$$Y = \hat{Y} = -R\, G^{m-\beta} / \sqrt{I_2} \; . \tag{5.59}$$

Thus, both $\hat{\Omega}$ and $\hat{Y}$ depend only on the back stress and the absolute temperature if a temperature dependence of the material parameters is accounted for. Then the dissipation inequality (5.3) takes the following form

$$\gamma = \gamma_P + \gamma_H + \gamma_R \geq 0 \tag{5.60}$$

where, observing the deviatoric character of the plastic strain rate,

$$\gamma_P = \sigma_{mn}\, \dot{\varepsilon}^P_{mn} = S_{mn}\, \dot{\varepsilon}^P_{mn} \tag{5.61}$$

is the plastic stress power, and

$$\gamma_H = -\Omega\, M\, a_{mn}\, \dot{\varepsilon}^P_{mn} \tag{5.62}$$

is the contribution due to the hardening term in the evolution equation for the back stresses; further,

$$\gamma_R = -Y\, M\, a_{mn}\, a_{mn} = +RM \frac{G^{m-\beta}}{\sqrt{I_2}}\, 2 I_2 \tag{5.63}$$

is the contribution due to the static recovery term.

Several observations can be made rather easily. The evolution equation for the plastic strain rate $\dot{\varepsilon}^P_{kl}$ is given by equ. (3.4). Then the plastic stress power reads

$$\gamma_p = S_{mn}\,\dot{\varepsilon}^{p}_{mn} = \begin{cases} A\,F^{n}\,\dfrac{S_{mn}\Sigma_{mn}}{\sqrt{J_2}}\,, & F>0 \;\wedge\; S_{mn}\Sigma_{mn}>0 \\ 0\,, & F\leq 0 \;\cup\; S_{mn}\Sigma_{mn}\leq 0\,; \end{cases} \tag{5.64}$$

obviously the Robinson model is designed in such a way that γ_p is non-negative, i.e.,

$$\gamma_p \geq 0 \tag{5.65}$$

for all processes. Here it should be noted that the thermodynamic requirement of non-negative entropy production actually does not require such a modeling since the plastic stress power is not the only term contributing to the intrinsic dissipation γ.

The Robinson model allows for an elastic region where the plastic strain rate vanishes:

$$\text{If} \quad F\leq 0 \;\cup\; S_{kl}\Sigma_{kl}\leq 0 \qquad \text{then} \qquad \dot{\varepsilon}^{p}_{mn} = 0\,.$$

In this region the intrinsic dissipation is completely determined by the static recovery of the back stress, and it is required to be non-negative

$$\gamma = \gamma_R = R_{(T)}\,M_{(T)}\,\frac{G^{m-\beta}}{\sqrt{I_2}}\,2I_2 \geq 0 \tag{5.66}$$

where

$$G = I_2/\mathcal{K}_0^{2}\,.$$

Since I_2, the second invariant of the back stress (see equ. (3.7)), is a non-negative quantity, the restriction (5.66) is satisfied if and only if the two material parameters R and M are constrained by

$$R_{(T)}\,M_{(T)} \geq \;, \quad \forall\, T. \tag{5.67}$$

This condition has already been obtained in the uniaxial case, equ. (5.10).

Equ. (5.67) simply requires that the parameters R and M have the same sign if none of them vanishes. The positiveness of R (as well as H and A) is, of course, an essential property of the Robinson model. Otherwise, the recovery of the back stress could be inverted or would be non-existing. Therefore, *inequality (5.67) yields the necessary requirement*

$$M \geq 0. \qquad (5.68)$$

If the plastic strain rates $\dot{\varepsilon}^{p}_{kl}$ are non-vanishing, that is, if the constraint conditions

$$F > 0 \qquad \text{and} \qquad S_{mn}\,\Sigma_{mn} > 0$$

are satisfied, then the contribution γ_H due to hardening is given by

$$\gamma_H = -\,\Omega\, M\, A\, F^{n}\,\frac{a_{mn}\,\Sigma_{mn}}{\sqrt{J_2}}\;; \qquad (5.69)$$

the product term $\Omega M A F^n$ is positive and the second invariant J_2 of the effective stress ε_{kl} is non-negative by definition. Thus, the sign of γ_H is controlled only by the sign of the tensor product $a_{mn}\,\Sigma_{mn}$ or by

$$K_2 := \frac{1}{2}\,a_{mn}\,\Sigma_{mn}\,. \qquad (5.70)$$

The scalar quantity K_2 is one of the mixed or simultaneous invariants of the two tensors a_{mn} and Σ_{mn} and it may be positive or negative. Here it should be noted that the irreducible system of invariants (integrity bases) of the two symmetric tensors a_{kl} and Σ_{kl} consists of the three invariants of each of the two tensors and of four simultaneous invariants [87, 89]:

$$\left.\begin{array}{llll}
tr\,\underline{a} = a_{kk} = 0 & & tr\,\underline{\Sigma} = \Sigma_{kk} = 0 & \\
tr\,\underline{a}\,\underline{a} = a_{kl}\,a_{lk} & & tr\,\underline{\Sigma}\,\underline{\Sigma} = \Sigma_{kl}\,\Sigma_{lk} & \\
tr\,\underline{a}\,\underline{a}\,\underline{a} = a_{kl}\,a_{lm}\,a_{mk} & & tr\,\underline{\Sigma}\,\underline{\Sigma}\,\underline{\Sigma} = \Sigma_{kl}\,\Sigma_{lm}\,\Sigma_{mk} &
\end{array}\right\} \qquad (5.71)$$

$$\left.\begin{array}{ll}
tr\,\underline{a}\,\underline{\Sigma} = a_{kl}\,\Sigma_{lk} & \\
tr\,\underline{a}\,\underline{\Sigma}\,\underline{\Sigma} = a_{kl}\,\Sigma_{lm}\,\Sigma_{mk} & \text{simultaneous invariants} \\
tr\,\underline{a}\,\underline{a}\,\underline{\Sigma} = a_{kl}\,a_{lm}\,\Sigma_{mk} & a_{kl} \text{ and } \Sigma_{kl}. \\
tr\,\underline{a}\,\underline{a}\,\underline{\Sigma}\,\underline{\Sigma} = a_{kl}\,a_{lm}\,\Sigma_{mn}\,\Sigma_{nk} &
\end{array}\right\} \qquad (5.72)$$

It is obvious that only three invariants of the irreducible system of the pair (a_{kl}, Σ_k) are involved in the representation of the intrinsic dissipation γ, i.e.,

$$\left.\begin{aligned} a_{kl}\, a_{lk} &= 2 I_2 \geq 0 \\ \Sigma_{kl}\, \Sigma_{lk} &= 2 J_2 \geq 0 \\ a_{kl}\, \Sigma_{lk} &= 2 K_2 \gtreqless 0 . \end{aligned}\right\} \qquad (5.73)$$

It is evident that the positiveness of the contribution γ_H, equ. (5.69), in the plastic region is only assured if K_2 is negative

$$K_2 < 0 \longrightarrow \gamma_H > 0 . \qquad (5.74)$$

In this case all three contributions to the intrinsic dissipation (5.60) are positive and thus the dissipation inequality is satisfied. However, if K_2 is positive the the contribution γ_H is negative, i.e.,

$$K_2 > 0 \longrightarrow \gamma_H < 0 \qquad (5.75)$$

and possibly the dissipation inequality may be violated by some set of state variables a_{kl}, Σ_{kl} and T. Therefore, the dissipation inequality has to be discussed only for a positive simultaneous invariant K_2.

In the plastic region the intrinsic dissipation takes now the following form

$$\begin{aligned} \gamma &= \gamma_P + \gamma_H + \gamma_R \\ &= \underbrace{S_{mn}\, \dot{\varepsilon}^p_{mn}}_{\gamma_P \geq 0} \underbrace{- \Omega M a_{mn}\, \dot{\varepsilon}^p_{mn}}_{\gamma_H \gtreqless 0} + \underbrace{\mathcal{Y} M a_{mn}\, a_{mn}}_{\gamma_R \geq 0} \\ &= \frac{A F^n}{\sqrt{J_2}} \left[S_{mn}\, \Sigma_{mn} - \Omega M a_{mn}\, \Sigma_{mn} \right] + \frac{R M G^{m-\beta}}{\sqrt{I_2}}\, a_{mn}\, a_{mn} \\ &= 2 \frac{A F^n}{\sqrt{J_2}} \left[J_2 + (1 - \Omega M) K_2 \right] + 2 R M G^{m-\beta} \sqrt{I_2} \end{aligned} \qquad (5.76)$$

where the relation

$$S_{mn}\, \Sigma_{mn} = (\Sigma_{mn} + a_{mn})\, \Sigma_{mn} = 2 (J_2 + K_2)$$

is applied.

With the introduction of dimensionless quantities

$$\left.\begin{array}{l}\tilde{s}_{kl} = s_{kl}/\varkappa_0 \quad , \quad \tilde{a}_{kl} = a_{kl}/\varkappa_0 \quad , \quad \tilde{\Sigma}_{kl} = \Sigma_{kl}/\varkappa_0 \\ \text{and} \\ \tilde{I}_2 = I_2/\varkappa_0^2 = G \quad , \quad \tilde{J}_2 = J_2/\varkappa_0^2 \quad , \quad \tilde{K}_2 = K_2/\varkappa_0^2 \end{array}\right\} \qquad (5.77)$$

where $\kappa_0 > 0$, equ. (5.76) reads

$$\gamma = \underbrace{\frac{2A(T)\,\tilde{F}(\tilde{J}_2)\,\varkappa_0}{\sqrt{\tilde{J}_2}}}_{\geq 0} \Big[\underset{\geq 0}{\tilde{J}_2} + \underset{\overset{>0}{(\gtrless)}}{\big(1 - \Omega(\tilde{I}_2;T)\,M(T)\big)}\,\tilde{K}_2 \Big]$$
$$+ \underbrace{2\,R(T)\,M(T)\,\varkappa_0\,\tilde{I}_2^{\,m-\beta+1/2}}_{\geq 0} \geq 0 \; ; \qquad (5.78)$$

here

$$\Omega(\tilde{I}_2;T) = \left\{\begin{array}{lll} H(T)/\tilde{I}_2^{\,\beta} & ; & \tilde{I}_2 > \underset{0}{\tilde{I}}_2 \;\cap\; \tilde{I}_2 + \tilde{K}_2 > 0 \\ H(T)/\underset{0}{\tilde{I}}_2^{\,\beta} & ; & \tilde{I}_2 \leq \underset{0}{\tilde{I}}_2 \;\cup\; \tilde{I}_2 + \tilde{K}_2 \leq 0 \end{array}\right\} \qquad (5.79)$$

with

$$\underset{0}{\tilde{I}}_2 = G_0 \; . \qquad (5.80)$$

Note that in inequality (5.78) only *positive* values $\tilde{K}_2$ are to be considered. Further, the admissible signs of other quantities are indicated.

Following the argument of Rational Thermodynamics, the dissipation inequality has to be satisfied for *all states* characterized by the deviatoric stress $\tilde{s}_{kl}$, the deviatoric back stress $\tilde{a}_{kl}$ and the temperature T. Equally well a state may be described by the effective stress $\tilde{\Sigma}_{kl} = \tilde{s}_{kl} - \tilde{a}_{kl}$, as well as $\tilde{a}_{kl}$ and T. The stress variables $\tilde{\Sigma}_{kl}$ and a_{kl} determine the invariants $\tilde{I}_2$, $\tilde{J}_2$, $\tilde{K}_2$. In general $\tilde{\Sigma}_{kl}$ and $\tilde{a}_{kl}$ may take *independently arbitrary* values and this property applies also to the second invariants $\tilde{J}_2$ and $\tilde{I}_2$, non-negative by definition.

According to the theory of invariants [87, 89], the simultaneous invariant $\tilde{K}_2$ cannot be expressed in terms of the other variants of the integrity base (5.71, 5.72). However, this does not mean that the invariant $\tilde{K}_2$ can take arbitrary values independent of the choice of $\tilde{I}_2$ and $\tilde{J}_2$. This is shown as follows.

First of all, the assumptions

$$\tilde{I}_2 = 0 \qquad \text{or} \qquad \tilde{J}_2 = 0$$

impose the condition that all components of the tensors $\tilde{a}_{kl}$ or $\tilde{\Sigma}_{kl}$ vanish

$$\tilde{a}_{kl} = 0 \qquad \text{or} \qquad \tilde{\Sigma}_{kl} = 0.$$

Thus, in both cases one has

$$\tilde{K}_2 = 0.$$

Further, consider the principal coordinate systems S' and S" which are associated with the tensors $\tilde{a}_{kl}$ and $\tilde{\Sigma}_{kl}$ and which are generally different. The principal values are $\tilde{a}_I$, $\tilde{a}_{II}$, $\tilde{a}_{III}$ and $\tilde{\Sigma}_I$, $\tilde{\Sigma}_{II}$, $\tilde{\Sigma}_{III}$ such that

$$\tilde{a}'_{kl} = \begin{bmatrix} \tilde{a}_I & 0 & 0 \\ 0 & \tilde{a}_{II} & 0 \\ 0 & 0 & \tilde{a}_{III} \end{bmatrix} \qquad \tilde{\Sigma}''_{kl} = \begin{bmatrix} \tilde{\Sigma}_I & 0 & 0 \\ 0 & \tilde{\Sigma}_{II} & 0 \\ 0 & 0 & \tilde{\Sigma}_{III} \end{bmatrix} \tag{5.81}$$

are the appropriate components in the two coordinate systems. The second invariants are given by

$$\left.\begin{aligned} \tilde{I}_2 &= \frac{1}{2}\left(\tilde{a}_I^2 + \tilde{a}_{II}^2 + \tilde{a}_{III}^2\right) \\ \tilde{J}_2 &= \frac{1}{2}\left(\tilde{\Sigma}_I^2 + \tilde{\Sigma}_{II}^2 + \tilde{\Sigma}_{III}^2\right). \end{aligned}\right\} \tag{5.82}$$

The relative orientation of the two systems of principal axes, both considered to be right handed, is completely determined by three angles. For definiteness the Eulerian angles

$$0 \leq \vartheta \leq \pi \;, \quad 0 \leq \varphi \leq 2\pi \;, \quad 0 \leq \psi \leq 2\pi$$

as defined by Goldstein [129] are used. If the cartesian coordinates of the principal systems S" and S' are denoted by x''_k and x'_k, then they are related by the transformation matrix A_{kl}

$$x'_k = A_{kl}\, x''_l$$

where A_{kl} contains the relative direction cosines

$$(A_{kl}) = \left(\cos(X_k', X_l'')\right) = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} . \qquad (5.83)$$

In terms of the Eulerian angles they are given by [129]

$$\left.\begin{aligned} \cos(X_1', X_1'') &= l_1 = \cos\psi \cos\varphi - \cos\vartheta \sin\varphi \sin\psi \\ \cos(X_2', X_1'') &= l_2 = -\sin\psi \cos\varphi - \cos\vartheta \sin\varphi \cos\psi \\ \cos(X_3', X_1'') &= l_3 = \sin\vartheta \sin\varphi \\ \cos(X_1', X_2'') &= m_1 = \cos\psi \sin\varphi - \cos\vartheta \cos\varphi \sin\psi \\ \cos(X_2', X_2'') &= m_2 = -\sin\psi \sin\varphi + \cos\vartheta \cos\varphi \cos\psi \\ \cos(X_3', X_2'') &= m_3 = -\sin\vartheta \cos\varphi \\ \cos(X_1', X_3'') &= n_1 = \sin\psi \sin\vartheta \\ \cos(X_2', X_3'') &= n_2 = \cos\psi \sin\vartheta \\ \cos(X_3', X_3'') &= n_3 = \cos\vartheta . \end{aligned}\right\} \qquad (5.84)$$

The back stress tensor $\tilde{\Sigma}'_{kl}$, given in the principal system S', is obtained from the transformation law

$$\tilde{\Sigma}'_{kl} = A_{km} \tilde{\Sigma}''_{mn} A_{nl} \qquad (5.85)$$

which yields, observing $(5.81)_2$,

$$\tilde{\Sigma}'_{kl} = A_{k1} \tilde{\Sigma}_{I} A_{1l} + A_{k2} \tilde{\Sigma}_{II} A_{2l} + A_{k3} \tilde{\Sigma}_{III} A_{3l} .$$

The simultaneous invariant $\tilde{K}_2$ is obtained from (5.73) with $(5.81)_1$

$$\begin{aligned} \tilde{K}_2 &= \tfrac{1}{2} \tilde{a}_{kl} \tilde{\Sigma}_{kl} = \tfrac{1}{2} \tilde{a}'_{kl} \tilde{\Sigma}'_{kl} \\ &= \tfrac{1}{2} \left[\tilde{a}_{I} \tilde{\Sigma}'_{11} + a_{II} \tilde{\Sigma}'_{22} + a_{III} \tilde{\Sigma}'_{33} \right] \end{aligned}$$

and thus

$$\begin{aligned}\tilde{K}_2 = \frac{1}{2}\Big[& \tilde{a}_I \tilde{\Sigma}_I (l_1)^2 + \tilde{a}_{II} \tilde{\Sigma}_{II} (m_2)^2 + \tilde{a}_{III} \tilde{\Sigma}_{III} (n_3)^2 \\ & + \left(\tilde{a}_I \tilde{\Sigma}_{II} + \tilde{a}_{II} \tilde{\Sigma}_I\right) m_1 l_2 \\ & + \left(\tilde{a}_I \tilde{\Sigma}_{III} + \tilde{a}_{III} \tilde{\Sigma}_I\right) n_1 l_3 \\ & + \left(\tilde{a}_{II} \tilde{\Sigma}_{III} + \tilde{a}_{III} \tilde{\Sigma}_{II}\right) n_2 m_3 \Big]. \end{aligned} \tag{5.86}$$

If one keeps the principal values of the effective stress and the back stress constant but chances the relative orientation of the two principal systems S' and S", then the invariants $\tilde{I}_2$ and $\tilde{J}_2$ are constant but the simultaneous invariant $\tilde{K}_2$ varies due to the change in the relative direction cosines. However, *this change is restricted due to the boundedness of the trigonometric functions* (5.72). This observation applies also two more complex situations where also the principal values change but $\tilde{I}_2$ and $\tilde{J}_2$ are kept constant. Thus, $\tilde{K}_2$ is not allowed to take *any* (positive) value if $\tilde{I}_2$ and $\tilde{J}_2$ are held constant. In other words, the property of arbitraryness and independence of the two deviatoric tensors $\tilde{a}_{kl}$ and $\tilde{\Sigma}_{kl}$ is not transferable to *all* of the three invariants $\tilde{I}_2$, $\tilde{J}_2$ and $\tilde{K}_2$. This has logical consequences for the restrictions to be obtained from the dissipation inequality (5.78).

Assume in contrast to the above conclusions that $\tilde{I}_2$, $\tilde{J}_2$ and $\tilde{K}_2 > 0$ could take independently arbitrary values. Then inequality (5.78) is satisfied *if and only if*

$$\left(1 - \Omega(\tilde{I}_2 ; T)\, M(T)\right) \geq 0 \qquad \forall\ \tilde{I}_2 , T \tag{5.87}$$

From (5.79) one finds than the maximum value for Ω is given by

$$\Omega_{max} = H(T) / \underset{o}{\tilde{I}}{}_2^{\beta} \, . \tag{5.88}$$

Thus, under the above assumption the restriction

$$1 - \frac{H(T)\; M(T)}{\underset{o}{\tilde{I}}{}_2^{\beta}} \geq 0 \quad , \quad \forall\ T \tag{5.89}$$

is a necessary and sufficient condition for the dissipation inequality to be satisfied for all values of $\tilde{I}_2$, $\tilde{J}_2$, $\tilde{K}_2$. However, since the invariants cannot take independently all positive values *the restriction (5.89) is only sufficient.*

Combining the two inequalities (5.68) and (5.89) yields

$$0 \leq M_{(T)} \leq \frac{\underset{\circ}{\tilde{I}}_2^{\,\beta}}{H_{(T)}} \tag{5.90}$$

This may be viewed as a lower and upper bound for the parameter M; the lower bound is necessary the upper only sufficient. If M is assumed to be given, inequality (5.90) represents a lower positive bound for $\underset{\circ}{\tilde{I}}_2\beta/H$.

It should be noted that condition (5.90) does not involve the state variables "stress" and "back stress" any more, but is a condition only on temperature dependent material parameters.

Because of the sufficiency of the inequality (5.89) or of the upper bound for M is inequality (5.90) values of M larger than $\underset{\circ}{\tilde{I}}_2\beta/H$ are possibly allowed without violation of the required non-negativeness of the intrinsic dissipation (5.78) for all values of the state variables $\tilde{a}_{kl}$ and $\tilde{\Sigma}_{kl}$. Therefore, the derivation of a *necessary and sufficient upper bound* on M or more generally *necessary and sufficient conditions* involving only the temperature dependent parameters would be rather valuable. This aspect is discussed in the following.

To begin with such an analysis requires a better understanding of the *boundedness* of the simultaneous invariant $\tilde{K}_2$ if the invariants $\tilde{J}_2$ and $\tilde{I}_2$ are prescribed. Here the *Schwarz inequality* applicable to vectors or matrices is the essential relation [64]. Applied to the two tensors $\tilde{a}_{kl}$ and $\tilde{\Sigma}_{kl}$ it reads

$$\left.\begin{aligned} &\tilde{K}_2 \leq \sqrt{\tilde{I}_2}\,\sqrt{\tilde{J}_2} \\ \text{or}\quad &\\ &\tilde{K}_2 = \xi\,\sqrt{\tilde{I}_2}\,\sqrt{\tilde{J}_2}\,, \quad \xi \leq 1\,; \end{aligned}\right\} \tag{5.91}$$

note that only positive K_2-values are of interest, thus, only the range $0 < \xi \leqq 1$ is to be considered. The dissipation inequality (5.78) then reads

$$
\begin{aligned}
\gamma = \underbrace{\frac{2A(T)\,\tilde{\tilde{F}}(\tilde{J}_2)\,\varkappa_0}{\sqrt{\tilde{J}_2}}}_{\geq 0}\Big[\underbrace{\tilde{J}_2}_{\geq 0} + \big(1 - \Omega(\tilde{I}_2;T)M(T)\big)\underbrace{\xi\sqrt{\tilde{I}_2}\sqrt{\tilde{J}_2}}_{\overset{>}{(=)}0}\Big] \\
+ \underbrace{2\,R(T)\,M(T)\,\varkappa_0\,\tilde{I}_2^{\,m-\beta+1/2}}_{\geq 0} \geq 0
\end{aligned}
\tag{5.92}
$$

where $\Omega(\tilde{I}_2; T)$ is the discontinuous function given by (5.79). We now consider constitutive models which permit the expression $[1 - \Omega(\tilde{I}_2; T)\,M(T)]$ to be negative in a region $\tilde{I}_2 < \tilde{I}_2^*$ such that (Fig. 10)

$$
[1-\Omega M] = \left\{
\begin{array}{lll}
1 - HM/\tilde{I}_2^{\beta} & > 0 \;, & \tilde{I}_2 > \tilde{I}_2^* \\
1 - HM/\tilde{I}_2^{*\beta} & = 0 \;, & \tilde{I}_2 = \tilde{I}_2^* \\
1 - HM/\tilde{I}_2^{\beta} & < 0 \;, & \tilde{I}_2 < \tilde{I}_2^* \\
1 - HM/\tilde{I}_{02}^{\beta} = \text{const.} & < 0 \;, & \tilde{I}_2 \leq \tilde{I}_{02} < \tilde{I}_2^*
\end{array}
\right\} .
\tag{5.93}
$$

In the region $\tilde{I}_2 \geqq \tilde{I}_2^*$ the dissipation γ is definitely positive but for $\tilde{I}_2 < \tilde{I}_2^*$ the dissipation may become negative for some set of values $\tilde{I}_2$, $\tilde{J}_2$ and ξ and a given temperature. However, the function

$$
\gamma = \hat{\gamma}(\tilde{I}_2, \tilde{J}_2, \xi, T)
\tag{5.94}
$$

is required to be non-negative for all values of its arguments in the regions

$$
\left.
\begin{array}{l}
0 \leq \tilde{I}_2 \leq \tilde{I}_2^* \\
1 \leq \tilde{J}_2 \\
0 < \xi \leq 1 \\
T > 0 \qquad ;
\end{array}
\right\}
\tag{5.95}
$$

note that for $\tilde{J}_2 \leqq 1$ the yield function $<F>$ vanishes (vanishing plastic strain increments) and then γ is definitely positive.

For a given set of values $\tilde{I}_2$, $\tilde{J}_2$, T and $[1 - \Omega M] < 0$ a *lower bound* γ_{LB} for the intrinsic dissipation is obtained by setting $\xi = 1$; thus

$$\begin{aligned}\gamma_{LB} &= \hat{\gamma}(\tilde{I}_2, \tilde{J}_2, 1; T)\\ &= \frac{2A(T)\,F(\tilde{J}_2)\,\varkappa_0}{\sqrt{\tilde{J}_2}}\left[\tilde{J}_2 + \left(1-\Omega(\tilde{I}_2;T)M(T)\right)\sqrt{\tilde{I}_2}\sqrt{\tilde{J}_2}\right]\\ &\quad + 2R(T)\,M(T)\,\varkappa_0\,\tilde{I}_2^{\,m-\beta+1/2}\\ &=: \hat{\gamma}_{LB}(\tilde{I}_2, \tilde{J}_2; T) \quad \leq \hat{\gamma}(\tilde{I}_2, \tilde{J}_2, \xi; T).\end{aligned} \tag{5.96}$$

Since the value $\xi=1$ *may indeed be attained* for some choice of $\tilde{\alpha}_{kl}$, $\tilde{\Sigma}_{kl}$ for given values of $\tilde{I}_2$ and $\tilde{J}_2$, the function $\hat{\gamma}_{LB}$ is subject to the constraint

$$\hat{\gamma}_{LB}(\tilde{I}_2, \tilde{J}_2; T) \geq 0, \quad \begin{array}{l}\forall\, 0 \leq \tilde{I}_2 < \tilde{I}_2^{*},\\ \tilde{J}_2 > 1,\\ T > 0.\end{array} \tag{5.97}$$

If this is assured, then $\hat{\gamma}$ is also non-negative and therefore it sufficies to study the lower bound $\hat{\gamma}_{LB}$.

With

$$F = J_2/\varkappa_0^2 - 1 = \tilde{J}_2 - 1$$

and the notation

$$x = \sqrt{\tilde{I}_2}, \quad y = \sqrt{\tilde{J}_2} \tag{5.98}$$

equ. (5.96) reads

$$\begin{aligned}\gamma_{LB} &= 2A\varkappa_0\,(y^2-1)^m\left[y + (1-\Omega(x)M)x^2\right]\\ &\quad + 2RM\varkappa_0\,(x^2)^{m-\beta+1/2} = \gamma_{LB}(x,y);\end{aligned} \tag{5.99}$$

here and in the following the dependence on temperature of various quantities is not indicated anymore. Then the lower bound dissipation (5.99) represents a surface in the (x, y, γ_{LB})-space. This permits to discuss the lower bound-dissipation function $\hat{\gamma}_{LB}$ in geometrical terms analogous to the analysis of the uniaxial stress (section 5.2); but it should be noted that the choice of the independent variables is different.

Since $[1 - \Omega M]$ is allowed to be negative for $x < x^* = \sqrt{\tilde{I}_2^*}$, a critical region exists in the first quadrant of the (x, y)-plane where γ_{LB} may become negative; this region has a closed boundary and outside this region the lower bound dissipation is definitely positive. This boundary is defined by (see also fig. 11a)

$$\left.\begin{aligned} &y = 1 &&; \text{boundary } \overline{AB'B''} \\ &y + \left(1 - \frac{HM}{x^{2\beta}}\right)x = 0 &&; \text{boundary } \overline{B''C} \\ &y + \left(1 - \frac{HM}{x_0^{2\beta}}\right)x = 0 &&; \text{boundary } \overline{CA} \end{aligned}\right\} \qquad (5.100)$$

where

$$x_0 = \sqrt{\tilde{I}_{2_0}} . \qquad (5.101)$$

In fig. 11a the boundary $\overline{B''C}$ is constructed assuming that

$$\beta \geq 1/2 \qquad (5.102)$$

in accordance with the data for the Robinson model [56]. *This limitation will be used throughout the following* (see also section 5.2); it implies that the shape of the critical region approximates a triangle.

Whenever a state (x, y) is on the boundary, the first term of the lower bound-dissipation (5.99) is zero either because $(y^2 - 1)$ or the term in [ ]-brackets vanishes; here it should be noted that $\Omega(x\cdot\cdot)$ is a continuous but non-smooth function. Thus, along the boundary the lower bound-dissipation is determined by the *positive* second term in (5.99), i.e., the dissipation γ_R due to recovery; this is only a function of the variable x (related to the back stresses):

$$\gamma_{LB} = \gamma_R = 2RM\,x_0\,(x^2)^{m-\beta+1/2} > 0 \text{ along } \overline{AB'B''CA} . \qquad (5.103)$$

The critical region exists if and only if point C is above point B' (fig. 11a); note that this is strictly true only for $\beta = 1/2$. Thus, the following criteria apply:

$$\text{A critical region} \left\{ \begin{array}{l} \text{exists} \\ \\ \text{does not exist} \end{array} \right\} \text{ if } \left\{ \begin{array}{l} y_c = \left(\frac{HM}{x_0^{2\beta}} - 1 \right) x_0 > 1 \\ \\ y_c = \left(\frac{HM}{x_0^{2\beta}} - 1 \right) x_0 \leq 1 \end{array} \right\}. \tag{5.104}$$

The non-existence of a critical region implies that the lower bound dissipation is positive for all values x, y and T. Therefore, the condition

$$\left(\frac{HM}{x_0^{2\beta}} - 1 \right) x_0 \leq 1 \tag{5.105}$$

assures compatibility of the thermodynamically extended Robinson model with the dissipation postulate; however, it is only a *sufficient condition*. Inequality (5.103) implies an upper bound for the parameter M

$$M \leq \frac{x_0^{2\beta}}{H} \left(1 + \frac{1}{x_0} \right). \tag{5.106$_A$}$$

With equ. (5.98) we have

$$x_0 = \sqrt{\underset{0}{\tilde{I}}_2}$$

and therefore equ. (5.106)$_A$ takes the alternative form

$$M \leq \frac{\underset{0}{\tilde{I}}_2^{\,\beta}}{H} \left(1 + \frac{1}{\sqrt{\underset{0}{\tilde{I}}_2}} \right). \tag{5.106$_B$}$$

A comparision with inequality (5.90) shows that (5.106)$_B$ is an improved upper bound which encloses the bound (5.90).

In principle *necessary and sufficient conditions* on the temperature dependent material parameters can be derived by requiring that the *absolute minimum value of the lower bound dissipation is non-negative in the critical region*. However, the classical extremal conditions

$$\frac{\partial \gamma_{LB}}{\partial x} = 0 \quad , \quad \frac{\partial \gamma_{LB}}{\partial y} = 0$$

cannot be simply applied since γ_{LB} is a non-smooth function: At $x = x_0$, i.e., along $\overline{B'C}$ (fig. 11a) the derivative $\partial \gamma_{LB}/\partial x$ is discontinuous. Therefore, analogous to section 5.2, a qualitative picture of the γ_{LB}-function in the two regions AB'C and B'B"C is derived.

Critical region B'B"C:

According to (5.99) with (5.79) the lower bound dissipation is

$$\gamma_{LB} = 2A\varkappa_0 \left(y^2-1\right)^m \left[y + \left(1 - \frac{HM}{x^{2\beta}}\right) x \right] + \gamma_R(x)$$

and thus

$$\left.\begin{aligned} \frac{\partial \gamma_{LB}}{\partial y} &= 2A\varkappa_0 \left(y^2-1\right)^{m-1} (1+2m) \left\{ y^2 + 2y\frac{m}{1+2m}\left(1-\frac{HM}{x^{2\beta}}\right)x \right. \\ &\qquad \left. - \frac{1}{1+2m} \right\} \\ \frac{\partial \gamma_{LB}}{\partial x} &= 2A\varkappa_0 \left(y^2-1\right)^m \left[(2\beta-1)\frac{HM}{x^{2\beta}} + 1 \right] \\ &\quad + 2RM\varkappa_0\, 2\left(m-\beta+\tfrac{1}{2}\right) \left(x^2\right)^{m-\beta}. \end{aligned}\right\} \quad (5.107)$$

Here it is to be observed that $\beta \leqq 1/2$ (5.102) and $m - \beta > 0$ (equ. (4.24)).

These functions take the following values at the boundary of the region B'B"C (table 4):

Table 4

	γ_{LB}	$\frac{\partial \gamma_{LB}}{\partial y}$	$\frac{\partial \gamma_{LB}}{\partial x}$
B' B"	$\gamma_R > 0$ progressive	0	$\frac{d\gamma_R}{dx} > 0$
B" C	$\gamma_R > 0$ progressive	> 0 (0 at B")	> 0
C B' $(x = x_0 + 0)$	$\substack{\leqq \\ >} 0$	$\substack{\leqq \\ >} 0$	> 0

Since the slope $\partial\gamma_{LB}/\partial y$ vanishes along B'B" and is positive along B"C there exists at least one extremal value of γ_{LB} along parallels to the y-axes. The condition

$$\frac{\partial \gamma_{LB}}{\partial y} \overset{!}{=} 0$$

yields $y_0 = 1$, i.e., the boundary B'B", and two roots

$$y_{1,2} = \frac{m}{1+2m}\left(\frac{HM}{x^{2\beta}} - 1\right)x \pm \left\{\frac{1}{1+2m} + \left[\frac{m}{1+2m}\left(\frac{HM}{x^{2\beta}} - 1\right)x\right]^2\right\}^{1/2}; \qquad (5.108)$$

only the first one is relevant, since the second one is outside the critical region.

Observing the derivatives along the boundary B'B" and B"C, it is evident that the root y_1 represents the locus of the *minimum* value of γ_{LB} along a parallel to the y-axes. Fig. 11b gives a qualitative picture of the γ_{LB}-distribution along any parallel in the critical region B'B"C except at point B" where P = Q.

Since the derivative $\partial\gamma_{LB}/\partial x$ is positive everywhere in the critical region B'B"C, the absolute minimum of γ_{LB} in this region must be on the boundary B'C. Therefore, the *absolute minimum* $(\gamma_{LB})_{Min}$ *of the region B'B"C* is at

$$\left.\begin{aligned} x &= x_0 \\ y &= y_0^1 = \frac{m}{1+2m}\left(\frac{HM}{x_0^{2\beta}} - 1\right)x_0 + \left\{\frac{1}{1+2m} + \left[\frac{m}{1+2m}\left(\frac{HM}{x_0^{2\beta}} - 1\right)x_0\right]^2\right\}^{1/2} \end{aligned}\right\} \qquad (5.109)$$

and is given by

$$(\gamma_{LB})_{Min} = \gamma_{LB}(x_0, y_0^1). \qquad (5.110)$$

With this geometrical results and the data of table 4 a qualitative picture of the surface $\gamma_{LB}(x, y)$ in the critical region B'B"C can be given: It is similar to the "sagging roof"-picture of fig. 8; of course, here the difference in the choice of the variables produces some distortion.

It is required that

$$(\gamma_{LB})_{Min} \geq 0 . \tag{5.111}$$

If this is satisfied, then the lower bound dissipation $\gamma_{LB}(x, y)$ is positive for all other states (x, y) in the region B'B"C; this follows from the derived geometry of the surface $\gamma_{LB}(x, y)$. Condition (5.111) is a somewhat limited restriction on the temperature dependent material parameters. Whenever the material parameters are chosen such that

$$\gamma_{LB}(x_0, y_0^1) < 0 ,$$

then it is *assured* that this set of parameters is *not thermodynamically consistent*: There exists a finite region in the critical region B'B"C where the lower bound dissipation is negative. However, if

$$\gamma_{LB}(x_0, y_0^1) \geq 0 ,$$

thermodynamically consistency is not yet assured since γ_{LB} still may possibly be negative in the critical region B'CA. This is discussed in the following.

<u>Critical region B'CA</u>

The lower bound dissipation is given by

$$\gamma_{LB} = 2A\varkappa_0 (y^2-1)^m \left[y - \left(\frac{HM}{x_0^{2\beta}} - 1 \right) x \right] + \gamma_R(x) \tag{5.112}$$

and the corresponding derivatives are

$$\left. \begin{aligned} \frac{\partial \gamma_{LB}}{\partial y} &= 2A\varkappa_0 (y^2-1)^{m-1} (1+2m) \Big\{ y^2 - 2y \frac{m}{1+2m} \left(\frac{HM}{x_0^{2\beta}} - 1 \right) x \\ &\qquad - \frac{1}{1+2m} \Big\} \\ \frac{\partial \gamma_{LB}}{\partial x} &= 2A\varkappa_0 (y^2-1)^m \left[- \left(\frac{HM}{x_0^{2\beta}} - 1 \right) \right] \\ &\qquad + 2RM\varkappa_0 \, 2(m-\beta+1/2) (x^2)^{m-\beta} \end{aligned} \right\} \tag{5.113}$$

At the boundary of region B'CA the above functions take the following values (table 5):

Table 5

	γ_{LB}	$\frac{\partial \gamma_{LB}}{\partial y}$	$\frac{\partial \gamma_{LB}}{\partial x}$
B'A	$\gamma_R > 0$ progressive	0	$\frac{d\gamma_R}{dx} > 0$
AC	$\gamma_R > 0$ progressive	> 0 (0 at B")	$\substack{\geqq \\ >} 0$
CB'	$\substack{\leqq \\ >} 0$	$\substack{\leqq \\ >} 0$	$\substack{\leqq \\ >} 0$

Similar to the region B'B"C the slope $\partial\gamma_{LB}/\partial y$ vanishes along $\overline{B'A}$ and is positive along $\overline{AC}$. Thus, there exists a relative minimum of the distribution of γ_{LB} along parallels to the y-axes. For a given x-value ($x_A < x < x_0$) the necessary condition for this minimum is given by

$$\frac{\partial \gamma_{LB}}{\partial y} = 0 \tag{5.114}$$

and this yields $y_0 = 1$ and two roots

$$y_{1,2} = \frac{m}{1+2m}\left(\frac{HM}{x_0^{2\beta}} - 1\right)x \overset{+}{(-)} \left\{\frac{1}{1+2m} + \left[\frac{m}{1+2m}\left(\frac{HM}{x_0^{2\beta}} - 1\right)x\right]^2\right\}^{1/2} \tag{5.115}$$

but only the root y_1 is relevant. Therefore, the γ_{LB}-distribution along any parallel to the y-axes is similar to that in region B'B"C (fig. 11b).

The second derivative of γ_{LB} along a parallel to the x-axes is given by

$$\frac{\partial^2 \gamma_{LB}}{\partial x^2} = 2RM\varkappa_0 \, 2(m-\beta+1/2) \, 2(m-\beta) \, x^{2(m-\beta)-1} > 0,$$

an intrinsically positive quantity and constant along parallels to the y-axes. This gives a qualitative idea of the shape of γ_{LB}-distributions along parallels to the x-axes. In principle the positiveness allows the occurrence of a relative minimum of this distribution.

The locus where the derivative $\partial\gamma_{LB}/\partial y$ vanishes in the critical regions B'CA and B'B''C is represented by the curve $\overline{\text{ADB''}}$ in the (x, y)-plane (fig. 12a). It is evident that a *relative minimum of the surface* $\gamma_{LB}(x, y)$ in the critical region B'CA, if it exists, is on the section $\overline{\text{AD}}$ and the necessary condition, in addition to (5.114), is

$$\frac{\partial \gamma_{LB}}{\partial x} = 0 \qquad \text{along} \qquad \overline{AD}. \tag{5.116}$$

This yields a second nonlinear relation for the coordinates x_1, y_1 of this extremum. Together with (5.114) this represents the determining system of equations. Instead of trying to solve this system of equations*, we prefer to study the question whether or not a relative minimum exists in the critical region B'CA.

The lower bound dissipation γ_{LB} along the section $\overline{\text{AD}}$ and $\overline{\text{DB''}}$ is given by

$$\check{\gamma}_{LB} = \gamma_{LB}(x, y_1(x)) = \check{\gamma}_{LB}(x) \tag{5.117}$$

where $y_1(x)$ is defined by (5.115) and (5.108), respectively. Fig. 12b gives a qualitative idea of this distribution. Along the section $\overline{\text{DB''}}$ the function (5.117) is monotonously increasing. Observing (5.107), the slope $d\gamma_{LB}/dx$ is given by

$$\frac{d\check{\gamma}_{LB}}{dx} = \left(\frac{\partial \gamma_{LB}(x,y)}{\partial x}\right)_{y=y_1} + \underbrace{\left(\frac{\partial \gamma_{LB}(x,y)}{\partial y}\right)_{y=y_1}}_{=0} \frac{dy_1}{dx} = \left(\frac{\partial \gamma_{LB}}{\partial x}\right)_{y=y_1}. \tag{5.118}$$

At point B'' one finds $y_1 = 1$ and thus

---

* In fact, an analytical solution appears not to be possible.

$$\frac{d\check{\gamma}_{LB}(x)}{dx} = \frac{d\gamma_R(x)}{dx} > 0.$$

Consequently, the function $\check{\gamma}_{LB}$ is tangential to γ_R at B" as shown in fig. 12b.

The second derivative is given by

$$\frac{d^2\check{\gamma}_{LB}}{dx^2} = \underset{>0}{\left(\frac{\partial^2\gamma_{LB}}{\partial x^2}\right)_{y_1}} + 2\underset{>0}{\left(\frac{\partial^2\gamma_{LB}}{\partial x\,\partial y}\right)_{y_1}}\underset{<0}{\frac{dy_1}{dx}} + \underset{>0}{\left(\frac{\partial^2\gamma_{LB}}{\partial y^2}\right)_{y_1}}\underset{>0}{\left(\frac{dy_1}{dx}\right)^2} + \underset{=0}{\left(\frac{\partial\gamma_{LB}}{\partial y}\right)_{y_1}}\frac{d^2y_1}{dx^2}$$

and consists of positive and negative contributions. Therefore, its sign is not immediately obvious. In fig. 12 different values are assumed; in any case a precise knowledge is not important since it is known that the absolute minimum in region B'B"C is at D ($x=x_0$, $y=y_1$).

At $x=x_0$ one finds

$$\left(\frac{d\check{\gamma}_{LB}}{dx}\right)_{x=x_0+0} = \underset{>0}{\left(\frac{\partial\gamma_{LB}}{\partial x}\right)_{\substack{x_0+0\\ y_1}}} + \underset{=0}{\left(\frac{\partial\gamma_{LB}}{\partial y}\right)_{\substack{x_0+0\\ y_1}}}\left(\frac{dy_1}{dx}\right)_{x_0+0} > 0$$

$$\left(\frac{d\check{\gamma}_{LB}}{dx}\right)_{x=x_0-0} = \underset{\gtrless 0}{\left(\frac{\partial\gamma_{LB}}{\partial x}\right)_{\substack{x_0-0\\ y_1}}} + \underset{=0}{\left(\frac{\partial\gamma_{LB}}{\partial y}\right)_{\substack{x_0-0\\ y_1}}}\left(\frac{dy_1}{dx}\right)_{x_0-0} \gtrless 0$$

such that

$$\left(\frac{d\check{\gamma}_{LB}}{dx}\right)_{x_0+0} - \left(\frac{d\check{\gamma}_{LB}}{dx}\right)_{x_0-0} = 2A\varkappa_0\left(y_1^2-1\right)^{-1} 2\beta\,\frac{HM}{x_0^{2\beta}} > 0.$$

Thus, the slope at x_0+0 is always larger than at x_0 - 0. This fact is accounted for in fig. 12b.

At $x=x_A$ one obtains

$$\left(\frac{d\check{\gamma}_{LB}}{dx}\right)_{x_A} = \left(\frac{\partial \gamma_{LB}}{\partial x}\right)_{\substack{x_A\\ y_1}} + \underset{=0}{\left(\frac{\partial \gamma_{LB}}{\partial y}\right)_{\substack{x_A\\ y_1}}} \left(\frac{dy_1}{dx}\right)_{x_A} = \frac{d\gamma_R}{dx} > 0.$$

Therefore, the distribution $\check{\gamma}_{LB}(x)$ along $\overline{AD}$ at $x=x_A$ is tangent to the function $\gamma_R(x)$ as shown in fig. 12b.

The second derivative of $\check{\gamma}_{LB}$ in the interval $x_A \geqq x < x_0$ is given by

$$\frac{d^2\check{\gamma}_{LB}}{dx^2} = \underset{>0}{\left(\frac{\partial^2 \gamma_{LB}}{\partial x^2}\right)_{y=y_1}} + 2\underset{<0}{\left(\frac{\partial^2 \gamma_{LB}}{\partial x\,\partial y}\right)_{y=y_1}} \underset{>0}{\frac{dy_1}{dx}} + \underset{>0}{\left(\frac{\partial^2 \gamma_{LB}}{\partial y^2}\right)} \underset{>0}{\left(\frac{dy_1}{dx}\right)^2}$$
$$+ \underset{=0}{\left(\frac{\partial \gamma_{LB}}{\partial y}\right)_{y=y_1}} \frac{d^2 y_1}{dx^2}.$$

The complexity of this expression makes it difficult to judge the possible occurrence of a change in sign. However, at $x=x_A$ a positive value is observed:

$$\left(\frac{d^2\check{\gamma}_{LB}}{dx^2}\right)_{x=x_A} = \left(\frac{d^2\gamma_R}{dx^2}\right)_{x=x_A}$$
$$= 2RM\, 2(m-\beta+1/2)\, 2(m-\beta)\, x_A^{2(m-\beta)-1} > 0.$$

Several distributions along the section $\overline{AD}$ are shown in fig. 12b (case 0 - 3). Case 3 implies the existence of a relative minimum. *Since the slope at $x=x_A$ is always positive* the presence of *a relative minimum* in the interval $x_A \leqq x \leqq x_0$ *implies the existence of two extrema at least.* Therefore, the necessary condition

$$\frac{d\check{\gamma}_{LB}}{dx} = 0 \quad , \quad x_A \leq x < x_0 \tag{5.119}$$

must yield at least two different roots in the interval. With (5.118) and (5.113)2 equ. (5.119) yields

$$\frac{d\check{\gamma}_{LB}}{dx} = -2A\varkappa_0 \left(y_1^2(x) - 1\right)^m \left[\frac{HM}{x_0^{2\beta}} - 1\right]$$
$$+ 2RM\varkappa_0\, 2(m-\beta+1/2)\left(x^2\right)^{m-\beta} = 0. \tag{5.120}$$

This problem is equivalent to the search for the points of intersection of two curves $z_1(x)$ and $z_2(x)$, i.e.,

$$z_1(x) = z_2(x) \tag{5.121}$$

where

$$\left.\begin{aligned} z_1(x) &= y_1^2(x) \\ z_2(x) &= 1 + \left(\frac{RM}{A}\,\frac{2(m-\beta+\frac{1}{2})}{\left[\frac{HM}{x_0^{2\beta}}-1\right]}\right)^{1/n} x^{2\frac{m-\beta}{n}} ; \end{aligned}\right\} \tag{5.122}$$

the function $y_1(x)$ is given by (5.115). A geometrical discussion of the two functions $z_1(x)$ and $z_2(x)$ will yield the required information.

The function $y_1(x)$ may be written as

$$y_1(x) = \mathcal{B}x + \left\{\mathcal{A} + \mathcal{B}^2x^2\right\}^{1/2} \tag{5.123}$$

with

$$\mathcal{A} = \frac{1}{1+2n} \leq \frac{1}{3} \quad , \quad \mathcal{B} = \frac{n}{1+2n}\left[\frac{HM}{x_0^{2\beta}} - 1\right] > 0 \tag{5.124}$$

where $\mathcal{B}$ is positive by assumption. Thus,

$$\left.\begin{aligned} \frac{dy_1}{dx} &= \mathcal{B} + \left\{\mathcal{A} + \mathcal{B}^2x^2\right\}^{-1/2}\mathcal{B}^2x > 0 \\ \frac{d^2y_1}{dx^2} &= 2\mathcal{A}\mathcal{B}^2\left\{\mathcal{A} + \mathcal{B}^2x^2\right\}^{-3/2} > 0 . \end{aligned}\right\} \tag{5.125}$$

For large values of x the function $y_1(x)$ approaches the straight line $2\mathcal{B}x$ from above; and for x-values close to zero $y_1(x)$ is tangent to $(\mathcal{A}^{1/2} + \mathcal{B}x)$. Its curvature is positive for all $0 < x$ and continuously decreasing. Fig. 13 gives a qualitative idea of the shape of $y_1(x)$ as well as its square, i.e., $z_1(x)$; this figure is constructed for some arbitrary x-scale and arbitrary value $\mathcal{B}$ but for $n = 5$.

The first and second derivative of $z_1(x)$ is given by

$$\left.\begin{aligned} \frac{dz_1}{dx} &= 2 y_1 \frac{dy_1}{dx} && > 0 \\ \frac{d^2 z_1}{dx^2} &= 2\left(\frac{dy_1}{dx}\right)^2 + 2 y_1 \frac{d^2 y_1}{dx^2} && > 0 \end{aligned}\right\} \quad (5.126)_A$$

and for $z_2(x)$ one obtains explicitely

$$\left.\begin{aligned} \frac{dz_2}{dx} &= \left(\frac{RM}{A} \frac{2(m-\beta+1/2)}{\left(\frac{HM}{x_0^{2\beta}}-1\right)}\right)^{1/n} \left(2\frac{m-\beta}{n}\right) x^{\left(2\frac{m-\beta}{n}-1\right)} && > 0 \\ \frac{d^2 z_2}{dx^2} &= \left(\qquad '' \qquad \right)^{1/n} \left(2\frac{m-\beta}{n}\right)\left(2\frac{m-\beta}{n}-1\right) x^{2\left(\frac{m-\beta}{n}-1\right)} && \gtreqless 0 \end{aligned}\right\} \quad (5.126)_B \; .$$

Obviously, the shape of the $z_2(x)$-function is determined by the ratio* $(m - \beta)/n > 0$ of the three exponents. If

$$\left\{\begin{array}{ll} \text{Case 1} & \frac{1}{2} < \frac{m-\beta}{n} \\ \text{Case 2} & \frac{1}{2} = \frac{m-\beta}{n} \\ \text{Case 3} & \frac{1}{2} > \frac{m-\beta}{n} \end{array}\right\} \quad \text{then} \quad \frac{d^2 z_2}{dx^2} \left\{\begin{array}{c} > 0 \\ = 0 \\ < 0 \end{array}\right\} \forall x > 0 . \quad (5.127)$$

Clearly, for cases 2 and 3 *there exists one and only one point of intersection* of the two curves $z_1(x)$ and $z_2(x)$ as shown in fig. 14. Case 1 requires some further analysis.

---

* The positiveness of $(m - \beta)$ and n in an inherent property of the Robinson model.

For large values of x the ratio of the two functions $z_1(x)$ and $z_2(x)$ is given by

$$\left(\frac{z_1}{z_2}\right)_{x\ large} = \frac{(2\mathcal{B}x)^2}{\mathcal{C}\, x^{2\frac{m-\beta}{m}}} = \frac{4\mathcal{B}^2}{\mathcal{C}}\, x^{2\left(1-\frac{m-\beta}{m}\right)}$$

where

$$\mathcal{C} := \left(\frac{RM}{A}\, \frac{2\left(m-\beta+\frac{1}{2}\right)}{\left(\frac{4M}{x_0^{2\beta}}-1\right)}\right)^{1/m}. \tag{5.128}$$

Hence, for *case 1* (i.e., $(m-\beta)/m > 1/2$)

$$\left.\begin{array}{ll} (a) & \lim\limits_{x\to\infty} \dfrac{z_1}{z_2} \longrightarrow \infty \\[2ex] (b) & \lim\limits_{x\to\infty} \dfrac{z_1}{z_2} = \dfrac{4\mathcal{B}^2}{\mathcal{C}} \\[2ex] (c) & \lim\limits_{x\to\infty} \dfrac{z_1}{z_2} \longrightarrow 0 \end{array}\right\} \text{ if and only if } \left\{\begin{array}{l} \dfrac{1}{2} < \dfrac{m-\beta}{m} < 1 \\[2ex] \dfrac{m-\beta}{m} = 1 \\[2ex] \dfrac{m-\beta}{m} > 1 \end{array}\right\} \tag{5.129}$$

At $x=0$ one observes $z_1(0)=\mathcal{A} < z_2(0) = 1$ and for large x-values *case 1(a)* implies $z_1(x) \gg z_2(x)$; this requires that there exists *at least one or a larger uneven number of intersections* of the two curves $z_1(x)$ and $z_2(x)$; if there is more than one intersection there should be at least three.

The case of one intersection, possibly in the interval $x_A < x < x_0$ as shown in fig. 12 (case 2), seems to be the most likely case. In fact, we will prove that for case 1(a) there is one and only one intersection in the range $x > x_A$.

As noted above the two functions $z_1(x)$ and $z_2(x)$ have at least one intersection for case 1(a). Assume that their first intersection is at $x = x_i > x_A$ (fig. 14). Consider the *lower bound function of $z_1(x)$* given by

$$\left(z_1(x)\right)_{lb} = (2\mathcal{B}x)^2 < z_1(x);$$

for large values of n or large values of x this is also a good approximation for $z_1(x)$. The intersection of the two functions $(z_1)_{lb}$ and z_2 is determined by

$$4 \mathcal{B}^2 \tilde{I}_2 = 1 + \mathcal{C} \tilde{I}_2^{\frac{m-\beta}{n}} , \quad \tilde{I}_2 = x^2 .$$

As long as $(m - \beta)/n < 1$ the shape of these two functions of $\tilde{I}_2$ shows that there exist one and only one intersection $(\tilde{I}_2)_i$

$$(\tilde{I}_2)_i = (x_i)_{ub}^2 > (x_i)^2 ,$$

which corresponds to the value $(x_i)_{ub}$, an upper bound for x_i. *Consequently, the actual functions z_1 and z_2 allow one and only one intersection in the range $x > x_A$ for case 1(a).*

On the other hand, case 1(c) implies that $z_1(x) \leqslant z_2(x)$ for large x-values. Thus, there exist none or at least two intersections depending on the factors $\mathcal{B}$ and $\mathcal{C}$.

For case 1(b) there exist one or two intersections depending also on the parameter $\mathcal{B}$ and $\mathcal{C}$. Some of these cases are illustrated in fig. 14.

It should be pointed out that the existence of a relative minimum in the critical region B'CA requires that at least two intersections of the functions $z_1(x)$ and $z_2(x)$ exist and are *located in the interval $x_A < x \leqq x_0$*. The above remark should be observed when studying the following conclusions.

Case 3 & 2 $(m-\beta)/n \leq 1/2$ and
Case 1(a) $1/2 < (m-\beta)/n < 1$

One and only point of intersection of $z_1(x)$ and $z_2(x)$ for $x_A < x$
⇒ relative minimum does not exist in critical region B'CA

Case 1(b) $(m-\beta)/n = 1$

One or two intersections for $x_A < x$
⇒ existence of relative minimum cannot be excluded; depends on parameter $\mathcal{B}$ and $\mathcal{C}$, too.

Case 1(c) $(m-\beta)/n > 1$

None or two intersections for $x_A < x$
⇒ existence of relative minimum not excluded; depends on parameter $\mathcal{B}$ and $\mathcal{C}$, too.

If $(m - \beta)/n < 1$, a relative minimum of the lower bound-dissipation γ_{LB} does not exist in the critical region B'CA and the distribution (3) in fig. 12b is not possible. Consequently, the absolute minimum of γ_{LB} is either at A or at D (distribution 0, 1 & 2, fig. 12b). At A the dissipation is positive a priori, but at D it may be positive or negative. Therefore, these results yield the following *theorem*:

The inequality

$$M \geq 0 \tag{5.130}$$

and, provided a critical region exists and $(m - \beta)/n < 1$, the inequality

$$\underset{0}{\gamma_{LB}} \geq 0 \tag{5.131}$$

with

$$\left.\begin{aligned} \underset{0}{\gamma_{LB}} &= \gamma_{LB}(\underset{0}{\chi}, \underset{0}{y_1}) \\ &= 2A\underset{0}{\varkappa}\,(\underset{0}{y_1}^2 - 1)^m \left[\underset{0}{y_1} - \left(\frac{HM}{\underset{0}{\chi}^{2\beta}} - 1\right)\underset{0}{\chi}\right] \\ &\quad + 2RM\underset{0}{\varkappa}\,(\underset{0}{\chi}^2)^{m-\beta+1/2} \\ \underset{0}{y_1} &= y_1(\underset{0}{\chi}) = \frac{m}{1+2m}\left(\frac{HM}{\underset{0}{\chi}^{2\beta}} - 1\right)\underset{0}{\chi} \\ &\quad + \left\{\frac{1}{1+2m} + \left[\frac{1}{1+2m}\left(\frac{HM}{\underset{0}{\chi}^{2\beta}} - 1\right)\underset{0}{\chi}\right]^2\right\}^{1/2} \end{aligned}\right\} \tag{5.132}$$

are *necessary and sufficient conditions* for the intrinsic dissipation γ to be positive whatever the variables $\tilde{I}_2$, $\tilde{J}_2$, $\tilde{K}_2$.

It should be noted that both inequalities do not involve the state variables $\tilde{I}_2$, $\tilde{J}_2$, $\tilde{K}_2$ anymore but only temperature dependent material parameters. Further, for the derivation of (5.130) and (5.131) use of the positiveness of the parameters A, H, R, $\underset{0}{\varkappa}$, m, n, β, $\underset{0}{\chi}$ as well as $(m - \beta)$ and $(2\beta - 1)$ has been made.

The above inequality (5.131) is a criterion which applies whenever a critical region exists that is if (see (5.102))

$$\left(\frac{HM}{x_0^{2\beta}} - 1\right) x_0^r > 1. \tag{5.133}$$

If, however,

$$\left(\frac{HM}{x_0^{2\beta}} - 1\right) x_0^r \leq 1 \tag{5.134}$$

a critical region does not exist and the intrinsic dissipation is positive everywhere. So inequality (5.134) is the first criterion to be tested.

Further, the above inequality (5.131) applies only if the ratio of the exponents is bounded by

$$0 < \frac{m-\beta}{n} < 1. \tag{5.135}$$

If $(m - \beta)/n \geqq 1$ the situation is much more complex; a necessary and sufficient condition has not been derived yet. Fortunately, in most applications of the Robinson model [56, 61] the exponents have the typical values $n \sim 4\text{-}5$, $m \sim 3.5\text{-}4$, $\frac{1}{2} < \beta < 1$ such that (5.135) is satisfied. Therefore, an analysis of the more complex situation $(m - \beta)/n \geq 1$ is superfluous.

The derivation of a strict upper bound criterion for the parameter M using (5.131) and (5.132) is a difficult task. However, a *sufficient upper bound* may be obtained as follows. The assumed existence of a critical region requires that

$$y_c = \left(\frac{HM}{x_0^{2\beta}} - 1\right) x_0^r > y_{01} > 1, \tag{5.136}$$

which implies

$$\frac{HM}{x_0^{2\beta}} - 1 > 0.$$

Consequently, equ. (5.131) yields

$$\left(\underset{0}{y}_1^{\,2}-1\right)^n\left[\left(\frac{HM}{\underset{0}{x}^{\alpha 2\beta}}-1\right)\underset{0}{x}^{\alpha}-\underset{0}{y}_1\right]\frac{1}{M}\le\frac{R}{A}\left(\underset{0}{x}^{\alpha 2}\right)^{m-\beta+1/2}\quad. \tag{5.137}$$

The basic trick is to find an expression which is an upper bound for the left hand side of (5.137) and which allows an explicit analytical solution for the parameter M. If $(\underset{0}{y}_1)_{ub}$ and $(\underset{0}{y}_1)_{lb}$ is an upper and a lower bound for $\underset{0}{y}_1$, respectively, then

$$\left(\left(\underset{0}{y}_1\right)_{ub}^{2}-1\right)^n\left[\left(\frac{HM}{\underset{0}{x}^{\alpha 2\beta}}-1\right)\underset{0}{x}^{\alpha}-\left(\underset{0}{y}_1\right)_{lb}\right]\frac{1}{M}$$

$$>\left(\underset{0}{y}_1^{\,2}-1\right)^n\left[\left(\frac{HM}{\underset{0}{x}^{\alpha 2\beta}}-1\right)\underset{0}{x}^{\alpha}-\underset{0}{y}_1\right]\frac{1}{M} \tag{5.138}$$

and the above left hand side is an upper bound of the r.h.s. Instead of requiring (5.137) the inequality

$$\left(\left(\underset{0}{y}_1\right)_{ub}^{2}-1\right)^n\left[\left(\frac{HM}{\underset{0}{x}^{\alpha 2\beta}}-1\right)\underset{0}{x}^{\alpha}-\left(\underset{0}{y}_1\right)_{lb}\right]\frac{1}{M}<\frac{R}{A}\left(\underset{0}{x}^{\alpha 2}\right)^{m-\beta+1/2} \tag{5.139}$$

is enforced. If the parameters satisfy (5.139), even more so inequality (5.137) is satisfied. Therefore, inequality (5.139) is only a sufficient criterion.

Dropping the term 1/(1 + 2n) in $(5.132)_2$, a lower bound for $\underset{0}{y}_1$ is

$$\left(\underset{0}{y}_1\right)_{lb}=\frac{2n}{1+2n}\left(\frac{HM}{\underset{0}{x}^{\alpha 2\beta}}-1\right)\underset{0}{x}^{\alpha}=2\mathcal{B}x^{\alpha}<\underset{0}{y}_1\quad. \tag{5.140}$$

Obviously, an upper bound is

$$\left(\underset{0}{y}_1\right)_{ub_1}=\left[\frac{HM}{\underset{0}{x}^{\alpha 2\beta}}-1\right]\underset{0}{x}^{\alpha}=\frac{1+2n}{n}\mathcal{B}\underset{0}{x}^{\alpha}=\left(2+\frac{1}{n}\right)\mathcal{B}\underset{0}{x}^{\alpha}-y_c \tag{5.141}$$

but a reduced upper bound can be obtained as follows. The function $y_1(x)$ in the critical region B'CA approaches the straight line $2\mathcal{B}x$ from above (fig. 13). Therefore, a parallel shift of this asymptote through point A gives an upper bound function for $y_1(x)$ which is below the boundary $\overline{AC}$ (fig. 13). Thus

$$\left(\underset{0}{y}_1\right)_{ub_2}=2\mathcal{B}\left(\underset{0}{x}^{\alpha}-x_A^{\alpha}\right)+1>\underset{0}{y}_1 \tag{5.142}$$

where

$$\tilde{x}_A = \frac{1}{\left[\frac{HM}{\tilde{x}_0^{2\beta}} - 1\right]} \cdot \tag{5.143}$$

Observing (5.124), one gets

$$\begin{aligned} (J_0^1)_{ub_2} &= \frac{2n}{1+2n}\left[\frac{HM}{\tilde{x}_0^{2\beta}} - 1\right]\left(\tilde{x}_0 - \frac{1}{\left[\frac{HM}{\tilde{x}_0^{2\beta}} - 1\right]}\right) + 1 \\ &= \frac{2n}{1+2n}\left[\frac{HM}{\tilde{x}_0^{2\beta}} - 1\right]\tilde{x}_0 + \frac{1}{1+2n} \\ &= 2B\tilde{x}_0 + \mathcal{A}. \end{aligned} \tag{5.144}$$

Combining (5.140), (5.141), and (5.144), one obtains

$$\underbrace{2B\tilde{x}_0}_{(J_0^1)_{lb}} < J_0^1 < \underbrace{2B\tilde{x}_0 + \frac{1}{1+2n}}_{(J_0^1)_{ub_2}} < \underbrace{\left(2+\frac{1}{n}\right)B\tilde{x}_0}_{(J_0^1)_{ub_1}} . \tag{5.145}$$

Thus, for large values of n the upper and lower bound are rather close. For a moderately large value, e.g., n = 4, inequality (5.145) gives

$$1 < \frac{J_0^1}{2B\tilde{x}_0} < 1{,}25 .$$

With the definition

$$\mu := \frac{HM}{\tilde{x}_0^{2\beta}} \tag{5.146}$$

and introducing (5.140) and (5.141) into (5.139), one obtains

$$\begin{aligned} \left((J_0^1)_{ub_2}^2 - 1\right)^n \left(1 - \frac{1}{\mu}\right) &< \frac{\tilde{x}_0^{2\beta}}{H}\,\frac{1+2n}{\tilde{x}_0}\,\frac{R}{A}\left(\tilde{x}_0^2\right)^{n-\beta+1/2} \\ &= \frac{R}{AH}(1+2n)\,\tilde{x}_0^{2n} \end{aligned} \tag{5.147}$$

where

$$(y_0)_{ub2} = \frac{2n}{1+2n}\left[\mu - 1\right]x_0 + \frac{1}{1+2n} \tag{5.148}$$

and

$$0 < \left(1 - \frac{1}{\mu}\right) < 1.$$

Still the l.h.s. of (5.147) does not allow an analytic solution for μ or M. But taking an upper bound of the l.h.s. of (5.147), such that

$$\left((y_0)^2_{ub2} - 1\right)^n < \frac{R}{AH}(1+2n)\, x_0^{2m} \tag{5.149}$$

is used instead of (5.145), greatly simplifies the problem and if the parameters satisfy (5.149), even more so they satisfy (5.147). Beyond that the l.h.s. of (5.149) may be a good approximation for the l.h.s. of (5.147) if $\mu \gg 1$; then

$$\left(1 - \frac{1}{\mu}\right) \approx 1 .$$

Note that the present analysis is done under the assumption that a critical region exists; this is the case if and only if

$$\mu = \frac{HM}{x_0^{2\beta}} > 1 + \frac{1}{x_0} . \tag{5.150}$$

If x_0 is a small number,

$$x_0 \ll 1 \tag{5.151}$$

one obtains

$$\left.\begin{array}{l} \mu \gg 1 \\ \text{and} \\ \dfrac{1}{1+x_0} < \left(1 - \dfrac{1}{\mu}\right) \lessapprox 1 . \end{array}\right\} \tag{5.152}$$

Hornberger and Stamm [58 - 60], when performing parametric calculations with the Robinson model, have used a relatively large value for x, i.e.,

$$x_0 \equiv G_0 = 0.5$$

However, Schwertel [61], when fitting the Robinson model to experimental data of a stainless steel, proposes to use a value less than a small limit, i.e.,

$$x_0 < 5\,10^{-2} \; ;$$

this is to improve the fit. On the other hand, an extremely small value for x_0, say $x_0 \sim 10^{-6}$, would cause trouble for the numerical calculation. Thus, this remark demonstrates that there exists materials which indeed require a relatively small value for x_0.

Irrespective of the magnitude of x_0, the inequality (5.149) is valid and from it the relation

$$(y_0^1)_{ub2} < \left\{ 1 + \left(\frac{R}{AH} (1+2n)\, x_0^{2m} \right)^{1/m} \right\}^{1/2}$$

follows; with (5.148) this yields

$$\mu < 1 + \frac{1}{x_0} \frac{1+2n}{2m} \left\{ 1 + \left(\frac{R}{AH} (1+2n) \right)^{1/n} x_0^{2\frac{m}{n}} \right\}^{1/2} - \frac{1}{2n x_0} \; . \qquad (5.153)$$

Note that this condition is *only a sufficient upper bound criterion for* μ. That is, provided a critical region exists, i.e.,

$$\mu > 1 + \frac{1}{x_0}$$

and a relative minimum is not located in the interval $x_A < x < x_0$, which is certain for

$$0 < (m-\beta)/m < 1 \; ,$$

the condition (5.153) *assures* that the intrinsic dissipation is positive. However, values of μ larger than the r.h.s. of (5.153) are not forbidden.

Obviously,

$$\left(1 + \frac{1}{x_0}\right) < 1 + \frac{1}{x_0} \frac{1+2n}{2m} \left\{ 1 + \left(\frac{R}{AH} (1+2n) \right)^{1/n} x_0^{2\frac{m}{n}} \right\}^{1/2} - \frac{1}{2n x_0}$$

Therefore, the sufficient condition (5.153) encompasses the condition (5.134) for the non-existence of a critical region.

For a *necessary and sufficient* condition one has to apply (5.131) and (5.132). However, if n is large, the upper and lower bounds $(y_1)_{ub2}$ and $(y_1)_{lb}$ are "close" to the exact value y_1, and if $x \ll 1$, then inequality (5.153) represents a "good" approximation to the necessary and sufficient condition (5.131) for μ (or M).

Provided

$$\omega := \left(\frac{R}{AH} (1+2n) \right)^{1/n} x_0^{2\frac{m}{n}} \ll 1 \tag{5.154}$$

the r.h.s. (5.153) may be simplified

$$\mu < 1 + \frac{1}{x_0} + \frac{1}{2} \frac{1+2m}{2m} \left(\frac{R}{AH} (1+2n) \right)^{1/n} x_0^{2\frac{m}{n}-1} . \tag{5.155}$$

With (5.146) one finally obtains

$$M < \frac{x_0^{2/\beta}}{H} \left\{ 1 + \frac{1}{x_0} + \frac{1}{2} \frac{1+2m}{2m} \left(\frac{R}{AH} (1+2n) \right)^{1/n} x_0^{2\frac{m}{n}-1} \right\} . \tag{5.156}$$

The shift of the upper bound for M to values larger than $(1 + 1/x_0)x_0^{2\beta}/H$ depends linearly on the ratio ω/x_0 of the two small quantities ω and x_0. It is obvious that for $\omega/x_0 \ll 1$ this shift is negligible and criterion $(5.106)_A$ is sufficiently accurate.

6. Stability Properties of Equilibrium States

In section (2.2) it has been pointed out that the evaluation of the residual entropy inequality at an equilibrium state yields important but limited restrictions on the constitutive functions if continuity and sufficient differentiability of the constitutive functions are assumed. Now, one should be aware of the fact that necessary and sufficient criteria for non-negative entropy production for *all states* in the state space have already be derived. Thus, the requirements (2.18) need not be discussed here.

We turn to the derivation of certain consequences following from stability requirements of the equilibrium states.

At first equilibrium states characterized by

$$\begin{aligned} F^E &= \frac{1}{2}\frac{s^E_{kl}\, s^E_{lk}}{\kappa_0^2} - 1 < 0 \\ a^E_{kl} &= 0 \end{aligned} \qquad (6.1)$$

are considered where s^E_{kl} and a^E_{kl} are equilibrium values. States defined by

$$F^E = \frac{1}{2}\frac{s^E_{kl}\, s^E_{lk}}{\kappa_0^2} - 1 = 0$$

are discussed later. In the neighborhood of the above equilibrium states (6.1) the plastic strain rate $\dot{\varepsilon}p_{kl}$ vanishes identically. Thus, the evolution equation (4.23) for the back stress reduces to

$$\dot{a}_{kl} = - R\left(\frac{I_2}{\kappa_0^2}\right)^{m-\beta} \frac{a_{kl}}{\sqrt{I_2}} . \qquad (6.2)$$

Consider an equilibrium state with the following equilibrium values

ε^E_{kl}	equilibrium value of total strain			
ε^{pE}_{kl}	"	"	"	plastic strain
$a^E_{kl} = 0$	"	"	"	back stress
T^E	"	"	"	temperature

Correspondingly, the equilibrium values of the stress and the deviatoric stress, i.e., σ^E_{kl} and s^E_{kl} are given by Hooke's law (3.2) such that $(6.1)_1$ is satisfied. We now consider a disturbed state such that all these values remain unchanged except the a_{kl}, which take new initial values $\overset{\circ}{a}_{kl}$. We require that this disturbance $\overset{\circ}{a}_{kl}$ is sufficiently small such that plastic strain rates do not occur, i.e., $(3.4)_2$ applies. If it is now required that the equilibrium state is locally asymptotically stable under constant total strain and temperature*, then it is obvious from the differential equations (6.2) that R must be positive, i.e.,

$$R > 0 \, . \tag{6.3}$$

The thermodynamic restriction $(5.95)_1$, which is a consequence of the evaluation of the residual dissipation inequality, then implies**

$$M > 0 \, . \tag{6.4}$$

This result can also be derived along the lines sketched in section 2.2. Starting from the same assumption as above, the residual dissipation inequality takes the following form during an isothermal relaxation process in the neighborhood of an equilibrium state where $\dot{\varepsilon}p_{kl} \equiv 0$

$$\begin{aligned} -\varrho \frac{\partial \hat{\psi}}{\partial a_{kl}} \dot{a}_{kl} &= -\varrho \frac{d}{dt} \hat{\psi}(\varepsilon^E_{mn}, T^E, \varepsilon^{pE}_{kl}, a_{pq}(t)) \\ &= -\varrho \frac{d}{dt} \check{\psi}(\varepsilon^{eE}_{mn}, T^E, a_{pq}(t)) \geq 0 \, . \end{aligned} \tag{6.5}$$

For the constitutive model under consideration the equality sign applies only at an equilibrium state, i.e., the model is strictly dissipative. Thus, equ. (6.5) implies along an isothermal relaxation process

$$\check{\psi}(\varepsilon^{eE}_{mn}, T^E, a_{pq}(t)) \geq \check{\psi}(\varepsilon^{eE}_{mn}, T^E, a^E_{pq}) \tag{6.6}$$

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* It should be kept in mind that different constraints could be applied, e.g., adiabatic enclosure at constant total strain.

** The case M = 0 is of no interest. It implies a vanishing entropy production due to the internal variables.

where

$$a^E_{pq} = 0 . \tag{6.7}$$

Since the equilibrium states satisfying (6.1) are locally asymptotically stable, inequality (6.6) may be written as

$$\check{\psi}(\varepsilon^{eE}_{mn}, T^E, a_{pq}) \geq \check{\psi}(\varepsilon^{eE}_{mn}, T^E, a^E_{pq}) \tag{6.8}$$

valid for *all* values a_{pq} in the neighborhood of a^E_{pq}. The equality sign applies only at the equilibrium state. Consequently, equ. (2.43) applies which yields

$$\left.\begin{aligned} (\delta_\alpha \check{\psi})_E &= \left(\frac{\partial \check{\psi}}{\partial a_{mn}}\right)_E \delta a_{mn} = \left(\frac{1}{\varrho} M_{(T)}\, a_{mn}\right)_E \delta a_{mn} = 0 \\ (\delta^2_\alpha \check{\psi})_E &= \left(\frac{\partial^2 \check{\psi}}{\partial a_{kl}\, \partial a_{mn}}\right) \delta a_{kl}\, \delta a_{mn} = \left(\frac{1}{\varrho} M_{(T)}\right)_E \delta a_{mn}\, \delta a_{mn} > 0 \end{aligned}\right\} \tag{6.9}$$

for all variations δa_{pq} subject to the constraint

$$\delta a_{kk} = 0$$

which is due to the deviatoric character of the back stress tensor. It is a trivial matter to show that $(6.9)_1$ yields (6.7), and $(6.9)_2$ implies

$$M_{(T^E)} > 0 . \tag{6.10}$$

We now turn to equilibrium states characterized by

$$\left.\begin{aligned} F^E = \frac{1}{2}\, \frac{s^E_{kl}\, s^E_{kl}}{\kappa_0^2} - 1 &= 0 \\ a^E_{kl} &= 0 \end{aligned}\right\} \tag{6.11}$$

It is obvious that a perturbation of these equilibrium states, such that the total strain ε_{kl} and temperature T are kept constant, may yield the production of plastic strain or not. Correspondingly, the following cases are considered:

Case (1): The perturbed state is such that not only strain and temperature are the same as in the equilibrium state but also the plastic strain. Consequently, the elastic strain and thus the stress remains unperturbated but the internal variables are affected. We assume that this perturbed state is within the region where $(3.4)_2$ applies but not on its boundary. Thus, during the following isothermal relaxation process no plastic strain is produced. Therefore, only the simple relation (6.2) applies and it is clear from the previous discussion that the internal variables relax to the equilibrium values $a^E_{kl} = 0$ with increasing time. Consequently, under such disturbances the equilibrium states are asymptotically stable.

Case (2): We assume that the perturbation is such that the total strain and temperature are the same as in the equilibrium state but that the equilibrium values of the plastic strain ε^p_{kl} and back stress a_{kl} are perturbed too. The response during the following isothermal relaxation process depends strongly on the fact whether plastic strains are produced or not. This can be decided on the basis of the constraint conditions given in (3.4). We assume the perturbed state satisfies the condition

$$F^{0} = \frac{1}{2}\,\frac{\overset{0}{\Sigma}_{kl}\,\overset{0}{\bar{\Sigma}}_{lk}}{\varkappa_0^{2}} - 1 < 0 \qquad \cup \qquad \overset{0}{S}_{kl}\,\overset{0}{\bar{\Sigma}}_{lk} < 0$$

so that the plastic strain rates are vanishing. Then relation (6.2) applies and the back stresses relax to the equilibrium values $a^E_{kl} = 0$. However, the other variables, i.e., plastic strain, elastic strain, and stress, remain at their perturbed values $\overset{0}{\varepsilon}{}^p_{kl}$, $\overset{0}{\varepsilon}{}^e_{kl}$, and $\overset{0}{s}_{kl}$. Thus, under this kind of disturbance the equilibrium state is not asymptotically stable. However, the response of the material under a bounded perturbation is bounded. Under the above conditions the equilibrium state may be considered to be indifferent.

Case (3): Again we consider a perturbation as in case (2), however, the perturbed state should satisfy the constraints

$$F^{0} > 0 \qquad \cap \qquad \overset{0}{S}_{kl}\,\overset{0}{\bar{\Sigma}}_{lk} > 0 .$$

Thus, during the following process at constant total strain $\varepsilon_{kl} = \varepsilon^E_{kl}$ and temperature $T = T^E$ the plastic strain rate is non-vanishing such that $(3.4)_1$ and $(4.23)_2$ apply. Note that $(4.23)_2$ is the appropriate form since the perturbation $\overset{\circ}{a}_{kl}$ is assumed to be sufficiently small such that

$$G = \frac{1}{2}\,\frac{\overset{\circ}{a}_{kl}\,\overset{\circ}{a}_{kl}}{\varkappa_0^2} < G_0 .$$

The ordinary differential equations governing the above process are derived as follows. Equ. (3.2) yields the deviatoric stress

$$s_{kl} = 2\mu\left(\varepsilon^e_{kl} - \frac{1}{3}\,\varepsilon^e_{mm}\,\delta_{kl}\right)$$

and with (3.1) one obtains after time differentiation

$$\dot{s}_{kl} = 2\mu\left[-\dot{\varepsilon}^p_{kl} + \left(\dot{\varepsilon}_{kl} - \frac{1}{3}\,\dot{\varepsilon}_{mm}\,\delta_{kl}\right) - \left(\dot{\varepsilon}^{th}_{kl} - \frac{1}{3}\,\dot{\varepsilon}^{th}_{mm}\,\delta_{kl}\right)\right]; \qquad (6.12)$$

note that μ is assumed constant and ε^p_{kl} is deviatoric. If total strain and temperature are held constant, then (6.12) reduces to

$$\dot{s}_{kl} = -2\mu\,\dot{\varepsilon}^p_{kl} . \qquad (6.13)$$

Consequently, observing $(3.4)_1$, (6.13), and $(4.23)_2$,

$$\left.\begin{aligned} \dot{s}_{kl} &= -2\mu\,A\,F^n\,\frac{s_{kl}-a_{kl}}{\sqrt{J_2}} \\ \dot{a}_{kl} &= +\frac{H}{G_0^{\beta}}\,A\,F^n\,\frac{s_{kl}-a_{kl}}{\sqrt{J_2}} - R\,G^{m-\beta}\,\frac{a_{kl}}{\sqrt{I_2}} \end{aligned}\right\} \quad (6.14)$$

and this represents the required set of differential equations for the deviatoric stress and back stress*. The associated initial conditions (perturbed values) are

$$\left.\begin{aligned} s_{kl}(0) &= \overset{\circ}{s}_{kl} \\ a_{kl}(0) &= \overset{\circ}{a}_{kl} \end{aligned}\right\} \quad (6.15)$$

---

* Equations (6.14) represent a system of 12-2 = 10 independent ordinary differential equations. Note that this system can be condensed to a system of three O.D.E.'s for the invariants $\tilde{I}_2$, $\tilde{J}_2$, $\tilde{K}_2$.

and these values are required to satisfy

$$F^0 = \frac{1}{2}\,\frac{(s^0_{kl} - a^0_{kl})(s^0_{lk} - a^0_{lk})}{\varkappa_0^2} - 1 > 0$$

and

$$s^0_{kl}\,(s^0_{lk} - a^0_{lk}) > 0 .$$

The initial values of the deviatoric stresses are given by

$$s^0_{kl} = 2\mu\,(\varepsilon^{e^0}_{kl} - \frac{1}{3}\,\varepsilon^{e^0}_{mm}\,\delta_{kl})$$

where

$$\varepsilon^{e^0}_{kl} = \varepsilon^{E}_{kl} - \varepsilon^{p0}_{kl} - \varepsilon^{uE}_{kl} .$$

The stability properties of the above nonlinear system of ordinary differential equations cannot be judged simply by inspection. Mathematical methods are required, and we will comment on two standard approaches [67]:

(1) Linearization at the equilibrium point
(2) The direct method of Lyapunov.

The first method consists of two parts. Firstly, the derivation of the linear variational equations of the nonlinear system (6.14); this requires a Taylor series developement of the right hand side at the equilibrium point

$$\left.\begin{aligned} s_{kl} &= s^E_{kl} \\ a_{kl} &= a^E_{kl} = 0 \end{aligned}\right\} \quad (6.16)$$

with

$$F^E = \frac{1}{2}\,\frac{s^E_{kl}\,s^E_{lk}}{\varkappa_0^2} - 1 = 0$$

and a reformulation of (6.14) in terms of the deviations ζ_{kl} and χ_{kl}

$$\left.\begin{aligned} \zeta_{kl} &= s_{kl} - s^E_{kl} \\ \chi_{kl} &= a_{kl} - a^E_{kl} = a_{kl} \end{aligned}\right\} \quad (6.17)$$

In the second part the stability analysis of the linear variational system is done. Turning to the first part, we note that generally n > 1 and, therefore, the first derivatives of the right hand sides of (6.14) vanish at the equilibrium point; this is true also for higher derivatives if n is large. Thus, if n is an integer and n > 1, linear variational equations cannot be set up. On the other hand, if n is not an integer, a Taylor series development at the equilibrium state is not possible. Consequently, the approach (1) is not applicable to the system (1).

For the discussion of Lyapunov's method some introductory remarks are necessary. We denote the deviations (6.17) by the letter ξ_K

$$\xi_K = \{ \xi_{kl} \, , \, \chi_{kl} \} . \tag{6.18}$$

Then the system (6.14) may be put in the general form

$$\dot{\xi}_K = X_K (\xi_M) \tag{6.19}$$

where

$$X_K (\xi_M) = 0 \qquad \text{for } \xi_M = 0 \tag{6.20}$$

represents the equilibrium point. The initial conditions (6.15) read

$$\xi_K (0) = \overset{\circ}{\xi}_K . \tag{6.21}$$

The solution $\xi_K(t)$ of (6.19) and (6.21) may be represented by a trajectory in the ξ_K-space (phase space) starting at the point $\xi_K = \overset{\circ}{\xi}_K$.

Lyapunov's theorem for a system of autonomous differential equations may be formulated as follows [67, 90]. Assume that a scalar function $V(\xi_K)$ can be prescribed with the following properties:

(a) V and its first partial derivatives are continuous in an open region Ω around the origin $\xi_K = 0$,

(b) V vanishes at the origin, i.e. $V_{(o)} = 0$,

(c) V is positive everywhere in Ω. Thus the origin is an isolated minimum of V.

A function with the properties (b) and (c) is called positive definite.

If in addition the change

$$\frac{d}{dt} V(\xi_K(t)) = \frac{\partial V}{\partial \xi_K} \dot{\xi}_K = \frac{\partial V}{\partial \xi_K} X_K(\xi_M)$$

is negative semidefinite or identically zero along all trajectories, i.e., if

$$\dot{V} = \frac{d}{dt} V(\xi_K(t)) \leq 0,$$

then the unperturbed solution, i.e., the equilibrium point $\xi_K = 0$, is stable. If V is negative definite

$$\dot{V} < 0,$$

except for $\xi_K = 0$ where $V = 0$, then the unperturbed solution is asymptotically stable.*

We recall the fact that the intrinsic dissipation must be non-negative during any process and, therefore, also for processes at constant total strain and temperature, i.e., along solutions of (6.14) and (6.16). Within the plastic region the intrinsic dissipation (2.99) reads

$$-\rho \frac{\partial \hat{\psi}}{\partial \varepsilon^p_{mn}} \dot{\varepsilon}^p_{mn} - \rho \frac{\partial \hat{\psi}}{\partial a_{mn}} \dot{a}_{mn} \geq 0 \tag{6.22}$$

where

$$\begin{aligned} \psi &= \hat{\psi}(\varepsilon_{kl}, T, \varepsilon^p_{mn}, a_{pq}) \\ &= \check{\psi}(\varepsilon^e_{kl}(\varepsilon_{rs}, \varepsilon^p_{rs}, \varepsilon^{th}_{rs}), T, a_{mn}). \end{aligned} \tag{6.23}$$

At constant strain and temperature

$$\varepsilon_{kl} = \text{const.}, \qquad T = \text{const.}$$

and consequently

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* For the strict definition of stability see ref. [67, 90]

$$\varepsilon^{th}_{kl} = const.$$

we obtain from inequality (6.22)

$$\dot{\hat{\psi}} = \frac{d\hat{\psi}}{dt} \leq 0 . \qquad (6.24)$$

The rate of the free energy is strictly negative in the plastic region but vanishes at the equilibrium state.

Condition (6.24) gives the motivation to check whether the free energy satisfies the requirements of Lyapunov's theorem observing that a constant term may always be added. Substracting the value at the equilibrium state gives

$$\Delta\psi = \psi - \psi^E \qquad (6.25)$$

such that requirement (b) is satisfied by $\Delta\psi$. Further, by assumption, requirement (a) is satisfied everywhere and since (6.24) applies to ψ it also applies to $\Delta\psi$.

Using Hooke's law (3.2), the free energy may also be expressed in terms of the stresses

$$\begin{aligned}
\Delta\psi &= \overset{x}{\psi}(\sigma_{kl}, T, a_{mn}) - \overset{x}{\psi}{}^E \\
&= \frac{1}{\rho}\,\frac{1}{2}\left(\frac{1+\nu}{E}\sigma_{kl}\sigma_{lk} - \frac{\nu}{E}\sigma_{mm}\sigma_{nn}\right) \\
&\quad + \frac{1}{\rho}\,\frac{1}{2} M a_{kl} a_{lk} + \hat{\psi}^T(T) - \overset{x}{\psi}{}^E \\
&= \overset{xx}{\psi}(s_{kl}, \sigma_{mm}, T, a_{pq}) - \overset{xx}{\psi}{}^E \qquad (6.26) \\
&= \frac{1}{\rho}\,\frac{1}{2}\left(\frac{1+\nu}{E}s_{kl}s_{lk} - \frac{1+4\nu}{3E}\sigma_{mm}\sigma_{nn}\right) \\
&\quad + \frac{1}{\rho}\,\frac{1}{2} M a_{kl} a_{lk} + \hat{\psi}^T(T) - \overset{xx}{\psi}{}^E
\end{aligned}$$

where

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \quad , \quad \nu = \frac{\lambda}{2(\lambda+\mu)} . \tag{6.27}$$

Note that

$$\sigma_{mm} = (3\lambda+2\mu)(\varepsilon_{mm} - \varepsilon^{th}_{mm});$$

thus, if the total strain and temperature are held constant, the trace σ_{mm} or the hydrostatic pressure

$$p = -\frac{1}{3}\sigma_{mm}$$

does not change. If the deviations (6.17) are introduced in $\Delta\psi$, equ. (6.26), it is seen that ξ_{kl} are quadratic but also linearly involved and the coefficients of the linear terms do not vanish identically:

$$s_{kl}s_{lk} - s^E_{kl}s^E_{lk} = \xi_{kl}\xi_{lk} + 2\xi_{kl}s^E_{lk} .$$

Consequently, the difference $\Delta\psi$ is not positive definite in the neighborhood of the considered equilibrium state. Thus, $\Delta\psi$ is not a Lyapunov function and stability cannot be assessed on this basis.

Presently, we will postpone the stability analysis of the general nonlinear system (6.14) and (6.15). We expect that the perturbated state will relax to the equilibrium state characterized by (6.11) if the material parameters A, H, and R are positive, μ taken to be positive anyway. But possibly this behaviour depends on the initial disturbance. In fact, it can be shown [91] for the restricted condition of a uniaxial state of stress that the equilibrium state (6.11) is not locally asymptotically stable: The equilibrium state is on the boundary of its domain of attraction (see fig. 2).

Summarizing the discussion of equilibrium states characterized by (6.11), it has been demonstrated that their stability depends on the kind of disturbance: For some disturbances asymptotic stability is assured, for others a kind of indifferent equilibrium is observed. If the perturbated state is in the plastic region and remains in this region at constant strain and temperature, asymptotic stability is not assured, but a general analysis is not yet available.

An important observation should be added. If the equilibrium state is locally asymptotically stable, the free energy has a relative minimum at the equilibrium state which implies (2.43)

$$\left(\frac{\partial \psi}{\partial \alpha_\gamma}\right)_E = 0$$

as shown in section (2.2).

Using the terminology of the classical Theory of Irreversible Processes [92, 93], the quantities $-\rho\, \partial\psi/\partial \alpha_\gamma$ represent the "affinities" or "thermodynamic forces" associated with the "thermodynamic fluxes" α_γ, i.e., the rates of the internal variables. Thus, under the above assumptions the "thermodynamic forces" vanish at equilibrium. Classically, this is considered as a part of the definition of thermodynamic equilibrium [93, 95]. For the equilibrium states in the non-plastic region (6.1) the thermodynamic forces $(-\rho\, \partial\psi/\partial a_{mn})_E$ indeed vanish as seen from equ. $(6.9)_1$. However, equilibrium states characterized by (6.11) are not locally asymptotically stable, and so the free energy has no relative minimum. This is seen also from (6.22) which can be written as

$$S_{mn}\, \dot{\varepsilon}^{p}_{mn} - \varrho \frac{\partial \psi}{\partial a_{mn}}\, \dot{a}_{mn} \geq 0$$

and obviously

$$S^{E}_{mn} \neq 0 , \qquad \left(\varrho \frac{\partial \psi}{\partial a_{mn}}\right)_E = 0 .$$

The "thermodynamic forces" of the plastic strain rates are given by the deviatoric stresses which do not vanish at the equilibrium states. Thus, following the discussion of Bowen [14], equ. (6.11) characterizes non-classical ("false") equilibrium states.

Finally, the following remark should be made. The assumed external constraints on the process following a perturbation, i.e., constant total strain and temperature, is just one set of constraints which involves heat exchange of the body with the surrounding but no exchange of mechanical work. Stability should also be studied under other constraints, e.g., when neither work nor heat are exchanged.

7. Discussion and Conclusions

The primary aim of the present study is to analyse whether or not a version of the Robinson model is consistent with thermodynamic principles as motivated in section 1. Here it should be noted that the Robinson model is a mechanical model not embedded in a truly thermodynamic frame although it involves the temperature. Therefore, a thermodynamic consistency analysis requires an extension of the model, i.e., an assumption about the structure of the free energy.

The analysis was done under a set of simplifications:

- Isotropic hardening was ignored, and, therefore, also the thermomechanical history dependence of the drag stress κ. This simplification reduces the number of evolution equations by one.
- The static recovery term in the evolution equation of the back stress a_{kl} was modified in the same way as done by Hornberger and Stamm [58-60]. This not only eliminates a discontinuity in this evolution equation but also allows for the existence of equilibrium states.
- The second law of thermodynamics is taken to be the classical Clausius-Duhem entropy inequality. Further, its evaluation was based on the Coleman-Noll argument. More advanced principles and concepts, e.g., the approach of Müller [62] (see also [63]), require a more elaborate analysis.

Starting point for the above analysis was the thermodynamics with internal state variables formulated by Coleman and Gurtin [13] within the frame of Rational Thermodynamics. This general frame was used by Perzyna and Wojno [19-22] and Kratochvil and Dillon [27, 28] to analyse the thermodynamic restrictions imposed on some elastic-viscoplastic or elastic-plastic material models. In these contributions the inelastic strains were interpreted as internal state variables such that the results of Coleman and Gurtin could be directly applied. This interpretation was also used in the present analysis.

Usually, the derivation of thermodynamic restrictions implies the assumption that processes are *smooth* functions of time; thus, the local *instant* form of the Clausius-Duhem entropy inequality is an adequate starting point. In fact, this is an assumption contained in the work [13, 63]. However, the Robinson model has the peculiar property that the evolution equations for the inelastic strain and the back stress involve *discontinuities* in the rates of these variables. Therefore, the

Robinson model requires a more general starting point, i.e., a time-integral formulation of the Clausius-Duhem entropy inequality and time-integral balance equations. Allowing for discontinuities in the evolution functions for the internal variables, a general internal variable model was set up such that the Robinson model (without thermomechanical history dependence) is a special case. The constitutive functions for internal energy, entropy, or free energy, however, were assumed to be continuous and differentiable up to any needed order. Under these conditions the instant Clausius-Duhem entropy inequality is adequate for all states except those where a discontinuity in the rates occurs; here the time-integral form is required. The instant form imposes restrictions as usual. It was shown then that the time-integral form does not impose additional constitutive restrictions at states where a discontinuity in the rates of the internal variables occurs. However, discontinuities in the process variables are induced. As a consequence, the restrictions obtained from the classical instant form of the Clausius-Duhem entropy inequality are applicable to all states, and no other constitutive restrictions are to be observed.

One may question whether the assumed differentiability of the free energy is compatible with the discontinuities in the evolution equations for the internal variables [113]. Some remarks on this subject are in place here.

In the classical linear Theory of Irreversible Processes (TIP) the rates $\dot{\alpha}_\gamma$ of the internal variables, i.e., the "thermodynamic fluxes", are assumed to be linearly related to the socalled "thermodynamic forces" A_γ so that

$$\dot{\alpha}_\gamma = L_{\gamma\delta} A_\delta .$$

The A_γ are the "work coefficients" of the fluxes in the Gibbs fundamental equation whose validity is a basic assumption of TIP. If the free energy ψ is differentiable with respect to the internal variables then it may be shown that

$$A_\gamma = - \varrho \frac{\partial \psi}{\partial \alpha_\gamma} .$$

Here one considers the "forces" as the causes of the "fluxes"; then, formally, a discontinuity in the "fluxes" is obtained if the "forces" are discontinuous or if the coefficients $L_{\gamma\delta}$ relating "forces" and "fluxes" are discontinuous. Thus, there appears to be no basic requirement that a discontinuity in the rates $\dot{\alpha}_\gamma$ - the fluxes - should always be connected only to a discontinuity in the derivatives $\partial\psi/\partial\alpha_\gamma$ of the free energy, i.e., the "forces", even in the linear Theory of Irreversible Processes. Furthermore, the present analysis is within the frame of Rational Thermodynamics which does not use the concept of forces and fluxes and which does not enforce a priory a Gibbs fundamental equation but which is fairly general with respect to the evolution and the state equations. Of course, the assumption of sufficiently smooth state functions (2.1) is a restriction on the constitutive behavior.

The present material model is a macroscopic phenomenological model and the internal variables used represent suitably chosen macroscopic mathematical entities to describe some of the observed material behavior. The understanding is that, in a certain *average sense*, they reflect essential features of microstructural changes. This interpretation is followed by many of the cited writers. Others have associated α_γ with special structural changes occuring at localized sites within the body, e.g., Rice and coworkers [32, 114, 115]. There the state functions are not explicitly required to be differentiable with respect to the history variables denoted by H; however, the notation in [32, 114, 115] is somewhat irritating since it is not always clear whether the existence of a limiting process for the state functions exists or not. In any case, the assumption of differentiability of the free energy is not just a formality which has also been pointed out by Nemat-Nasser [116]; in general this assumption may exclude certain effects.

The constitutive restrictions obtained for the Robinson model will now be discussed. Here we will first turn to the thermodynamically extended model; later the consequences for the purely mechanical model will be considered.

The thermodynamic extension requires an assumption about the free energy. The elastic response is described by Hooke's law where the elastic constants are possibly temperature dependent but independent on the plastic strains and back stresses. Therefore, the free energy consists of a term depending only on the first and second invariant of the elastic strains such that the elastic strains (or the stresses) are quadratically involved. This term is entirely analogous to classical thermoelastic assumptions. However, this term has to be supplemented additively by two terms representing the influence of the back stress and, of course, the purely thermal effects. The essential assumption is that the second invariant of the deviatoric back stress is involved linearly so that the free energy depends only quadratically on the back stress; this is the most simple assumption compatible with isotropy and differentiability.

Expecting mathematical simplicity, the uniaxial case was subjected to a consistency analysis first. But since the multiaxial case is the more embracing one, we will restrict the discussion to this case.

The requirement that the intrisinc dissipation is non-negative for all admissible processes has yielded a set of conditions with different logical properties. The *necessary condition* (5.97), i.e.,

$$R_{(T)}\; M_{(T)} \geq 0$$

gives with R > 0 (7.1)

$$M_{(T)} \geq 0\,.$$

Further, implying (7.1)$_2$ and assuming a critical region does not exist, i.e.,

$$\left(\frac{HM}{\breve{X}_0^{2/\beta}} - 1\right)\breve{X}_0 \leq 1 \quad \text{or equivalently} \quad M \leq \frac{\breve{X}_0^{2/\beta}}{H}\left(1 + \frac{1}{\breve{X}_0}\right), \tag{7.2)$_A$ (}$$

the intrinsic dissipation is assured to be non-negative; thus, inequality (7.2)$_A$ is only a *sufficient condition*. However, if a critical region exists, that is

$$\left(\frac{HM}{\breve{X}_0^{2/\beta}} - 1\right)\breve{X}_0 > 1\,, \tag{7.2)$_B$ (}$$

and if

$$0 < \frac{m-\beta}{m} < 1\,, \tag{7.3}$$

condition (5.131), i.e.,

$$\left.\begin{aligned} &\underset{0}{\gamma}_{LB} \geq 0 \\ &\text{where} \\ &\underset{0}{\gamma}_{LB} = 2A\varkappa_0\left(\mathcal{Y}_0^2 - 1\right)^m\left[\mathcal{Y}_0 - \left(\frac{HM}{\breve{X}_0^{2/\beta}} - 1\right)\breve{X}_0\right] \\ &\qquad + 2RM\varkappa_0\left(\breve{X}_0^2\right)^{m-\beta+1/2} \end{aligned}\right\} \tag{7.4}$$

is a *necessary and sufficient requirement* for a non-negative intrinsic dissipation.

These conditions involve only the temperature dependent material parameters but not the variables "stress", "back stress", and "plastic strain".

It is important to note that, when deriving the above restrictions, it was understood that the material parameters satisfy the following requirements:
Firstly,

$$\left.\begin{array}{l} A,\ \varkappa_0,\ n \quad > 0 \\ H,\ R,\ m,\ \beta \quad > 0 \end{array}\right\} \qquad (7.5)$$

and secondly and more specific,

$$m-\beta > 0 \quad , \quad \beta \geq 1/2 \quad , \quad \frac{m-\beta}{n} < 1. \qquad (7.6)$$

The condition $(m - \beta) > 0$ assures that an equilibrium state for vanishing back stresses exists (see section 4.2). The condition $\beta \geqq 1/2$ implies that a critical region in the (x,y)-plane has a quasi triangular shape such that the existence or non-existence of a critical region can be characterized by the simple criteria $(7.2)_{A\&B}$. Finally, when the condition $(m - \beta)/n < 1$ applies it is certain that a *relative* minimum of the lower bound-dissipation does not exist in the critical region B'CA.

Practically relevant material data are consistent with the conditions (7.6). Therefore, the analysis was not extended to the range $(m - \beta)/n < 1$. In this range, strict conditions are more complex.

Whereas inequality $(7.1)_2$ represents a *necessary lower bound criterion* for the free energy parameter M and $(7.2)_A$ a *sufficient upper bound criterion*, condition $(7.4)_1$ should yield a necessary and sufficient upper bound for M. However, a strict criterion is difficult to obtain. Nevertheless, it was possible to develop from (7.4) an *improved sufficient upper bound for M* using bounding estimates of various terms in (7.4). This bound is given by (5.153) or explicitly

$$M < \frac{\varkappa_0^{2\beta}}{H}\left[1 + \frac{1}{\varkappa_0}\,\frac{1+2n}{2m}\left\{1 + \left(\frac{R}{AH}(1+2n)\right)^{1/n} \varkappa_0^{\frac{2m}{n}}\right\}^{1/2}\right] - \frac{1}{2n\varkappa_0}; \qquad (7.7)$$

obviously, this condition includes the sufficient condition $(7.2)_A$. Further, it may be shown that for large values of the exponent n and small values $\underset{o}{x}$ ($\underset{o}{x} \ll 1$) the above condition represents a good approximation for $(7.4)_I$.

The separation of the dissipation inequality into the two conditions $(7.1)_I$ and $(7.4)_I$ is due to the fact that the Robinson model involves a "switch" which controls the transition between regions where the plastic strain rate is either vanishing or not. If the plastic strain rate vanishes, then dissipation is only due to the static recovery characterized by the parameter R. Thus, inequality $(7.1)_I$ expresses the fact that this contribution to the total dissipation must be non-negative and this is a necessary and sufficient condition whenever the plastic strain rates vanish.

It should be noted that the derivation of $(7.1)_I$ actually did not involve any assumptions on the two parameters R and M. Thus, for $R \neq 0$ and $M \neq 0$ condition $(7.1)_I$ simply requires that R and M have the same sign. Of course, $M = 0$ is mathematically permissible but this would imply that the internal variables a_{kl} - the back stresses - do not contribute to the intrinsic entropy production which appears to be unreasonable. Therefore, assuming $R_{(T)} > 0$, condition $(7.1)_I$ simply implies $M_{(T)} > 0$. The positiveness of the recovery parameter R as well as the hardening parameter H is an essential property of the Robinson model. The equality of sign of both R and H is necessary for the existence of stationary states, e.g., secondary creep. Otherwise the hardening and the "static" recovery term in the evolution equations for the back stress could not compensate each other to yield a vanishing rate in the back stress. Also, if both R and H where negative, primary creep with a decreasing creep rate could not be modeled. Furthermore, the positiveness of the parameter R alone implies that equilibrium states in the region of vanishing plastic strain rates are asymptotically stable under constant strain and temperature (section 6). This is a rather reasonable property since otherwise equilibrium states could not exist in nature. Thus, we conclude that a unique assignment of either a positive or negative sign to each of the parameters R and M cannot be based on thermodynamic considerations alone but other physically reasonable requirements, e.g., certain simple response properties of the model or stability of equilibrium states, are necessary too.

In this connection it should be pointed out that in the present analysis the requirement of positiveness of the entropy production (second law) does not impose stability of equilibrium states by itself, but stability is a separate demand. Similar observations were made by Perzyna [96], Podio-Guidugli [97], Anthony [98], and especially Serrin [100] in his illuminating article; in fact, Coleman and Gurtin [13] introduced the asymptotic stability of equilibrium states as an extra requirement.

On the other hand, one may easily get the impression in textbooks that existence and stability of equilibrium states is a consequence of the first and second law alone (e.g. [102], [103], [62]). In fact, there exist several indications pointing to an intimate relationship between the second law and stability. Notably, the work of Ericksen [104, 105], Koiter [106], Coleman and Dill [108], and Gurtin [109] has revealed that the Clausius-Duhem inequality induces Ljapunov stability of equilibrium states for a variety of materials. Dafermos [110] established a different connection between stability and the second law, in that stability is interpreted as continuous dependence of thermodynamic processes on the initial state and supply terms. He showed within the context of thermoelasticity theory without heat conduction that, whenever they exist, smooth processes are stable within the class of (not necessarily smooth) proper processes [110] that satisfy the Clausius-Duhem inequality in its traditional form. However, this result was established under certain assumptions on material response, relating to the convexity of internal energy. Thus, as pointed out by Dafermos [110], it is not generally to be expected that the second law will by itself induce stability, unless it is supplemented by appropriate restrictions on constitutive relations. Here one may add the almost trivial remark that primary assumptions and derived theorems obtained in one course of theoretical deduction may interchange their role in another course of deduction. We illustrate this for the thermodynamically extended Robinson model. Lame's parameter λ, μ are positive by assumption. If the free energy ψ is assumed to be a positive function of the variables $\mathbb{T}^{**}$ and vanishing at the reference state $\mathbb{T}^{**} = \{0, 0, T_0\}$, then this implies that the temperature dependent functions $M_{(T)}$ and $\psi^T{}_{(T)}$ are positive too. Then, according to the dissipation inequality condition, equ. $(7.1)_1$ is required which yields the positiveness of the parameter R. This in turn implies the asymptotic stability of equilibrium states in a regime with vanishing plastic strain rates.

After this digression we proceed to discuss condition (7.4). It has been demonstrated that the derivation of (7.4), especially its logical status, is less trivial than (7.1) since this requires the *identification of the absolute minimum* of the intrinsic dissipation in the critical region whenever such a region in the (x, y)-plane exists.

The non-linearity of the problem forbids a formal analytic solution. Also a numerical approach would be of a very limited value. Rather, a geometric interpretation of the relations is used to a large extend. This allows for a limited range of the parameters, i.e., $(m - \beta)/n < 1$ as well as $\beta \geqq 1/2$, to exclude the existence of a relative minimum of the intrinsic dissipation in the critical region. This in turn permits to localize the absolute minimum. Condition $(7.4)_1$ simply states that the intrinsic dissipation (more accurately: its lower attainable bound) at this location is non-negative.

The inequality $(7.4)_1$ can easily be used for a test of the thermodynamic consistency of the extended Robinson model. Further, the sufficient upper bound condition (7.7) is a handy criterion for the free energy parameter M: It allows to define a range for M which assures that the extended Robinson model is compatible with the dissipation inequality.

The derivation of necessary and sufficient conditions for the extended Robinson model has demonstrated that such an analysis is relatively complex; by no means it is certain that conditions, which involve only the temperature dependent material parameters, can be derived in explicit terms. This complexity makes it understandable that various authors propose the concept of a positive convex dissipation potential which determines the evolution equations for the interval variables and which assures a positive intrinsic dissipation [e.g., 9, 32, 42, 95, 112, 114, 145]. This approach has been criticized by Onat and Leckie [126].

Two further remarks are added.

Firstly, in the spirit of Rational Thermodynamics, which requires that the Clausius-Duhem entropy inequality is *identically satisfied for all processes* every extended Robinson model with exponents satisfying (7.6) but whose parameters are not compatible with $(7.4)_1$ is thermodynamically inconsistent. Consequently, the model has to be modified. Here we will not elaborate on this subject in detail but ad hoc-possibilities are indicated. For some initial choice of material parameters the condition

$$\gamma_{initial}\,(\tilde{I}_2, \tilde{J}_2, \tilde{K}_2, T) = 0$$

together with (5.91) and (5.95) defines a surface in the $(\tilde{I}_2, \tilde{J}_2, \tilde{K}_2)$-space; here the temperature T is assumed to be constant. This surface devides the $(\tilde{I}_2, \tilde{J}_2, \tilde{K}_2)$-space into an admissible region ($\gamma_{initial} \geqq 0$) and into an inadmissible region ($\gamma_{initial} < 0$). The surface may be considered as a kind of switch condition which controls a change in the constitutive model such that *processes are allowed to transmit* the initial surface $\gamma_{initial} = 0$. A very simple adjustment is obtained if one of the material parameters, e.g., $H_{(T)}$, $R_{(T)}$, $M_{(T)}$ or $\underset{o}{\tilde{I}}_2$, is modified such that the intrinsic dissipation is non-negative. A specific choice would be the one where the intrinsic dissipation vanishes identically (non-dissipative process). Consequently, the parameter modified will become a function not only of the temperature but also of the invariants $\tilde{I}_2$, $\tilde{J}_2$, $\tilde{K}_2$.

On the other hand, the surface ($\gamma_{\text{initial}} = 0$) may be considered as a kind of "limiting surface" so that points beyond this surface cannot be reached. Thus, the limiting surface represents a kind of "thermodynamic yield surface". Further analysis of this idea is required but we will not proceed here.

One must admit that all these approaches are purely formal and ad hoc and it is not excluded that the adjusted models fail when confronted with experimental results.

It should be mentioned that a concept of a "thermodynamic yield surface" was set up by Lubliner [111]; starting from a set of evolution equations, it was possible to derive a yield condition from the Clausius-Duhem inequality. However, the structure of the evolution equations in [111]* is very different from the one considered in this report such that Lubliner's concept of a thermodynamic yield surface is not comparable with the one indicated above.

We now turn to the primary question of the thermodynamic consistency of the *mechanical* Robinson model without isotropic hardening whose parameters satisfy (7.5) and (7.6). The two thermodynamic conditions (7.1) and (7.4) contain the thermodynamic parameter M which is not part of the purely mechanical Robinson model. Thus, the parameter M is free and its choice has no consequences for the purely mechanical predictions.

Both thermodynamic conditions (7.1) and (7.4) are formally satisfied if the parameter M is chosen such that

$$0 \leq M < \frac{X_0^{2\beta}}{H}\left[1 + \frac{1}{X_0}\frac{1+2n}{2n}\left\{1+\left(\frac{R}{AH}(1+2n)\right)^{1/n} X_0^{2\frac{m}{n}}\right\}^{1/2} - \frac{1}{2nX_0}\right]; \qquad (7.8)_A$$

note that the upper bound is solely defined by the parameters of the mechanical Robinson model.

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* All evolution equations contain a scalar function ϕ, a priory restricted to be a non-negative function of the state variables. As a consequence, the intrinsic dissipation is proportional to ϕ.

This result yields the following conclusions:

- *There exist a thermodynamically extended version of the Robinson model which can be designed in such a way that the intrinsic entropy production is assured to be non-negative whatever the parameters*[*] *of the mechanical Robinson model are.* Of course, the extension chosen here is not the only one but obviously the simplest. Since the parameters per se are found not to be subject to thermodynamic restrictions, *the mechanical Robinson model is formally consistent with the second law.*

- Consider the case that the mechanical Robinson model with a given set of mechanical parameters is in agreement with the data of isothermal mechanical experiments (e.g. creep, relaxation etc.). The thermodynamic parameter M is not required to predict these tests. Therefore, a choice consistent with $(7.8)_A$ cannot be tested with these experiments. According to section 4.1, the parameter M plays a quantitative role only in the energy balance equation; therefore, *caloric measurements are also required for the test of the validity of the thermodynamically extended model and, obviously the extended model may possibly fail in these tests.*

Remark:
Another sufficient condition is given by (5.89) such that inequality (5.90) applies, i.e.,

$$0 \leq M \leq \frac{X_0^{2/\beta}}{H} . \qquad (7.8)_B$$

Obviously, this inequality is contained in $(7.8)_A$ but more important it is also valid when (7.5) and $(7.6)_1$ but not $(7.6)_2$ and $(7.6)_3$ are satisfied. This situation has not been analyzed in this report but likely there exist an improved upper bound for M less restrictive than the one in $(7.8)_B$. In any case, the above interpretations formally apply also when the exponents of the Robinson model do not satisfy $(7.6)_2$ and $(7.6)_3$.

These observations clearly demonstrate that a *thermodynamic consistency* analysis of a *purely mechanical model* can only yield the assurance that the model is the isothermal derivative of one or the other thermodynamically extended (i.e. non-isothermal) model which is compatible with the second law. This extended model may still fail energetically even under isothermal conditions.

The generally accepted viewpoint that a purely mechanical material model should be thermodynamically consistent, i.e., compatible with the second law, has obvi-

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[**] However, the parameters are required to satisfy (7.5) and (7.6).

ously a quality which ranks above the requirement that the model should be compatible with some caloric test results which are not in the centre of interest of the intended application. Two reasons are indicated: Firstly, this priority assures at least that the mechanical model is *thermodynamically consistent in principle*, and secondly, restrictions may be obtained which *limit the range of possibilities* for setting up the mechanical model. This last aspect is frequently realized within a purely mechanical context by requiring that the mechanical material model should satisfy certain *work-inequalities*; they are or appear to be related to irreversibility and the second law [119 - 128]. An analysis of this relation and especially the relation to the concept used in this report must be postponed.

We will close the discussion by pointing out the required future theoretical work to enhance understanding of the phenomenological Robinson model.

- So far only a simplified version of the Robinson model has been analysed. The thermodynamical consistency analysis must be extended to include isotropic hardening as well as the thermomechanical path dependence of the drag stress. This last aspect is modeled such that the evolution equation for the drag stress involves the rate of the observable variable "temperature" in a non-integrable form [56]. This requires special attention when the Clausius-Duhem entropy inequality is evaluated.

- In this analysis the Clausius-Duhem inequality has been used in its traditional form in connection with the Coleman-Noll argument. Both concepts are subject to criticism [112, 63]. A more advanced and less restrictive entropy principle and a physically more realistic method for exploiting the entropy principle should be used [62].

- It is suggested to provide some evidence whether the choice of the free energy is acceptable in the light of available caloric experiments.

- Consequences of work-inequalities proposed in the literature to restrict purely mechanical material models should be applied to the Robinson model and the results should be compared with the conclusions obtained in this report.

- Last but not least it is mentioned that various visco-plastic models use the concept of a dissipation potential to satisfy the Clausius-Duhem inequality. The relation of the Robinson model to this concept should be clarified.

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Nomenclature*

A	material parameter in the evolution law for the visco-plastic strain, equ. (3.4)
$\mathcal{A}$	lumped parameter, equ. (5.124)
a_{kl}	deviatoric back stress, equ. (3.7)
$\mathcal{B}$	lumped parameter, equ. (5.124)
$C_1, C_2, C_j^{(i)}$	constraint functions, equ. (2.45), (2.94)
$\mathcal{C}^*, C_o^*$	function characterizing boundary of critical regime, equ. (5.14)
$\mathcal{C}$	lumped parameter, equ. (5.128)
$\mathcal{D}^*$	intrinsic dissipation due to recovery along boundary of critical regime, equ. (5.14)
E	internal energy, equ. (2.67) or Young's modulus, equ. (3.3)
f_γ	evolution function for the internal variables, equ. (2.2)
F	yield function, equ. (3.7)
$G \equiv \tilde{I}_2$	dimensionless second invariant of back stress, equ. (3.7)
$G_o \equiv \tilde{I}_{o2}$	value defining switch in evolution equation for the back stress, equ. (3.5)
$g_k = \partial T/\partial x_k$	temperature gradient, equ. (2.1) & page 6
H	total heating, equ. (2.67), or hardening parameter, equ. (3.5)

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* The number of the equation indicates where the symbol appears for the first time. The list contains primarily those mathematical symbols which are used throughout the text.

h_γ	evolution function for internal variables, equ. (2.45)
$I_2\,,\ \tilde{I}_2$	second invariant of back stress (internal stress), equ. (3.7); dimensionless second invariant, equ. (5.77)
$J_2\,,\ \tilde{J}_2$	second invariant of effective stress, equ. (3.7); dimensionless second invariant, equ. (5.77)
K	kinetic energy, equ. (2.67)
$K_2\,,\ \tilde{K}_2$	2nd order simulaneous invariant of back stress and effective stress, equ. (5.63); dimensionless simultaneous invariant, equ. (5.77)
l_γ	evolution function for internal variables, equ. (2.45)
M	material parameter characterizing the contribution of the back stress to the free energy (stored energy parameter), equ. (4.5)
m	material parameter: exponent in the evolution law for the back stress, equ. (3.5)
n	material parameter: exponent in the evolution law for the visco-plastic strain, equ. (3.4)
$\bar{n}\,,\ \bar{n}_R$	external unit vectors in current and reference configuration, equ. (2.67), (2.70)
$O\,,\ O_R$	current and reference surface of body
P	total mechanical power, equ. (2.67)
p	hydrostatic stress, pressure, page 116
q_k	component of (current) heat flux vector
$\bar{q}\,,\ \bar{q}_R$	heat flux vector in current or reference configuration

r	external heat source per unit mass, equ. (2.3)
$\mathbb{R}$	set of rate variables, equ. (2.52)
R	recovery parameter, equ. (3.5)
$(\cdot)_R$	index denoting undeformed reference configuration, equ. (2.70)
$\mathcal{R}$	dimensionless intrinsic dissipation due to recovery, equ. (5.75)
s_{kl}	deviatoric stress, equ. (3.7)
t, t_1, t_2, t_c	time
$\bar{t}\ ,\ \bar{t}_R$	stress vectors referred to current or reference configuration, equ. (2.67)
T	absolute temperature, equ. (2.8)
$\bar{u}\ ,\ u_k$	displacement vector and its components
$\bar{v}$	velocity vector, equ. (2.67)
V, V_R	volume in the current and reference configuration
W^p	plastic work per unit volume, equ. (3.9)
x_M, x_k	cartesian coordinate; a distinction between Eulerian and Lagrangian coordinates need not to be made since infinitesimal displacement gradients are assumed in this study
$x = \sqrt{\tilde{I}_2}$	variable, equ. (5.98)
$x_0 = \sqrt{\tilde{I}_{02}} = \sqrt{G_0}$	parameter, equ. (5.101)
x_A	lumped parameter, equ. (5.143)

$y = \sqrt{\tilde{J}_2}$	variable, equ. (5.98)
$y_1(x)$	function, solution of $\partial\gamma/\partial y = 0$, equ. (5.108) & (5.115)
$y_0 = y_1(x_0)$	y-position of γ-minimum along $\overline{B'C}$ at $x = x_0$, equ. (5.132)
$z_1(x)$, $z_2(x)$	functions, equ. (5.122)
α_γ	general internal variables, equ. (2.1), (2.2)
α^*	dimensionless internal variable characterizing the uniaxial back stress, equ. (3.16)
α_0	value defining switch in evolution equations for the back stress, equ. (3.18)
α_0^*	dimensionless value of α_0, equ. (3.18)
$\alpha_{kl}^{(i)}$	tensorial internal variables, equ. (2.94)
α_{kl}	tensorial internal variable defining the back stress tensor (kinematic hardening variables), equ. (2.98), (2.99), (3.7)
β	material parameter: exponent in evolution equation for the back stress, equ. (3.20)
Γ, $\hat{\Gamma}$	constitutive quantity and the related function, equ. (3.6), (3.8)
γ	dissipation (entropy production rate times absolute temperature): total dissipation, equ. (2.3), (2.5), & intrinsic dissipation, equ. (5.13) ff
γ_P, γ_H, γ_R	plastic, hardening and recovery contribution to intrinsic dissipation, equ. (5.60)
γ_{LB}	lower bound-dissipation, equ. (5.96)

$\underset{0}{\gamma}_{LB}$	lower bound-dissipation at $x = x_0$, $y = y_0^{\gamma}$, equ. (5.132)
$\theta, \hat{\theta}$	constitutive function, equ. (3.6), (3.8)
ϵ	internal energy per unit mass, equ. (2.1)
ε_{kl}	total strain tensor, equ. (2.1)
ε^{e}_{kl}	elastic strain tensor, equ. (3.1)
ε^{p}_{kl}	visco-plastic strain tensor, equ. (3.1), (3.4)
ε^{th}_{kl}	thermal strain tensor, equ. (3.1), (3.2)
ϱ_{kl}	deviation from equilibrium stress, equ. (6.17)
η	entropy per unit mass, equ. (2.1)
$\varkappa$	drag stress (isotropic hardening variables), equ. (2.95), (3.6), (3.7)
$\varkappa_0$	initial value of drag stress, proportional to yield stress, equ. (3.10)
Λ	heat conductivity, equ. (4.13)
λ_K	eigenvalues, equ. (4.13)
λ, μ	Lame's constants, equ. (3.3)
μ	dimensionless free energy parameter, equ. (5.146)
ν	Poisson's number, equ. (6.27)
ξ_γ, ξ_κ	variables denoting deviation from equilibrium values, equ. (2.33), (6.18)
$\Pi, \overset{*}{\Pi}, \overset{**}{\Pi}, \overset{\Delta}{\Pi}$	various sets of independent variables characterizing a point in different state spaces, equ. (2.11), (2.97), (2.102), (4.2), (5.1)

ϱ	density, equ. (2.67)
Σ_{kl}	effective stress, equ. (3.6), (3.7)
σ_{kl}	symmetric stress tensor; a distinction between the Cauchy stress and the Kirchhoff-Piola stress of 2nd kind is not necessary, since infinitesimal displacement gradients are assumed, equ. (2.1)
σ	entropy production rate, equ. (2.3), or uniaxial stress, table 1, equ. (3.23)
σ^*	dimensional uniaxial stress, equ. (3.16)
τ	increment of time (page 35) or shear stress, table 1, equ. (3.23)
τ^*	dimensionless shear stress, equ. (3.24)
$y, \overset{\vartriangle}{y}$	general constitutive function, equ. (2.103), (5.1)
χ_{kl}	deviation of the back stress from its equilibrium value, equ. (6.17)
$\psi, \hat{\psi}, \tilde{\psi}, \check{\psi}$	free energy per unit mass, page 7, equ. (2.106), (4.2)
$\Omega, \overset{\vartriangle}{\Omega}$	general constitutive function, equ. (2.103), (5.1)

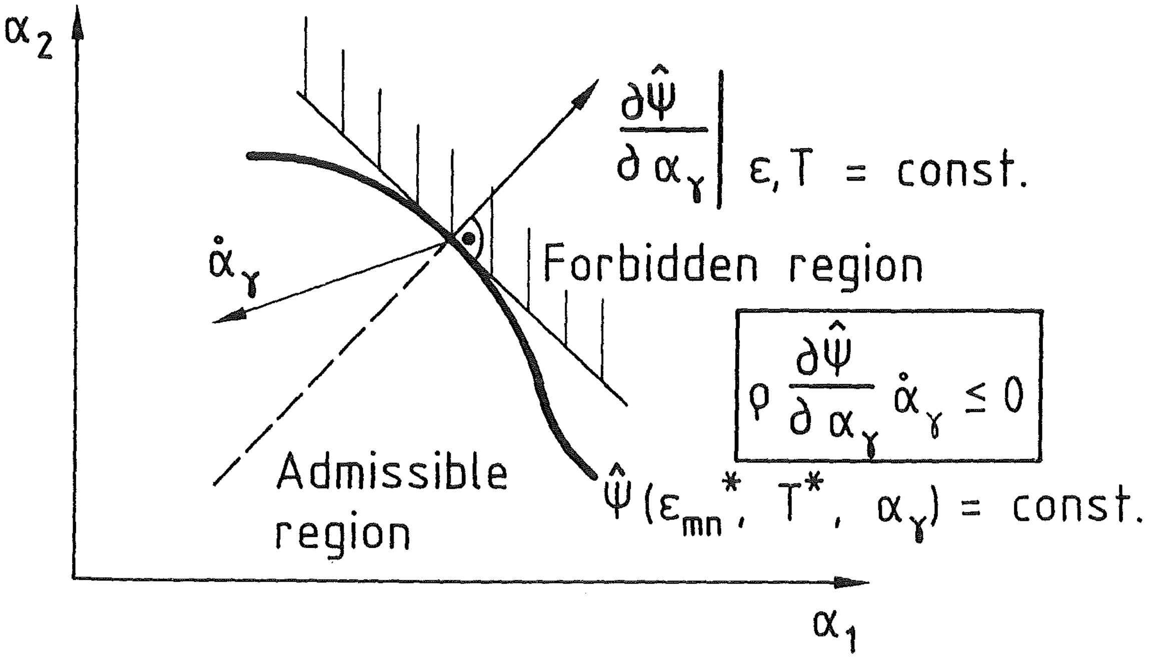


Fig. 1: Relation Between the Normal of the Constant Free Energy-Surface and the Evolution Function $\mathring{\alpha}_\gamma = \mathcal{F}_\gamma^*$

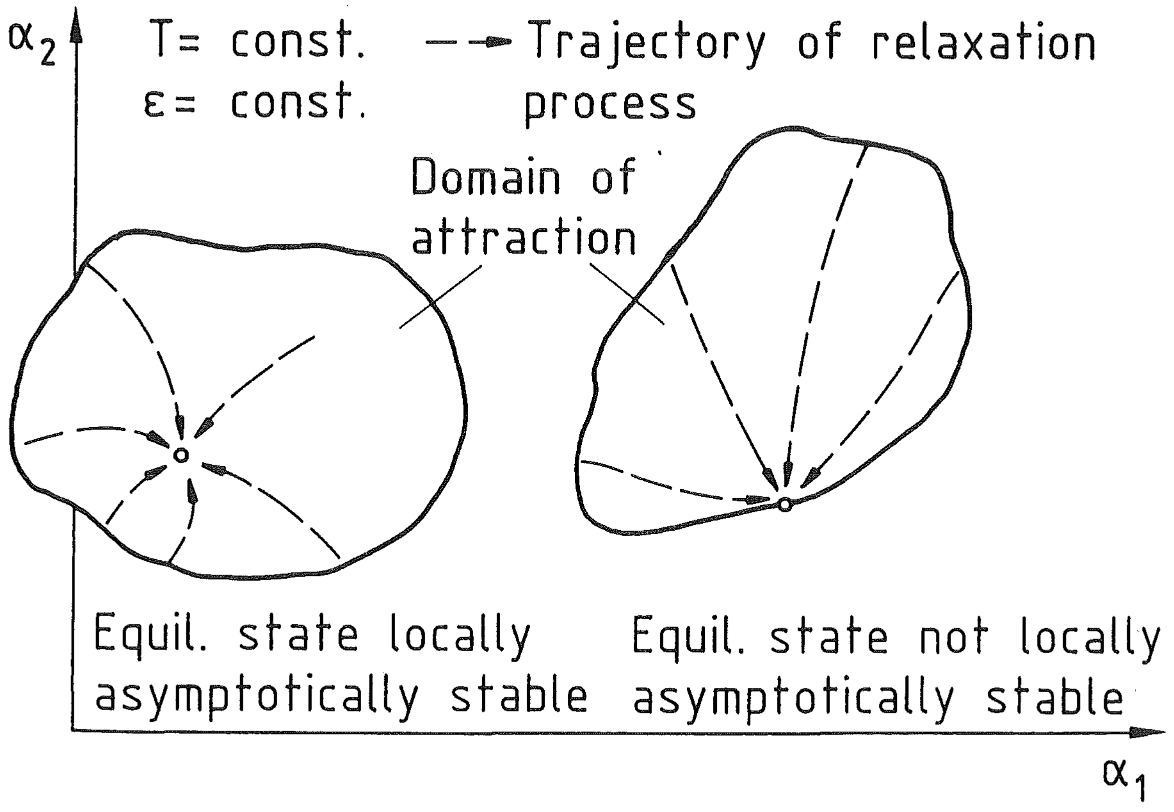


Fig. 2: Equilibrium States and Asymptotic Stability at Constant Strain and Temperature for the Case of Two Internal Variables

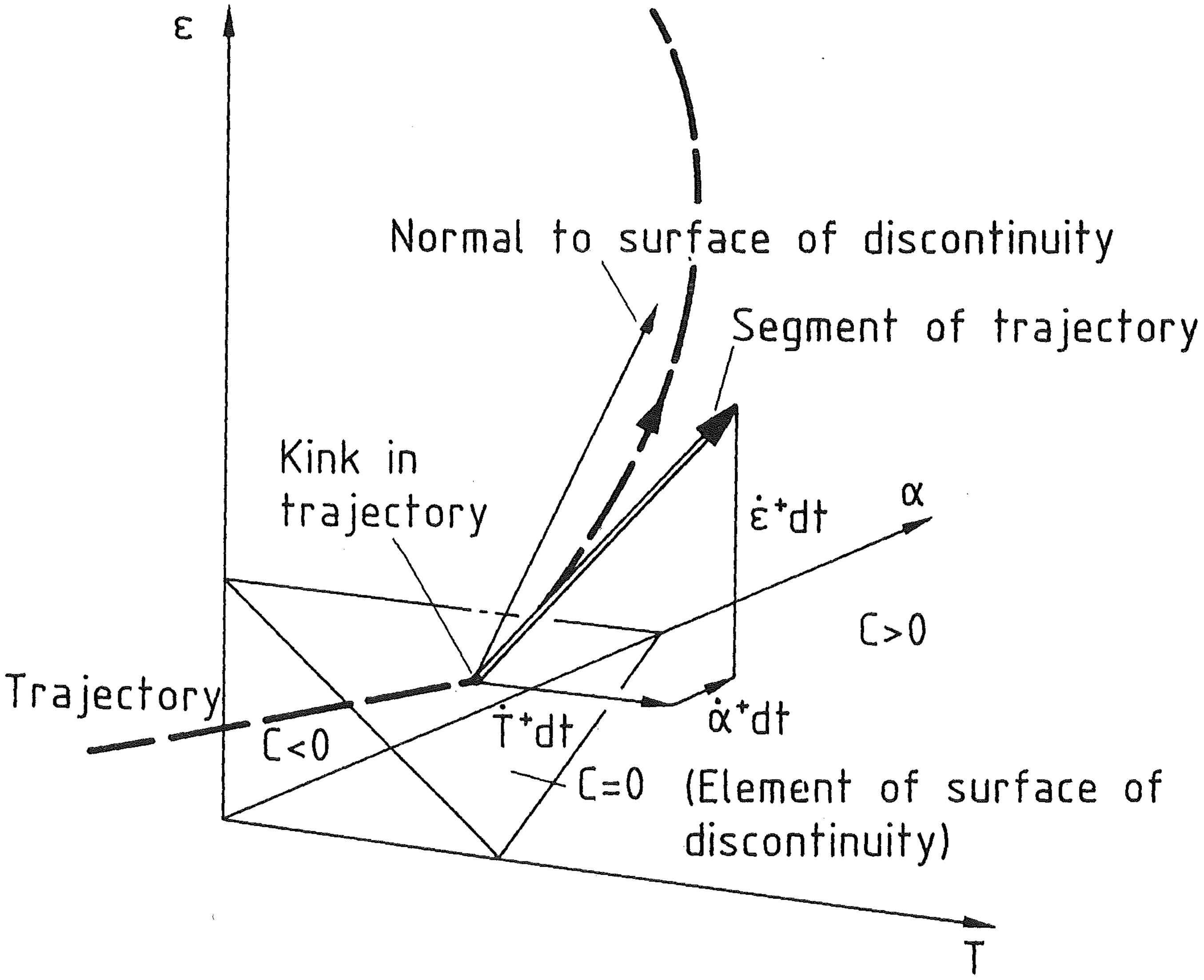


Fig. 3: Intersection of a Trajectory (process path) with a Surface of Discontinuity C=0, Simplified Version for a Three-Parameter Model

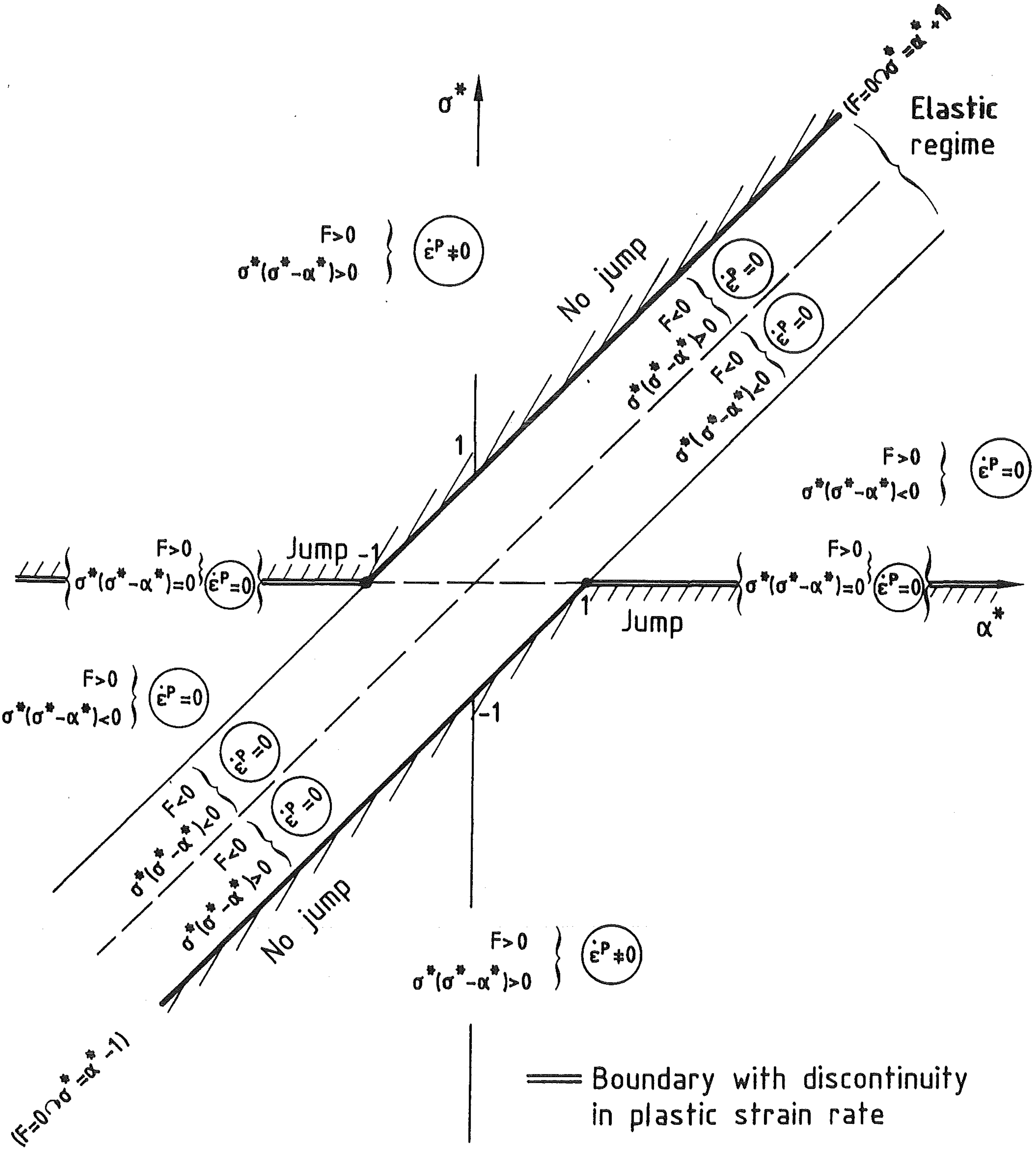


Fig. 4: Regions in the (σ^*,α^*)-Space for the Plastic Strain Rate $\dot{\varepsilon}_p$, Continuous and Discontinuous Boundaries

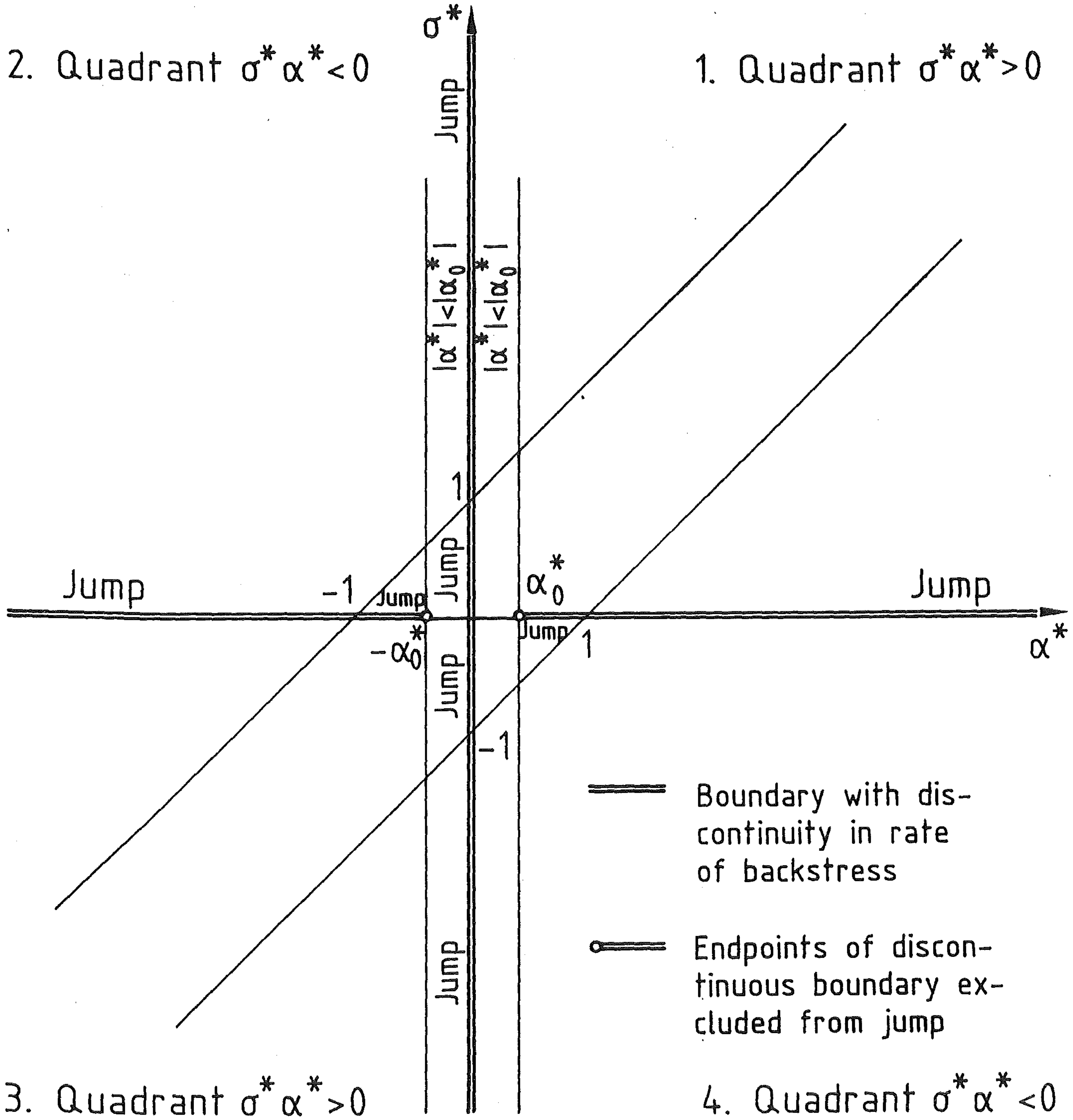


Fig. 5: Regions in the (σ^*, α^*)-Space for the Backstress Rate $\dot{\alpha}^*$, Continuous and Discontinuous Boundaries

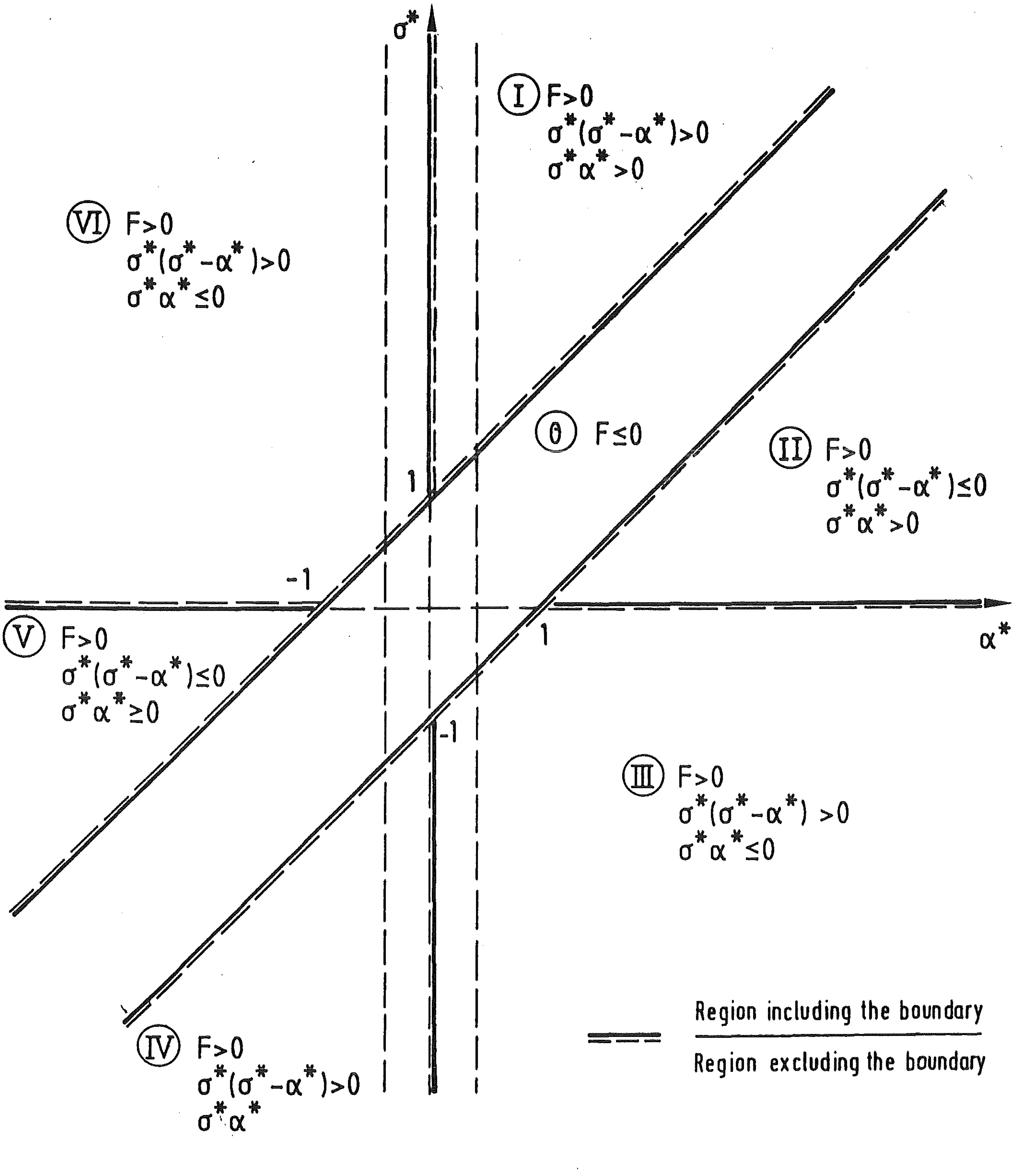


Fig. 6: Regions in the (σ^*, α^*)-Plane for the Evaluation of the Intrinsic Dissipation Inequality

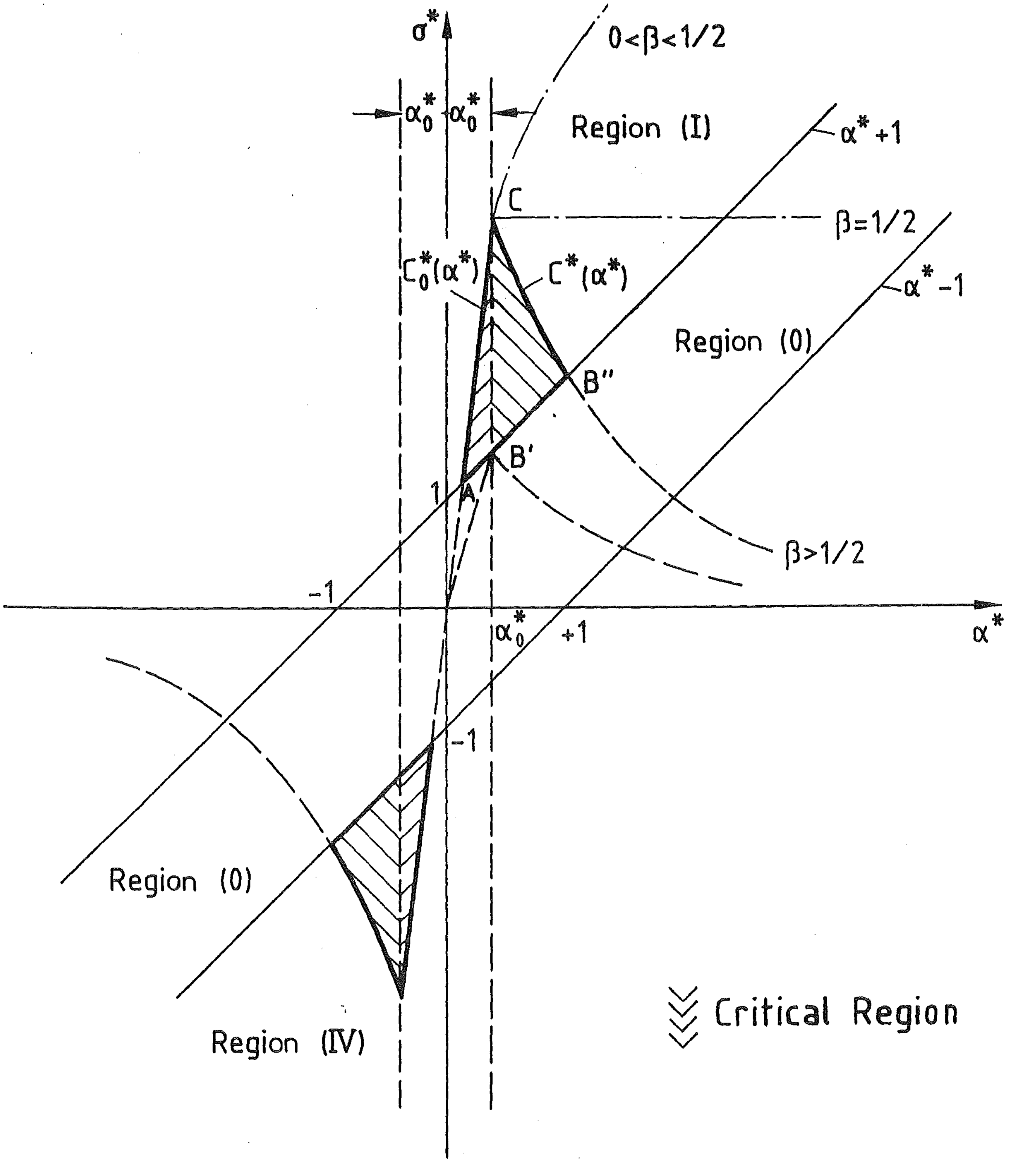


Fig. 7: Critical Regions in the (σ^*, α^*)-Plane

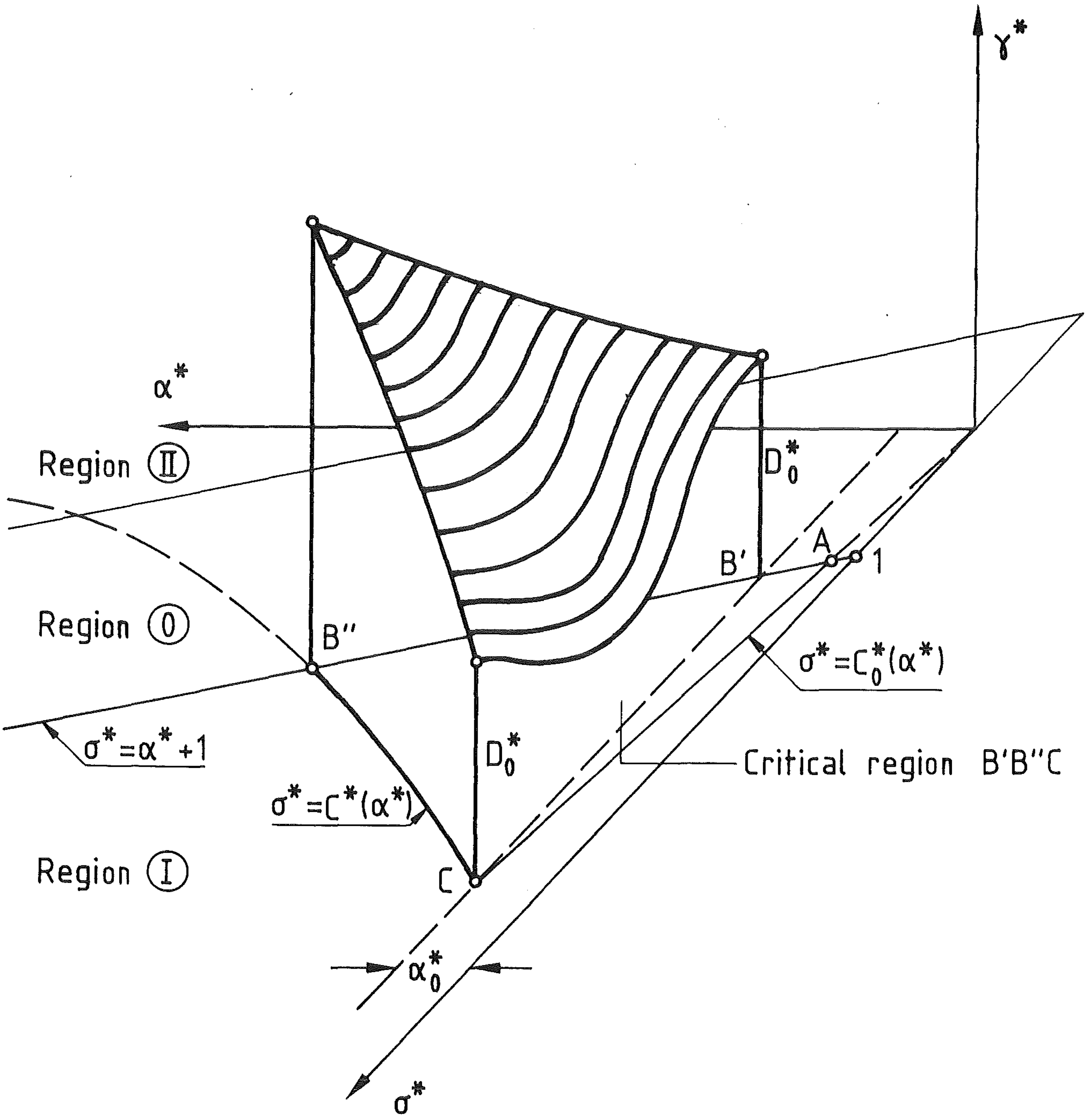


Fig. 8: Intrinsic Entropy Production Rate in the Critical Region B'B''C

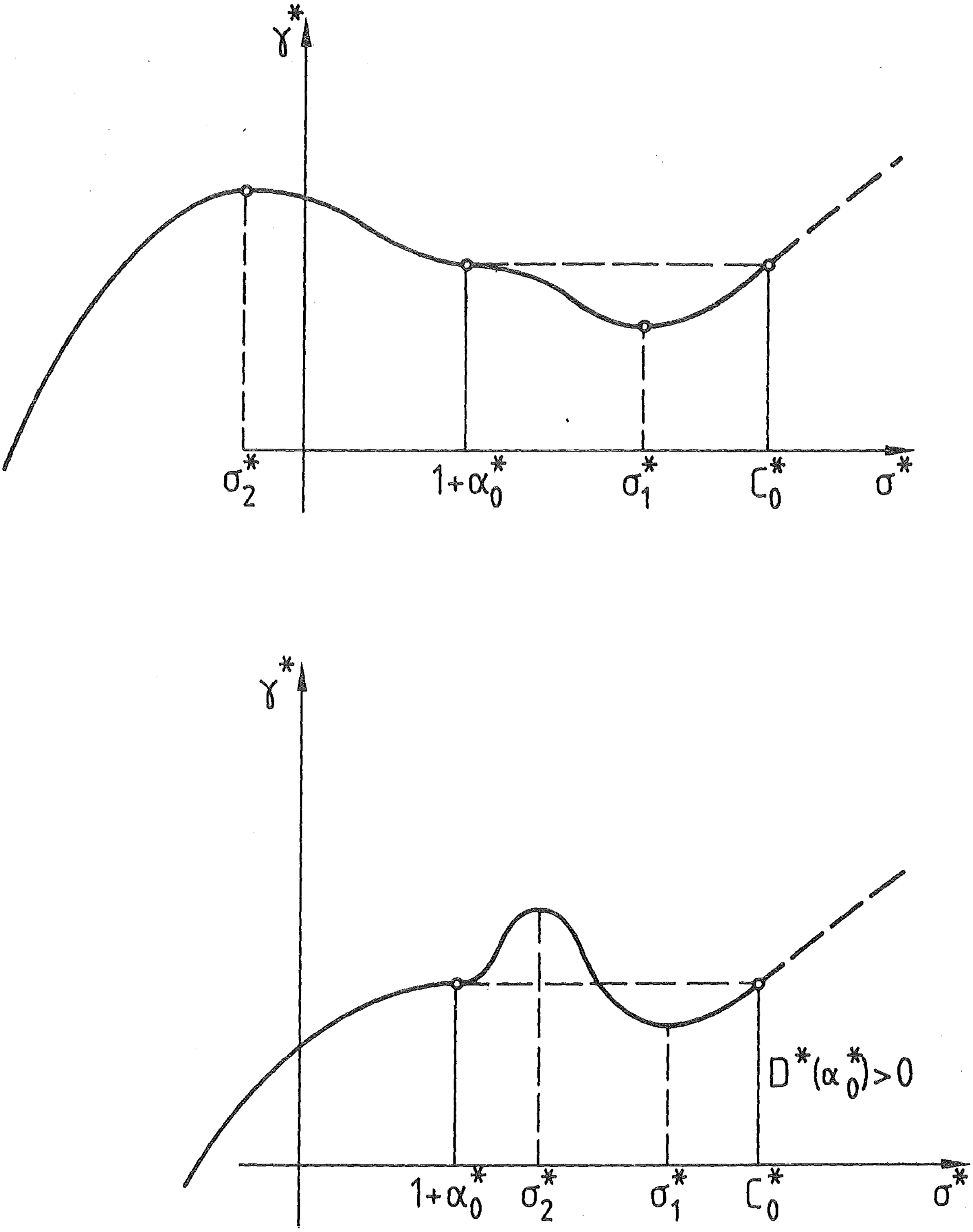
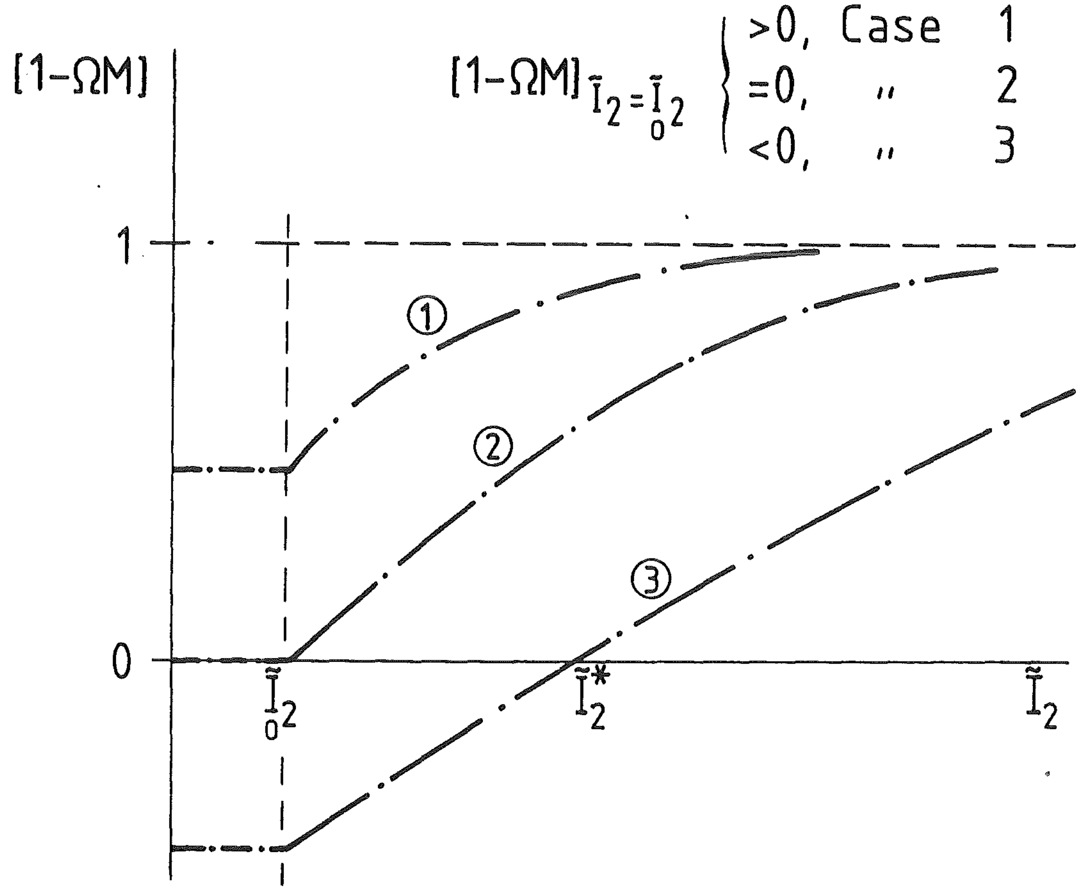


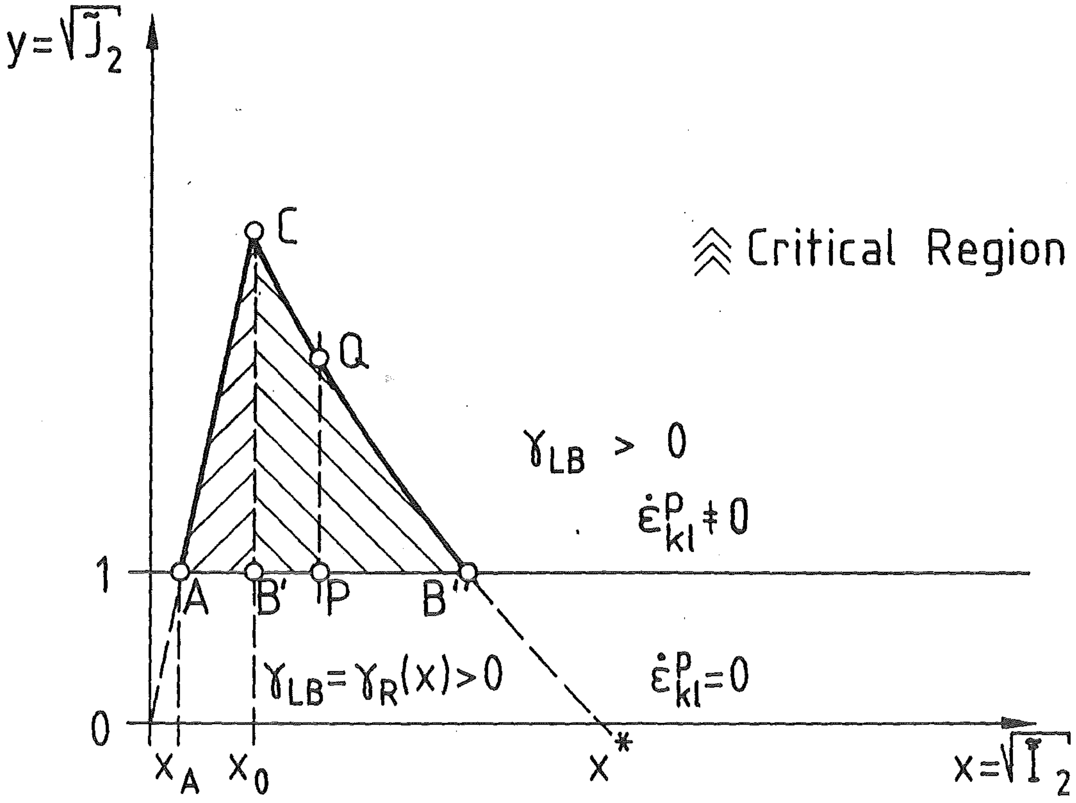
Fig. 9: Distribution of the Intrinsic Dissipation along B'C (see Fig. 8)



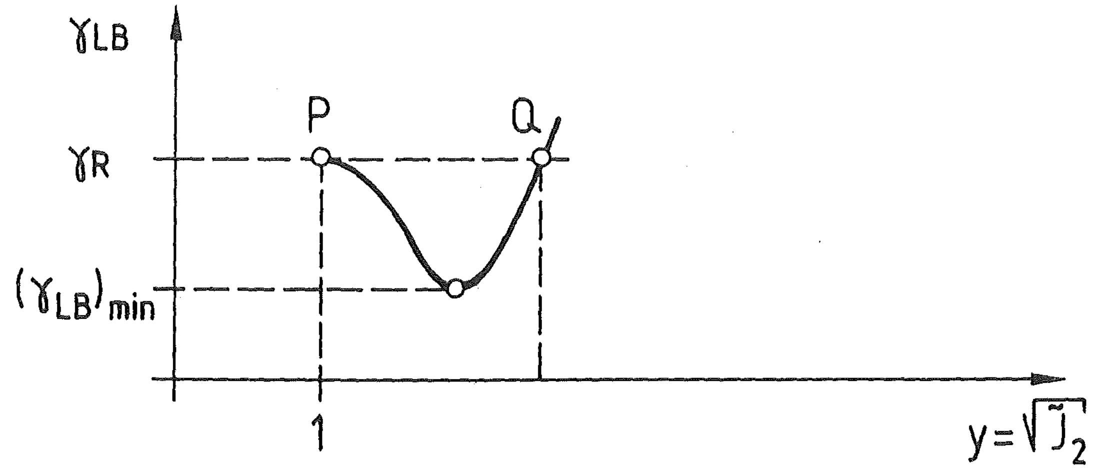
Case 1 and 2 : Non-Negativeness of Intrinsic Dissipation

Case 3 : Critical Case

Fig. 10 Distribution of [1-ΩM] as a Function of $\tilde{I}_2$ (Second Invariant of Back Stress Tensor)



(a) Critical Regions in the (x, y)-Plane



(b) Distribution of the Lower Bound Dissipation along $\overline{PQ}$

Fig. 11 Critical Regions and Distribution of Lower Bound Dissipation along Parallel to the y-Axes in Region B'B''C

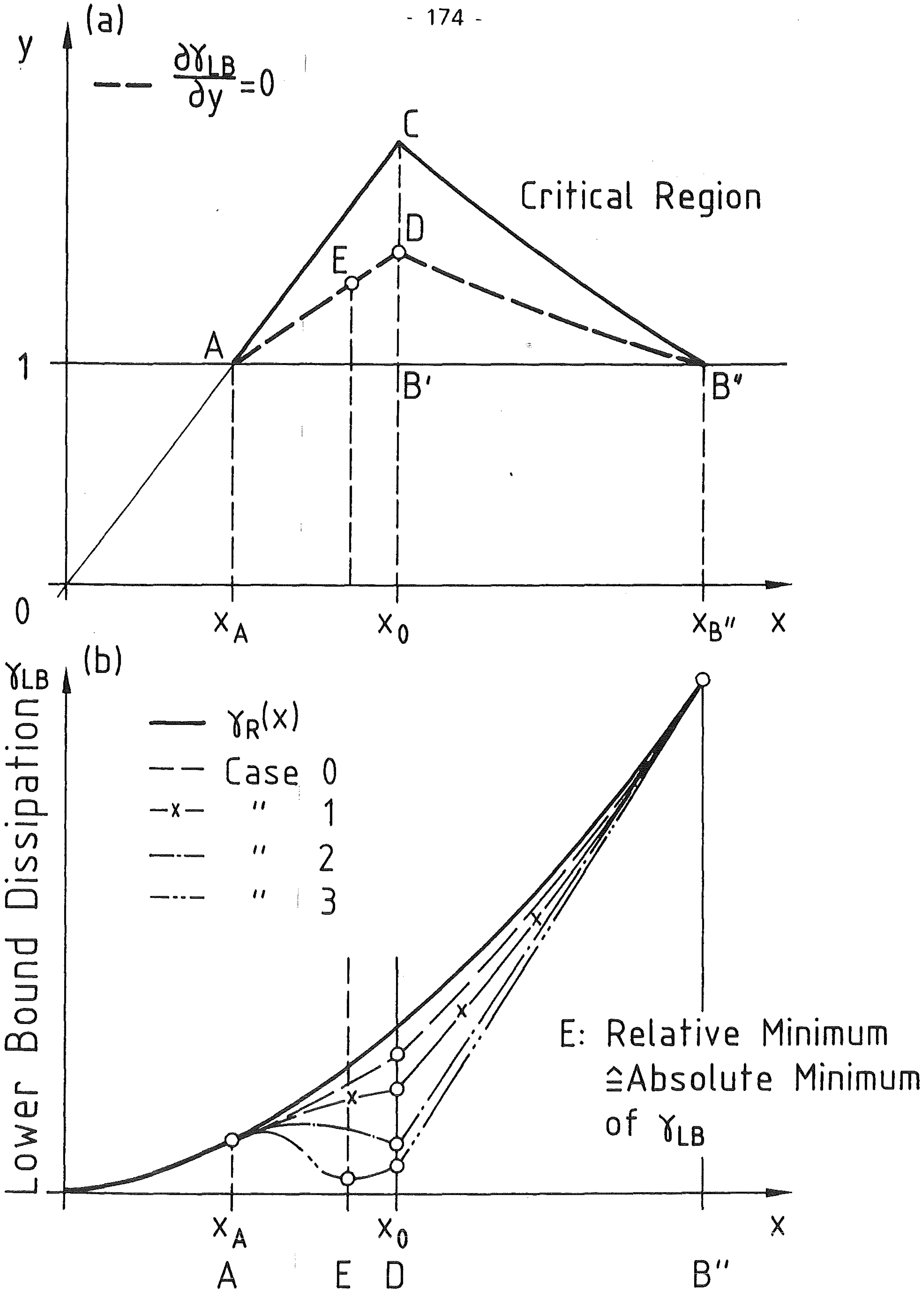


Fig. 12 Locus ($\partial \gamma_{LB}/\partial y=0$) in the (x,y)-Plane, Distributions of Lower Bound Dissipation along this Locus (Schematic)

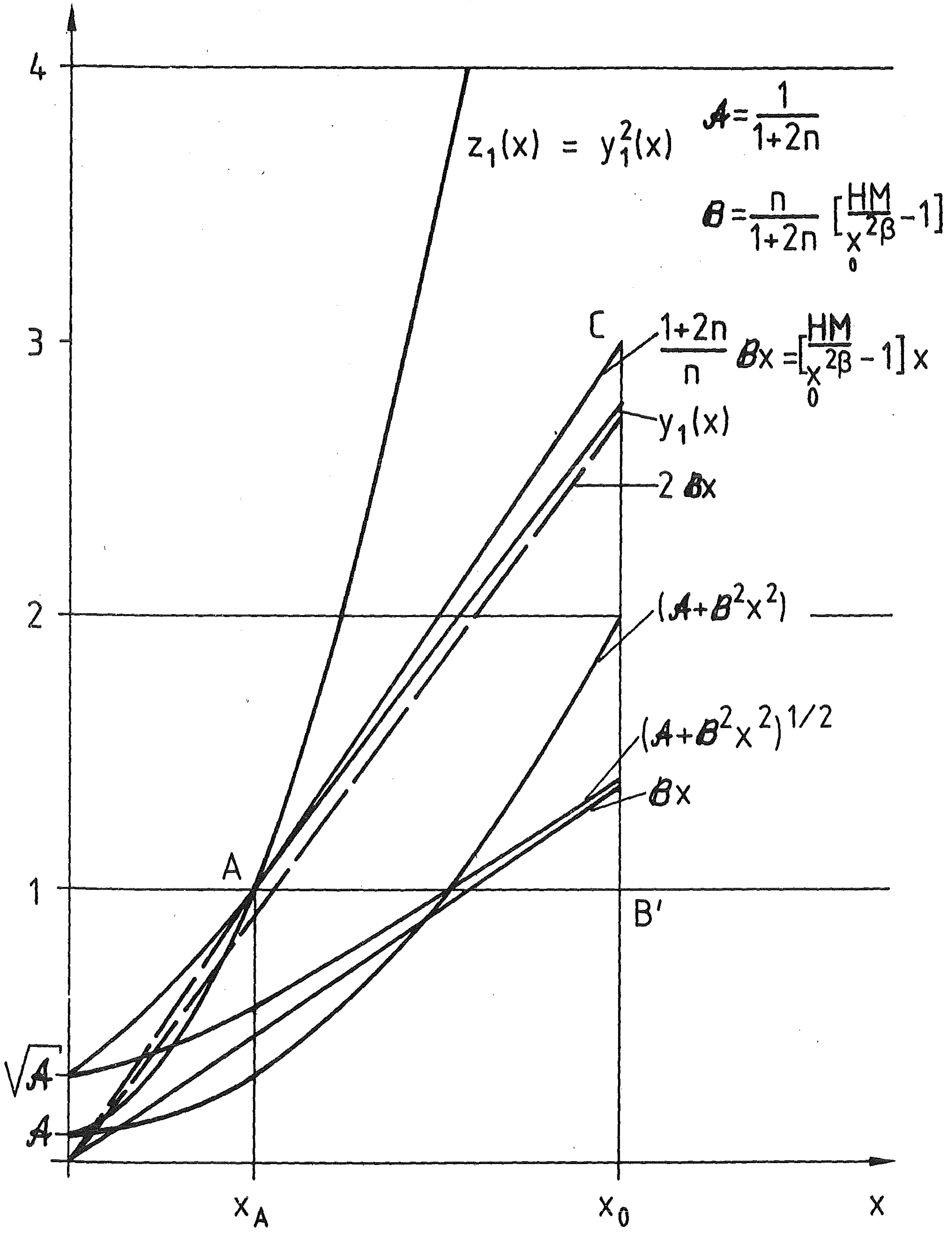


Fig. 13 Form of the y_1-Function in the Critical Region B'CA (for n=5)

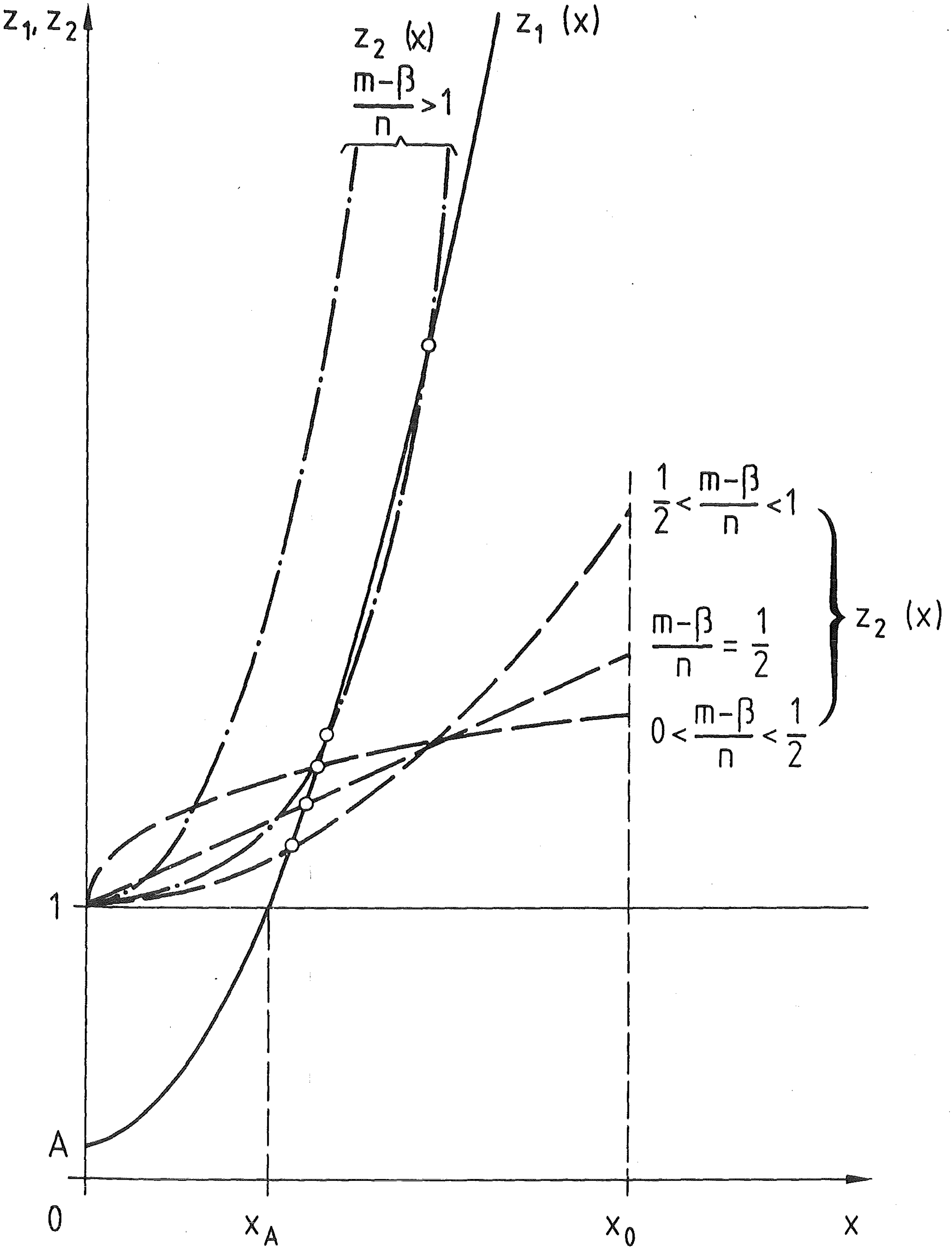


Fig. 14 Schematic Representation of the Functions $z_1(x)$ and $z_2(x)$ for Different Exponents and Possible Intersections in the Interval $x_A < x < x_0$