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Review of Mathematical and Physical Basis of Two-Phase Flow Modelling

M. Bottoni, W. Sengpiel
Institut für Reaktorsicherheit
Projekt Nukleare Sicherheitsforschung

Kernforschungszentrum Karlsruhe

KERNFORSCHUNGSZENTRUM KARLSRUHE

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M. Bottoni*
W. Sengpiel

*Argonne National Laboratory, Argonne, IL60439, USA

Kernforschungszentrum Karlsruhe GmbH, Karlsruhe

Kernforschungszentrum Karlsruhe GmbH
Postfach 3640, 7500 Karlsruhe 1

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Abstract

Starting from a continuum-mechanical approach, this report gives a detailed overview of the deduction of conservation equations for the analytical description of two-phase flows by means of an adequate averaging process resulting in a two-fluid model and a homogeneous mixture model. The mathematical process of averaging leads to macroscopic formulations of stress terms and interfacial interaction terms. These terms depend on microscopic variables and thus give some helpful insight into the physical processes which have to be described by constitutive relations.

Überblick über die mathematischen und physikalischen Grundlagen der Modellierung von Zweiphasenströmungen

Zusammenfassung

Ausgehend von einer kontinuumsmechanischen Betrachtungsweise wird in diesem Bericht eine ausführliche Darstellung der Herleitung der Erhaltungsgleichungen für ein 2-Fluid-Modell sowie ein homogenes Mischungsmodell mit Hilfe eines geeigneten Mittelungsprozesses zur analytischen Beschreibung von Zweiphasenströmungen gegeben. Der mathematische Mittelungsprozeß führt auf makroskopische Formulierungen der Spannungsterme und der Wechselwirkungsterme an den Phasengrenzflächen. Diese Terme sind von mikroskopischen Variablen abhängig und gestatten somit einen hilfreichen Einblick in die physikalischen Zusammenhänge, die durch konstitutive Modelle beschrieben werden müssen.

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Nomenclature

Remark:

Dimensionless quantities are denoted by (-). Dimensions are not given for symbols used for different physical quantities.

$\mathbf{a}$	generalized vector field, see eq. s (3.2.2.7) ff.	
a_i	interfacial area per unit volume	(m-1)
A	area	(m2)
c	mean curvature of surface	(m-2)
C	curve length	(m)
$\overline{\mathbf{D}}_k^x$	mean viscous dissipation of phase k, defined by eq. (4.3.5)	(W/m2)
$\overline{D}_k$	turbulent energy dissipation of phase k, defined by. eq. (4.7.4)	(W/m3)
$\overline{D}_{sk}$	turbulent entropy source of phase k, defined by eq. (4.7.8)	(W/m3 K)
D_m	turbulent energy disspation of mixture, defined by eq. (6.1.1.4)	(W/m3)
D_m^{st}	turbulent entropy source in the mixture, defined by eq. (6.1.1.5)	(W/m3 K)
D_m^s	auxialary definition of entropy source in the mixture, defined by eq. (6.2.7.6)	(W/m3 K)
e	total energy	(J/kg)
$\mathbf{e}$	unit vector	(-)
e_i^σ	energy source at interfacial surface between phases	(W/m2)
$\overline{e}_k^{x,\rho}$	mean total energy of phase k, defined by eq. (4.1.8)	(J/kg)
$\overline{e}_{ki}$	mean total energy associated to interfacial total energy flux of phase k, defined by eq. (4.5.7)	(J/kg)
$\tilde{\overline{e}}_k$	auxiliary definition given by eq. (5.2.2.2)	(J/kg)
$\overline{E}_{kin,k}^{x,\rho}$	mean kinetic energy of phase k, defined by eq. (4.1.7)	(J/kg)

$\overline{E_k}$	interfacial heat source of phase k, defined by eq. (4.6.6)	(W/m³)
f	source density for quantity ψ (see eq. (2.1.1))	
F	general function, defined by eq. (3.1.19)	
g	weight function, see eq. (3.1.7) general scalar function, see eq. (3.2.3.2)	(-)
$\mathbf{g}$	gravity acceleration	(m/s²)
h	enthalpy	(J/kg)
$\overline{h_k^{x,\rho}}$	mean enthalpy of phase k, defined by eq. (4.1.5)	(J/kg)
$\overline{h}_{ki}$	mean enthalpy associated to interfacial enthalpy flux of phase k, defined by eq. (4.5.4)	(J/kg)
$\tilde{h}_k$	difference between mean enthalpy of phase k and mean enthalpy of mixture, defined by eq. (6.1.2.3)	(J/kg)
h_m	enthalpy of mixture, defined by eq. (6.1.2.10) or alternatively by eq. (6.3.3.1a)	(J/kg)
$\underline{\mathbf{I}}$	Identity tensor	(-)
$\mathbf{J}$	diffusive flux (see eq. (2.1.1))	
κ	spatial coordinate	(m)
m	interfacial source of ψ (see eq. (2.2.1))	
$\mathbf{m}_i^{\sigma}$	momentum source at interface between phases	(N/m²)
$\overline{\mathbf{m}}$	total momentum of both phases, defined by eq. (5.3.3)	(N/m³)
$\overline{\mathbf{M}}_k$	momentum exchange of phase k per unit volume and time, defined by eq. (4.6.2)	(N/m³)
$\overline{\mathbf{M}_k'}$	turbulent interfacial force of phase k per unit length, defined by eq. (4.7.5)	(N/m³)
$\overline{\mathbf{M}_k^d}$	interfacial force in phase k , defined by eq. (5.1.2.9)	(N/m³)
$\mathbf{n}$	unit vector normal to interface surface between phases	(-)

N	number of elements in a set	(-)
p	thermodynamic pressure defined by (2.3.45)	(N/m2)
$\overline{p}_k^x$	mean normal stress of phase k, defined by eq. (4.3.3)	(N/m2)
$\tilde{\overline{p}}_{ki}$	interfacial pressure of phase k per unit length, defined by eq. (4.6.3)	(N/m3)
$\overline{p_{ki}}$	alternative definition of interfacial pressure of phase k, given by eq. (4.6.12)	(N/m2)
p_m	pressure of mixture, defined by eq. (6.1.1.3)	(N/m2)
$\mathbf{q}$	heat flux	(W/m2)
q_i	heat flux at interface	(W/m2)
q_{si}	source of entropy at interface	(W/m2 K)
$\overline{\mathbf{q}}_k^x$	mean energy flux of phase k, defined by eq. (4.3.6)	(W/m2)
$\hat{\hat{\mathbf{q}}}_k^{Re}$	turbulent enthalpy flux of phase k, defined by eq. (4.4.3)	(W/m2)
$\hat{\mathbf{q}}_k^{Re}$	mean turbulent internal energy flux of phase k, defined by eq. (4.4.4)	(W/m2)
$\overline{\mathbf{q}}_k^{kin}$	mean turbulent kinetic energy flux of phase k, defined by eq. (4.4.5)	(W/m2)
$\overline{\mathbf{q}}_k^{tot}$	mean turbulent total energy flux of phase k, defined by eq. (4.4.6)	(W/m2)
$\overline{\mathbf{q}}_k^{pt}$	mean turbulent velocity-pressure correlation of phase k, defined by eq. (4.4.8)	(W/m2)
$\overline{\mathbf{q}}_k^{p}$	turbulent velocity-pressure correlation of phase k, defined by eq. (4.7.2)	(W/m2)
$\overline{\mathbf{q}}_k^{\tau}$	turbulent shear work of phase k, defined by eq. (4.7.3)	(W/m2)
$\overline{\mathbf{q}_k^{Re}}$	total turbulent energy flux of phase k, defined by eq. (4.9.1)	(W/m2)
$\mathbf{q}_m^x$	energy flux of mixture, defined by eq. (6.1.2.19)	(W/m2)

$\hat{\hat{q}}_m^{Re}$	turbulent enthalpy flux of mixture, defined by eq. (6.1.2.20)	(W/m2)
q_m	auxiliary definition of energy flux of mixture, defined by eq. (6.2.3.5)	(W/m2)
$\tilde{q}_m$	auxiliary definition of enery flux of mixture, defined by eq. (6.2.3.9)	(W/m2)
$\hat{q}_m$	auxiliary definition of energy flux of mixture, defined by eq. (6.2.5.4)	(W/m2)
$\hat{\hat{q}}_m$	auxiliary definition of enery flux of mixture, defined by eq. (6.2.6.3)	(W/m2)
$\hat{q}_m^{Re}$	mean turbulent internal energy flux of mixture, implicitly defined by eq. (6.3.2.12)	(W/m2)
r	specific heat source	(W/kg)
$\bar{r}_k^{x,\rho}$	mean energy source of phase k, defined by eq. (4.2.3)	(W/kg)
r_m	energy source of mixture, defined by eq. (6.1.2.15)	(W/m3)
R	radius of curvature	(m)
	subspace	(-)
s	entropy	(J/kg K)
Δs	specific entropy generation rate, defined by eq. (2.3.38) (remark that Δs is not the increment of s, but is used as a symbol in itself)	(W/m3 K) (W/m2)
$\bar{s}_k^{x,\rho}$	mean entropy of phase k, defined by eq. (4.1.9)	(J/kg K)
$\bar{\bar{s}}_{ki}$	mean entropy associated to interfacial entropy flux of phase k, defined by eq. (4.5.8)	(J/kg K)
$\bar{s}_k$	interfacial entropy source of phase k, defined by eq. (4.6.7)	(W/m3 K)
$\bar{\bar{s}}_{Tk}$	entropy source due to temperature gradient, defined by eq. (4.8.3)	(W/m3 K)

s_m	entropy of mixture, definbey by eq. (6.1.2.11)	(J/kg K)
Δs_m	auxiliary definition of mixture entropy, given by eq. (6.2.7.3)	(W/m^3 K)
s_{Tm}	entropy source of mixture due to temperature gradient, defined by eq. (6.2.7.5)	(W/m^3 K)
S	surface area	(m^2)
t	time	(s)
t'	time	(s)
$\underline{\mathbf{T}}$	stress tensor	(N/m^2)
T	thermodynamic temperature, defined by (2.3.44) time interval	(K) (s)
$\overline{T_k^{x,\rho,s}}$	mean temperature of phase k, defined by eq. (4.8.2)	(K)
$\underline{\mathbf{T}}_k^x$	mean stress tensor of phase k, defined by eq. (4.3.2)	(N/m^2)
$\underline{\mathbf{T}}_k^{Re}$	mean Reynolds stress of phase k, defined by eq. (4.4.2)	(N/m^2)
$\overline{\underline{\mathbf{T}}_{k,mm}^{Re}}$	alternative definition of mean Reynolds stress of phase k, given by eq. (4.4.9)	(N/m^2)
$\underline{\mathbf{T}}_m^x$	total stress of mixture, defined by eq. (6.1.2.17)	(N/m^2)
$\underline{\mathbf{T}}_m^{Re}$	Reynolds stress of mixture, defined by eq. (6.1.2.18)	(N/m^2)
$\underline{\mathbf{T}}_m$	auxiliary definition of total stress of mixture, given by eq. (6.2.2.4)	(N/m^2)
u	internal energy	(J/kg)
$\overline{u_k^{x,\rho}}$	mean internal energy of phase k, defined by eq. (4.1.6)	(J/kg)
$\overline{u_k^{Re}}$	turbulent kinetic energy of phase k, defined by eq. (4.7.1)	(J/kg)
$\overline{u}_{ki}$	mean internal energy associated with internal energy flux of phase k, defined by eq. (4.5.5)	(J/kg)

$\tilde{u}_k$	difference between mean internal energy of phase k and mean internal energy of mixture, defined by eq. (6.1.2.2)	(J/kg)
u_m	internal energy of mixture, defined by eq. (6.1.2.9)	(J/kg)
u_m^{Re}	turbulent kinetic energy of mixture, defined by eq. (6.1.2.16)	(J/m^3)
υ	specific volume	(m^3/kg)
$\mathbf{v}$	velocity vector	(m/s)
$\overline{\mathbf{v}}_k^{x,p}$	mean velocity of phase k, defined by eq. (4.1.4)	(m/s)
$\overline{\mathbf{v}}_{ki}$	mean velocity associated to interfacial momentum flux of phase k, defined by eq. (4.5.3)	(m/s)
$\tilde{\mathbf{v}}_k$	"phase velocity difference" or difference between mean velocity of phase k and mixture velocity, defined by eq. (6.1.2.1)	(m/s)
$\mathbf{v}_m$	mixture velocity, defined by eq. (6.1.2.4)	(m/s)
$\tilde{\mathbf{v}}'_k$	difference between turbulent fluctuation of phase velocity and turbulent fluctuation of mixture velocity, defined by eq. (6.1.3.7)	(m/s)
$\hat{\mathbf{v}}_m$	auxiliary definition of mixture velocity, given by eq. (6.2.6.6)	(m/s)
$\mathbf{v}_{Sl}$	slip velocity or velocity difference between the phases, defined by eq. (6.3.1.1)	(m/s)
V	volume	(m^3)
$\overline{W}_k$	interfacial work of phase k, defined by eq. (4.6.5)	(W/m^3)
$\overline{W''_k}$	turbulent interfacial work of phase k, defined by eq. (4.7.6)	(W/m^3)
$\overline{W'_k}$	alternative defintion of turbulent interfacial work of phase k, given by eq. (4.7.7)	(W/m^3)
W_m	interfacial work of mixture, defined by eq. (6.2.5.7)	(W/m^3)

$\hat{W}_m$	auxiliary definition of interfacial work of mixture, given by eq. (6.2.6.4)	(W/m^3)
$\hat{\hat{W}}_m$	auxiliary definition of interfacial work of mixture, given by eq. (6.2.6.7)	(W/m^3)
x	coordinate direction thermodynamic quality, defined by eq. (6.3.3.1b)	(m) (-)
x'	coordinate direction	(m)
$\mathbf{x}$	$= (x_1, x_2, x_3)$ vector denoting spatial coordinates	(m)
X	spatial coordinate	(m)
X_k	phase indicator function	(-)
y	coordinate direction	(m)
z	coordinate direction	(m)

special symbols

$\bar{f}$	mean value of function f
$< >$	generalized definition of average, satisfying conditions (3.1.15) through (3.1.18)
$< f >_1$	line average of function f, defined by eq. (3.1.4)
$< f >_2$	area average of function f, defined by eq. (3.1.3)
$< f >_3$	volume average of function f, defined by eq. (3.1.2)
$\bar{f}^e$	ensemble average of function f, defined by eq. (3.1.5)

Greek

$\bar{\alpha}_k$	volume fraction of phase k, defined by eq. (4.1.1)	(-)
$\bar{\Gamma}_k$	mean mass production rate of phase k, defined by eq. (4.5.2)	(kg/m^3 s)
δ	dissipation, defined by (2.3.40) Dirac delta function	(W/kg)
$\boldsymbol{\delta}$	unit vector	(-)
$\bar{\varepsilon}$	auxiliary definition of power density of both phases, given by eq. (5.3.4)	(W/m^3)
ρ	density	(kg/m^3)
ρ_m	mixture density, defined by eq. (6.1.1.2)	(kg/m^3)
σ	surface tension	(N/m)
$\bar{\sigma}_k^{x,\rho}$	mean entropy source of phase k, defined by eq. (4.2.4)	(W/kg K)
σ_m	entropy source of mixture, defined by eq. (6.1.2.12)	(W/kg K)
τ	time	(s)
$\underline{\tau}$	shear stress	(N/m^2)
$\bar{\underline{\tau}}_k^x$	mean shear stress of phase k, defined by eq. (4.3.4)	(N/m^2)
$\bar{\underline{\tau}}_{sk}^x$	mean entropy source due to shear stress, for phase k, defined by eq. (4.3.8)	(N/m^2 K)
$\tilde{\bar{\underline{\tau}}}_{ki}$	interfacial shear stress of phase k per unit length, defined by eq. (4.6.4)	(N/m^3)
$\bar{\underline{\tau}}_{ki}$	alternativ defintion of interfacial shear stress of phase k, defined by eq. (4.6.13)	(N/m^2)
$\underline{\tau}_m$	shear stress of mixture, defined by eq. (6.1.2.14)	(N/m^2)
Φ	test function, defined by eq. (3.2.1)	

$\overline{\Phi}_k^x$	mean entropy flux of phase k, defined by eq. (4.3.7)	($W/m^2 K$)
$\overline{\Phi}_k^{Re}$	turbulent entropy flux of phase k, defined by eq. (4.4.7)	($W/m^2 K$)
Φ_m	entropy flux of mixture, defined by eq. (6.1.2.21)	($W/m^2 K$)
Φ_m^{Re}	turbulent entropy flux of mixture, defined by eq. (6.1.2.22)	($W/m^2 K$)
$\tilde{\Phi}_m$	auxiliary definition of entropy flux of mixture, given by eq. (6.2.7.10)	($W/m^2 K$)
ψ	general physical quantity (see eq. (2.1.1))	
ω	element of a set Ω of observations, (see eq. (3.1.6))	(-)
Ω	set of observations	(-)

Indices

e	ensemble
g	weighted average
i	interface dummy index
j	general coordinate direction dummy index
k	phase index dummy index
kin	kinetic
l	dummy index
m	mixture
n	element of a set
Re	Reynolds
s	entropy
t	transpose

1. Introduction

Multiphase flow phenomena involve a widespread domain of technical applications ranging from the safety analysis of reactor cores under hypothetical accidents to pneumatic transport lines or optimization of combustion processes. Since the basic book by R.B. Bird, W. Stewart and E.N. Lightfoot [4] on transport phenomena has found wide distribution, the analytical treatment of the fundamental equations describing the conservation of mass, momentum and enthalpy has become the basic tool for the investigation of these processes, leading to computational codes which are in general less expensive to run than the operation of demonstration plants.

In the more restricted domain of two-phase flow, a state of the art of the ongoing research has been given in the well known monograph by M. Ishii [6] where the fundamental theory of thermo-fluid dynamic problems is presented in a self-consistent and mathematically rigorous treatment. In this monograph special emphasis has been put on the rigorous formulation of time averaging processes which, starting from the local instantaneous formulation of the conservation equations, yield a system of macroscopic models for the practical treatment of three-dimensional engineering systems.

A more recent reference text on the thermohydraulics of two-phase systems is the van Karman Institute Book edited by J.M. Delhaye, M. Giot, M.L. Riethmuller [7], which is particularly oriented towards nuclear engineering applications. Among other significant results, it gives the conceptually important proof by J.M. Delhaye of equivalence of space-time and time-space averaging procedures. Thus the same composite-averaged equations can be obtained

i) by time-averaging the local conservation equations and then taking the space averaging over a given domain, or

ii) by space-averaging over a given domain and then time-averaging.

The theoretical importance of this result is matched by the practical circumstances which make experimental results independent of the particular technique used (sequence of time and space averaging) for recording mean-valued data.

Meanwhile, the theoretical effort of establishing the basic mathematical description of the physical processes involved in two-phase flow has been further refined. It does not concern only the derivation of macroscopic averaged equations, but also the modelling of terms describing the details of transport processes of mass, momentum and enthalpy between the phases, taking into account the physical characteristics of the phase boundary. The mathematical formulation of these terms is a particularly challenging task when turbulent exchange phenomena must be accounted for, as it is customary for most technical problems.

A state of the art of this ongoing research has been presented recently in the monograph by D.A. Drew and R.T. Wood [1]. This reference gives also a very valuable overview of the constitutive equations used as a closure of the system of fundamental equations describing two-phase flow systems and of the numerical methods currently applied for their solutions in computer programmes.

Development of computer codes describing thermo-fluid dynamic problems in LMFBRs bundles has been made at IRE during the past 15 to 20 years. The trend has gone from one-dimensional heterogeneous flow description to three-dimensional programmes based on a slip model or a separated-phases model. The latter implies a detailed description of the physical phenomena underlying the exchange processes between the phases.

This report aims at providing a theoretical basis for the mathematical description of these physical processes, which should be both self-consistent and detailed enough to be read currently without much additional analytical work needed. This presentation of the subject relies heavily upon the basic work of reference [1], but it is hoped that the more detailed analytical treatment presented here makes this report a more readable reference as a background for practical applications.

Chapter 2 presents conservation equations and jump conditions at the phase boundary in the local (microscopic) formulation. The mathematical averaging procedures are presented in chapter 3, where the general form of averaged equations is derived. An original contribution, if any, with respect to the reference [1], is the attempt of a systematic classification of physical quantities for either phase made in chapter 4. This classification should help in getting physical insight into the relevance of exchange terms between the phases, insight which remains sometimes obscured if a mere mathematical definition, without physical background, is used. In chapter 5 a detailed form of the averaged conservation equations and jump conditions is derived for either phase both in the Eulerian and in the Lagrangian formulation. Eventually, in chapter 6 the conservation equations for the mixture are derived. They form the basis for a description of two-phase flow based on a "slip-model" approach.

This report is intended to be the first part of a comprehensive documentation consisting of three parts. Parts II and III should deal with constitutive equations and numerical methods, respectively.

2. Instantaneous local equations and jump conditions

2.1 General form of conservation equations

The canonical form of the equations for the flow of a pure compressible Newtonian fluid (or for the flow within the region consisting entirely of one phase in a multiphase mixture) is [1], [2], [3],

$$\frac{\partial(\rho\psi)}{\partial t} + \nabla \cdot (\rho\,\psi\,\mathbf{v}) - \nabla \cdot \mathbf{J} - \rho f = 0 \qquad [2.1.1]$$

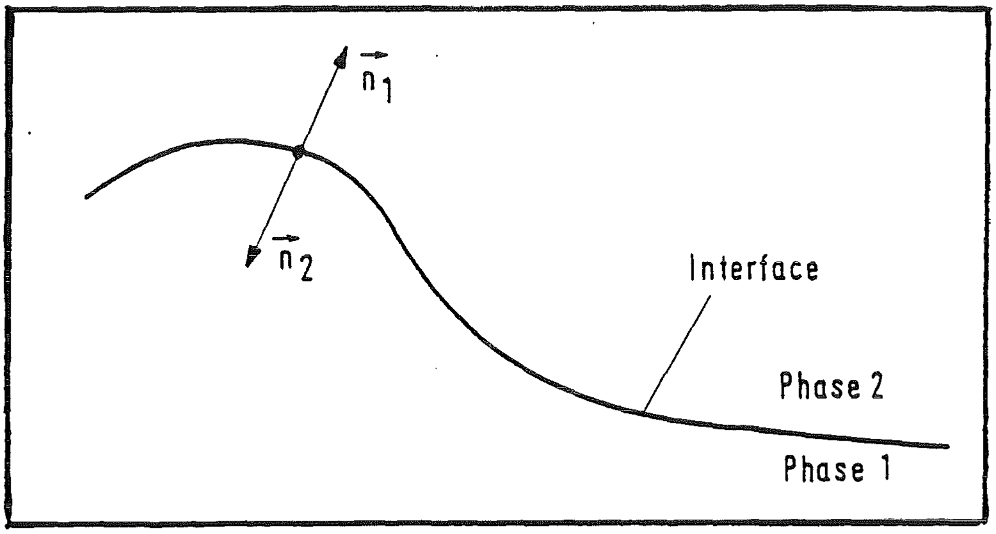
where: ρ density, $\mathbf{v}$ velocity vector, ψ quantity conserved, $\mathbf{J}$ diffusive flux, f source density for ψ.

In case of specific entropy as conserved quantity the left-hand side of (2.1.1) is not-negative.

We denote with brackets [ ] equations and formulas appearing for the first time in the text.

2.2 General form of jump conditions at the interface between phases

With reference to the following sketch, the jump conditions at the phase interface are



$$\rho_1\psi_1(\mathbf{v}_1 - \mathbf{v}_i)\cdot\mathbf{n}_1 - \mathbf{J}_1\cdot\mathbf{n}_1 + \rho_2\psi_2(\mathbf{v}_2 - \mathbf{v}_i)\cdot\mathbf{n}_2 - \mathbf{J}_2\cdot\mathbf{n}_2 = m \qquad [2.2.1]$$

where: $\mathbf{v}_i$ velocity vector of the interface, $\mathbf{n}_k$ unit normal outward directed, m interfacial source of ψ.

Equation 2.2.1 can be written in compact form:

$$\left[\left[\left(\rho_k \psi_k (\mathbf{v}_k - \mathbf{v}_i) - \mathbf{J}_k\right) \cdot \mathbf{n}_k\right]\right] = m \qquad [2.2.2]$$

The summation convention holds for the left hand side. The symbol [[ ]] denotes the jump across the phase interface.

The several equations considered in the conservation form (2.1.1) and the respective values of ψ, $\mathbf{J}$, f and m are given in Table I.

Conservation Principle	ψ	$\mathbf{J}$	f	m
Mass	1	0	0	0
Momentum	$\mathbf{v}$	$\underline{\mathbf{T}}$	$\mathbf{g}$	$\mathbf{m}_i^\sigma = \sigma\, c\, \mathbf{n}$
Total Energy	$e = u + v^2/2$	$\underline{\mathbf{T}} \cdot \mathbf{v} - \mathbf{q}$	$\mathbf{g} \cdot \mathbf{v} + r$	$e_i^\sigma + q_i$
Mechanical Energy	$v^2/2$	$\underline{\mathbf{T}} \cdot \mathbf{v}$	$\mathbf{g} \cdot \mathbf{v} - 1/\rho\, (\underline{\mathbf{T}} : \nabla \mathbf{v})$	e_i^σ
Internal Energy	u	$- \mathbf{q}$	$1/\rho\, (\underline{\mathbf{T}} : \nabla \mathbf{v}) + r$	q_i
Enthalpy	h	$- \mathbf{q}$	$1/\rho\, (\partial p/\partial t + \mathbf{v} \cdot \nabla p + (\underline{\tau} : \nabla \mathbf{v})) + r$	q_i
Entropy	s	$- \mathbf{q}/T$	r/T	$q_{si} \geqq 0$

σ = surface tension [N/m]
c = mean curvature of the interface [1/m] (1/R with radius R of curvature)
$\mathbf{n}$ = unit normal pointing out of a concave curvature
e_i^σ = surface energy source term at the interface [W/m2]
q_i = distributed heating source at the interface [W/m2]
q_{si} = source of entropy at the interface [W/(m2 K)]
r = specific heat source [W/kg]

Table I: Variables in Generic Conservation and Jump Equations

2.3 Detailed form of Conservation Equations

In this section the detailed form of the conservation equations [4] and of the jump conditions is given. They are summarized in Table II.

A. Mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \ \mathbf{v}) = 0 \qquad [2.3.1]$$

B. Momentum

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = \nabla \cdot \underline{T} + \rho \mathbf{g} \qquad [2.3.2]$$

$\underline{T}$ is the symmetric stress tensor ($\underline{T} = \underline{T}^t$, t denotes the transpose). It is convenient to write the stress tensor in terms of pressure p and shear stresses $\underline{\tau}$. Thus,

$$\underline{T} = -p\underline{I} + \underline{\tau} = -p \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} . \qquad [2.3.3]$$

In rectangular coordinates ($x = x_1, y = x_2, z = x_3$) we have for convective momentum transport term

$$\nabla \cdot \rho \mathbf{v} \mathbf{v} = \sum_i \frac{\partial}{\partial x_i} \delta_i \cdot \sum_j \sum_k \rho \, v_j \, v_k \, \delta_j \, \delta_k =$$

$$[2.3.4]$$

$$= \sum_i \sum_j \sum_k \frac{\partial}{\partial x_i} \rho \, v_j \, v_k \, \delta_{ij} \, \delta_k = \sum_k \sum_i \frac{\partial}{\partial x_i} \rho \, v_i \, v_k \, \delta_k \, .$$

Hence for the components of the vector $\nabla \cdot \rho \mathbf{v} \mathbf{v}$ we obtain

$$\left(\nabla \cdot \rho \mathbf{v} \mathbf{v}\right)_1 = \left(\sum_i \frac{\partial}{\partial x_i} \rho \, v_i \, v_1 \right) \delta_1 = \left(\frac{\partial}{\partial x_1} \rho \, v_1 v_1 + \frac{\partial}{\partial x_2} \rho \, v_2 \, v_1 + \frac{\partial}{\partial x_3} \rho \, v_3 \, v_1 \right) \delta_1 \qquad [2.3.5a]$$

$$\left(\nabla \cdot \rho \mathbf{v} \mathbf{v}\right)_2 = \left(\sum_i \frac{\partial}{\partial x_i} \rho \, v_i \, v_2 \right) \delta_2 = \left(\frac{\partial}{\partial x_1} \rho \, v_1 v_2 + \frac{\partial}{\partial x_2} \rho \, v_2 \, v_2 + \frac{\partial}{\partial x_3} \rho \, v_3 \, v_2 \right) \delta_2 \qquad [2.3.5b]$$

$$\left(\nabla \cdot \rho \mathbf{v} \mathbf{v}\right)_3 = \left(\sum_i \frac{\partial}{\partial x_i} \rho \, v_i \, v_3 \right) \delta_3 = \left(\frac{\partial}{\partial x_1} \rho \, v_1 v_3 + \frac{\partial}{\partial x_2} \rho \, v_2 \, v_3 + \frac{\partial}{\partial x_3} \rho \, v_3 \, v_3 \right) \delta_3 \qquad [2.3.5c]$$

Furthermore, for the stress term we have

$$\nabla \cdot \underline{T} = \sum_i \frac{\partial}{\partial x_i} \delta_i \cdot \sum_j \sum_k \delta_j \, \delta_k \, T_{jk} =$$

[2.3.6]

$$= \sum_i \sum_j \sum_k \frac{\partial}{\partial x_i} T_{jk} \, \delta_{ji} \, \delta_k = \sum_k \left(\sum_i \frac{\partial}{\partial x_i} T_{ik} \right) \delta_k \; .$$

Hence, the components of the vector $\nabla \cdot \underline{T}$ are

$$(k = 1) \; (\nabla \cdot \underline{T})_1 = \left(\sum_i \frac{\partial}{\partial x_i} T_{i1} \right) \delta_1 = \left(\frac{\partial}{\partial x_1} T_{11} + \frac{\partial}{\partial x_2} T_{21} + \frac{\partial}{\partial x_3} T_{31} \right) \delta_1 = \sigma_1 \delta_1 \qquad [2.3.7a]$$

$$(k = 2) \; (\nabla \cdot \underline{T})_2 = \left(\sum_i \frac{\partial}{\partial x_i} T_{i2} \right) \delta_2 = \left(\frac{\partial}{\partial x_1} T_{12} + \frac{\partial}{\partial x_2} T_{22} + \frac{\partial}{\partial x_3} T_{32} \right) \delta_2 = \sigma_2 \delta_2 \qquad [2.3.7b]$$

$$(k = 3) \; (\nabla \cdot \underline{T})_3 = \left(\sum_i \frac{\partial}{\partial x_i} T_{i3} \right) \delta_3 = \left(\frac{\partial}{\partial x_1} T_{13} + \frac{\partial}{\partial x_2} T_{23} + \frac{\partial}{\partial x_3} T_{33} \right) \delta_3 = \sigma_3 \delta_3 \qquad [2.3.7c]$$

This can be written symbolically

$$\nabla \cdot \underline{T} = \; < \frac{\partial}{\partial x} \; \frac{\partial}{\partial y} \; \frac{\partial}{\partial z} > \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \; < \sigma_1 \;\; \sigma_2 \;\; \sigma_3 > \qquad [2.3.8]$$

with

$$\sigma_1 = \frac{\partial T_{11}}{\partial x} + \frac{\partial T_{21}}{\partial y} + \frac{\partial T_{31}}{\partial z} = - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$\sigma_2 = \frac{\partial T_{12}}{\partial x} + \frac{\partial T_{22}}{\partial y} + \frac{\partial T_{32}}{\partial z} = - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \qquad [2.3.9]$$

$$\sigma_3 = \frac{\partial T_{13}}{\partial x} + \frac{\partial T_{23}}{\partial y} + \frac{\partial T_{33}}{\partial z} = - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \, .$$

C. Total Energy

The detailed form of the conservation equation for the total energy

$$e = u + \frac{v^2}{2} \qquad [2.3.10]$$

with u internal energy [J/kg] and $\upsilon^2/2$ kinetic energy is

$$\frac{\partial}{\partial t}\left[\rho\left(u+\frac{1}{2}\,\upsilon^2\right)\right]+\nabla\cdot\left[\rho\left(u+\frac{1}{2}\,\upsilon^2\right)\mathbf{v}\right]= \qquad [2.3.11]$$

$$=\nabla\cdot(\underline{\mathbf{T}}\cdot\mathbf{v})-\nabla\cdot\mathbf{q}+\rho\,\mathbf{g}\cdot\mathbf{v}+\rho\,r$$

where $\mathbf{q}$ is the heat flux [W/m2] and r the specific heat source [W/kg].

One has

$$\underline{\mathbf{T}}\cdot\mathbf{v}=\sum_i\sum_j T_{ij}\,\boldsymbol{\delta}_i\,\boldsymbol{\delta}_j\cdot\sum_k \upsilon_k\,\boldsymbol{\delta}_k= \qquad [2.3.12]$$

$$=\sum_i\sum_j\sum_k T_{ij}\,\upsilon_k\,\boldsymbol{\delta}_i\,\delta_{jk}=\sum_i\sum_j T_{ij}\,\upsilon_j\,\boldsymbol{\delta}_i\,.$$

Hence, in rectangular coordinates we obtain

$$\nabla\cdot(\underline{\mathbf{T}}\cdot\mathbf{v})=\sum_i\frac{\partial}{\partial x_i}\,\boldsymbol{\delta}_i\cdot\sum_j\sum_k T_{jk}\,\upsilon_k\,\boldsymbol{\delta}_j= \qquad [2.3.13]$$

$$=\sum_i\sum_j\sum_k\frac{\partial}{\partial x_i}\,T_{jk}\,\upsilon_k\,\delta_{ij}=\sum_i\sum_k\frac{\partial}{\partial x_i}\,(T_{ik}\,\upsilon_k)\,.$$

$$\nabla\cdot(\underline{\mathbf{T}}\cdot\mathbf{v})=\frac{\partial}{\partial x}\left[T_{11}\,\upsilon_x+T_{12}\,\upsilon_y+T_{13}\,\upsilon_z\right]+$$

$$+\frac{\partial}{\partial y}\left[T_{21}\,\upsilon_x+T_{22}\,\upsilon_y+T_{23}\,\upsilon_z\right]+ \qquad [2.3.14]$$

$$+\frac{\partial}{\partial z}\left[T_{31}\,\upsilon_x+T_{32}\,\upsilon_y+T_{33}\,\upsilon_z\right].$$

Thus, using

$$\underline{\mathbf{T}}=-p\,\underline{\mathbf{I}}+\underline{\tau} \qquad (2.3.3)$$

we obtain

$$\nabla\cdot(\underline{\mathbf{T}}\cdot\mathbf{v})=-\left(\frac{\partial}{\partial x}\,p\,\upsilon_x+\frac{\partial}{\partial y}\,p\,\upsilon_y+\frac{\partial}{\partial z}\,p\,\upsilon_z\right)+\frac{\partial}{\partial x}\left(\tau_{xx}\,\upsilon_x+\tau_{xy}\,\upsilon_y+\tau_{xy}\,\upsilon_z\right) \qquad [2.3.15]$$

$$+\frac{\partial}{\partial y}\left(\tau_{yx}\,\upsilon_x+\tau_{yy}\,\upsilon_y+\tau_{yz}\,\upsilon_z\right)+\frac{\partial}{\partial z}\left(\tau_{zx}\,\upsilon_x+\tau_{zy}\,\upsilon_y+\tau_{zz}\,\upsilon_z\right).$$

D. Mechanical Energy

We start to multiply the momentum equation (2.3.2) by v:

$$\mathbf{v} \cdot \left[\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho\, \mathbf{v}\, \mathbf{v}) \right] = \mathbf{v} \cdot (\nabla \cdot \underline{\mathbf{T}}) + \mathbf{v} \cdot \rho\, \mathbf{g} \;. \qquad [2.3.16]$$

Rearrangement of the term within square brackets with help of the mass equation yields

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho\, \mathbf{v}\, \mathbf{v}) = \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \frac{\partial \rho}{\partial t} + (\rho\, \mathbf{v} \cdot \nabla)\, \mathbf{v} + \mathbf{v}\, (\nabla \cdot \rho\, \mathbf{v}) \qquad [2.3.17]$$

$$= \rho \frac{\partial \mathbf{v}}{\partial t} + (\rho\, \mathbf{v} \cdot \nabla)\, \mathbf{v} = \rho \frac{D\mathbf{v}}{Dt}$$

having used the continuity equation (2.3.1).

Thus, equation (2.3.16) yields

$$\mathbf{v} \cdot \rho \frac{D\mathbf{v}}{Dt} = \mathbf{v} \cdot (\nabla \cdot \underline{\mathbf{T}}) + \mathbf{v} \cdot \rho\, \mathbf{g} \qquad [2.3.18]$$

Using the identity

$$\frac{D\, (\mathbf{v} \cdot \mathbf{v})}{Dt} = 2\, \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} \qquad [2.3.19]$$

one derives

$$\rho \frac{D}{Dt} \left(\frac{1}{2}\, \upsilon^2 \right) = \mathbf{v} \cdot (\nabla \cdot \underline{\mathbf{T}}) + \mathbf{v} \cdot \rho\, \mathbf{g} . \qquad [2.3.20]$$

Using

$$\rho \frac{D}{Dt} \left(\frac{1}{2}\, \upsilon^2 \right) = \rho \left[\frac{\partial}{\partial t} \left(\frac{1}{2}\, \upsilon^2 \right) + \mathbf{v} \cdot \nabla \left(\frac{1}{2}\, \upsilon^2 \right) \right] =$$

$$= \frac{\partial}{\partial t} \left(\frac{1}{2}\, \rho\, \upsilon^2 \right) - \frac{\upsilon^2}{2} \frac{\partial \rho}{\partial t} + \rho\, \mathbf{v} \cdot \nabla \left(\frac{1}{2}\, \upsilon^2 \right) =$$

$$= \frac{\partial}{\partial t} \left(\frac{1}{2}\, \rho\, \upsilon^2 \right) + \frac{\upsilon^2}{2} \left(\nabla \cdot \rho\, \mathbf{v} \right) + \rho\, \mathbf{v} \cdot \nabla \left(\frac{1}{2}\, \upsilon^2 \right) =$$

$$= \frac{\partial}{\partial t} \left(\frac{1}{2}\, \rho\, \upsilon^2 \right) + \nabla \cdot \left(\frac{1}{2}\, \upsilon^2\, \rho\, \mathbf{v} \right) \qquad [2.3.21]$$

and the identity

$$\mathbf{v} \cdot (\nabla \cdot \underline{\mathbf{T}}) = \nabla \cdot (\underline{\mathbf{T}} \cdot \mathbf{v}) - (\underline{\mathbf{T}} : \nabla\, \mathbf{v}) \qquad [2.3.22]$$

which holds for a symmetric tensor $\underline{\mathbf{T}}$ (see proof below), we get the final equation of mechanical energy:

$$\frac{\partial}{\partial t} \left(\frac{1}{2}\, \rho\, \upsilon^2 \right) + \nabla \cdot \left(\frac{1}{2}\, \rho\, \upsilon^2\, \mathbf{v} \right) = \nabla \cdot (\underline{\mathbf{T}} \cdot \mathbf{v}) - (\underline{\mathbf{T}} : \nabla\, \mathbf{v}) + (\mathbf{v} \cdot \rho\, \mathbf{g}) \;. \qquad [2.3.23]$$

In rectangular coordinates, one has

$$\mathbf{v} \cdot \nabla \mathbf{v} = \sum_i v_i \, \boldsymbol{\delta}_i \cdot \sum_j \sum_k \frac{\partial v_k}{\partial x_j} \, \boldsymbol{\delta}_j \, \boldsymbol{\delta}_k =$$

[2.3.24]

$$= \sum_i \sum_j \sum_k v_i \frac{\partial v_k}{\partial x_j} \, \delta_{ij} \, \boldsymbol{\delta}_k = \sum_k \sum_i v_i \frac{\partial v_k}{\partial x_i} \, \boldsymbol{\delta}_k$$

which can be written, symbolically

$$\mathbf{v} \cdot \nabla \mathbf{v} = < v_x \quad v_y \quad v_z > \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix} \qquad [2.3.25]$$

or

$$\mathbf{v} \cdot \nabla \mathbf{v} = < v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \qquad [2.3.26]$$

$$v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}$$

$$v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} > .$$

Proof of the identity

$$\mathbf{v} \cdot (\nabla \cdot \underline{\mathbf{T}}) = \nabla \cdot (\underline{\mathbf{T}} \cdot \mathbf{v}) - (\underline{\mathbf{T}} : \nabla \mathbf{v}) \qquad (2.3.22)$$

for a symmetric tensor $\underline{\mathbf{T}}$:

By (2.3.6)

$$\nabla \cdot \underline{\mathbf{T}} = \sum_j \sum_k \frac{\partial}{\partial x_k} T_{kj} \, \boldsymbol{\delta}_j$$

Hence, for the left side (LS) we have

$$\text{LS} = \mathbf{v} \cdot (\nabla \cdot \underline{\mathbf{T}}) = \sum_i v_i \, \boldsymbol{\delta}_i \cdot \sum_j \sum_k \frac{\partial T_{kj}}{\partial x_k} \, \boldsymbol{\delta}_j =$$

$$= \sum_i \sum_j \sum_k v_i \frac{\partial T_{kj}}{\partial x_k} \, \delta_{ij} = \sum_i \sum_k v_i \frac{\partial T_{ki}}{\partial x_k} \, .$$

By (2.3.13) and (2.3.30), which is proved hereafter independently, one has, for the right side (RS):

$$RS = \sum_i \sum_k \frac{\partial}{\partial x_i} (T_{ik} \upsilon_k) - \sum_i \sum_k T_{ik} \frac{\partial \upsilon_i}{\partial x_k}$$

$$= \sum_i \sum_k \left(\upsilon_k \frac{\partial T_{ik}}{\partial x_i} + T_{ik} \frac{\partial \upsilon_k}{\partial x_i} - T_{ki} \frac{\partial \upsilon_k}{\partial x_i} \right)$$

$$= \sum_i \sum_k \upsilon_i \frac{\partial T_{ik}}{\partial x_k} \quad \left(because\ of\ T_{ik} = T_{ki} \right)$$

q.e.d.

E. Internal Energy

Subtracting the Mechanical Energy equation (2.3.23) from the Total Energy equation (2.3.11) we get

$$\frac{\partial}{\partial t} \left[\rho \left(u + \frac{1}{2} \upsilon^2 \right) \right] + \nabla \cdot \left[\rho \left(u + \frac{1}{2} \upsilon^2 \right) \mathbf{v} \right] - \frac{\partial}{\partial t} \left(\rho \frac{1}{2} \upsilon^2 \right) - \nabla \cdot \left(\rho \frac{1}{2} \upsilon^2 \mathbf{v} \right) = \qquad [2.3.27]$$

$$= \nabla \cdot (\underline{\mathbf{T}} \cdot \mathbf{v}) - \nabla \cdot \mathbf{q} + \rho \mathbf{g} \cdot \mathbf{v} + \rho r - \nabla \cdot (\underline{\mathbf{T}} \cdot \mathbf{v}) + (\underline{\mathbf{T}} : \nabla \mathbf{v}) - \rho \mathbf{g} \cdot \mathbf{v},$$

hence

$$\frac{\partial}{\partial t} (\rho u) + \nabla \cdot (\rho u \mathbf{v}) = - \nabla \cdot \mathbf{q} + (\underline{\mathbf{T}} : \nabla \mathbf{v}) + \rho r . \qquad [2.3.28]$$

With

$$\underline{\mathbf{T}} = - p \underline{\mathbf{I}} + \underline{\boldsymbol{\tau}} \qquad (2.3.3)$$

we get

$$\frac{\partial}{\partial t} (\rho u) + \nabla \cdot (\rho u \mathbf{v}) = - \nabla \cdot \mathbf{q} - p (\nabla \cdot \mathbf{v}) + (\underline{\boldsymbol{\tau}} : \nabla \mathbf{v}) + \rho r . \qquad [2.3.29]$$

In rectangular coordinates one has

$$\underline{\mathbf{T}} : \nabla \mathbf{v} = \sum_i \sum_j T_{ij} \delta_i \delta_j : \sum_k \sum_l \frac{\partial}{\partial x_k} \upsilon_l \delta_k \delta_l$$

$$= \sum_i \sum_j \sum_k \sum_l T_{ij} \frac{\partial \upsilon_l}{\partial x_j} \delta_{jk} \delta_i \cdot \delta_l$$

$$= \sum_i \sum_j \sum_l T_{ij} \frac{\partial v_l}{\partial x_j} \delta_{il} = \sum_i \sum_j T_{ij} \frac{\partial v_i}{\partial x_j} = \qquad [2.3.30]$$

$$= \sum_j T_{1j} \frac{\partial v_1}{\partial x_j} + \sum_j T_{2j} \frac{\partial v_2}{\partial x_j} + \sum_j T_{3j} \frac{\partial v_3}{\partial x_j} =$$

$$= T_{11} \frac{\partial v_1}{\partial x_1} + T_{12} \frac{\partial v_1}{\partial x_2} + T_{13} \frac{\partial v_1}{\partial x_3} +$$

$$+ T_{21} \frac{\partial v_2}{\partial x_1} + T_{22} \frac{\partial v_2}{\partial x_2} + T_{23} \frac{\partial v_2}{\partial x_3} +$$

$$+ T_{31} \frac{\partial v_3}{\partial x_1} + T_{32} \frac{\partial v_3}{\partial x_2} + T_{33} \frac{\partial v_3}{\partial x_3} .$$

Hence, using

$$\underline{T} = - p \, \underline{I} + \underline{\tau} \qquad (2.3.3)$$

we have

$$(\underline{T} : \nabla \mathrm{v}) = - p \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) +$$

$$+ \tau_{xx} \left(\frac{\partial v_x}{\partial x} \right) + \tau_{yy} \left(\frac{\partial v_y}{\partial y} \right) + \tau_{zz} \left(\frac{\partial v_z}{\partial z} \right) +$$

$$+ \tau_{xy} \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) + \tau_{xz} \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) + \tau_{yz} \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) \qquad [2.3.31]$$

$$= - p \, (\nabla \cdot \mathrm{v}) + (\underline{\tau} : \nabla \mathrm{v}) .$$

F. <u>Enthalpy</u>

With $u = h - p/\rho$, it follows from the internal energy equation (2.3.29)

$$\frac{\partial}{\partial t} (\rho h - p) + \nabla \cdot \left[\left(\rho h - p \right) \mathrm{v} \right] = - \nabla \cdot \mathrm{q} - p \, (\nabla \cdot \mathrm{v}) + (\underline{\tau} : \nabla \mathrm{v}) + \rho r , \qquad [2.3.32]$$

hence

$$\frac{\partial}{\partial t} (\rho h) + \nabla \cdot (\rho h \, \mathrm{v}) = \frac{\partial p}{\partial t} + \nabla \cdot p \, \mathrm{v} - \nabla \cdot \mathrm{q} - p \, (\nabla \cdot \mathrm{v}) + (\underline{\tau} : \nabla \mathrm{v}) + \rho r . \qquad [2.3.33]$$

Using the identities

$$\nabla \cdot (p \, \mathrm{v}) = p \, (\nabla \cdot \mathrm{v}) + \mathrm{v} \cdot \nabla p \qquad [2.3.34]$$

one derives

$$\frac{\partial}{\partial t}(\rho h) + \nabla\cdot(\rho h \mathbf{v}) = \frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p - \nabla\cdot\mathbf{q} + (\underline{\tau}:\nabla\mathbf{v}) + \rho r \; . \qquad [2.3.35]$$

With

$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p = \frac{Dp}{Dt} \qquad [2.3.36]$$

eq. (2.3.35) yields

$$\frac{\partial}{\partial t}(\rho h) + \nabla\cdot(\rho h \mathbf{v}) = -\nabla\cdot\mathbf{q} + \frac{Dp}{Dt} + (\underline{\tau}:\nabla\mathbf{v}) + \rho r \; . \qquad [2.3.37]$$

G. Entropy

i) Entropy inequality

With $\psi = s$ equation (2.1.1) yields the Clausius-Duhem entropy inequality

$$\frac{\partial(\rho s)}{\partial t} + \nabla\cdot(\rho s \mathbf{v}) + \nabla\cdot\left(\frac{\mathbf{q}}{T}\right) - \frac{\rho r}{T} = \Delta s \geq 0 \; . \qquad [2.3.38]$$

For reversible processes $\Delta s = 0$. Remark that Δs must not be considered as the increment of s, but as a new symbol in itself. Δs (W/m^3 K) has in fact different dimensions from s (J/kg K).

Using the operator identity (2.3.48) and the continuity equation (2.3.1) the previous equation can be written in terms of the substantial derivative as

$$\rho\frac{Ds}{Dt} + \nabla\cdot\left(\frac{\mathbf{q}}{T}\right) - \frac{\rho r}{T} = \Delta s \geq 0 \; . \qquad [2.3.39]$$

Δs can be expressed in terms of the dissipation δ by letting

$$\Delta s = \frac{\rho}{T}\,\delta \; . \qquad [2.3.40]$$

Thus one derives from equation (2.3.39)

$$\delta = T\frac{Ds}{Dt} + \frac{T}{\rho}\,\nabla\cdot\left(\frac{\mathbf{q}}{T}\right) - r \geq 0 \; . \qquad [2.3.41]$$

If T is constant the Clausius-Duhem inequality reduces to the Clausius-Planck inequality

$$\delta = T\frac{Ds}{Dt} - \frac{1}{\rho}\,(\rho r - \nabla\cdot\mathbf{q}) \geq 0 \; . \qquad [2.3.42]$$

The dissipation δ is thus the amount by which the entropy increase rate Ds/Dt, multiplied by the absolute temperature, exceeds the heating (diffusion and intrinsic generation).

In the following we want to derive an analytical expression of the term Δs, hence of the dissipation δ. This can be obtained by combining the local internal energy equation (2.3.28) with the Gibbs equation. Therefore the Gibbs equation is derived first. When the term Δs is known analytically, an entropy equation can be derived as shown.

ii) Derivation of the Gibbs equation

Differentiation with respect to time of the thermodynamic fundamental relation expressing the internal energy as function of the specific entropy and density, u = u (s, ρ), yields

$$\frac{Du}{Dt} = \left(\frac{\partial u}{\partial s}\right)_\rho \frac{Ds}{Dt} + \left(\frac{\partial u}{\partial \rho}\right)_s \frac{D\rho}{Dt} . \qquad [2.3.43]$$

Using the thermodynamic definitions of temperature and pressure

$$T = \left(\frac{\partial u}{\partial s}\right)_\rho \qquad [2.3.44]$$

$$p = -\left(\frac{\partial u}{\partial v}\right)_s = \rho^2 \left(\frac{\partial u}{\partial \rho}\right)_s \qquad [2.3.45]$$

(with v = 1/ρ = specific volume), equation (2.3.43) becomes:

$$\frac{Du}{Dt} = T \frac{Ds}{Dt} + \frac{p}{\rho^2} \frac{D\rho}{Dt} . \qquad [2.3.46]$$

Using D (1/ρ) = - Dρ /ρ² one has the Gibbs equation (1st form)

$$\frac{Du}{Dt} = T \frac{Ds}{Dt} - p \frac{D(1/\rho)}{Dt} . \qquad [2.3.47]$$

With the operator identity

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \qquad [2.3.48]$$

equation (2.3.47) becomes

$$\frac{Du}{Dt} = T \frac{Ds}{Dt} - p \left[\frac{\partial}{\partial t} (1/\rho) + \mathbf{v} \cdot \nabla \left(\frac{1}{\rho}\right) \right]$$

$$= T \frac{Ds}{Dt} + \frac{p}{\rho^2} \left(\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho \right) \qquad [2.3.49]$$

$$= T \frac{Ds}{Dt} + \frac{p}{\rho^2} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) - \rho (\nabla \cdot \mathbf{v}) \right] .$$

Using the continuity equation for the continuum (2.3.1) one obtains a second form of the Gibbs equation

$$\frac{Du}{Dt} = T\frac{Ds}{Dt} - \frac{p}{\rho}(\nabla \cdot \mathbf{v}). \qquad [2.3.50]$$

iii) <u>Derivation of the entropy equation</u>

Let us recall the local internal energy equation

$$\frac{\partial(\rho u)}{\partial t} + \nabla\cdot(\rho u \mathbf{v}) = \underset{(a)}{-\nabla\cdot\mathbf{q}} + \underset{(b)}{(\underline{\mathbf{T}}:\nabla\mathbf{v})} + \underset{(c)}{\rho r}. \qquad (2.3.28)$$

Term (a) represents the internal energy input by heat conduction per unit volume and time; term (b) represents both reversible and irreversible internal energy sources per unit volume and time due to compression and viscous dissipation; term (c) represents an intrinsic source of internal energy.

Using (2.3.1) and (2.3.48) equation (2.3.28) becomes

$$\rho\frac{Du}{Dt} = -\nabla\cdot\mathbf{q} + \underline{\mathbf{T}}:\nabla\mathbf{v} + \rho r. \qquad [2.3.51]$$

Expanding the stress term in pressure and shear stress components

$$\underline{\mathbf{T}} = -p\,\underline{\mathbf{I}} + \underline{\boldsymbol{\tau}} \qquad (2.3.3)$$

and using the identity

$$-p\,\underline{\mathbf{I}}:\nabla\mathbf{v} = -p\,(\nabla\cdot\mathbf{v}) \qquad \text{(from (2.3.31))}$$

equation (2.3.51) becomes

$$\rho\frac{Du}{Dt} = -\nabla\cdot\mathbf{q} - p\,(\nabla\cdot\mathbf{v}) + \underline{\boldsymbol{\tau}}:\nabla\mathbf{v} + \rho r. \qquad [2.3.52]$$

Replacing Du/Dt by means of the Gibbs equation (2.3.50) one has

$$\rho\left[T\frac{Ds}{Dt} - \frac{p}{\rho}(\nabla\cdot\mathbf{v})\right] = -\nabla\cdot\mathbf{q} - p\,(\nabla\cdot\mathbf{v}) + \underline{\boldsymbol{\tau}}:\nabla\mathbf{v} + \rho r, \qquad [2.3.53]$$

hence the entropy equation in terms of the substantial time derivative

$$\rho\frac{Ds}{Dt} = -\frac{1}{T}(\nabla\cdot\mathbf{q}) + \frac{1}{T}(\underline{\boldsymbol{\tau}}:\nabla\mathbf{v}) + \frac{\rho r}{T}. \qquad [2.3.54]$$

Using again the identity (2.3.48) and the continuity equation (2.3.1) one derives the entropy equation in terms of the partial time derivative

$$\frac{\partial (\rho s)}{\partial t} + \nabla \cdot (\rho s \mathbf{v}) = -\frac{1}{T} (\nabla \cdot \mathbf{q}) + \frac{1}{T} (\underline{\tau} : \nabla \mathbf{v}) + \frac{\rho}{T} r . \qquad [2.3.55]$$

Using

$$\frac{1}{T} (\nabla \cdot \mathbf{q}) = \nabla \cdot \left(\frac{\mathbf{q}}{T} \right) - \mathbf{q} \cdot \nabla \left(\frac{1}{T} \right) \qquad [2.3.56]$$

one obtains:

$$\frac{\partial (\rho s)}{\partial t} + \nabla \cdot (\rho s \mathbf{v}) + \nabla \cdot \left(\frac{\mathbf{q}}{T} \right) - \frac{\rho r}{T} = \mathbf{q} \cdot \nabla \left(\frac{1}{T} \right) + \frac{1}{T} (\underline{\tau} : \nabla \mathbf{v}) . \qquad [2.3.57]$$

Comparing (2.3.57) with (2.3.38) one derives the required analytical expression for Δs:

$$\Delta s = \mathbf{q} \cdot \nabla \left(\frac{1}{T} \right) + \frac{1}{T} (\underline{\tau} : \nabla \mathbf{v}) \geq 0 . \qquad [2.3.58]$$

Independently of Δs being zero or not one can write the entropy equation, which must always be satisfied identically for reversible and irreversible processes, in the form:

$$\frac{\partial}{\partial t} (\rho s) + \nabla \cdot (\rho s \mathbf{v}) + \nabla \cdot \left(\frac{\mathbf{q}}{T} \right) - \frac{\rho r}{T} - \mathbf{q} \cdot \nabla \left(\frac{1}{T} \right) - \frac{1}{T} (\underline{\tau} : \nabla \mathbf{v}) = 0 . \qquad [2.3.59]$$

2.4 Detailed form of Jump Conditions

The jump conditions can easily be deduced from the generic equation (2.2.2) and from Table I. The derived detailed forms are as follows:

i) Jump condition for mass

Letting $\psi = 1$, $\mathbf{J} = 0$ and $m = 0$ in eq. 2.2.2 one derives

$$\left[\!\left[\rho_k (\mathbf{v}_k - \mathbf{v}_i) \cdot \mathbf{n}_k \right]\!\right] = 0 . \qquad [2.4.1]$$

(the subscript k is a summation index).

ii) Jump condition for momentum:

Letting $\psi = \mathbf{v}$, $\mathbf{J} = \underline{\mathbf{T}}$ and $m = \mathbf{m}_i^\sigma$ in eq. (2.2.2) one obtains

$$\left[\!\left[\left[\rho_k \mathbf{v}_k (\mathbf{v}_k - \mathbf{v}_i) - \underline{\mathbf{T}}_k \right] \cdot \mathbf{n}_k \right]\!\right] = \mathbf{m}_i^\sigma \qquad [2.4.2]$$

iii) Jump condition for total energy

Letting $\psi = u + v^2/2$ and $\mathbf{J} = \underline{\mathbf{T}} \cdot \mathbf{v} - \mathbf{q}$ and $m = e_i^\sigma + q_i$ in eq. (2.2.2) one has

$$\left\|\left[\rho_k\left(u_k + \frac{1}{2}\, v_k^2\right)(\mathbf{v}_k - \mathbf{v}_i) - (\underline{\mathbf{T}}_k \cdot \mathbf{v}_k - \mathbf{q}_k)\right] \cdot \mathbf{n}_k\right\| = e_i^\sigma + q_i \qquad [2.4.3]$$

iv) Jump condition for mechanical energy

Letting $\psi = v^2/2$ and $\mathbf{J} = \underline{\mathbf{T}} \cdot \mathbf{v}$ and $m = e_i^\sigma$ in eq. (2.2.2) one derives

$$\left\|\left[\rho_k\, \frac{v_k^2}{2}\, (\mathbf{v}_k - \mathbf{v}_i) - (\underline{\mathbf{T}}_k \cdot \mathbf{v}_k)\right] \cdot \mathbf{n}_k\right\| = e_i^\sigma \qquad [2.4.4]$$

v) Jump condition for internal energy

Letting $\psi = u$ and $\mathbf{J} = -\mathbf{q}$ and $m = q_i$ in eq. (2.2.2) one has:

$$\left\|\left[\rho_k\, u_k\, (\mathbf{v}_k - \mathbf{v}_i) + \mathbf{q}_k\right] \cdot \mathbf{n}_k\right\| = q_i \qquad [2.4.5]$$

vi) Jump condition for enthalpy

Letting $\psi = h$ and $\mathbf{J} = -\mathbf{q}$ and $m = q_i$ in eq. (2.2.2) one has:

$$\left\|\left[\rho_k\, h_k\, (\mathbf{v}_k - \mathbf{v}_i) + \mathbf{q}_k\right] \cdot \mathbf{n}_k\right\| = q_i \qquad [2.4.6]$$

vii) Jump condition for entropy

Letting $\psi = s$ and $\mathbf{J} = -\mathbf{q}/T$ and $m = 0$ in eq. (2.2.2) one derives the entropy jump condition:

$$\left\|\left[\rho_k\, s_k\, (\mathbf{v}_k - \mathbf{v}_i) + \frac{\mathbf{q}_k}{T}\right] \cdot \mathbf{n}_k\right\| = q_{si} \geq 0\ . \qquad [2.4.7]$$

GENERAL FORM

Conservation equation

$$\frac{\partial(\rho\psi)}{\partial t} + \nabla \cdot (\rho\ \psi\ \mathbf{v}) - \nabla \cdot \mathbf{J} - \rho f = 0 \qquad (2.1.1)$$

Jump conditions at phase interface

$$\left[\left[\ \left[\rho_k \psi_k \left(\mathbf{v}_k - \mathbf{v}_i\right) - \mathbf{J}_k\right] \cdot \mathbf{n}_k\ \right]\right] = m \qquad (2.2.2)$$

DETAILED FORM

Conservation equations

Conservation of mass

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\ \mathbf{v}) = 0 \qquad (2.3.1)$$

Conservation of momentum

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla \cdot (\rho\ \mathbf{v}\ \mathbf{v}) = \nabla \cdot \underline{\mathbf{T}} + \rho\ \mathbf{g} \qquad (2.3.2)$$

Conservation of total energy

$$\frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho\ e\ \mathbf{v}) = \nabla \cdot (\underline{\mathbf{T}} \cdot \mathbf{v}) - \nabla \cdot \mathbf{q} + \rho\mathbf{g} \cdot \mathbf{v} + \rho\ r \qquad (2.3.11)$$

Conservation of mechanical energy:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\ \rho\ v^2\right) + \nabla \cdot \left(\frac{1}{2}\ \rho\ v^2\ \mathbf{v}\right) = \nabla \cdot (\underline{\mathbf{T}} \cdot \mathbf{v}) - \underline{\mathbf{T}} : \nabla\mathbf{v} + \mathbf{v} \cdot \rho\ \mathbf{g} \qquad (2.3.24)$$

Conservation of internal energy:

$$\frac{\partial}{\partial t}(\rho\ u) + \nabla \cdot (\rho\ u\ \mathbf{v}) = -\nabla \cdot \mathbf{q} - p\ (\nabla \cdot \mathbf{v}) + \underline{\tau} : \nabla\mathbf{v} + \rho\ r \qquad (2.3.29)$$

Conservation of specific enthalpy:

$$\frac{\partial(\rho h)}{\partial t} + \nabla \cdot (\rho\ h\ \mathbf{v}) = -\nabla \cdot \mathbf{q} + \frac{Dp}{Dt} + \underline{\tau} : \nabla\mathbf{v} + \rho\ r \qquad (2.3.37)$$

Table II - Summary of local instantaneous equations

Entropy inequality

$$\frac{\partial(\rho s)}{\partial t} + \nabla \cdot (\rho\, s\, \mathbf{v}) + \nabla \cdot \left(\frac{\mathbf{q}}{T}\right) - \frac{\rho r}{T} = \Delta s \geq 0 \qquad (2.3.38)$$

$$\Delta s = \mathbf{q} \cdot \nabla\left(\frac{1}{T}\right) + \frac{1}{T}\,(\underline{\tau} : \nabla\, \mathbf{v}) \geq 0 \qquad (2.3.58)$$

Entropy equation

$$\frac{\partial(\rho s)}{\partial t} + \nabla \cdot (\rho\, s\, \mathbf{v}) + \nabla \cdot \frac{\mathbf{q}}{T} - \frac{\rho r}{T} - \mathbf{q} \cdot \nabla\left(\frac{1}{T}\right) - \frac{1}{T}\,(\underline{\tau} : \nabla\, \mathbf{v}) = 0 \qquad (2.3.59)$$

Jump conditions:

mass

$$\Big[\Big[\quad \rho_k\, (\mathbf{v}_k - \mathbf{v}_i) \cdot \mathbf{n}_k \quad \Big]\Big] = 0 \qquad (2.4.1)$$

momentum

$$\Big[\Big[\quad \Big[\rho_k\, \mathbf{v}_k (\mathbf{v}_k - \mathbf{v}_i) - \underline{\mathbf{T}}_k\Big] \cdot \mathbf{n}_k \quad \Big]\Big] = \mathbf{m}_i^{\sigma} \qquad (2.4.2)$$

total energy

$$\Big[\Big[\quad \Big[\rho_k \left(u_k + \frac{v_k^2}{2}\right)(\mathbf{v}_k - \mathbf{v}_i) - \underline{\mathbf{T}}_k \cdot \mathbf{v}_k + \mathbf{q}_k\Big] \cdot \mathbf{n}_k \quad \Big]\Big] = \mathbf{e}_i^{\sigma} + \mathbf{q}_i \qquad (2.4.3)$$

mechanical energy

$$\Big[\Big[\quad \Big[\rho_k\, \frac{v_k^2}{2}\, (\mathbf{v}_k - \mathbf{v}_i) - \underline{\mathbf{T}}_k \cdot \mathbf{v}_k\Big] \cdot \mathbf{n}_k \quad \Big]\Big] = e_i^{\sigma} \qquad (2.4.4)$$

internal energy

$$\Big[\Big[\quad \Big[\rho_k\, \mathrm{u}_k\, (\mathbf{v}_k - \mathbf{v}_i) + \mathbf{q}_k\Big] \cdot \mathbf{n}_k \quad \Big]\Big] = q_i \qquad (2.4.5)$$

enthalpy

$$\Big[\Big[\quad \Big[\rho_k\, h_k\, (\mathbf{v}_k - \mathbf{v}_i) + \mathbf{q}_k\Big] \cdot \mathbf{n}_k \quad \Big]\Big] = q_i \qquad (2.4.6)$$

entropy

$$\Big[\Big[\quad \Big[\rho_k\, s_k\, (\mathbf{v}_k - \mathbf{v}_i) + \frac{\mathbf{q}_k}{T}\Big] \cdot \mathbf{n}_k \quad \Big]\Big] = q_{si} \geq 0\,. \qquad (2.4.7)$$

TABLE II, continued

3. General form of averaged conservation equations and jump conditions

3.1 Classification of averaging procedures

Let us consider a given function of some independent variables, for instance time t and spatial coordinates x. According to which of the independent variables the definition of average is applied, averaging procedures can be classified into three main classes, namely Eulerian, Lagrangian and Boltzmann statistical averaging.

i) Eulerian averaging

Let f be a function of time t and of the spatial coordinates $\mathbf{x} = (x_1, x_2, x_3)$. We define the following averages:

Time average

$$\bar{f} = \frac{1}{\Delta t} \int_t^{t+\Delta t} f(t, \mathbf{x})\, dt \qquad [3.1.1]$$

Volume average

$$< f >_3 = \frac{1}{\Delta V} \int_{\Delta V} f(t, \mathbf{x})\, dV \qquad [3.1.2]$$

Area average

$$< f >_2 = \frac{1}{\Delta A} \int_{\Delta A} f(t, \mathbf{x})\, dA \qquad [3.1.3]$$

Line average

$$< f >_1 = \frac{1}{\Delta C} \int_{\Delta C} f(t, \mathbf{x})\, dC \qquad [3.1.4]$$

Statistical mean value or ensemble averaging

$$\bar{f}^e = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f_n(t, \mathbf{x}) \qquad [3.1.5]$$

In the case of time averaging, the time interval Δt must be carefully chosen, large enough compared to the time scale of turbulent fluctuations, and small enough compared to the scale of overall flow fluctuations for transient flow conditions. For steady-state flow conditions, we may principally average for $\Delta t \to \infty$ [i.e. $\bar{f} \to \lim_{\Delta t \to \infty} f(t)$]. In practice, however, the limit in the definitions of time and ensemble averages are replaced by a finite time interval and a finite set of functions f_n, respectively. The Eulerian time average represents the local viewpoint, by holding the space coordinates fixed. For instance, if f represents the concentration of particles (bubbles) transported by the fluid, f represents its average, over the time interval Δt, at a given space point.

The concept of ensemble averaging deserves some clarification as to how the set of functions f_n are defined. The function f may be considered as a depending variable undergoing (turbulent) fluctuations. Then f_n can be defined as a time average of the fluctuations with amplitudes lying in a given range. The arithmetic mean of these samples f_n gives the ensemble average. However, different definitions of the set $\{f_n\}$ are possible.

If f is also function of some parameter ω, a generalization of the definition of discrete averaging (3.1.5) to the continuous case is

$$\bar{f}^e = \int_\Omega f(t, \mathbf{x}, \omega)\, dm(\omega) \qquad [3.1.6]$$

where $m(\omega)$ is the probability of obtaining the value (specimen) $f(t, \mathbf{x}, \omega)$ in the continuous spectrum of possible values within the set Ω of all observations.

Weighted space averages can be defined over a subspace ΔR ($= \Delta V, \Delta A, \Delta C$) by

$$< f >_{3g} = \int_{\Delta R} g(\mathbf{x}) f(t, \mathbf{x})\, dR \qquad [3.1.7]$$

where the weight function $g(\mathbf{x})$ must satisfy the condition

$$\int_{\Delta R} g(\mathbf{x})\; dR = 1 \;. \qquad [3.1.8]$$

Note that with the change of variable $t = t' - \tau$ ($0 \le \tau \le T \equiv \Delta t$), $dt = -d\tau$, the definition (3.1.1) can be written

$$\bar{f} = -\frac{1}{T} \int_{t'-T}^{t'} f(t' - \tau, \mathbf{x})\, d\tau \qquad [3.1.9]$$

Similarly, the definition

$$< f >_1 = \frac{1}{\Delta X} \int_x^{x + \Delta X} f(t, x)\, dx \qquad [3.1.10]$$

with the change of variable $x = x' - \kappa$ ($0 \le \kappa \le \Delta X$), $dx = -d\kappa$, becomes

$$< f >_1 = -\frac{1}{\Delta X} \int_{x' - \Delta X}^{x'} f(t, x' - \kappa)\, d\kappa \,. \qquad [3.1.11]$$

ii) Lagrangian averaging

Let f be a function of time t and of the generalized coordinates $X = X(\mathbf{x}, t)$. We consider the following averages:

Time average

$$\bar{f} = \frac{1}{\Delta t} \int_{t}^{t+\Delta t} f(t, X)\, dt \; ; \qquad [3.1.12]$$

Statistical mean value or ensemble averaging

$$\bar{f}^{e} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f_n (t, X) \; . \qquad [3.1.13]$$

The Lagrangian time average represents the mean value of f in a fluid element considered following its motion over the time interval Δt.

iii) Boltzmann statistical averaging

Let $\psi = \psi$ (t, x, v) be a "density function" of time, space coordinates x and velocity v of particles whose distribution is described and let $f = f(t, \mathbf{x}, \mathbf{v})$ be some property of these particles. The Boltzmann statistical average is defined by

$$\bar{f} = \frac{\int f(t, \mathbf{x}, \mathbf{v})\, \psi(t, \mathbf{x}, \mathbf{v})\, d\mathbf{v}}{\int \psi(t, \mathbf{x}, \mathbf{v})\, d\mathbf{v}} \; . \qquad [3.1.14]$$

In the following we consider the Eulerian volume average, unless otherwise specified.

The above defined averages, denoted in general by the symbol < >, satisfy the following conditions

$$< f + g > \; = \; < f > + < g > \qquad [3.1.15]$$

$$< < f > g > \; = \; < f > \; < g > \qquad [3.1.16]$$

$$< \frac{\partial f}{\partial t} > \; = \; \frac{\partial}{\partial t} < f > \qquad [3.1.17]$$

$$< \nabla f > \; = \; \nabla < f > \qquad [3.1.18]$$

The last two relations are justified by the theorem of derivation under the integral sign. This states that, given a function of the parameter y defined by

$$F(y) = \int_a^b f(x, y)\, dx, \qquad [3.1.19]$$

with the constant limits *a*, *b*, its derivative is

$$F'(y) = \frac{d}{dy} F(y) = \int_a^b \frac{\partial f(x, y)}{\partial y} dx = \int_a^b f_y'(x, y)\, dx \qquad [3.1.20]$$

Applying this theorem to the definition of mean value in the form (3.1.9)

$$\bar{f}(t) = -\frac{1}{T} \int_{t-T}^{t} f(t-\tau)\, d\tau \qquad [3.1.21]$$

yields

$$\frac{\partial \bar{f}(t)}{\partial t} = -\frac{1}{T} \int_{t-T}^{t} \frac{\partial f(t-\tau)}{\partial t} d\tau = \frac{\overline{\partial f}}{\partial t} \qquad [3.1.22]$$

which is relation (3.1.17).

Similarly, taking the derivative of the function

$$\bar{f}(x) = -\frac{1}{\Delta X} \int_{x-\Delta X}^{x} f(t, x-x')\, dx' \qquad [3.1.23]$$

yields

$$\frac{\partial \bar{f}(x)}{\partial x} = -\frac{1}{\Delta X} \int_{x-\Delta X}^{x} \frac{\partial f(x-x')}{\partial x} dx' = \frac{\overline{\partial f(x)}}{\partial x}\,. \qquad [3.1.24]$$

This holds for every coordinate direction, thus (3.1.18) is proved.

3.2 The phase indicator function X_k as a generalized function

3.2.1 General properties of the generalized function X_k

In the process of averaging the conservation equations of each phase over a control volume difficulties arise because of the phase interfaces within the volume. To cope with these difficulties a phase indicator function X_k is defined with value 1 or 0 according to whether the point x considered lays within phase *k* or not:

$$X_k(\mathbf{x}, t) = \begin{cases} 0 & \mathbf{x} \text{ not in phase } k \\ 1 & \mathbf{x} \text{ in phase } k \end{cases}$$

From a mathematical viewpoint the function X_k is dealt with as a generalized function (see ref. [5]) with the following properties with respect to its differentiation. Let $\Phi(x, t)$ be a test function which vanishes at infinity ($x \to \pm\infty$ and $t \to \pm\infty$), and otherwise arbitrary. Then the derivatives $\partial X_k/\partial t$ and $\partial X_k/\partial x_j$ are defined by:

$$\int_V \int_T \frac{\partial X_k(\mathbf{x}, t)}{\partial t} \Phi(\mathbf{x}, t)\, d\mathbf{x}\, dt = -\int_V \int_T X_k(\mathbf{x}, t) \frac{\partial}{\partial t} \Phi(\mathbf{x}, t)\, d\mathbf{x}\, dt , \qquad [3.2.1]$$

$$\int_V \int_T \frac{\partial X_k(\mathbf{x}, t)}{\partial x_j} \Phi(\mathbf{x}, t)\, d\mathbf{x}\, dt = -\int_V \int_T X_k(\mathbf{x}, t) \frac{\partial \Phi(\mathbf{x}, t)}{\partial x_j}\, d\mathbf{x}\, dt . \qquad [3.2.2]$$

If $f(\mathbf{x}, t)$ is a smooth function, except at the phase interface, then the products $f \cdot (\partial X_k/\partial_t)$ and $f \cdot (\partial X_k/\partial x_j)$ are defined by:

$$\int_V \int_T f \frac{\partial X_k(\mathbf{x}, t)}{\partial t} \Phi(\mathbf{x}, t)\, d\mathbf{x}\, dt = -\int_V \int_T X_k(\mathbf{x}, t) \frac{\partial}{\partial t} \left[f\Phi(\mathbf{x}, t) \right] d\mathbf{x}\, dt , \qquad [3.2.3]$$

$$\int_V \int_T f \frac{\partial X_k(\mathbf{x}, t)}{\partial x_j} \Phi(\mathbf{x}, t)\, d\mathbf{x}\, dt = -\int_V \int_T X_k(\mathbf{x}, t) \frac{\partial}{\partial x_j} \left[f\Phi(\mathbf{x}, t) \right] d\mathbf{x}\, dt . \qquad [3.2.4]$$

According to the definition of X_k, one has for any given function $f(x, t)$

$$\int_V \int_T X_k(\mathbf{x}, t)\, f(\mathbf{x}, t)\, d\mathbf{x}\, dt = \int_{V_k} \int_T f(\mathbf{x}, t)\, d\mathbf{x}\, dt \qquad [3.2.5]$$

where V_k is the subset of V where $X_k = 1$.

3.2.2 Generalization of relation $\overline{\nabla f} = \nabla \bar{f}$

By means of the phase indicator function X_k we derive first a generalization of the relation (3.1.18), $\overline{\nabla f} = \nabla \bar{f}$, which holds only in absence of phase interfaces (i.e. within each phase). When both phases are present within a control volume, we define the phase averages of a function f and of its gradient by:

$$\bar{f} = \overline{X_1 f + X_2 f} \qquad [3.2.2.1]$$

$$\overline{\nabla f} = \overline{X_1 \nabla f + X_2 \nabla f} = \overline{X_1 \nabla f} + \overline{X_2 \nabla f} \qquad [3.2.2.2]$$

From the identity

$$\overline{\nabla\left(X_k f\right)} = \overline{X_k \nabla f + f \nabla X_k} = \overline{X_k \nabla f} + \overline{f \nabla X_k} \qquad [3.2.2.3]$$

one derives, applying (3.1.18) to each phase

$$\overline{X_k \nabla f} = \overline{\nabla\left(X_k f\right)} - \overline{f \nabla X_k} = \nabla\left(\overline{X_k f}\right) - \overline{f \nabla X_k} \ . \qquad [3.2.2.4]$$

Hence, from (3.2.2.2) one obtains

$$\overline{\nabla f} = \nabla\left(\overline{X_1 f}\right) - \overline{f \nabla X_1} + \nabla\left(\overline{X_2 f}\right) - \overline{f \nabla X_2} =$$

$$= \nabla\left(\overline{X_1 f + X_2 f}\right) - \overline{\left(f \nabla X_1 + f \nabla X_2\right)} \qquad [3.2.2.5]$$

$$\overline{\nabla f} = \nabla \bar{f} - \overline{\left\{\left[f\right]\right\} \nabla X_k}$$

where the jump $f_{1i} - f_{2i}$ of the function f across the interface i statisfies the relation

$$f \nabla X_1 + f \nabla X_2 = f_{1i} \frac{\partial X_1}{\partial n_1} + f_{2i} \frac{\partial X_2}{\partial n_2} \qquad (\mathbf{n}_1 = \mathbf{n}, \quad \mathbf{n}_2 = -\mathbf{n})$$

$$= f_{1i} \frac{\partial X_1}{\partial n_1} - f_{2i} \frac{\partial X_2}{\partial n_1} \equiv \left\{\left[f\right]\right\} \nabla X_k \ \textit{(by definition)} \ . \qquad [3.2.2.6]$$

Thus, the relation (3.2.2.5) with the definition (3.2.2.6) is a generalization of (3.1.18).

Similar relationships hold for a vector field. Let a be a vector defined in the region occupied by both phases. We define the volume averages

$$\bar{\mathbf{a}} = \overline{X_1 \mathbf{a} + X_2 \mathbf{a}} \qquad [3.2.2.7]$$

$$\overline{\nabla \cdot \mathbf{a}} = \overline{X_1 \nabla \cdot \mathbf{a} + X_2 \nabla \cdot \mathbf{a}} = \overline{X_1 \nabla \cdot \mathbf{a}} + \overline{X_2 \nabla \cdot \mathbf{a}} \qquad [3.2.2.8]$$

From the identity

$$\overline{\nabla\left(X_k \cdot \mathbf{a}\right)} = \overline{X_k \nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla X_k} = \overline{X_k \nabla \cdot \mathbf{a}} + \overline{\mathbf{a} \cdot \nabla X_k} \qquad [3.2.2.9]$$

one derives, applying (3.1.18) to each phase

$$\overline{X_k \nabla \cdot \mathbf{a}} = \overline{\nabla \cdot X_k \mathbf{a}} - \overline{\mathbf{a} \cdot \nabla X_k} = \nabla \cdot \overline{X_k \mathbf{a}} - \overline{\mathbf{a} \cdot \nabla X_k} \qquad [3.2.2.10]$$

Hence, from (3.2.2.8)

$$\overline{\nabla \cdot \mathbf{a}} = \nabla \cdot \overline{X_1 \mathbf{a}} - \overline{\mathbf{a} \cdot \nabla X_1} + \nabla \cdot \overline{X_2 \mathbf{a}} - \overline{\mathbf{a} \cdot \nabla X_2}$$

$$= \nabla \cdot \overline{\left(X_1 \mathbf{a} + X_2 \mathbf{a}\right)} - \overline{\left(\mathbf{a} \cdot \nabla X_1 + \mathbf{a} \cdot \nabla X_2\right)} \qquad [3.2.2.11]$$

$$\overline{\nabla \cdot \mathbf{a}} = \nabla \cdot \bar{\mathbf{a}} - \overline{\left\{\left\{\mathbf{a}\right\}\right\} \cdot \nabla X_k}$$

with the definition of the jump condition

$$\overline{\left\{\left\{\mathbf{a}\right\}\right\} \nabla X_k} \equiv \overline{\left(\mathbf{a} \cdot \nabla X_1 + \mathbf{a} \cdot \nabla X_2\right)} . \qquad [3.2.2.12]$$

3.2.3 Properties of the gradient ∇X_k of the phase function

We show next that the term $f\nabla X_k$ is a measure of the surface average of the function f over the phase interface, evaluated on the side of the phase k. Using (3.2.4) and (3.2.5) one has

$$\int_V \int_T \Phi(\mathbf{x}, t)\, f\, \nabla X_k \, d\mathbf{x}\, dt = -\int_V \int_T X_k \nabla \left(\Phi(\mathbf{x}, t) f\right) d\mathbf{x}\, dt$$

$$= -\int_{V_k} \int_T \nabla \left(\Phi(\mathbf{x}, t) f\right) d\mathbf{x}\, dt . \qquad [3.2.3.1]$$

Applying the Gauss-Ostrogradskii theorem for the gradient of a scalar g

$$\int_V \nabla g \, dV = \int_S \mathbf{n}\, g\, dS \qquad [3.2.3.2]$$

where $\mathbf{n}$ is the outwardly directed unit vector, to the above integral yields:

$$\int_V \int_T \Phi(\mathbf{x},t)\, f\, \nabla X_k\, d\mathbf{x}\, dt = -\int_S \int_T \Phi(\mathbf{x},t)\, f_{ki}\, \mathbf{n}\, dS\, dt \qquad [3.2.3.3]$$

where f_{ki} is the limiting value of f at the side of the interface occupied by phase k. In particular for $f \equiv 1$ one has

$$\int_V \int_T \Phi(\mathbf{x},t)\, \nabla X_k\, d\mathbf{x}\, dt = -\int_S \int_T \Phi(\mathbf{x},t)\, \mathbf{n}\, dS\, dt = -\int_S \int_T \Phi(\mathbf{x},t)\, \mathbf{n}_k\, dS\, dt \qquad [3.2.3.4]$$

with $\mathbf{n}_k$ directed outwards of phase k as shown in the sketch of section 2. ∇X_k is zero everywhere, except at the interface. It therefore gives a measure of the area of the interface, as justified by the surface integral in (3.2.3.4).

Moreover one has

$$\nabla X_k = \left(\mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z}\right) X_k \qquad [3.2.3.5]$$

$$\nabla X_k \cdot \mathbf{n} = \frac{\partial X_k}{\partial x}\, \mathbf{e}_x \cdot \mathbf{n} + \frac{\partial X_k}{\partial y}\, \mathbf{e}_y \cdot \mathbf{n} + \frac{\partial X_k}{\partial z}\, \mathbf{e}_z \cdot \mathbf{n} \equiv \frac{\partial X_k}{\partial n} \qquad [3.2.3.6]$$

where $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ are the unit vectors in the x, y, z directions, respectively. Because ∇X_k is directed along $\mathbf{n}_k$, we can write

$$\nabla X_k = \mathbf{n}_k \frac{\partial X_k}{\partial n_k} = \mathbf{n}_k\, \delta(\mathbf{x} - \mathbf{x}_i) \qquad [3.2.3.7]$$

where δ is the Dirac delta function.

$\partial X_k / \partial n$ is itself a scalar valued generalized function for which, similarly to (3.2.2), the following holds

$$\int_V \int_T \frac{\partial X_k(\mathbf{x},t)}{\partial n} \Phi(\mathbf{x},t)\, d\mathbf{x}\, dt = -\int_V \int_T X_k(\mathbf{x},t) \frac{\partial \Phi(\mathbf{x},t)}{\partial n}\, d\mathbf{x}\, dt$$

$$= -\int_{V_k} \int_T \frac{\partial}{\partial n} \Phi(\mathbf{x},t)\, d\mathbf{x}\, dt \qquad [3.2.3.8]$$

$$= -\int_S \int_T \Phi(\mathbf{x},t)\, \mathbf{n}\, dS\, dt\ ,$$

which yields again (3.2.3.4). Thus, it is justified to consider the mean value of $\partial X_k/\partial n$ over the interface as a measure of the area of the interface, and define

$$\overline{\frac{\partial X_k}{\partial n_k}} \equiv \overline{a_i} \qquad [3.2.3.9]$$

with a_i as the average interfacial area per unit volume (m2/m3).

We remark furthermore that $\nabla X_1 = -\nabla X_2$, hence from (3.2.3.7) we obtain

$$\nabla X_1 = \mathbf{n}_1 \frac{\partial X_1}{\partial n_1} = -\mathbf{n}_2 \frac{\partial X_2}{\partial n_2} = -\nabla X_2 \qquad [3.2.3.10]$$

Letting

$$\overline{\frac{\partial X_1}{\partial n_1}} = \overline{a_i} = \overline{\frac{\partial X_2}{\partial n_2}} \qquad [3.2.3.11]$$

it holds

$$\overline{\nabla X_1} = \overline{a_i}\, \mathbf{n}_1 ,$$
$$\overline{\nabla X_2} = \overline{a_i}\, \mathbf{n}_2 . \qquad [3.2.3.12]$$

Let $\mathbf{v}_i$ be the velocity of the phase interface. From

$$\mathbf{v}_i \cdot \mathbf{n}_2 = -\mathbf{v}_i \cdot \mathbf{n}_1 \qquad [3.2.3.13]$$

one derives

$$\mathbf{v}_i \cdot \frac{1}{\overline{a_i}} \overline{\nabla X_2} = -\mathbf{v}_i \cdot \frac{1}{\overline{a_i}} \overline{\nabla X_1} , \qquad [3.2.3.14]$$

$$\mathbf{v}_i \cdot \overline{\nabla X_1} + \mathbf{v}_i \cdot \overline{\nabla X_2} = 0 . \qquad [3.2.3.15]$$

Multiplying (3.2.3.15) by $\rho\psi$ one has

$$\rho_1 \psi_1 \mathbf{v}_i \cdot \overline{\nabla X_1} + \rho_2 \psi_2 \mathbf{v}_i \cdot \overline{\nabla X_2} = 0, \qquad [3.2.3.16]$$

because

$$\rho \psi \mathbf{v}_i \cdot \overline{\nabla X_k} = \rho_k \psi_k \mathbf{v}_i \cdot \overline{\nabla X_k} \qquad (k = 1, 2). \qquad [3.2.3.17]$$

3.2.4 Proof of the identity $DX_k/Dt = 0$

Applying the definitions of derivatives (3.2.1) and (3.2.4) and spreading the time domain to infinity one has

$$I = \int_V \int_T \frac{DX_k}{Dt} \Phi(\mathbf{x}, t) \, d\mathbf{x} \, dt = \int_V \int_T \left(\frac{\partial X_k}{\partial t} + \mathbf{v}_i \cdot \nabla X_k \right) \Phi(\mathbf{x}, t) \, d\mathbf{x} \, dt$$

$$= - \int_V \int_T X_k(\mathbf{x}, t) \left\{ \frac{\partial \Phi(\mathbf{x}, t)}{\partial t} + \frac{\partial \left[(v_i)_j \, \Phi(\mathbf{x}\ t) \right]}{\partial x_j} \right\} d\mathbf{x} \, dt$$

$$= - \int_{-\infty}^{+\infty} \int_{V_k} \left[\frac{\partial \Phi(\mathbf{x}, t)}{\partial t} + \Phi(\mathbf{x}, t)(\nabla \cdot \mathbf{v}_i) + \mathbf{v}_i \cdot \nabla \Phi(\mathbf{x}, t) \right] d\mathbf{x} \, dt$$

[3.2.4.1]

$$= - \int_{-\infty}^{+\infty} \int_{V_k} \frac{D\Phi(\mathbf{x}, t)}{Dt} \, d\mathbf{x} \, dt - \int_{-\infty}^{+\infty} \int_{V_k} \Phi(\mathbf{x}, t) \left(\nabla \cdot \mathbf{v}_i \right) d\mathbf{x} \, dt \, .$$

Because $\nabla \cdot \mathbf{v}_i = 0$ there remains

$$I = - \int_{-\infty}^{+\infty} \left[\frac{D}{Dt} \left(\int_{V_k} \Phi(\mathbf{x}, t) \, d\mathbf{x} \right) \right] dt = \left[\int_{V_k} \Phi(\mathbf{x}, t) \, d\mathbf{x} \right]_{-\infty}^{+\infty} = 0 \qquad [3.2.4.2]$$

because of the hypothesis that Φ (x, t) vanishes at infinity. Being the test function Φ (x, t) otherwise arbitrary one has

$$\frac{DX_k}{Dt} = \frac{\partial X_k}{\partial t} + \mathbf{v}_i \cdot \nabla X_k = 0 \, . \qquad [3.2.4.3]$$

3.3 General form of the averaged conservation equation for phase *k*

The average conservation equation for phase *k* is obtained formally by multiplying each term of equation (2.1.1) by X_k and taking the average over a control volume. Thus one derives:

$$\frac{\partial}{\partial t}\overline{X_k\,\rho\psi} + \nabla\cdot\overline{X_k\,\rho\psi\mathbf{v}} - \nabla\cdot\overline{X_k\,\mathbf{J}} - \overline{X_k\,\rho f} =$$

$$= \overline{\rho\psi\frac{\partial X_k}{\partial t}} + \overline{\rho\psi\mathbf{v}\cdot\nabla X_k} - \overline{J\cdot\nabla X_k} + \qquad [3.3.1a]$$

$$+ \overline{X_k\frac{\partial}{\partial t}\rho\psi} + \overline{X_k\,\nabla\cdot\rho\psi\mathbf{v}} - \overline{X_k\nabla\cdot J} - \overline{X_k\rho f}\,.$$

The last four terms are identically zero because of equation (2.1.1). We have also applied the relation $\nabla\cdot\overline{a} = \overline{\nabla\cdot a}$ which holds within each phase. Multiplying equation (3.2.4.3) by $\rho\,\psi$ taking the average and substracting from the right side of equation (3.3.1a) yields

$$\frac{\partial}{\partial t}\overline{X_k\,\rho\psi} + \nabla\cdot\overline{X_k\rho\psi\mathbf{v}} - \nabla\cdot\overline{X_k\,J} - \overline{X_k\rho f} =$$

$$= \overline{\rho\psi\frac{\partial X_k}{\partial t}} + \overline{\rho\psi\mathbf{v}\cdot\nabla X_k} - \overline{J\cdot\nabla X_k} - \overline{\rho\psi\frac{\partial X_k}{\partial t}} - \overline{\rho\psi\mathbf{v}_i\cdot\nabla X_k} = \qquad [3.3.1b]$$

$$= \overline{\rho\psi\left(\mathbf{v}-\mathbf{v}_i\right)\cdot\nabla X_k} - \overline{J\cdot\nabla X_k}\,.$$

Thus equation (3.3.1a) becomes:

$$\frac{\partial}{\partial t}\overline{X_k\,\rho\psi} + \nabla\cdot\overline{X_k\,\rho\psi\mathbf{v}} - \nabla\cdot\overline{X_k\,J} - \overline{X_k\,\rho f} = \qquad [3.3.2]$$

$$= \overline{\left[\rho\psi\left(\mathbf{v}-\mathbf{v}_i\right)_k - \mathbf{J}\right]\cdot\nabla X_k}\,.$$

which is the basic equation sought. Remark that while the sum of terms

$$\frac{\partial}{\partial t}(\rho\,\psi) + \nabla\cdot(\rho\,\psi\,\mathbf{v}) - \nabla\cdot\mathbf{J} - \rho\,f$$

is identically zero within each phase, the left hand side of equation (3.3.1b) is not zero in general. In fact it equals the interfacial source of ψ due to two contributions:

i) due to the phase change, if

$$\rho\,\psi\left(\mathbf{v}-\mathbf{v}_i\right)\cdot\nabla X_k \;=\; \rho\,\psi\left(\mathbf{v}-\mathbf{v}_i\right)\cdot\mathbf{n}_k\frac{\partial X_k}{\partial n}\neq 0$$

ii) due to the diffusive flux $-J\cdot\nabla X_k$ across the interface.

Writing equation (3.3.2) for both phases and summing one obtains

$$\frac{\partial}{\partial t}\overline{X_1\,\rho_1\psi_1}+\nabla\cdot\overline{X_1\,\rho_1\psi_1\mathbf{v}_1}-\nabla\cdot\overline{X_1\,J_1}-\overline{X_1\,\rho_1 f_1}+$$

$$+\frac{\partial}{\partial t}\overline{X_2\,\rho_2\psi_2}+\nabla\cdot\overline{X_2\,\rho_2\psi_2\mathbf{v}_2}-\nabla\cdot\overline{X_2\,J_2}-\overline{X_2\,\rho_2 f_2}= \qquad [3.3.3]$$

$$=\overline{\left[\rho_1\,\psi_1\left(\mathbf{v}_1-\mathbf{v}_i\right)-\mathbf{J}_1\right]\cdot\nabla X_1}+\overline{\left[\rho_2\,\psi_2\left(\mathbf{v}_2-\mathbf{v}_i\right)-\mathbf{J}_2\right]\cdot\nabla X_2}\;.$$

Equation (3.3.3) can also be obtained by means of the following alternative way. Let us consider the average of each term of equation (2.1.1) taken over a volume containing both phases, then apply equation (3.2.2.11) to each vector term:

$$\frac{\partial}{\partial t}\overline{(\rho\,\psi)}+\overline{\nabla\cdot(\rho\,\psi\,\mathbf{v})}-\overline{\nabla\cdot J}-\overline{\rho f}=0 \qquad [3.3.4]$$

$$\frac{\partial\,\overline{(\rho\,\psi)}}{\partial t}+\nabla\cdot\overline{(\rho\,\psi\,\mathbf{v})}-\overline{\left\{\left\{\rho\,\psi\,\mathbf{v}\right\}\right\}\cdot\nabla X_k}-\nabla\cdot\overline{J}+\overline{\left\{\left\{J\right\}\right\}\cdot\nabla X_k}-\overline{\rho f}=0. \qquad [3.3.5]$$

Applying the definitions of averages one has:

$$\frac{\partial}{\partial t}\overline{\left(X_1\,\rho_1\psi_1+X_2\,\rho_2\psi_2\right)}+\nabla\cdot\overline{\left(X_1\,\rho_1\psi_1\,\mathbf{v}_1+X_2\,\rho_2\psi_2\mathbf{v}_2\right)}-\nabla\cdot\overline{\left(X_1 J_1+X_2 J_2\right)}-$$

$$[3.3.6]$$

$$-\overline{\left(X_1\,\rho_1 f_1+X_2\,\rho_2\,f_2\right)}=\overline{\left\{\left\{\rho\,\psi\,\mathbf{v}\right\}\right\}\cdot\nabla X_k}-\overline{\left\{\left\{J\right\}\right\}\cdot\nabla X_k}\;.$$

Using the definition (3.2.2.12) the right hand side (r.h.s.) becomes:

$$(r.h.s.) = \overline{\left\{\left[\rho\,\psi\,\mathbf{v}\right]\right\}\cdot\nabla X_k} - \overline{\left\{\left[J\right]\right\}\cdot\nabla X_k} =$$

$$= \overline{\left(\rho_1\psi_1\,\mathbf{v}_1\cdot\nabla X_1 + \rho_2\psi_2\mathbf{v}_2\cdot\nabla X_2\right)} - \overline{\left(J_1\cdot\nabla X_1 + J_2\cdot\nabla X_2\right)}\,. \qquad [3.3.7]$$

Recalling (3.2.3.16), equation (3.3.7) becomes

$$(r.h.s.) = \overline{\left[\rho_1\,\psi_1\left(\mathbf{v}_1 - \mathbf{v}_i\right)\cdot\nabla X_1} + \overline{\rho_2\,\psi_2\left(\mathbf{v}_2 - \mathbf{v}_i\right)\cdot\nabla X_2} - \overline{\left(J_1\cdot\nabla X_1 + J_2\cdot\nabla X_2\right)} = \qquad [3.3.8]$$

$$= \overline{\left[\rho_1\,\psi_1\left(\mathbf{v}_1 - \mathbf{v}_i\right) - J_1\right]\cdot\nabla X_1} + \overline{\left[\rho_2\,\psi_2\left(\mathbf{v}_2 - \mathbf{v}_i\right) - J_2\right]\cdot\nabla X_2}\,,$$

and therefore we find that equation (3.3.6) coincides with equation (3.3.3).

3.4 General form of the averaged jump conditions

The averaged jump condition is obtained in its general form by multiplying equation (2.2.2) by $\partial X_k/\partial n_k$ and averaging. Thus one obtains:

$$\overline{\left[\left[\left(\rho_k\,\psi_k\,(\mathbf{v}_k - \mathbf{v}_i) - \mathbf{J}_k\right)\cdot\mathbf{n}_k\frac{\partial X_k}{\partial n_k}\right]\right]} = \overline{m\,\frac{\partial X_k}{\partial n_k}} \qquad [3.4.1]$$

or, using (3.2.3.7)

$$\overline{\left[\left[\left(\rho_k\,\psi_k\,(\mathbf{v}_k - \mathbf{v}_i) - \mathbf{J}_k\right)\cdot\nabla X_k\right]\right]} = \overline{\left(\mathbf{n}_k\cdot\nabla X_k\right)m}\;. \qquad [3.4.2]$$

Remark on the physical significance and on the mathematical derivation of the jump conditions (3.4.2).

Equation (3.4.2) allows for the actual calculation of the jump condition and shows its physical meaning given by the expression at the right hand side according to the values of m in Table I. The analytical derivation of section 3.3 shows - on the other hand - how the jump conditions appear explicitly in the conservation equations when these are subjected to the volume averaging process. This can be accomplished in either of two ways:

i) Multiplying each term of equation (2.1.1) by X_k and averaging,

ii) Averaging equation (2.1.1) itself and applying the formulas (3.2.2.5) and (3.2.2.11).

Both approaches yield, as shown, equation (3.3.3).

$$\overline{\nabla f} = \nabla \bar{f} - \overline{\{\{f\}\} \nabla X_k} = \nabla \bar{f} - \overline{\left(f \nabla X_1 + f \nabla X_2\right)} \quad (f \text{ scalar}) \qquad (3.2.2.5)$$

$$\overline{\nabla \cdot \mathbf{a}} = \nabla \cdot \bar{\mathbf{a}} - \overline{\{\{\mathbf{a}\}\} \nabla X_k} = \nabla \cdot \bar{\mathbf{a}} - \overline{\left(\mathbf{a} \cdot \nabla X_1 + \mathbf{a} \cdot \nabla X_2\right)} \quad (\mathbf{a} \text{ vector}) \qquad (3.2.2.11)$$

$$\nabla X_k = \mathbf{n}_k \frac{\partial X_k}{\partial n_k} = \mathbf{n}_k \, \delta\left(\mathbf{x} - \mathbf{x}_i\right) \qquad (3.2.3.7)$$

$$\overline{\frac{\partial X_k}{\partial n}} = \bar{a}_i \quad (\text{interfacial surface area}) \qquad (3.2.3.9)$$

$$\overline{\frac{\partial X_k}{\partial t}} + \mathbf{v}_i \cdot \nabla X_k = \frac{D X_k}{D t} = 0 \qquad (3.2.4.3)$$

General form of volume averaged conservation equation for phase k:

$$\frac{\partial}{\partial t}\overline{\left(X_k \rho \psi\right)} + \nabla \cdot \overline{\left(X_k \rho \psi \mathbf{v}\right)} - \nabla \cdot \overline{\left(X_k J\right)} - \overline{\left(X_k \rho f\right)} = \overline{\left[\rho \psi \left(\mathbf{v} - \mathbf{v}_i\right) - \mathbf{J}\right] \cdot \nabla X_k} \qquad (3.3.2)$$

General form of volume averaged jump condition

$$\overline{\left[\left[\left(\rho_k \psi_k \left(\mathbf{v}_k - \mathbf{v}_i\right) - \mathbf{J}_k\right) \cdot \nabla X_k\right]\right]} = \overline{\left(\mathbf{n}_k \cdot \nabla X_k\right) m} \qquad (3.4.2)$$

TABLE III - Summary of main formulas in Section 3

4. Definition of averaged variables for phase *k*

Overview

Given a physical variable Ψ (or f or J) (as a subcase $\Psi \equiv 1$) we define in this section the following classes of averages:

Class a_1 - Mass weighted averages, definded by:

$$\overline{\psi}_k^{x,\rho} = \frac{\overline{X_k \rho \psi}}{\overline{\alpha}_k \overline{\rho}_k^x}$$

Class a_2 - Mass weighted averages, definded by:

$$\overline{f}_k^{x,\rho} = \frac{\overline{X_k \rho f}}{\overline{\alpha}_k \overline{\rho}_k^x}$$

Class b_1 - Phasic (or volume weighted) averages, defined by:

$$\overline{J}_k^x = \frac{\overline{X_k J}}{\overline{\alpha}_k}$$

Class b_2 - Phasic (or volume weighted) averages of turbulent fluctuations, defined by:

$$\overline{J}_k^{Re} = \frac{\overline{X_k \rho v' \psi'}}{\overline{\alpha}_k}$$

Class c_1 - Average source of ψ due to phase change, defined by:

$$S_\psi = \overline{\rho \psi \left(v - v_i\right) \cdot \nabla X_k}$$

Class c_2 - Average source of ψ due to diffusive flux J, defined by:

$$F_k = \overline{J \cdot \nabla X_k} \quad or \quad F_k = \overline{J \, |\nabla X_k|}$$

Class d - Further turbulent correlations

Class e - Miscellaneous definitions

Class f - Comprehensive definitions

A further class of mass flux weighted averages is defined in section 6.1.3.

4.1 Class a_1 - "Mass-weighted" averages of scalar or vector Ψ in Table I

Let first

$$\overline{\alpha}_k = \overline{X}_k \qquad [4.1.1]$$

be, by definition, the volume fraction of phase *k*.

For the given physical variable Ψ, the mass weighted average is defined by

$$\overline{\psi}_k^{x,\rho} = \frac{\overline{X_k \rho \psi}}{\overline{\alpha}_k \overline{\rho}_k^x} \qquad [4.1.2]$$

with the average density defined by

$$\overline{\rho}_k^x = \frac{\overline{X_k \rho}}{\overline{\alpha}_k} \quad . \quad (kg/m^3) \qquad [4.1.3]$$

Definition (4.1.3) is obtained by (4.1.2) setting $\psi = 1$. Superscripts x, ρ denote that the average of ψ is taken over void fraction and density. Furthermore assuming for ψ the values

$$\psi = \left\{ \mathrm{v}, h, u, \frac{v^2}{2}, e = u + \frac{v^2}{2}, s \right\}$$

we obtain the definitions for the averages of:

Velocity ($\Psi = \mathrm{v}$)

$$\overline{\mathrm{v}}_k^{x,\rho} = \frac{\overline{X_k \rho \mathrm{v}}}{\overline{\alpha}_k \overline{\rho}_k^x} \qquad (m/s) \qquad [4.1.4]$$

Enthalpy ($\Psi = h$)

$$\overline{h}_k^{x,\rho} = \frac{\overline{X_k \rho h}}{\overline{\alpha}_k \overline{\rho}_k^x} \qquad (J/kg) \qquad [4.1.5]$$

Internal energy ($\Psi = u$)

$$\overline{u}_k^{x,\rho} = \frac{\overline{X_k \rho u}}{\overline{\alpha}_k \overline{\rho}_k^x} \qquad (J/kg) \qquad [4.1.6]$$

Kinetic energy ($\Psi = v^2/2$)

$$\overline{E}_{kin,k}^{x,\rho} = \frac{1}{2} \frac{\overline{X_k \rho v^2}}{\overline{\alpha}_k \overline{\rho}_k^x} \qquad (J/kg) \qquad [4.1.7]$$

Total energy ($\Psi = e = u + v^2/2$)

$$\overline{e}_k^{x,\rho} = \frac{\overline{X_k \rho e}}{\overline{\alpha}_k \overline{\rho}_k^x} \qquad (J/kg) \qquad [4.1.8]$$

Entropy ($\Psi = s$)

$$\overline{s}_k^{x,\rho} = \frac{\overline{X_k \rho s}}{\overline{\alpha}_k \overline{\rho}_k^x} \qquad (J/kg\ K) \qquad [4.1.9]$$

From the definition $e = u + v^2/2$ one has obviously

$$\bar{e}_k^{x,\rho} = \bar{u}_k^{x,\rho} + \bar{E}_{kin,k}^{x,\rho} \qquad (J/kg) \qquad [4.1.10]$$

or

$$\overline{X_k \rho e} = \overline{X_k \rho u} + \overline{X_k \rho \frac{v^2}{2}} \quad . \qquad [4.1.11]$$

4.2 Class a_2 - "Mass weighted" averages of scalar f in Table I

We apply the definition

$$\bar{f}_k^{x,\rho} = \frac{\overline{X_k \rho f}}{\bar{\alpha}_k \bar{\rho}_k^x} \qquad [4.2.1]$$

for the following cases of practical interest:

work by gravity forces

$$(f = \mathbf{g} \cdot \mathbf{v}) \qquad \mathbf{g} \cdot \bar{\mathbf{v}}_k^{x,\rho} = \frac{\overline{X_k \rho \mathbf{g} \cdot \mathbf{v}}}{\bar{\alpha}_k \bar{\rho}_k^x} \qquad (W/kg) \qquad [4.2.2]$$

energy source

$$(f = r) \qquad \bar{r}_k^{x,\rho} = \frac{\overline{X_k \rho r}}{\bar{\alpha}_k \bar{\rho}_k^x} \qquad (W/kg) \qquad [4.2.3]$$

entropy source

$$(f = r/T) \qquad \bar{\sigma}_k^{x,\rho} = \frac{\overline{X_k \rho r/T}}{\bar{\alpha}_k \bar{\rho}_k^x} \quad . \qquad (W/kg\,K) \qquad [4.2.4]$$

4.3 Class b_1 - "Phasic" averages of vector (or tensor) J in Table I

Let us apply the phasic average definition

$$\bar{J}_k^x = \frac{\overline{X_k J}}{\bar{\alpha}_k} \qquad [4.3.1]$$

to the set of physical quantities:

$$\mathbf{J} = \left\{ \underline{\mathbf{T}} = -p\underline{\mathbf{I}} + \underline{\tau},\ -p\underline{\mathbf{I}},\ \underline{\tau},\ \underline{\mathbf{T}} \cdot \mathbf{v},\ -\mathbf{q},\ -\mathbf{q}/T,\ \frac{\underline{\tau}}{T} \right\}$$

We obtain the following phasic averages:

Stress Tensor (J = $\underline{T}$)

$$\underline{\bar{T}}_k^x = \frac{\overline{X_k \underline{T}}}{\bar{\alpha}_k} \qquad (N/m^2) \qquad [4.3.2]$$

Normal stress (J = - p$\underline{I}$)

$$-\ \underline{\bar{p}}_k^x = -\ \frac{\overline{X_k p \underline{I}}}{\bar{\alpha}_k} \qquad (N/m^2) \qquad [4.3.3]$$

Shear stress (J = $\underline{\tau}$)

$$\underline{\bar{\tau}}_k^x = \frac{\overline{X_k \underline{\tau}}}{\bar{\alpha}_k} \qquad (N/m^2) \qquad [4.3.4]$$

Viscous dissipation (J = $\underline{T} \cdot$ v)

$$\bar{D}_k^x = -\ \frac{\overline{X_k \underline{T} \cdot \mathbf{v}}}{\bar{\alpha}_k} \qquad (W/m^2) \qquad [4.3.5]$$

Energy flux (J = - q)

$$-\ \bar{\mathbf{q}}_k^x = -\ \frac{\overline{X_k \mathbf{q}}}{\bar{\alpha}_k} \qquad (W/m^2) \qquad [4.3.6]$$

Entropy flux (J = - q/T)

$$-\ \bar{\Phi}_k^x = -\ \frac{\overline{X_k \mathbf{q}/T}}{\bar{\alpha}_k} \qquad (W/m^2 K) \qquad [4.3.7]$$

Entropy source due to shear stress (J = $\underline{\tau}/T$)

$$\underline{\bar{\tau}}_{sk}^x = \frac{\overline{X_k \frac{\underline{\tau}}{T}}}{\bar{\alpha}_k} \ . \qquad (N/m^2 K) \qquad [4.3.8]$$

4.4 Class b$_2$ - "Phasic" averages of turbulent fluctuations of scalar (or vector) Ψ in Table I

Let us apply the phasic average definition

$$-\ \bar{J}_k^{Re} = \frac{\overline{X_k \rho \mathbf{v}_k' \psi_k'}}{\bar{\alpha}_k} \qquad [4.4.1]$$

to the set of physical quantities, representing turbulent fluctuations:

$$\bar{\Psi}_k' = \left\{ \mathbf{v}_k',\ h_k',\ u_k',\ \frac{v_k'^2}{2},\ e_k' = u_k' + \frac{v_k'^2}{2},\ s_k',\ \frac{p'}{\rho} \right\}$$

We obtain the following phasic averages:

<u>Reynolds stress</u> $(\Psi_k' = v_k')$

$$-\overline{\underline{T}}_k^{Re} = \frac{\overline{X_k \rho \, v_k' v_k'}}{\overline{\alpha}_k} \qquad \left(\frac{J}{m^3}\right) \tag{4.4.2}$$

<u>Turbulent enthalpy flux</u> $(\Psi_k' = h_k')$

$$\hat{\overline{q}}_k^{Re} = \frac{\overline{X_k \rho \, v_k' h_k'}}{\overline{\alpha}_k} \qquad \left(\frac{W}{m^2}\right) \tag{4.4.3}$$

<u>Turbulent internal energy flux</u> $(\Psi_k' = u_k')$

$$\hat{q}_k^{Re} = \frac{\overline{X_k \rho \, v_k' u_k'}}{\overline{\alpha}_k} \qquad \left(\frac{W}{m^2}\right) \tag{4.4.4}$$

<u>Turbulent kinetic energy flux</u> $(\Psi_k' = \upsilon_k'^2/2)$

$$\overline{q}_k^{kin} = \frac{1}{2} \frac{\overline{X_k \rho \, v_k' \upsilon_k'^2}}{\overline{\alpha}_k} \qquad \left(\frac{W}{m^2}\right) \tag{4.4.5}$$

<u>Turbulent total energy flux</u> $(\Psi_k' = e_k')$

$$\overline{q}_k^{tot} = \frac{\overline{X_k \rho \, v_k' e_k'}}{\overline{\alpha}_k} \qquad \left(\frac{W}{m^2}\right) \tag{4.4.6}$$

<u>Turbulent entropy flux</u> $(\Psi_k' = s_k')$

$$\overline{\Phi}_k^{Re} = - \frac{\overline{X_k \rho \, v_k' s_k'}}{\overline{\overline{\alpha}}_k} \qquad \left(\frac{W}{m^2 \, K}\right) \tag{4.4.7}$$

<u>Turbulent velocity-pressure correlation</u> $(\Psi_k' = p'/\rho)$

$$\overline{q}_k^{pt} = \frac{\overline{X_k \rho \, v_k' \frac{p'}{\rho}}}{\overline{\alpha}_k} \, . \qquad \left(\frac{W}{m^2}\right) \tag{4.4.8}$$

Alternative definitions of Reynolds stresses can be introduced with reference to different definitions (to be precised later) of turbulent velocity fluctuations, for instance:

$$-\overline{\underline{T}}_{k,mm}^{Re} = \frac{\overline{X_k \rho \, v_{km}' v_{km}'}}{\overline{\alpha}_k} = 2 \, \overline{u}_{km}^{Re} \qquad \left(\frac{J}{m^3}\right) \tag{4.4.9}$$

where $\overline{u}^{Re}_{km}$ represents the turbulent kinetic energy (see also 4.7.1).

Remark - Justification of definition (4.4.1)

Definition (4.4.1) is consistent with the definition of turbulent fluctuations as differences between instantaneous and mean values. Let us introduce the definitions

$$\psi = \overline{\psi}_k^{x,\rho} + \psi_k' \qquad [4.4.10]$$

$$v = \overline{v}_k^{x,\rho} + v_k' \qquad [4.4.11]$$

into the mean value $X_k \rho \psi v$:

$$\overline{X_k \rho \psi v} = \overline{X_k \rho \left(\overline{\psi}_k^{x,\rho} + \psi_k' \right)\left(\overline{v}_k^{x,\rho} + v_k' \right)} =$$

$$= \overline{X_k \rho \, \overline{\psi}_k^{x,\rho} \, \overline{v}_k^{x,\rho}} + \overline{X_k \rho \, \psi_k' v_k'} + \overline{X_k \rho \, \overline{\psi}_k^{x,\rho} v_k'} + \overline{X_k \rho \, \psi_k' \, \overline{v}_k^{x,\rho}} . \qquad [4.4.12]$$

The third and fourth terms at the right side vanish because the mean values of turbulent fluctuations are zero. Using (4.1.3) one has

$$\overline{X_k \rho \psi v} = \overline{\psi_k^{x,\rho}} \; \overline{v_k^{x,\rho}} \; \overline{X_k \rho} + \overline{X_k \rho \psi_k' \, v_k'}$$

$$= \overline{\psi_k^{x,\rho}} \; \overline{v_k^{x,\rho}} \; \overline{\alpha_k} \; \overline{\rho_k^x} + \overline{X_k \rho \psi_k' \, v_k'} \qquad [4.4.13]$$

$$= \overline{\psi_k^{x,\rho}} \; \overline{v_k^{x,\rho}} \; \overline{\alpha_k} \; \overline{\rho_k^x} - \overline{\alpha_k} \; \overline{J_k^{Re}} ,$$

where we set

$$- \overline{\alpha_k} \; \overline{J_k^{Re}} = \overline{X_k \rho \psi_k' \, v_k'} ,$$

consistently with (4.4.1).

In a further class we consider the averages of the sources of ψ due to phase change (class c_1) and due to the diffusive flux J (class c_2). These are of interest for the evaluation of the second member of equation (3.3.3).

4.5 Class c_1 - Average source of ψ due to phase change

The average source of ψ due to phase change is defined by:

$$S_\psi = \left(Average\,Source\,of\;\psi \right) = \overline{\rho\,\psi\left(\mathrm{v} - \mathrm{v}_i\right) \cdot \nabla X_k}\;. \qquad [4.5.1]$$

By application of this definition to the set of physical quantities

$$\psi = \left\{ 1, \mathrm{v}, h, u, \frac{\upsilon^2}{2}, e = u + \frac{\upsilon^2}{2}, s \right\}$$

one finds the following averages:

Mass generation rate

$$\left(\psi = 1\right) \qquad \overline{\Gamma_k} = \overline{\rho\left(\mathrm{v} - \mathrm{v}_i\right) \cdot \nabla X_k} \qquad \left(\frac{kg}{m^3\,s}\right) \qquad [4.5.2]$$

Interfacial momentum flux

$$\left(\psi = \mathrm{v}\right) \qquad \overline{\mathrm{v}_{ki}}\;\overline{\Gamma_k} = \overline{\rho\,\mathrm{v}\left(\mathrm{v} - \mathrm{v}_i\right) \cdot \nabla X_k} \qquad \left(\frac{kg}{m^2\,s^2}\right) \qquad [4.5.3]$$

Interfacial enthalpy flux

$$\left(\psi = h\right) \qquad \overline{h_{ki}}\;\overline{\Gamma_k} = \overline{\rho\,h\left(\mathrm{v} - \mathrm{v}_i\right) \cdot \nabla X_k} \qquad \left(\frac{W}{m^3}\right) \qquad [4.5.4]$$

Interfacial internal energy flux

$$\left(\psi = u\right) \qquad \overline{u_{ki}}\;\overline{\Gamma_k} = \overline{\rho\,u\left(\mathrm{v} - \mathrm{v}_i\right) \cdot \nabla X_k} \qquad \left(\frac{W}{m^3}\right) \qquad [4.5.5]$$

Interfacial kinetic energy flux

$$\left(\psi = \frac{\upsilon^2}{2}\right) \qquad \frac{1}{2}\;\overline{\upsilon^2_{ki}}\;\overline{\Gamma_k} = \frac{1}{2}\;\overline{\rho\,\upsilon^2\left(\mathrm{v} - \mathrm{v}_i\right) \cdot \nabla X_k} \qquad \left(\frac{W}{m^3}\right) \qquad [4.5.6]$$

Interfacial total energy flux

$$\left(\psi = e\right) \qquad \overline{e_{ki}}\;\overline{\Gamma_k} = \overline{\rho\,e\left(\mathrm{v} - \mathrm{v}_i\right) \cdot \nabla X_k} \qquad \left(\frac{W}{m^3}\right) \qquad [4.5.7]$$

Interfacial entropy flux

$$\left(\psi = s\right) \qquad \overline{s_{ki}}\;\overline{\Gamma_k} = \overline{\rho\,s\left(\mathrm{v} - \mathrm{v}_i\right) \cdot \nabla X_k}\;. \qquad \left(\frac{W}{m^3\,K}\right) \qquad [4.5.8]$$

4.6 Class c_2 - Average source of Ψ due to the molecular flux J

The average source of Ψ due to diffusive flux is defined by

$$F_k = \overline{J \cdot \nabla X_k} \quad or \quad F_k = \overline{J \,|\, \nabla X_k|} \ . \qquad [4.6.1]$$

Applying this definition to the set of physical quantities

$$J = \left\{ \ \underline{\mathbf{T}} = -p\,\underline{\mathbf{I}} + \underline{\tau}, -p\,\underline{\mathbf{I}}, \underline{\tau}, \underline{\mathbf{T}} \cdot \ \mathbf{v}, -\mathbf{q}, -\mathbf{q}/T \ \right\}$$

one has the following averages:

Interfacial force per unit length (or momentum exchange per unit volume and time)

$$\left(J = \underline{\mathbf{T}}\right) \qquad -\overline{\mathbf{M}_k} = \overline{\underline{\mathbf{T}} \cdot \nabla X_k} \qquad \left(\frac{N}{m^3}\right) \qquad [4.6.2]$$

Interfacial pressure per unit length

$$\left(J = -p\underline{\mathbf{I}}\right) \qquad -\overline{\tilde{p}_{ki}} = -\,|\,\overline{p\underline{\mathbf{I}} \cdot \nabla X_k}\,| = -\,|\,\overline{p \ \nabla X_k}\,| \qquad \left(\frac{N}{m^3}\right) \qquad [4.6.3]$$

Interfacial shear stress per unit length

$$\left(J = \underline{\tau}\right) \qquad \overline{\tilde{\underline{\tau}}_{ki}} = \overline{\underline{\tau} \ |\nabla X_k|} \qquad \left(\frac{N}{m^3}\right) \qquad [4.6.4]$$

Interfacial work

$$\left(J = \underline{\mathbf{T}} \cdot \mathbf{v}\right) \qquad -\overline{\mathbf{W}_k} = \overline{\left(\underline{\mathbf{T}} \cdot \mathbf{v}\right) \cdot \nabla X_k} \qquad \left(\frac{W}{m^3}\right) \qquad [4.6.5]$$

Interfacial heat source

$$\left(J = -\,\mathbf{q}\right) \qquad -\overline{E_k} = \overline{-\,\mathbf{q} \cdot \nabla X_k} \qquad \left(\frac{W}{m^3}\right) \qquad [4.6.6]$$

Interfacial entropy source

$$\left(J = -\,\mathbf{q}/T\right) \qquad -\overline{s_k} = \overline{-\frac{\mathbf{q}}{T} \cdot \nabla X_k} \ . \qquad \left(\frac{W}{m^3 K}\right) \qquad [4.6.7]$$

In the following we use the mathematical identities

$$\underline{\mathbf{I}} \cdot \nabla X_k = \sum_i \sum_j \delta_i \, \delta_j \, \delta_{ij} \cdot \sum_l \delta_l \frac{\partial X_k}{\partial x_l} = \sum_i \sum_j \sum_l \delta_i \, \delta_{ij} \, \delta_{jl} \frac{\partial X_k}{\partial x_l} =$$

$$= \sum_i \sum_j \delta_i \, \delta_{ij} \, \frac{\partial X_k}{\partial x_j} = \sum_i \delta_i \, \frac{\partial X_k}{\partial x_i} = \nabla X_k \, . \qquad [4.6.8]$$

Consequently,

$$p\underline{\mathbf{I}} \cdot \nabla X_k = p \, \nabla X_k \, , \qquad [4.6.9]$$

$$p\underline{\mathbf{I}} \cdot \mathbf{v} = p \, \mathbf{v} \, . \qquad [4.6.10]$$

We introduced in section 3.2.3 the concept of interfacial area density, defined by

$$\overline{a_i} = \overline{\nabla X_k} = \overline{\frac{\partial X_k}{\partial n} \mathbf{n}_k} = \overline{\mathbf{n}_k \, \delta \, (\mathbf{x} - \mathbf{x}_s)} \quad \left(\frac{m^2}{m^3}\right) \qquad [4.6.11]$$

where, in a two-phase mixture, $\mathbf{x}_s$ is a point on the interface between the phases, with normal vektor $\mathbf{n}_k$ directed towards phase k. Thus, $\delta\,(\mathbf{x} - \mathbf{x}_s) = \partial X_k/\partial_n$ is different from zero only on the interface and gives a measure of its area. Using (4.6.11) we redefine the interfacial pressure and shear stress by

$$\overline{p_{ki}} = \frac{\overline{\tilde{p}_{ki}}}{\overline{a_i}} = \frac{\overline{p \nabla X_k}}{\overline{\nabla X_k}} \quad \left(\frac{N}{m^2}\right) \qquad [4.6.12]$$

$$\overline{\underline{\tau}_{ki}} = \frac{\overline{\underline{\tilde{\tau}}_{ki}}}{\overline{a_i}} = \frac{\overline{\underline{\tau} \, \nabla X_k}}{\overline{\nabla X_k}} \quad \left(\frac{N}{m^2}\right). \qquad [4.6.13]$$

4.7 Class d - Further turbulent correlations

Turbulent kinetic energy

$$\overline{u_k^{Re}} = \frac{1}{2} \, \frac{\overline{X_k \, \rho \, v_k'^2}}{\overline{\alpha_k} \, \overline{\rho_k^x}} \qquad \left(\frac{m^2}{s^2}\right) \qquad [4.7.1]$$

Turbulent pressure-velocity correlation

$$\overline{\mathbf{q}_k^p} = \frac{\overline{X_k \, p \, \mathbf{v}_k'}}{\overline{\alpha_k}} \qquad \left(\frac{W}{m^2}\right) \qquad [4.7.2]$$

Turbulent shear work

$$\overline{\mathbf{q}_k^\tau} = - \frac{\overline{X_k \underline{\tau} \cdot \mathbf{v}_k'}}{\overline{\alpha_k}} \qquad \left(\frac{W}{m^2}\right) \qquad [4.7.3]$$

Turbulent energy dissipation

$$\overline{D_k} = - \frac{\overline{X_k \underline{\tau} : \nabla v_k'}}{\overline{\alpha_k}} \qquad \left(\frac{W}{m^3}\right) \qquad [4.7.4]$$

Turbulent interfacial force

$$- \overline{\mathbf{M}_k'} = \overline{\underline{\mathbf{T}}_{ki}' \cdot \nabla X_k} \qquad \left(\frac{N}{m^3}\right) \qquad [4.7.5]$$

Turbulent interfacial work

$$- \overline{\mathbf{W}_k''} = \overline{\left(\underline{\mathbf{T}}_{ki}' \cdot \mathbf{v}\right) \cdot \nabla X_k} \qquad \left(\frac{W}{m^3}\right)$$

$$= \overline{-\left(p_{ki}' \underline{\mathbf{I}} \cdot \mathbf{v}\right) \cdot \nabla X_k + \left(\underline{\tau}_{ki}' \cdot \mathbf{v}\right) \cdot \nabla X_k} \qquad [4.7.6]$$

$$= \overline{- p_{ki}' \mathbf{v}_i \cdot \nabla X_k + \left(\underline{\tau}_{ki}' \cdot \mathbf{v}\right) \cdot \nabla X_k} \quad .$$

The above definition of turbulent interfacial work $\overline{W_k}''$ takes into account the turbulent fluctuations of both pressure (p') and shear stresses ($\underline{\tau}'$). It is however convenient to introduce another definition which takes into account the turbulent fluctuations of pressure only:

$$- \overline{\mathbf{W}_k'} = \overline{- p_{ki}' \left(\mathbf{v}_i \cdot \nabla X_k\right) + \left(\underline{\tau} \cdot \mathbf{v}\right) \cdot \nabla X_k} \quad . \qquad [4.7.7]$$

A basic relationship between the interfacial force M_k (4.6.2) and the turbulent force M_k'(4.7.5) will be derived in the following (see equation 4.10.9).

Similarly, a relationship between the interfacial work $\overline{W_k}$ (equation 4.6.5) and its turbulent counterpart $\overline{W_k}'$(equation 4.7.7) is given by equation (4.11.8).

Turbulent entropy source

$$\overline{D_{sk}} = \frac{\overline{X_k \frac{\underline{\tau}}{T} : \nabla v_k'}}{\overline{\alpha_k}} \qquad \left(\frac{W}{m^3 K}\right) . \qquad [4.7.8]$$

4.8 Class e - Miscellaneous definitions

Furthermore, the following definitions are of interest in the following:

Pressure time derivative:

$$\frac{\partial \overline{p_k^x}}{\partial t} = \frac{\overline{X_k \partial p/\partial t}}{\overline{\alpha_k}} \qquad \left(\frac{N}{m^2\, s}\right) \qquad [4.8.1]$$

Temperature

$$\overline{T_k^{x,\rho,s}} = \frac{\overline{X_k \rho s T}}{\overline{\alpha_k}\; \overline{\rho_k^x}\; \overline{s_k^{x,\rho}}} \qquad (K) \qquad [4.8.2]$$

Entropy source due to temperature gradient

$$\overline{s_{Tk}} = \overline{X_k \;\; \mathbf{q} \cdot \nabla\left(\frac{1}{T}\right)} \;. \qquad \left(\frac{W}{m^3\, K}\right) \qquad [4.8.3]$$

4.9 Class f - Comprehensive definitions

The total turbulent energy flux is defined by

$$\overline{\mathbf{q}_k^{Re}} = \widehat{\mathbf{q}}_k^{Re} + \overline{\mathbf{q}_k^{p}} + \overline{\mathbf{q}_k^{\tau}} + \overline{\mathbf{q}_k^{kin}} =$$

$$= \frac{\overline{X_k \rho\, \mathbf{v}_k'\, u_k'}}{\overline{\alpha_k}} + \frac{\overline{X_k p\, \mathbf{v}_k'}}{\overline{\alpha_k}} - \frac{\overline{X_k \underline{\tau} \cdot \mathbf{v}_k'}}{\overline{\alpha_k}} + \frac{1}{2}\frac{\overline{X_k \rho\, \mathbf{v}_k'\, v_k'^2}}{\overline{\alpha_k}} \;. \qquad [4.9.1]$$

(4.4.4)	(4.7.2)	(4.7.3)	(4.4.5)
Turbulent internal energy flux	Turbulent pressure-velocity correlation	Turbulent shear work	Turbulent kinetic energy flux

Using (4.4.3) and $\rho h'_k = \rho u'_k + p' = \rho u'_k + p - \overline{p}$, one derives:

$$\widehat{\mathbf{q}}_k^{Re} = \frac{\overline{X_k \rho\, \mathbf{v}_k'\, h_k'}}{\overline{\alpha_k}} = \frac{\overline{X_k \rho\, \mathbf{v}_k'\, u_k'}}{\overline{\alpha_k}} + \frac{\overline{X_k p\, \mathbf{v}_k'}}{\overline{\alpha_k}} = \widehat{\mathbf{q}}_k^{Re} + \overline{\mathbf{q}_k^{p}} \;, \qquad [4.9.2]$$

because $\overline{X_k \overline{p} \mathbf{v}'_k} = 0$.

Hence, from (4.9.1) we obtain

$$\overline{\mathbf{q}_k^{Re}} = \widehat{\mathbf{q}}_k^{Re} + \overline{\mathbf{q}_k^{\tau}} + \overline{\mathbf{q}_k^{kin}} \;. \qquad \left(\frac{W}{m^2}\right) \qquad [4.9.3]$$

4.10 Relationship between $\overline{\mathbf{M}_k}$ (4.6.2) and $\overline{\mathbf{M}_k'}$ (4.7.5) (instantaneous value and turbulent fluctuation of interfacial forces)

$$- \overline{\mathbf{M}_k} = \overline{\underline{\mathbf{T}} \cdot \nabla X_k} = \overline{\left(-p\,\underline{\mathbf{I}} + \underline{\tau}\right) \cdot \nabla X_k} = \overline{-p\,\underline{\mathbf{I}} \cdot \nabla X_k} + \overline{\underline{\tau} \cdot \nabla X_k} =$$

$$= \overline{-p\,\nabla X_k} + \overline{\underline{\tau} \cdot \nabla X_k} \;. \qquad [4.10.1]$$

Let

$$p = \overline{p}^x + p' \qquad [4.10.2]$$

$$\underline{\tau} = \overline{\underline{\tau}}^x + \underline{\tau}' . \qquad [4.10.3]$$

Introducing (4.10.2) and (4.10.3) into (4.10.1) one derives:

$$- \overline{\mathbf{M}_k} = - \overline{\left(\overline{p}^x + p'\right) \nabla X_k} + \overline{\left(\overline{\underline{\tau}}^x + \underline{\tau}'\right) \cdot \nabla X_k}$$

$$= - \overline{\overline{p}^x\,\nabla X_k} - \overline{p'\,\nabla X_k} + \overline{\overline{\underline{\tau}}^x \cdot \nabla X_k} + \overline{\underline{\tau}' \cdot \nabla X_k} \qquad [4.10.4]$$

The first and third terms at the right side are computed by

$$\overline{p_{ki}}\;\overline{\nabla X_k} = \overline{\overline{p}^x\,\nabla X_k} \qquad [4.10.5]$$

$$\overline{\underline{\tau}_{ki}} \cdot \overline{\nabla X_k} = \overline{\overline{\underline{\tau}}^x \cdot \nabla X_k} \;, \qquad [4.10.6]$$

where $\overline{p_{ki}}$, $\overline{\underline{\tau}_{ki}}$ have been defined by (4.6.12), (4.6.13).

Using

$$\underline{\mathbf{T}}' = -p'\,\underline{\mathbf{I}} + \underline{\tau}' \qquad [4.10.7]$$

$$\overline{\underline{\mathbf{T}}'_{ki} \cdot \nabla X_k} = \overline{-p'\,\underline{\mathbf{I}} \cdot \nabla X_k} + \overline{\underline{\tau}' \cdot \nabla X_k} = \overline{-p'\,\nabla X_k} + \overline{\underline{\tau}' \cdot \nabla X_k} \qquad [4.10.8]$$

and introducing into (4.10.4) one derives

$$-\overline{\mathbf{M}_k} = \overline{-p_{ki}} \;\; \overline{\nabla X_k} + \overline{\underline{\tau}_{ki}} \cdot \overline{\nabla X_k} + \overline{\underline{\mathbf{T}}'_{ki} \cdot \nabla X_k}$$

$$= \overline{-p_{ki}} \;\; \nabla \overline{\alpha_k} + \overline{\underline{\tau}_{ki}} \cdot \nabla \overline{\alpha_k} + \overline{\underline{\mathbf{T}}'_{ki} \cdot \nabla X_k} ,$$

hence

$$\overline{\mathbf{M}_k} = \overline{p_{ki}} \;\; \nabla \overline{\alpha_k} - \overline{\underline{\tau}_{ki}} \cdot \nabla \overline{\alpha_k} + \overline{\mathbf{M}'_k} . \qquad [4.10.9]$$

4.11 Relationship between $\overline{W_k}$ (4.6.5) and $\overline{W_k}'$(4.7.7) (Interfacial work and turbulent interfacial work)

$$-\overline{W_k} = \overline{\left(\underline{\mathbf{T}} \cdot \mathbf{v}\right) \cdot \nabla X_k} = \overline{\left(-p\,\underline{\mathbf{I}} \cdot \mathbf{v}\right) \cdot \nabla X_k} + \overline{\left(\underline{\tau} \cdot \mathbf{v}\right) \cdot \nabla X_k} =$$

$$= -\overline{p\,\mathbf{v} \cdot \nabla X_k} + \overline{\left(\underline{\tau} \cdot \mathbf{v}\right) \cdot \nabla X_k} , \qquad [4.11.1]$$

having used the identity $p\,\underline{\mathbf{I}} \cdot \mathbf{v} = p\,\mathbf{v}$. Adding and substracting $\mathbf{v}_i$ in the first term at the right side one obtaines

$$-\overline{W_k} = -\overline{p\,\mathbf{v}_i \cdot \nabla X_k} - \overline{p\left(\mathbf{v} - \mathbf{v}_i\right) \cdot \nabla X_k} + \overline{\left(\underline{\tau} \cdot \mathbf{v}\right) \cdot \nabla X_k} . \qquad [4.11.2]$$

Setting $p = \overline{p^x} + p'$ in the first term at the right side, yields

$$-\overline{W_k} = -\overline{\overline{p^x}\,\mathbf{v}_i \cdot \nabla X_k} - \overline{p'\,\mathbf{v}_i \cdot \nabla X_k} - \overline{p\left(\mathbf{v} - \mathbf{v}_i\right) \cdot \nabla X_k} + \overline{\left(\underline{\tau} \cdot \mathbf{v}\right) \cdot \nabla X_k} . \qquad [4.11.3]$$

From (4.10.5) and (4.10.6) we derive at the phase interface the following relationships:

$$\overline{p_{ki}} \; \overline{\left(\mathbf{v}_i \cdot \nabla X_k\right)} = \overline{\overline{p^x}\,\mathbf{v}_i \cdot \nabla X_k} \qquad [4.11.4]$$

$$\overline{p'_{ki}} \; \overline{\left(\mathbf{v}_i \cdot \nabla X_k\right)} = \overline{p'\,\mathbf{v}_i \cdot \nabla X_k} . \qquad [4.11.5]$$

Thus, replacing the first and second term at the right side, one has:

$$- \overline{W_k} = - \overline{p_{ki}} \overline{\left(v_i \cdot \nabla X_k \right)} - \overline{p_{ki}'} \overline{\left(v_i \cdot \nabla X_k \right)} - \overline{p \left(v - v_i \right) \cdot \nabla X_k} + \overline{\left(\underline{\tau} \cdot v \right) \cdot \nabla X_k} \,. \qquad [4.11.6]$$

Equations (3.2.4.3) and (4.1.1) yield

$$\overline{\left(v_i \cdot \nabla X_k \right)} = - \overline{\frac{\partial X_k}{\partial t}} = - \frac{\partial \overline{\alpha_k}}{\partial t} \,. \qquad [4.11.7]$$

Using this identity in the first term at the right side, and with the definition (4.7.7) of $\overline{W_k}'$one has finally:

$$\overline{W_k} = - \overline{p_{ki}} \, \frac{\partial \overline{\alpha_k}}{\partial t} + \overline{p \left(v - v_i \right) \cdot \nabla X_k} + \overline{W_k'} \,. \qquad [4.11.8]$$

The definitions of average variables introduced in section 4.1 through 4.9 are summarized in Table IV and rearranged in Table V according to the physical quantity the averages are associated to.

$$\overline{\alpha_k} = \overline{X_k}$$

Class a$_1$ - Mass weighted averages:

$$\psi = \left\{ 1, \mathbf{v}, h, u, \frac{v^2}{2}, \quad e = u + \frac{v^2}{2}, s \right\}$$

$$\overline{\psi}_k^{x,\rho} = \frac{\overline{X_k \rho \psi}}{\overline{\alpha}_k \overline{\rho}_k^{x}} \qquad \left\{ \overline{\rho}_k^{x}, \overline{\mathbf{v}}_k^{x,\rho}, \overline{h}_k^{x,\rho}, \overline{u}_k^{x,\rho}, \overline{E}_{kin,k}^{x,\rho}, \overline{e}_k^{x,\rho}, \overline{s}_k^{x,\rho} \right\}$$

Class a$_2$ - Further mass weighted averages:

$$f = \left\{ \mathbf{g} \cdot \mathbf{v}, r, r/T \right\}$$

$$\overline{f}_k^{x,\rho} = \frac{\overline{X_k \rho f}}{\overline{\alpha}_k \overline{\rho}_k^{x}} \qquad \left\{ \mathbf{g} \cdot \overline{\mathbf{v}}_k^{x,\rho}, \overline{r}_k^{x,\rho}, \overline{\sigma}_k^{x,\rho} \right\}$$

Class b$_1$ - Phasic (or volume weighted) averages:

$$J = \left\{ \underline{\mathbf{T}}, p\underline{\mathbf{I}}, \underline{\tau}, \underline{\mathbf{T}} \cdot \mathbf{v}, \mathbf{q}, \mathbf{q}/T, \frac{\underline{\tau}}{T} \right\}$$

$$\overline{J}_k^{x} = \frac{\overline{X_k J}}{\overline{\alpha}_k} \qquad \left\{ \overline{\underline{\mathbf{T}}}_k^{x}, \overline{\mathbf{p}}_k^{x}, \overline{\underline{\tau}}_k^{x}, \overline{\mathbf{D}}_k^{x}, \overline{\mathbf{q}}_k^{x}, \overline{\Phi}_k^{x}, \overline{\underline{\tau}}_{sk}^{x} \right\}$$

Class b$_2$ - Phasic (or volume weighted) averages of turbulent fluctuations:

$$\overline{\psi'}_k = \left\{ \mathbf{v}'_k, h'_k, u'_k, \frac{v'^2_k}{2}, e'_k = u'_k + \frac{v'^2_k}{2}, s'_k, \frac{p'}{\rho} \right\}$$

$$-\overline{J}_k^{Re} = \frac{\overline{X_k \rho \mathbf{v}'_k \psi'_k}}{\overline{\alpha}_k} \qquad \left\{ -\overline{\underline{\mathbf{T}}}_k^{Re}, \hat{\hat{\mathbf{q}}}_k^{Re}, \hat{\mathbf{q}}_k^{Re}, \overline{\mathbf{q}}_k^{kin}, \overline{\mathbf{q}}_k^{tot}, \overline{\Phi}_k^{Re}, \overline{\mathbf{q}}_k^{pt} \right\}$$

TABLE IV: Summary of definitions of averaged variables for phase k

Class c_1 - Average source of Ψ due to phase change:

$$\psi = \left\{ 1, \mathbf{v}, h, u, \frac{v^2}{2}, \quad e = u + \frac{v^2}{2}, s \right\}$$

$$S_\psi = \overline{\rho\,\psi\left(\mathbf{v} - \mathbf{v}_i\right)\cdot\nabla X_k} \qquad \left\{ \overline{\Gamma_k}, \overline{\mathbf{v}_{ki}}\ \overline{\Gamma_k}, \overline{h_{ki}}\ \overline{\Gamma_k}, \overline{u_{ki}}\ \overline{\Gamma_k}, \frac{1}{2}\overline{v_{ki}^2}\ \overline{\Gamma_k}, \overline{e_{ki}}\ \overline{\Gamma_k}, \overline{s_{ki}}\ \overline{\Gamma_k} \right\}$$

Class c_2 - Average source of Ψ due to diffusive flux J:

$$J = \left\{ \underline{\mathbf{T}} = -p\underline{\mathbf{I}} + \underline{\boldsymbol{\tau}} - p\underline{\mathbf{I}}, \underline{\boldsymbol{\tau}}, \underline{\mathbf{T}}\cdot\mathbf{v}, -\mathbf{q}, -\mathbf{q}/T \right\}$$

$$F_K = \overline{J\cdot\ \nabla X_k} \qquad \left\{ \overline{\mathbf{M}_k}, \overline{\tilde{p}_{ki}}, \overline{\tilde{\underline{\tau}}_{ki}}, \overline{\mathbf{W}_k}, \overline{E_k}, \overline{s_k} \right\}$$

Class d - Further turbulent correlations:

$$\left\{ \overline{u_k^{Re}}, \overline{\mathbf{q}_k^{p}}, \overline{\mathbf{q}_k^{\tau}}, \overline{D_k}, \overline{\mathbf{M}_k'}, \overline{\mathbf{W}_k''}, \overline{\mathbf{W}_k'}, \overline{D_{sk}} \right\}$$

Class e - Miscellaneous definitions:

$$\left\{ \frac{\partial \overline{p_k^x}}{\partial t}, \overline{T}_k^{x,\rho,s}, \bar{s}_{Tk} \right\}$$

Class f - Comprehensive definitions:

$$\overline{\mathbf{q}_k^{Re}} = \hat{\mathbf{q}}_k^{Re} + \overline{\mathbf{q}_k^{p}} + \overline{\mathbf{q}_k^{\tau}} + \overline{\mathbf{q}_k^{kin}}$$

$$\overline{q_k^{Re}} = \hat{\hat{q}}_k^{Re} + \overline{q_k^{\tau}} + \overline{q_k^{kin}}$$

TABLE IV - continued.

Averaged values derived from

<u>Density</u> ρ (kg/m3):

$$\overline{\rho_k^x} = \frac{\overline{X_k \rho}}{\overline{\alpha_k}} \qquad \left(\frac{kg}{m^3}\right) \qquad (4.1.3)$$

$$\overline{\Gamma_k} = \overline{\rho\,(v - v_i)\cdot \nabla X_k} \qquad \left(\frac{kg}{m^3 s}\right) \qquad (4.5.2)$$

<u>Total energy</u> *e* (J/kg)

$$\overline{e_k^{x,\rho}} = \frac{\overline{X_k \rho e}}{\overline{\alpha_k}\ \overline{\rho_k^x}} \qquad \left(\frac{J}{kg}\right) \qquad (4.1.8)$$

$$\overline{q_k^{tot}} = \frac{\overline{X_k \rho v_k' e_k'}}{\overline{\alpha_k}} \qquad \left(\frac{W}{m^2}\right) \qquad (4.4.6)$$

$$\overline{e_{ki}}\ \ \overline{\Gamma_k} = \overline{\rho e\,(v - v_i)\cdot \nabla X_k} \qquad \left(\frac{W}{m^3}\right) \qquad (4.5.7)$$

<u>Internal energy</u> *u* (J/kg)

$$\overline{u_k^{x,\rho}} = \frac{\overline{X_k \rho u}}{\overline{\alpha_k}\ \overline{\rho_k^x}} \qquad \left(\frac{J}{kg}\right) \qquad (4.1.6)$$

$$\hat{q}_k^{Re} = \frac{\overline{X_k \rho v_k' u_k'}}{\overline{\alpha_k}} \qquad \left(\frac{W}{m^2}\right) \qquad (4.4.4)$$

$$\overline{u_{ki}}\ \ \overline{\Gamma_k} = \overline{\rho u\,(v - v_i)\cdot \nabla X_k} \qquad \left(\frac{W}{m^3}\right) \qquad (4.5.5)$$

<u>Kinetic energy</u> $v^2/2$ (J/kg)

$$\overline{E_{kin,k}^{x,\rho}} = \frac{1}{2}\ \frac{\overline{X_k \rho v^2}}{\overline{\alpha_k}\ \overline{\rho_k^x}} \qquad \left(\frac{J}{kg}\right) \qquad (4.1.7)$$

TABLE V - Rearrangement of Table IV with definitions of averaged variables for phase k

$$\overline{\mathbf{q}_k^{kin}} = \frac{1}{2} \frac{\overline{X_k \rho \mathbf{v}_k' v_k'^2}}{\overline{\alpha}_k} \qquad \left(\frac{W}{m^2}\right) \qquad (4.4.5)$$

$$\frac{1}{2} \overline{v_{ki}^2} \; \overline{\Gamma}_k = \frac{1}{2} \overline{\rho v^2 (\mathbf{v} - \mathbf{v}_i) \cdot \nabla X_k} \qquad \left(\frac{W}{m^3}\right) \qquad (4.5.6)$$

Enthalpy h (J/kg)

$$\overline{h_k^{x,\rho}} = \frac{\overline{X_k \rho h}}{\overline{\alpha}_k \; \overline{\overline{\rho_k^x}}} \qquad \left(\frac{J}{kg}\right) \qquad (4.1.5)$$

$$\hat{\hat{\mathbf{q}}}_k^{Re} = \frac{\overline{X_k \rho \mathbf{v}_k' h_k'}}{\overline{\alpha}_k} \qquad \left(\frac{W}{m^2}\right) \qquad (4.4.3)$$

$$\overline{h_{ki}} \quad \overline{\Gamma}_k = \overline{\rho h (\mathbf{v} - \mathbf{v}_i) \cdot \nabla X_k} \qquad \left(\frac{W}{m^3}\right) \qquad (4.5.4)$$

Entropy s (J/kg K)

$$\overline{s_k^{x,\rho}} = \frac{\overline{X_k \rho s}}{\overline{\alpha}_k \; \overline{\overline{\rho_k^x}}} \qquad \left(\frac{J}{kg\ K}\right) \qquad (4.1.9)$$

$$\overline{\boldsymbol{\phi}_k^x} = -\frac{\overline{X_k \mathbf{q}/T}}{\overline{\overline{\alpha}}_k} \qquad \left(\frac{W}{m^2\ K}\right) \qquad (4.3.7)$$

$$\overline{\boldsymbol{\phi}_k^{Re}} = -\frac{\overline{X_k \rho \mathbf{v}_k' s_k'}}{\overline{\overline{\alpha}}_k} \qquad \left(\frac{W}{m^2\ K}\right) \qquad (4.4.7)$$

$$\overline{s_{ki}} \quad \overline{\Gamma}_k = \overline{\rho s (\mathbf{v} - \mathbf{v}_i) \cdot \nabla X_k} \qquad \left(\frac{W}{m^3\ K}\right) \qquad (4.5.8)$$

$$\overline{s}_k = \overline{\frac{\mathbf{q}}{T} \cdot \nabla X_k} \qquad \left(\frac{W}{m^3\ K}\right) \qquad (4.6.7)$$

Velocity $\mathbf{v}$ (m/s)

$$\overline{\mathbf{v}_k^{x,\rho}} = \frac{\overline{X_k \rho \overline{\mathbf{v}}}}{\overline{\alpha}_k \; \overline{\overline{\rho_k^x}}} \qquad (m/s) \qquad (4.1.4)$$

TABLE V - continued

$$\mathbf{g} \cdot \overline{\mathbf{v}_k^{x,\rho}} = \frac{\overline{X_k \rho \mathbf{g} \cdot \mathbf{v}}}{\overline{\alpha_k}\ \overline{\rho_k^x}} \qquad \left(\frac{W}{kg}\right) \qquad (4.2.2)$$

$$\overline{\underline{T}_k^{Re}} = -\frac{\overline{X_k \rho \mathbf{v}_k' \mathbf{v}_k'}}{\overline{\alpha_k}} \qquad \left(\frac{N}{m^2}\right) \qquad (4.4.2)$$

$$\overline{\underline{T}_{k,mm}^{Re}} = -\frac{\overline{X_k \rho \mathbf{v}_{km}' \mathbf{v}_{km}'}}{\overline{\alpha_k}} \qquad \left(\frac{N}{m^2}\right) \qquad (4.4.9)$$

$$\overline{\mathbf{v}_{ki}}\ \overline{\Gamma_k} = \overline{\rho \mathbf{v} (\mathbf{v} - \mathbf{v}_i) \cdot \nabla X_k} \qquad \left(\frac{kg}{m^2 s^2}\right) \qquad (4.5.3)$$

$$\overline{u_k^{Re}} = \frac{1}{2} \frac{\overline{X_k \rho v_k'^2}}{\overline{\alpha_k}\ \overline{\rho_k^x}} \qquad \left(\frac{m^2}{s^2}\right) \qquad (4.7.1)$$

$$\overline{\mathbf{q}_k^{Re}} = \hat{\mathbf{q}}_k^{Re} + \overline{\mathbf{q}_k^p} + \overline{\mathbf{q}_k^\tau} + \overline{\mathbf{q}_k^{kin}} \qquad \left(\frac{W}{m^3}\right) \qquad (4.9.1)$$

<u>Heat flux</u> q (W/m2)

$$\overline{\mathbf{q}_k^x} = \frac{\overline{X_k \mathbf{q}}}{\overline{\alpha_k}} \qquad \left(\frac{W}{m^2}\right) \qquad (4.3.6)$$

$$\overline{E_k} = \overline{\mathbf{q} \cdot \nabla X_k} \qquad \left(\frac{W}{m^3}\right) \qquad (4.6.6)$$

$$\overline{s_{Tk}} = \overline{X_k \mathbf{q} \cdot \nabla \left(\frac{1}{T}\right)} \qquad \left(\frac{W}{m^3} K\right) \qquad (4.8.3)$$

<u>Energy source</u> r (W/kg)

$$\overline{r_k^{x,\rho}} = \frac{\overline{X_k \rho r}}{\overline{\alpha_k}\ \overline{\rho_k^x}} \qquad \left(\frac{W}{kg}\right) \qquad (4.2.3)$$

$$\overline{\sigma_k^{x,\rho}} = \frac{\overline{X_k \rho r/T}}{\overline{\alpha_k}\ \overline{\rho_k^x}} \qquad \left(\frac{W}{kg\ K}\right) \qquad (4.2.4)$$

TABLE V - continued

Temperature T (K)

$$\overline{T_k^{x,\rho,s}} = \frac{\overline{X_k \rho s T}}{\overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{s_k^{x,\rho}}} \qquad \left(K\right) \qquad (4.8.2)$$

Stress tensor $\underline{\mathrm{T}}$ (N/m2)

$$\overline{\underline{\mathrm{T}}_k^x} = \frac{\overline{X_k \underline{\mathrm{T}}}}{\overline{\alpha_k}} \qquad \left(\frac{N}{m^2}\right) \qquad (4.3.2)$$

$$\overline{\mathrm{D}_k^x} = \frac{\overline{X_k \underline{\mathrm{T}} \cdot \mathrm{v}}}{\overline{\alpha_k}} \qquad \left(\frac{W}{m^2}\right) \qquad (4.3.5)$$

$$\overline{\mathrm{M}_k} = -\overline{\underline{\mathrm{T}} \cdot \nabla X_k} \qquad \left(\frac{N}{m^3}\right) \qquad (4.6.2)$$

$$\overline{\mathrm{W}_k} = -\overline{(\underline{\mathrm{T}} \cdot \mathrm{v}) \cdot \nabla X_k} \qquad \left(\frac{W}{m^3}\right) \qquad (4.6.5)$$

$$\overline{\mathrm{M}_k'} = -\overline{\underline{\mathrm{T}}_{ki}' \cdot \nabla X_k} \qquad \left(\frac{N}{m^3}\right) \qquad (4.7.5)$$

$$\overline{\mathrm{W}_k''} = -\overline{\left(\underline{\mathrm{T}}_{ki}' \cdot \mathrm{v}\right) \cdot \nabla X_k} \qquad \left(\frac{W}{m^3}\right) \qquad (4.7.6)$$

$$\overline{\mathrm{W}_k'} = -\left[\overline{-p_{ki}' \mathrm{v}_i \cdot \nabla X_k + (\underline{\tau} \cdot \mathrm{v}) \cdot \nabla X_k}\right] \qquad \left(\frac{W}{m^3}\right) \qquad (4.7.7)$$

Relation between interfacial forces (instantaneous and turbulent fluctuations)

$$\overline{\mathrm{M}_k} = \overline{p_{ki}}\ \nabla \overline{\alpha_k} - \overline{\underline{\tau}_{ki}} \cdot \nabla \overline{\alpha_k} + \overline{\mathrm{M}_k'} \qquad (4.10.9)$$

Relation between interfacial work

$$\overline{\mathrm{W}_k} = -\overline{p_{ki}} \frac{\partial \overline{\alpha_k}}{\partial t} + \overline{p\,(\mathrm{v} - \mathrm{v}_i) \cdot \nabla X_k} + \overline{\mathrm{W}_{ki}'} \qquad (4.11.8)$$

TABLE V - continued

<u>Shear stress</u> $\underline{\tau}$ (N/m2)

$$\overline{\underline{\tau}_k^x} = \frac{\overline{X_k \underline{\tau}}}{\overline{\alpha_k}} \qquad \left(\frac{N}{m^2}\right) \qquad (4.3.4)$$

$$\overline{\underline{\tau}_{sk}^x} = \frac{\overline{X_k \frac{\underline{\tau}}{T}}}{\overline{\alpha_k}} \qquad \left(\frac{N}{m^2\ K}\right) \qquad (4.3.8)$$

$$\overline{\tilde{\underline{\tau}}_{ki}} = \overline{\underline{\tau}\,|\nabla X_k|} \qquad \left(\frac{N}{m^3}\right) \qquad (4.6.4)$$

$$\overline{\underline{\tau}_{ki}} = \frac{\overline{\tilde{\underline{\tau}}_{ki}}}{\overline{a_i}} = \frac{\overline{\underline{\tau}\,\nabla X_k}}{\overline{\nabla X_k}} \qquad \left(\frac{N}{m^2}\right) \qquad (4.6.13)$$

$$\overline{\mathbf{q}_k^{\tau}} = -\frac{\overline{X_k \underline{\tau} \cdot \mathbf{v}_k'}}{\overline{\alpha_k}} \qquad \left(\frac{W}{m^3}\right) \qquad (4.7.3)$$

$$\overline{D_k} = \frac{\overline{X_k \underline{\tau} : \nabla \mathbf{v}_k'}}{\overline{\alpha_k}} \qquad \left(\frac{W}{m^3}\right) \qquad (4.7.4)$$

$$\overline{D_{sk}} = \frac{\overline{X_k \frac{\underline{\tau}}{T} : \nabla \mathbf{v}_k'}}{\overline{\alpha_k}} \qquad \left(\frac{W}{m^3\ K}\right) \qquad (4.7.8)$$

<u>Pressure</u> p (N/m2)

$$\overline{\underline{\mathbf{p}}_k^x} = \frac{\overline{X_k p \underline{\mathbf{I}}}}{\overline{\alpha_k}} \qquad \left(\frac{N}{m^2}\right) \qquad (4.3.3)$$

$$\overline{\tilde{p}_{ki}} = |\,\overline{p \underline{\mathbf{I}} \cdot \nabla X_k}\,| = |\,\overline{p \nabla X_k}\,| \qquad \left(\frac{N}{m^3}\right) \qquad (4.6.3)$$

$$\overline{p_{ki}} = \frac{\overline{\tilde{p}_{ki}}}{\overline{a_i}} = \frac{\overline{p \nabla X_k}}{\overline{\nabla X_k}} \qquad \left(\frac{V}{m^2}\right) \qquad (4.6.12)$$

$$\overline{\mathbf{q}_k^p} = \frac{\overline{X_k p \mathbf{v}_k'}}{\overline{\alpha_k}} \qquad \left(\frac{W}{m^2}\right) \qquad (4.7.2)$$

$$\frac{\partial \overline{p_k^x}}{\partial t} = \frac{\overline{X_k \partial p/\partial t}}{\overline{\alpha_k}} \qquad \left(\frac{N}{m^2 s}\right) \qquad (4.8.1)$$

TABLE V - continued

5. Detailed form of the averaged conservation equations and jump conditions

5.1 Eulerian form of averaged three-dimensional conservation equations

5.1.1 Continuity equation

We multiply the local instantaneous conservation equation for phase k by X_k and average. The procedure is the same for all conservation equations considered in this chapter.

From the equation of conservation of mass, equation (2.3.1),we obtain

$$\overline{X_k \frac{\partial \rho}{\partial t}} + \overline{X_k (\nabla \cdot \rho \, \mathrm{v})} = 0 \, . \qquad [5.1.1.1]$$

Noting that

$$X_k \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(X_k \rho) - \rho \frac{\partial X_k}{\partial t} \, , \qquad [5.1.1.2]$$

$$X_k (\nabla \cdot \rho \, \mathrm{v}) = \nabla \cdot (X_k \rho \, \mathrm{v}) - \rho \, \mathrm{v} \cdot \nabla X_k \, , \qquad [5.1.1.3]$$

recalling

$$\frac{\partial X_k}{\partial t} + \mathrm{v}_i \cdot \nabla X_k = 0 \qquad (3.2.4.3)$$

and using the rules (3.1.17) and (3.1.18), we obtain

$$\frac{\partial}{\partial t}(\overline{X_k \rho}) + \nabla \cdot (\overline{X_k \rho \, \mathrm{v}}) = \overline{\rho (\mathrm{v} - \mathrm{v}_i) \cdot \nabla X_k} \, . \qquad [5.1.1.4]$$

The terms $\overline{X_k \rho}$, $\overline{X_k \rho \mathrm{v}}$ and $\overline{\rho (\mathrm{v} - \mathrm{v}_i) \cdot \nabla X_k}$ follow from definitions (4.1.3), (4.1.4) and (4.5.2), respectively. Thus, we have

$$\frac{\partial}{\partial t}\left(\overline{\alpha_k} \, \overline{\rho_k^x} \right) + \nabla \cdot \left(\overline{\alpha_k} \, \overline{\rho_k^x} \, \overline{\mathrm{v}_k^{x,\rho}} \right) = \overline{\Gamma_k} \, . \qquad [5.1.1.5]$$

Γ_k is a mass generation rate due to phase change processes at the interface (evaporation and condensation).

5.1.2 Momentum equation

Considering equation (2.3.2) and following the same procedure as in section 5.1.1, we obtain

$$\overline{X_k \frac{\partial (\rho \mathrm{v})}{\partial t}} + \overline{X_k (\nabla \cdot \rho \, \mathrm{v}\mathrm{v})} = \overline{X_k (\nabla \cdot \underline{\mathrm{T}})} + \overline{X_k \rho \, \mathrm{g}} \, . \qquad [5.1.2.1]$$

We note the identities

$$X_k \frac{\partial(\rho \mathbf{v})}{\partial t} = \frac{\partial}{\partial t} (X_k \rho \mathbf{v}) - \rho \mathbf{v} \frac{\partial X_k}{\partial t} , \qquad [5.1.2.2]$$

$$X_k (\nabla \cdot \rho \mathbf{vv}) = \nabla \cdot (X_k \rho \mathbf{vv}) - \rho \mathbf{vv} \cdot \nabla X_k , \qquad [5.1.2.3]$$

$$X_k (\nabla \cdot \underline{\mathrm{T}}) = \nabla \cdot (X_k \underline{\mathrm{T}}) - \underline{\mathrm{T}} \cdot \nabla X_k . \qquad [5.1.2.4]$$

Before continuing to consider equation (5.1.2.1), we want to proof the validity of (5.1.2.3) and (5.1.2.4):

For a symmetric tensor $\underline{\mathrm{T}}$ one has

$$\nabla \cdot X_k \underline{\mathrm{T}} = X_k (\nabla \cdot \underline{\mathrm{T}}) + \underline{\mathrm{T}} \cdot \nabla X_k ,$$

which can be proved als follows

$$\nabla \cdot X_k \underline{\mathrm{T}} = \sum_i \frac{\partial}{\partial x_i} \boldsymbol{\delta}_i \cdot \sum_j \sum_l X_k T_{jl} \boldsymbol{\delta}_j \boldsymbol{\delta}_l =$$

$$= \sum_i \sum_j \sum_l \frac{\partial}{\partial x_i} \boldsymbol{\delta}_i \cdot X_k T_{jl} \boldsymbol{\delta}_j \boldsymbol{\delta}_l =$$

$$= \sum_i \sum_j \sum_l \frac{\partial}{\partial x_i} X_k T_{jl} \delta_{ij} \boldsymbol{\delta}_l = \sum_l \sum_i \frac{\partial}{\partial x_i} X_k T_{il} \boldsymbol{\delta}_l = \qquad [5.1.2.4b]$$

$$= \sum_l \left[\sum_i \left(X_k \frac{\partial T_{il}}{\partial x_i} + T_{il} \frac{\partial X_k}{\partial x_i} \right) \right] \boldsymbol{\delta}_l ;$$

$$X_k (\nabla \cdot \underline{\mathrm{T}}) = X_k \left(\sum_i \frac{\partial}{\partial x_i} \boldsymbol{\delta}_i \cdot \sum_j \sum_l T_{jl} \boldsymbol{\delta}_j \boldsymbol{\delta}_l \right)$$

$$= X_k \left(\sum_i \sum_j \sum_l \frac{\partial}{\partial x_i} T_{jl} \delta_{ij} \boldsymbol{\delta}_l \right) = X_k \left(\sum_l \sum_i \frac{\partial}{\partial x_i} T_{il} \right) \boldsymbol{\delta}_l ; \qquad [5.1.2.4c]$$

$$\underline{\mathrm{T}} \cdot \nabla X_k = \sum_i \sum_j T_{ij} \boldsymbol{\delta}_i \boldsymbol{\delta}_j \cdot \sum_l \frac{\partial}{\partial x_l} X_k \boldsymbol{\delta}_l =$$

$$= \sum_i \sum_j \sum_l T_{ij} \frac{\partial}{\partial x_l} X_k \boldsymbol{\delta}_i \delta_{jl} = \sum_i \left(\sum_j T_{ij} \frac{\partial}{\partial x_j} X_k \right) \boldsymbol{\delta}_i . \qquad [5.1.2.4d]$$

From the last two expressions one derives:

$$X_k\,(\nabla\cdot\underline{T}) + \underline{T}\cdot\nabla X_k = X_k\left(\sum_l\sum_i \frac{\partial}{\partial x_i}T_{il}\right)\delta_l + \sum_i\left(\sum_j T_{ij}\frac{\partial}{\partial x_j}X_k\right)\delta_i$$

$$= \sum_l\left(\sum_i X_k\frac{\partial}{\partial x_i}T_{il}\right)\delta_l + \sum_l\left(\sum_i T_{li}\frac{\partial}{\partial x_i}X_k\right)\delta_l \qquad [5.1.2.4e]$$

$$= \sum_l\left[\sum_i\left(X_k\frac{\partial T_{il}}{\partial x_i} + T_{li}\frac{\partial}{\partial x_i}X_k\right)\right]\delta_l\,.$$

The second member of (5.1.2.4d) is identical with the second member of (5.1.2.4b) because $T_{il} = T_{li}$. Hence one derives

$$X_k\,(\nabla\cdot\underline{T}) = \nabla\cdot X_k\underline{T} - \underline{T}\cdot\nabla X_k$$

which is equation (5.1.2.4).

If $\underline{T} = \rho\,\mathbf{v}\,\mathbf{v}$, which is also a symmetric tensor, this identity becomes:

$$X_k\,(\nabla\cdot\rho\,\mathbf{v}\mathbf{v}) = \nabla\cdot(X_k\,\rho\,\mathbf{v}\mathbf{v}) - \rho\,\mathbf{v}\mathbf{v}\cdot\nabla X_k\,. \qquad (5.1.2.3)$$

Thus using again equation (3.2.4.3), we obtain from equation (5.1.2.1)

$$\frac{\partial}{\partial t}\,(\overline{X_k\,\rho\,\mathbf{v}}) + \nabla\cdot(\overline{X_k\,\rho\,\mathbf{v}\mathbf{v}}) = \nabla\cdot\,\overline{(X_k\underline{T})} + \overline{X_k\,\rho\,\mathbf{g}} + \overline{\rho\,\mathbf{v}(\mathbf{v}-\mathbf{v}_i)\cdot\nabla X_k} - \overline{\underline{T}\cdot\nabla X_k}\,.$$

Using the definitions

$$\overline{\mathbf{v}_{ki}}\;\overline{\Gamma_k} = \overline{\rho\,\mathbf{v}\,(\mathbf{v}-\mathbf{v}_i)\cdot\nabla X_k}\,, \qquad (4.5.3)$$

$$\overline{\alpha_k}\;\overline{\underline{T}_k^x} = \overline{X_k\underline{T}}\,, \qquad (4.3.2)$$

$$\overline{\mathbf{M}_k} = -\,\overline{\underline{T}\cdot\nabla X_k} \qquad (4.6.2)$$

and

$$\overline{\alpha_k}\;\overline{\rho_k^x} = \overline{X_k\rho} \qquad (4.1.3)$$

one derives:

$$\frac{\partial}{\partial t}\,\overline{(X_k\,\rho\,\mathbf{v})} + \nabla\cdot\overline{(X_k\,\rho\,\mathbf{v}\,\mathbf{v})} = \nabla\cdot\overline{\alpha}_k\;\overline{\underline{\mathbf{T}}_k^x} + \overline{\alpha}_k\;\overline{\rho_k^x}\;\mathbf{g} + \overline{\mathbf{v}_{ki}}\;\overline{\Gamma}_k + \overline{\mathbf{M}_K}\,. \qquad [5.1.2.6]$$

Let us now consider the instantaneous velocity vector as sum of the mean value and of the turbulent fluctuation

$$\mathbf{v} = \overline{\mathbf{v}_k^{x,\rho}} + \mathbf{v}_k'\,. \qquad (4.4.10)$$

The left side of (5.1.2.6) becomes:

$$\frac{\partial}{\partial t}\,\overline{(X_k\,\rho\,\mathbf{v})} + \nabla\cdot\overline{(X_k\,\rho\,\mathbf{v}\,\mathbf{v})} =$$

$$= \frac{\partial}{\partial t}\,\overline{\left[X_k\,\rho\left(\overline{\mathbf{v}_k^{x,\rho}} + \mathbf{v}_k'\right)\right]} + \nabla\cdot\,\overline{\left[X_k\,\rho\left(\overline{\mathbf{v}_k^{x,\rho}} + \mathbf{v}_k'\right)\left(\overline{\mathbf{v}_k^{x,\rho}} + \mathbf{v}_k'\right)\right]} =$$

$$[5.1.2.7]$$

$$= \frac{\partial}{\partial t}\,\overline{\left(X_k\,\rho\,\overline{\mathbf{v}_k^{x,\rho}}\right)} + \frac{\partial}{\partial t}\,\overline{\left(X_k\,\rho\,\mathbf{v}_k'\right)} +$$

$$+ \nabla\cdot\overline{\left(X_k\,\rho\,\overline{\mathbf{v}_k^{x,\rho}}\;\overline{\mathbf{v}_k^{x,\rho}}\right)} + \nabla\cdot\overline{\left(X_k\,\rho\,\mathbf{v}_k'\;\mathbf{v}_k'\right)} + 2\,\nabla\cdot\overline{\left(X_k\,\rho\,\overline{\mathbf{v}_k^{x,\rho}}\;\mathbf{v}_k'\right)}\,.$$

The second and fifth terms at the right hand side vanish because they are mean values of turbulent fluctuations.

Using now again the definition

$$\overline{\rho_k^x} = \overline{X_k\rho}\,/\,\overline{\alpha}_k \qquad (4.1.3)$$

and

$$-\;\overline{\alpha}_k\;\overline{\underline{\mathbf{T}}_k^{Re}} = \overline{X_k\rho\,\mathbf{v}_k'\,\mathbf{v}_k'}\,, \qquad (4.4.2)$$

one derives from (5.1.2.6) the detailed form of the averaged momentum equation

$$\frac{\partial}{\partial t}\left(\overline{\alpha}_k\;\overline{\rho_k^x}\;\overline{\mathbf{v}_k^{x,\rho}}\right) + \nabla\cdot\left(\overline{\alpha}_k\;\overline{\rho_k^x}\;\overline{\mathbf{v}_k^{x,\rho}}\;\overline{\mathbf{v}_k^{x,\rho}}\right) = \nabla\cdot\;\overline{\alpha}_k\left(\overline{\underline{\mathbf{T}}_k^x} + \overline{\underline{\mathbf{T}}_k^{Re}}\right) +$$

$$[5.1.2.8]$$

$$+\;\overline{\alpha}_k\;\overline{\rho_k^x}\;\mathbf{g} + \overline{\mathbf{v}_{ki}}\;\overline{\Gamma}_k + \overline{\mathbf{M}}_k\,.$$

In order to separate shear stress from pressure forces as well as mean field effects from local effects in this integral interfacial force $\overline{\mathbf{M}}_k$, we decompose it in the following way. Because of

$$\underline{T} = -p\underline{I} + \underline{\tau} \tag{2.3.3}$$

we write

$$\overline{M_k} = -\overline{\underline{T}\cdot\nabla X_k} = \overline{p\underline{I}\cdot\nabla X_k} - \overline{\underline{\tau}\cdot\nabla X_k} \tag{4.10.1}$$

and obtain, following the procedure outlined in section 4.10:

$$\overline{M_k} = \overline{p_{ki}}\,\nabla\overline{\alpha_k} - \overline{\underline{\tau}_{ki}}\cdot\nabla\overline{\alpha_k} + \overline{M_k'}. \tag{4.10.9}$$

$\overline{M_k'} = \overline{p'\nabla X_k} - \overline{\underline{\tau}'\cdot\nabla X_k}$ is the interfacial force due to local pressure and shear stress effects; $\overline{p_{ki}}\,\nabla\overline{\alpha_k}$ and $-\overline{\underline{\tau}_{ki}}\cdot\nabla\overline{\alpha_k}$ are averaged interfacial pressure and shear stress forces, respectively.

With respect to modelling interfacial forces through use of constitutive equations, it will be useful to introduce the interfacial force $\overline{M_k^d}$, defined by

$$\begin{aligned}\overline{M_k^d} &= \overline{M_k'} - \overline{\underline{\tau}_{ki}}\cdot\nabla\overline{\alpha_k} = \\ &= \overline{p_k'\nabla X_k} - \overline{\underline{\tau}_k'\cdot\nabla X_k} - \overline{\underline{\tau}_{ki}}\cdot\nabla\overline{\alpha_k}\ ,\end{aligned} \tag{5.1.2.9}$$

which contains all interfacial interactions (e.g. viscous drag, lift forces, virtual mass forces etc.) except for the mean interfacial pressure, $\overline{p_{ki}}\,\nabla\overline{\alpha_k}$ and the momentum exchange term due to interfacial mass transfer $\overline{v_{ki}^m}\ \overline{\Gamma_k}$.

Therefore, from (4.10.9) one derives

$$\overline{M_k} = \overline{p_{ki}}\,\nabla\overline{\alpha_k} + \overline{M_k^d}\ . \tag{5.1.2.10}$$

Thus, equation (5.1.2.8) becomes:

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{v_K^{x,\rho}}\right) + \nabla\cdot\left(\overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{v_k^{x,\rho}}\ \overline{v_k^{x,\rho}}\right) = \nabla\cdot\overline{\alpha_k}\left(\overline{\underline{T}_k^x} + \overline{\underline{T}_k^{Re}}\right) + \\ &+ \overline{\alpha_k}\ \overline{\rho_k^x}\ g + \overline{v_{ki}}\ \overline{\Gamma_k} + \overline{p_{ki}}\ \nabla\overline{\alpha_k} + \overline{M_k^d}\ .\end{aligned} \tag{5.1.2.11}$$

Eventually, using equation (2.3.3) $\underline{T} = -p\underline{I} + \underline{\tau}$ and the identities

$$\nabla\cdot\left(\overline{\alpha_k}\ \overline{\underline{p}_k^x}\right) = \overline{\underline{p}_k^x}\cdot\nabla\overline{\alpha_k} + \overline{\alpha_k}\ \nabla\cdot\overline{\underline{p}_k^x}\ ,$$

$$\nabla\cdot\overline{\alpha_k \underline{T}_k^x} = \nabla\cdot\left[-\overline{\alpha_k}\,\overline{\underline{p}_k^x} + \overline{\alpha_k}\,\overline{\underline{\tau}_k^x}\right] = -\nabla\cdot\left(\overline{\alpha_k}\,\overline{\underline{p}_k^x}\right) + \nabla\cdot\left(\overline{\alpha_k}\,\overline{\underline{\tau}_k^x}\right) =$$

$$= -\overline{\alpha_k}\,\nabla\cdot\overline{\underline{p}_k^x} - \overline{\underline{p}_k^x}\cdot\nabla\overline{\alpha_k} + \nabla\cdot\left(\overline{\alpha_k}\,\overline{\underline{\tau}_k^x}\right)$$

we obtain

$$\frac{\partial}{\partial t}\left(\overline{\alpha_k}\,\overline{\rho_k^x}\,\overline{\mathbf{v}_k^{x,\rho}}\right) + \nabla\cdot\left(\overline{\alpha_k}\,\overline{\rho_k^x}\,\overline{\mathbf{v}_k^{x,\rho}}\,\overline{\mathbf{v}_k^{x,\rho}}\right) = \qquad [5.1.2.12]$$

$$= -\overline{\alpha_k}\nabla\overline{p_k^x} + \nabla\cdot\overline{\alpha_k}\left(\overline{\underline{\tau}_k^x} + \overline{\underline{T}_k^{Re}}\right) + \overline{\alpha_k}\,\overline{\rho_k^x}\mathbf{g} + \overline{\mathbf{v}_{ki}^m}\,\overline{\Gamma_k} + \left(\overline{p_{ki}} - \overline{p_k^x}\right)\nabla\overline{\alpha_k} + \overline{\mathbf{M}_k^d}\ .$$

In the above derivation we used

$$\nabla\cdot\overline{\underline{p}_k^x} = \nabla\cdot\overline{p_k^x}\,\underline{I} = \nabla\overline{p_k^x}\ . \qquad [5.1.2.13]$$

5.1.3 Total energy equation

We multiply equation (2.3.11) by X_k and, after operations similar to those performed for the equations of conservation of mass and momentum, obtain

$$\frac{\partial}{\partial t}\underbrace{\overline{\left[X_k\rho\left(u+\frac{1}{2}v^2\right)\right]}}_{①} + \nabla\cdot\underbrace{\overline{\left[X_k\rho\left(u+\frac{1}{2}v^2\right)\mathbf{v}\right]}}_{②} =$$

$$= \underbrace{\overline{\rho\left(u+\frac{1}{2}v^2\right)\left(\mathbf{v}-\mathbf{v}_i\right)\cdot\nabla X_k}}_{③} + \nabla\cdot\underbrace{\overline{\left[X_k\left(\underline{T}\cdot\mathbf{v}\right)\right]}}_{④} - \qquad (5.1.3.1)$$

$$-\underbrace{\overline{\left(\underline{T}\cdot\mathbf{v}\right)\cdot\nabla X_k}}_{⑤} - \underbrace{\overline{\nabla\cdot X_k\mathbf{q}} + \overline{\mathbf{q}\cdot\nabla X_k}}_{⑥} + \underbrace{\overline{X_k\rho\mathbf{g}\cdot\mathbf{v}}}_{⑦} + \underbrace{\overline{X_k\rho r}}_{⑧}\ .$$

Now, we account for the effects of turbulence on the velocity, pressure and internal energy fields by applying:

$$\mathbf{v} = \overline{\mathbf{v}_k^{x,\rho}} + \mathbf{v}_k',$$

$$p = \overline{p_k^x} + p_k',$$

$$u = \overline{u_k^{x,\rho}} + u_k',$$

according to definitions (4.4.10) and (4.4.11).

Then, with

$$\upsilon^2 = \overline{\mathbf{v}_k^{x,\rho}} \cdot \overline{\mathbf{v}_k^{x,\rho}} + 2 \ \overline{\mathbf{v}_k^{x,\rho}} \ \cdot \ \mathbf{v}_k^{'} + \mathbf{v}_k^{'} \cdot \mathbf{v}_k^{'} \qquad [5.1.3.2]$$

and noting that

$$\overline{X_k \rho \overline{\mathbf{v}_k^{x,\rho}} \cdot \mathbf{v}_k^{'}} = 0 \ ,$$

$$\overline{X_k \rho u_k^{'}} = 0 \ , \qquad [5.1.3.3]$$

we write for term ① of equation (5.1.3.1)

$$\frac{\partial}{\partial t} \overline{\left[X_k \rho \left(u + \frac{1}{2} \upsilon^2\right)\right]} = \frac{\partial}{\partial t} \left[\overline{X_k \rho} \ \ \overline{u_k^{x,\rho}} + \frac{1}{2} \ \overline{X_k \rho \left(\overline{\upsilon_k^{x,\rho}}\right)^2} + \frac{1}{2} \ \overline{X_k \rho \left(\upsilon_k^{'}\right)^2}\right]. \qquad [5.1.3.4]$$

By means of definitions (4.1.3) and (4.7.1) we have finally

$$\frac{\partial}{\partial t} \overline{\left[X_k \rho \left(u + \frac{1}{2} \upsilon^2\right)\right]} = \frac{\partial}{\partial t} \overline{\alpha_k} \ \overline{\rho_k^x} \left[\overline{u_k^{x,\rho}} + \frac{1}{2} \left(\overline{\upsilon_k^{x,\rho}}\right)^2 + \overline{u_k^{Re}}\right] . \qquad [5.1.3.5]$$

For term ② we write

$$\nabla \cdot \overline{\left[X_k \rho \left(u + \frac{1}{2} \upsilon^2\right) \mathbf{v}\right]} =$$

$$= \nabla \cdot \overline{\left[X_k \rho \left(\overline{\mathbf{u}_k^{x,\rho}} + u_k^{'}\right)\left(\overline{\mathbf{v}_k^{x,\rho}} + \mathbf{v}_k^{'}\right)\right]} + \nabla \cdot \overline{\left\{X_k \rho \frac{1}{2} \left[\left(\overline{\upsilon_k^{x,\rho}}\right)^2 + 2 \overline{\mathbf{v}_k^{x,\rho}} \cdot \mathbf{v}_k^{'} + \left(\upsilon_k^{'}\right)^2\right]\left[\overline{\mathbf{v}_k^{x,\rho}} + \mathbf{v}_k^{'}\right]\right\}} =$$

$$= \nabla \cdot \overline{\left(X_k \rho \ \overline{u_k^{x,\rho}} \ \overline{\mathbf{v}_k^{x,\rho}}\right)} + \nabla \cdot \overline{\left(X_k \rho \ u_k^{'} \ \mathbf{v}_k^{'}\right)} +$$

$$+ \nabla \cdot \overline{\left(X_k \rho \frac{1}{2} \left(\overline{\upsilon_k^{x,\rho}}\right)^2\right) \overline{\mathbf{v}_k^{x,\rho}}} + \nabla \cdot \overline{\left(X_k \rho \frac{1}{2} \left(\upsilon_k^{'}\right)^2\right) \overline{\mathbf{v}^{x,\rho}}} +$$

$$+ \nabla \cdot \overline{\left[X_k \rho \left(\overline{\mathbf{v}_k^{x,\rho}} \cdot \mathbf{v}_k^{'}\right)\right] \mathbf{v}_k^{'}} + \nabla \cdot \overline{\left(X_k \rho \frac{1}{2} \left(\upsilon_k^{'}\right)^2\right) \mathbf{v}_k^{'}} \ . \qquad [5.1.3.6]$$

Using the definitions

$$\overline{\rho_k^x} = \overline{X_k \rho} \ / \ \overline{\alpha_k} \qquad (4.1.3)$$

$$\widehat{\mathbf{q}_k^{Re}} = \overline{X_k \rho \, u_k^{'} \mathbf{v}_k^{'}} \ / \ \overline{\alpha_k} \ , \qquad (4.4.4)$$

$$\overline{\mathbf{u}_k^{Re}} = \frac{1}{2} \ \overline{X_k \rho \left(\upsilon_k^{'}\right)^2} \ / \left(\overline{\alpha_k} \ \ \overline{\rho_k^x}\right) , \qquad (4.7.1)$$

$$- \ \overline{\underline{\mathbf{T}}_k^{Re}} = \overline{X_k \rho \ \mathbf{v}_k^{'} \mathbf{v}_k^{'}} \ / \ \overline{\alpha_k} \ , \qquad (4.4.2)$$

$$\overline{\mathbf{q}_k^{kin}} = \frac{1}{2}\,\overline{X_k\,\rho\,\left(\upsilon_k'\right)^2 \mathbf{v}_k'}\,/\,\;\overline{\alpha_k}\,, \qquad (4.4.5)$$

term ② becomes after some rearrangements

$$\nabla\cdot\overline{\left[X_k\,\rho\left(u+\frac{1}{2}\,\upsilon^2\right)\mathbf{v}\right]} = \nabla\cdot\overline{\alpha_k}\;\overline{\rho_k^x}\;\overline{\mathbf{v}^{x,\rho}}\;\left[\overline{u_k^{x,\rho}}+\frac{1}{2}\left(\overline{\upsilon_k^{x,\rho}}\right)^2+\overline{u_k^{Re}}\right]+ \qquad [5.1.3.7]$$

$$+\,\nabla\cdot\overline{\alpha_k}\;\widehat{\mathbf{q}_k^{Re}}+\nabla\cdot\overline{\alpha_k}\;\overline{\mathbf{q}_k^{kin}}-\nabla\cdot\left(\overline{\alpha_k}\;\overline{\underline{\mathbf{T}}_k^{Re}}\cdot\overline{\mathbf{v}_k^{x,\rho}}\right).$$

With respect to the derivation of the Reynolds stress term

$$\nabla\cdot\overline{\left[X_k\,\rho\left(\overline{\mathbf{v}_k^{x,\rho}}\cdot\mathbf{v}_k'\right)\right]\mathbf{v}_k'} = -\;\nabla\cdot\left(\overline{\alpha_k}\;\overline{\underline{\mathbf{T}}_k^{Re}}\cdot\overline{\mathbf{v}_k^{x,\rho}}\right) \qquad [5.1.3.8]$$

we used the identity [4]

$$\left(\overline{\mathbf{v}_k^{x,\rho}}\cdot\mathbf{v}_k'\right)\mathbf{v}_k' = \overline{\mathbf{v}_k^{x,\rho}}\cdot\mathbf{v}_k'\mathbf{v}_k'\,. \qquad [5.1.3.9]$$

Because the tensor $\mathbf{v}_k'\,\mathbf{v}_k'$ is symmetric, we can write

$$\left(\overline{\mathbf{v}_k^{x,\rho}}\cdot\mathbf{v}_k'\right)\mathbf{v}_k' = \mathbf{v}_k'\mathbf{v}_k'\cdot\overline{\mathbf{v}_k^{x,\rho}}\,. \qquad [5.1.3.10]$$

For term ③ of the total energy equation (5.1.3.1) we write

$$\overline{\rho\left(u+\frac{1}{2}\,\upsilon^2\right)\left(\mathbf{v}-\mathbf{v}_i\right)\cdot\nabla X_k} = \overline{\rho\,u\left(\mathbf{v}-\mathbf{v}_i\right)\cdot\nabla X_k} + \overline{\rho\,\frac{1}{2}\,\upsilon^2\left(\mathbf{v}-\mathbf{v}_i\right)\cdot\nabla X_k} \qquad [5.1.3.11]$$

and we obtain by means of definitions (4.5.5) and (4.5.6)

$$\overline{\rho\left(u+\frac{1}{2}\,\upsilon^2\right)\left(\mathbf{v}-\mathbf{v}_i\right)\cdot\nabla X_k} = \overline{u_{ki}}\;\overline{\Gamma_k}+\frac{1}{2}\,\overline{\upsilon_{ki}^2}\;\overline{\Gamma_k} = \left(\overline{u_{ki}}+\frac{1}{2}\,\overline{\upsilon_{ki}^2}\right)\overline{\Gamma_k}\,. \qquad [5.1.3.12]$$

Term ④ is decomposed as follows:

$$\nabla\cdot\overline{X_k\left(\underline{\mathbf{T}}\cdot\mathbf{v}\right)} = \nabla\cdot\overline{X_k\left(\underline{\mathbf{T}}\cdot\overline{\mathbf{v}_k^{x,\rho}}\right)} + \nabla\cdot\overline{X_k\left(\underline{\mathbf{T}}\cdot\mathbf{v}_k'\right)}\,. \qquad [5.1.3.13]$$

With definition (4.3.2) we have for the first term on the right hand-side of (5.1.3.13)

$$\nabla\cdot\overline{X_k\left(\underline{\mathbf{T}}\cdot\overline{\mathbf{v}_k^{x,\rho}}\right)} = \nabla\cdot\left(\overline{\alpha_k}\;\overline{\underline{\mathbf{T}}_k^x}\cdot\overline{\mathbf{v}_k^{x,\rho}}\right)\,. \qquad [5.1.3.14]$$

For the second term on the right hand-side of (5.1.3.13) we use eq. (2.3.3) and obtain

$$\nabla \cdot \overline{X_k\left(\underline{T} \cdot v_k'\right)} = - \nabla \cdot \overline{X_k\left(p\underline{I} \cdot v_k'\right)} + \nabla \cdot \overline{X_k\left(\underline{\tau} \cdot v_k'\right)} . \qquad [5.1.3.15]$$

With definitions (4.7.2) and (4.7.3) we write

$$\nabla \cdot \overline{\left(X_k \underline{T} \cdot v_k'\right)} = - \nabla \cdot \overline{\alpha_k}\ \overline{q_k^p} - \nabla \cdot \overline{\alpha_k}\ \overline{q_k^\tau} \qquad [5.1.3.16]$$

and thus, term ④ becomes

$$\nabla \cdot \overline{X_k\left(\underline{T} \cdot v\right)} = \nabla \cdot \left(\overline{\alpha_k}\overline{\underline{T}_k^x} \cdot \overline{v_k^{x,\rho}}\right) - \nabla \cdot \overline{\alpha_k}\ \overline{q_k^p} - \nabla \cdot \overline{\alpha_k}\ \overline{q_k^\tau} . \qquad [5.1.3.17]$$

For term ⑤ we have by definition

$$\overline{W_k} = - \overline{\left(\underline{T} \cdot v\right) \cdot \nabla X_k} . \qquad (4.6.5)$$

According to definitions (4.3.6) and (4.6.6), the energy flux is

$$\overline{q_k^x} = \overline{X_k q} \ / \ \overline{\alpha_k} \qquad (4.3.6)$$

and the interfacial heat source is

$$\overline{E_k} = \overline{q \cdot \nabla X_k} . \qquad (4.6.6)$$

Thus, terms ⑥ become

$$- \overline{\nabla \cdot \left(X_k q\right)} + \overline{q \cdot \nabla X_k} = - \nabla \cdot \overline{\alpha_k}\ \overline{q_k^x} + \overline{E_k} . \qquad [5.1.3.18]$$

For terms ⑦ and ⑧ we find by means of definitions (4.1.3) and (4.2.3)

$$\overline{X_k \rho g \cdot v} = \overline{\alpha_k}\ \overline{\rho_k^x}\ g \cdot \overline{v_k^{x,\rho}} ,$$

$$\overline{X_k \rho r} = \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{r_k^{x,\rho}} .$$

Eventually we obtain the final averaged equation of conservation of total energy in the following form:

$$\frac{\partial}{\partial t}\left\{\overline{\alpha_k}\ \overline{\rho_k^x}\left[\overline{u_k^{x,\rho}} + \frac{1}{2}\left(\overline{v_k^{x,\rho}}\right)^2 + \overline{u_k^{Re}}\right]\right\} + \nabla \cdot \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{v_\kappa^{x,\rho}}\left[\overline{u_k^{x,\rho}} + \frac{1}{2}\left(\overline{v_k^{x,\rho}}\right)^2 + \overline{u_k^{Re}}\right] =$$

$$= \nabla \cdot \overline{\alpha_k}\left[\left(\overline{\underline{T}_k^x} + \overline{\underline{T}_k^{Re}}\right) \cdot \overline{v_k^{x,\rho}} - \overline{q_k^x} - \overline{q_k^{Re}}\right] + \qquad [5.1.3.19]$$

$$+ \left(\overline{u_{ki}} + \frac{1}{2}\overline{v_{ki}^2}\right)\overline{\Gamma_k} + \overline{E_k} + \overline{W_k} + \overline{\alpha_k}\ \overline{\rho_k^x}\ g \cdot \overline{v_k^{x,\rho}} + \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{r_k^{x,\rho}} ,$$

with

$$\overline{q_k^{Re}} = \hat{q}_k^{Re} + \overline{q_k^{kin}} + \overline{q_k^{p}} + \overline{q_k^{\iota}} \ . \qquad (4.9.1)$$

Proof of the identity:

$$(\mathbf{w} \cdot \mathbf{v})\ \mathbf{v} = \mathbf{w} \cdot \mathbf{v}\ \mathbf{v}\ . \qquad [5.1.3.20]$$

Left side:

$$\mathbf{w} \cdot \mathbf{v} = \sum_i w_i\ \boldsymbol{\delta}_i \cdot \sum_j v_j\ \boldsymbol{\delta}_j = \sum_i \sum_j w_i\, v_j\ \delta_{ij} = \sum_i w_i\ v_i$$

$$(\mathbf{w} \cdot \mathbf{v})\mathbf{v} = \left(\sum_i w_i\ v_i\right) \sum_j v_j\ \boldsymbol{\delta}_t = \sum_j \left(\sum_i w_i\ v_i\right) v_j\ \boldsymbol{\delta}_j\ ;$$

Right side:

$$\mathbf{w} \cdot \mathbf{v}\ \mathbf{v} = \sum_i w_i\ \boldsymbol{\delta}_\iota \cdot \sum_j \sum_k v_j\ v_k\ \boldsymbol{\delta}_j\ \boldsymbol{\delta}_k =$$

$$= \sum_i \sum_j \sum_k w_i\ v_j\ v_k\ \delta_{ij}\ \boldsymbol{\delta}_k = \sum_i \sum_k w_i\ v_i\ v_k\ \boldsymbol{\delta}_k =$$

$$= \sum_k \left(\sum_i w_i\ v_i\right) v_k\ \boldsymbol{\delta}_k \quad q.e.d.\ .$$

5.1.4 Mechanical energy equation

Multiplying equation (2.3.23) by X_k and averaging yields

$$\overline{X_k \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right)} + \overline{X_k \nabla \cdot \left(\frac{1}{2}\rho v^2 \mathbf{v}\right)} =$$

$$[5.1.4.1]$$

$$= \overline{X_k \nabla \cdot \left(\underline{\mathbf{T}} \cdot \mathbf{v}\right)} - \overline{X_k \left(\underline{\mathbf{T}} : \nabla \mathbf{v}\right)} + \overline{\left(X_k\ \rho\ \mathbf{g} \cdot \mathbf{v}\right)}\ .$$

By means of the identities

$$X_k \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = \frac{\partial}{\partial t}\left(X_k \frac{1}{2}\rho v^2\right) - \frac{1}{2}\rho v^2 \frac{\partial}{\partial t} X_k\ , \qquad [5.1.4.2]$$

$$X_k \nabla \cdot \left(\frac{1}{2}\rho v^2 \mathbf{v}\right) = \nabla \cdot \left(X_k \frac{1}{2}\rho v^2 \mathbf{v}\right) - \frac{1}{2}\rho v^2\ \mathbf{v} \cdot \nabla X_k\ , \qquad [5.1.4.3]$$

$$X_k \nabla \cdot \left(\underline{\mathbf{T}} \cdot \mathbf{v}\right) = \nabla \cdot \left(X_k \underline{\mathbf{T}} \cdot \mathbf{v}\right) - \left(\underline{\mathbf{T}} \cdot \mathbf{v}\right) \cdot \nabla X_k\ , \qquad [5.1.4.4]$$

because of the relationship

$$X_k \left(\underline{T} : \nabla \mathrm{v}\right) = X_k \left(\underline{T} : \nabla \overline{\mathrm{v}^{x,\rho}}\right) + X_k \left(\underline{T} : \nabla \mathrm{v}'\right) \qquad [5.1.4.5]$$

and with equation (3.2.4.3) we obtain

$$\frac{\partial}{\partial t} \overline{\left(X_k \frac{1}{2} \rho v^2\right)} + \nabla \cdot \overline{\left(X_k \frac{1}{2} \rho v^2 \mathrm{v}\right)} = \nabla \cdot \overline{\left(X_k \underline{T} \cdot \mathrm{v}\right)} +$$

$$+ \overline{\frac{1}{2} \rho v^2 \left(\mathrm{v} - \mathrm{v}_i\right) \cdot \nabla X_k} - \overline{\left(\underline{T} \cdot \mathrm{v}\right) \cdot \nabla X_k} - \overline{X_k \left(\underline{T} : \nabla \overline{\mathrm{v}_k^{x,\rho}}\right)} - \qquad [5.1.4.6]$$

$$- \overline{X_k \left(\underline{T} : \nabla \mathrm{v}_k'\right)} + \overline{\left(X_k \rho \mathbf{g} \cdot \mathrm{v}\right)} .$$

The left hand-side of equation (5.1.4.6) has been considered in the frame of the deduction of the total energy equation, in equations (5.1.3.2) through (5.1.3.7), just setting $u = 0$. Thus, we write

$$\frac{\partial}{\partial t} \overline{\left(X_k \frac{1}{2} \rho v^2\right)} + \nabla \cdot \overline{\left(X_k \frac{1}{2} \rho v^2 \mathrm{v}\right)} =$$

$$= \frac{\partial}{\partial t} \overline{\alpha_k} \, \overline{\rho_k^x} \left[\frac{1}{2} \left(\overline{v_k^{x,\rho}}\right)^2 + \overline{u_k^{Re}}\right] + \nabla \cdot \overline{\alpha_k} \, \overline{\rho_k^x} \, \overline{\mathrm{v}^{x,\rho}} \left[\frac{1}{2} \left(\overline{v_k^{x,\rho}}\right)^2 + \overline{u_k^{Re}}\right] - \qquad [5.1.4.7]$$

$$- \nabla \cdot \overline{\alpha_k} \left(\overline{\underline{T}_k^{Re}} \cdot \overline{\mathrm{v}_k^{x,\rho}}\right) + \nabla \cdot \overline{\alpha_k} \, \overline{\mathrm{q}_k^{kin}} .$$

According to equation (5.1.3.17), the first term on the right hand-side of equation (5.1.4.6) is

$$\nabla \cdot \overline{\left(X_k \underline{T} \cdot \mathrm{v}\right)} = \nabla \cdot \left(\overline{\alpha_k} \, \overline{\underline{T}_k^x} \cdot \overline{\mathrm{v}_k^{x,\rho}}\right) - \nabla \cdot \overline{\alpha_k} \left(\overline{\mathrm{q}_k^p} + \overline{\mathrm{q}_k^\tau}\right) . \qquad (5.1.3.17)$$

Furthermore, we use the definitions

$$\frac{1}{2} \overline{v_{ki}^2} \, \overline{\Gamma_k} = \overline{\frac{1}{2} \rho v^2 \left(\mathrm{v} - \mathrm{v}_i\right) \cdot \nabla X_k} , \qquad (4.5.6)$$

$$\overline{W_k} = - \overline{\left(\underline{T} \cdot \mathrm{v}\right) \cdot \nabla X_k} . \qquad (4.6.5)$$

For the fourth term on the right hand-side of equation (5.1.4.6), we can write by means of definition (4.3.2)

$$- \overline{X_k \left(\underline{T} : \nabla \overline{v_k^{x,\rho}} \right)} = - \overline{\alpha_k} \; \overline{\underline{T}_k^x} : \nabla \overline{v_k^{x,\rho}} \; . \qquad [5.1.4.8]$$

With equation (2.3.3) and the definition of the turbulent energy dissipation $\overline{D}_k$ (4.7.4), we obtain for the fifth term on the right hand-side of equation (5.1.4.6) the following form:

$$- \overline{X_k \left(\underline{T} : \nabla v_k' \right)} = \overline{X_k \, p \left(\nabla \cdot v_k' \right)} - \overline{\alpha_k} \; \overline{D_k} \; . \qquad [5.1.4.9]$$

We used here the identity

$$p \, \underline{I} : \nabla v = p \, (\nabla \cdot v) \; . \qquad [5.1.4.10]$$

Thus we obtain

$$\frac{\partial}{\partial t} \overline{\alpha_k} \; \overline{\rho_k^x} \left[\frac{1}{2} \left(\overline{v_k^{x,\rho}} \right)^2 + \overline{u_k^{Re}} \right] + \nabla \cdot \overline{\alpha_k} \; \overline{\rho_k^x} \; \overline{v_k^{x,\rho}} \left[\frac{1}{2} \left(\overline{v_k^{x,\rho}} \right)^2 + \overline{u_k^{Re}} \right] =$$

$$= \nabla \cdot \overline{\alpha_k} \left[\left(\overline{\underline{T}_k^x} + \overline{\underline{T}_k^{Re}} \right) \cdot \overline{v_k^{x,\rho}} - \overline{q_k^{kin}} - \overline{q_k^{p}} - \overline{q_k^{\tau}} \right] + \qquad [5.1.4.11]$$

$$+ \frac{1}{2} \overline{v_{ki}^2} \; \overline{\Gamma_k} + \overline{W_k} - \overline{\alpha_k} \; \overline{\underline{T}_k^x} : \nabla \overline{v_k^{x,\rho}} + \overline{X_k \, p \left(\nabla \cdot v_k' \right)} - \overline{\alpha_k} \; \overline{D_k} + \overline{\alpha_k} \; \overline{\rho_k^x} \; g \cdot \overline{v_k^{x,\rho}} \; .$$

5.1.5 Internal energy equation

We subtract the mechanical energy equation (5.1.4.11) from the total energy equation (5.1.3.19) to deduce the internal energy equation. Thus we obtain

$$\frac{\partial}{\partial t} \left(\overline{\alpha_k} \; \overline{\rho_k^x} \; \overline{u_k^{x,\rho}} \right) + \nabla \cdot \left(\overline{\alpha_k} \; \overline{\rho_k^x} \; \overline{v_k^{x,\rho}} \; \overline{u_k^{x,\rho}} \right) =$$

$$= - \nabla \cdot \overline{\alpha_k} \left(\overline{q_k^x} + \widehat{\overline{q_k^{Re}}} \right) + \overline{u_{ki}} \; \overline{\Gamma_k} + \overline{E_k} + \overline{\alpha_k} \; \overline{D_k} + \qquad [5.1.5.1]$$

$$+ \overline{\alpha_k} \; \overline{\underline{T}_k^x} : \nabla \overline{v_k^{x,\rho}} - \overline{X_k \, p \, \nabla \cdot v_k'} + \overline{\alpha_k} \; \overline{\rho_k^x} \; \overline{r_k^{x,\rho}} \; .$$

5.1.6 Enthalpy equation

Multiplying equation (2.3.35) by X_k and averaging yields

$$\overline{X_k \frac{\partial}{\partial t}(\rho\, h)} + \overline{X_k \nabla \cdot (\rho\, h\, \mathbf{v})} =$$

$$= \overline{X_k \frac{\partial p}{\partial t}} + X_k\, \overline{\mathbf{v} \cdot \nabla p} - \overline{X_k \nabla \cdot q} + \overline{X_k \left(\underline{\mathrm{T}} : \nabla \mathbf{v}\right)} + \overline{X_k\, \rho\, r}\ . \qquad [5.1.6.1]$$

With the identities

$$X_k \frac{\partial\, \rho\, h}{\partial t} = \frac{\partial \left(X_k\ \rho\, h\right)}{\partial t} - \rho\, h \frac{\partial X_k}{\partial t}\ , \qquad [5.1.6.2]$$

$$X_k \nabla \cdot (\rho\, h\ \mathbf{v}) = \nabla \cdot X_k\, \rho\, h\ \mathbf{v} - \rho\, h\ \mathbf{v} \cdot \nabla X_k\ , \qquad [5.1.6.3]$$

$$X_k \nabla \cdot \mathbf{q} = \nabla \cdot X_k\, \mathbf{q} - \mathbf{q} \cdot \nabla X_k\ , \qquad [5.1.6.4]$$

together with equation

$$\frac{\partial X_k}{\partial t} + \mathbf{v}_i \cdot \nabla X_k = 0 \qquad (3.2.4.3)$$

and noting that

$$\mathbf{v}_k = \overline{\mathbf{v}_k^{x,\rho}} + \mathbf{v}_k'\ ,$$

$$p_k = \overline{p_k^{x}} + p_k'\ ,$$

$$h_k = \overline{h_k^{x,\rho}} + h_k'\ ,$$

equation (5.1.6.1) becomes

$$\frac{\partial}{\partial t}\overline{\left(X_k\, \rho\, h\right)} + \nabla \cdot \overline{\left(X_k\, \rho \left(\overline{h_k^{x,\rho}} + h_k' \right)\left(\overline{\mathbf{v}_k^{x,\rho}} + \mathbf{v}_k' \right)\right)} = \overline{\rho\, h \left(\mathbf{v} - \mathbf{v}_i\right) \cdot \nabla X_k} +$$

$$+ \overline{X_k \left(\frac{\partial\, \overline{p_k^{x}}}{\partial t} + \frac{\partial p_k'}{\partial t} \right)} + \overline{X_k \left(\overline{\mathbf{v}_k^{x,\rho}} + \mathbf{v}_k' \right) \cdot \nabla \left(\overline{p_k^{x}} + p_k' \right)} - \overline{\nabla \cdot X_k \mathbf{q}} + \qquad [5.1.6.5]$$

$$+ \overline{\mathbf{q} \cdot \nabla X_k} + \overline{X_k\, \underline{\mathrm{T}} : \nabla\, \overline{\mathbf{v}_k^{x,\rho}}} + \overline{X_k\, \underline{\mathrm{T}} : \nabla\, \mathbf{v}_k'} + \overline{X_k\, \rho\, r}\ .$$

By means of the definitions

$$\overline{h_k^{x,\rho}} = \overline{X_k \rho h} \,/\, \overline{\alpha_k}\ \overline{\rho_k^x} \ , \tag{4.1.5}$$

$$\hat{\hat{\mathbf{q}}}_k^{Re} = \overline{X_k \rho h_k^{'} \mathbf{v}_k^{'}} \,/\, \overline{\alpha_k} \ , \tag{4.4.3}$$

$$\overline{h_{ki}}\ \overline{\Gamma_k} = \overline{\rho h \left(\mathbf{v} - \mathbf{v}_i\right) \cdot \nabla X_k} \ , \tag{4.5.4}$$

$$\overline{\mathbf{q}_k^x} = \overline{X_k \mathbf{q}} \,/\, \overline{\alpha_k} \ , \tag{4.3.6}$$

$$\overline{E_k} = \overline{\mathbf{q} \cdot \nabla X_k} \ , \tag{4.6.6}$$

$$\overline{D_k} = \overline{X_k \underline{\tau} : \nabla \mathbf{v}_k^{'}} \,/\, \overline{\alpha_k} \ , \tag{4.7.4}$$

$$\overline{r_k^{x,\rho}} = \overline{X_k \rho r} \,/\, \overline{\alpha_k}\ \overline{\rho_k^x} \ , \tag{4.2.3}$$

we obtain after some rearrangements

$$\frac{\partial}{\partial t}\left(\overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{h_k^{x,\rho}}\right) + \nabla \cdot \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{h_k^{x,\rho}}\ \overline{\mathbf{v}_k^{x,\rho}} =$$

$$= - \nabla \cdot \overline{\alpha_k}\left(\overline{\mathbf{q}_k^x} + \hat{\hat{\mathbf{q}}}_k^{Re}\right) + \overline{h_{ki}}\ \overline{\Gamma_k} + \overline{\alpha_k}\left(\frac{\partial \overline{p_k^x}}{\partial t} + \overline{\mathbf{v}_k^{x,\rho}} \cdot \nabla \overline{p_k^x}\right) + \tag{5.1.6.6}$$

$$+ \overline{X_k\left(\frac{\partial p_k^{'}}{\partial t} + \mathbf{v}_k \cdot \nabla p_k^{'}\right)} + \overline{E_k} + \overline{\alpha_k}\,\overline{\underline{T}_k^x} : \nabla \overline{\mathbf{v}_k^{x,\rho}} + \overline{\alpha_k}\ \overline{D_k} - \overline{X_k p \nabla \cdot \mathbf{v}_k^{'}} + \overline{\alpha_k}\,\overline{\rho_k^x}\ \overline{r_k^{x,\rho}} \,.$$

5.1.7 Entropy inequality and entropy equation

i) Entropy inequality

We multiply equations (2.3.38) and (2.3.58) with X_k and average:

$$\overline{X_k \frac{\partial}{\partial t}(\rho s)} + \overline{X_k \nabla \cdot (\rho s \mathbf{v})} + \overline{X_k \nabla \cdot \frac{\mathbf{q}}{T}} - \overline{X_k \frac{\rho r}{T}} = \overline{X_k \Delta s} \geq 0 \ , \tag{5.1.7.1}$$

$$\overline{X_k\ \Delta s} = \overline{X_k\, \mathbf{q} \cdot \nabla\left(\frac{1}{T}\right)} + \overline{\frac{X_k}{T}\ (\underline{\tau} : \nabla \mathbf{v})} \geq 0\ . \qquad [5.1.7.2]$$

Using the identities

$$X_k \frac{\partial}{\partial t} (\rho\, s) = \frac{\partial}{\partial t} (X_k\, \rho\, s) - \rho\, s \frac{\partial X_k}{\partial t}\ , \qquad [5.1.7.3]$$

$$X_k\ \nabla \cdot (\rho\, s\, \mathbf{v}) = \nabla \cdot (X_k\, \rho\, s\, \mathbf{v}) - \rho\, s\, \mathbf{v} \cdot \nabla X_k\ , \qquad [5.1.7.4]$$

$$X_k\ \nabla \cdot \frac{\mathbf{q}}{T} = \nabla \cdot \left(X_k \frac{\mathbf{q}}{T}\right) - \frac{\mathbf{q}}{T} \cdot \nabla X_k , \qquad [5.1.7.5]$$

and equation (3.2.4.3) we obtain

$$\frac{\partial}{\partial t}\ \overline{(X_k\, \rho\, s)} + \nabla \cdot \left[\overline{X_k\, \rho\, s\left(\overline{\mathbf{v}_k^{x,\rho}} + \mathbf{v}_k'\right)}\right] - \overline{\rho\, s\left(\mathbf{v} - \mathbf{v}_i\right) \cdot \nabla X_k} + \nabla \cdot \overline{\left(X_k \frac{\mathbf{q}}{T}\right)} -$$

$$- \overline{\frac{\mathbf{q}}{T} \cdot \nabla X_k} - \overline{X_k \frac{\rho r}{T}} = \overline{X_k \Delta s} \geq 0 \qquad [5.1.7.6]$$

and

$$\overline{X_k\ \Delta s} = \overline{X_k \mathbf{q} \cdot \nabla\left(\frac{1}{T}\right)} + \overline{\frac{X_k}{T} \left[\underline{\tau} : \left(\overline{\mathbf{v}_k^{x,\rho}} + \mathbf{v}_k'\right)\right]}\ . \qquad [5.1.7.7]$$

Using in equation (5.1.7.6) $s_k = \overline{s_k^{x,\rho}} + s_k'$ and the definitions

$$\overline{s_k^{x,\rho}} = \overline{X_k\, \rho\, s} \,/\, \left(\overline{\alpha_k}\ \overline{\rho_k^x}\right), \qquad (4.1.6)$$

$$\overline{\Phi_k^{Re}} = \overline{- X_k\, \rho\, s'\, \mathbf{v}'} \,/\, \overline{\alpha_k}\ , \qquad (4.4.7)$$

$$\overline{s_{ki}}\ \overline{\Gamma_k} = \overline{\rho\, s\left(\mathbf{v} - \mathbf{v}_i\right) \cdot \nabla X_k}\ , \qquad (4.5.8)$$

$$\overline{\Phi_k^x} = - \overline{X_k\, \mathbf{q}/T} \,/\, \overline{\alpha_k}\ , \qquad (4.3.7)$$

$$\overline{s_k} = \overline{\frac{\mathbf{q}}{T} \cdot \nabla X_k}\ , \qquad (4.6.7)$$

$$\overline{\sigma_k^{x,\rho}} = \overline{X_k\, \rho\, r/T} \,/\, \left(\overline{\alpha_k}\ \overline{\rho_k^x}\right), \qquad (4.2.4)$$

using in equation (5.1.7.7) the definitions

$$\overline{s_{Tk}} = \overline{X_k \; \mathbf{q} \cdot \nabla\left(\frac{1}{T}\right)} \,, \tag{4.8.3}$$

$$\overline{\underline{\tau}^x_{sk}} = \frac{\overline{X_k \frac{\underline{\underline{\tau}}}{T}}}{\alpha_k} \,, \tag{4.3.8}$$

$$\overline{D_{sk}} = \overline{\frac{X_k}{T} \underline{\tau} : \nabla \mathbf{v}_k' \; / \alpha_k} \tag{4.7.8}$$

and letting

$$\overline{\Delta s_k} = \overline{X_k \; \Delta s} \tag{5.1.7.8}$$

one derives the entropy inequality for phase k

$$\frac{\partial}{\partial t}\left(\overline{\alpha_k} \; \overline{\rho^x_k} \; \overline{s^{x,\rho}_k}\right) + \nabla\cdot\left(\overline{\alpha_k} \; \overline{\rho^x_k} \; \overline{s^{x,\rho}_k} \; \overline{\mathbf{v}^{x,\rho}_k}\right) - \overline{s_{ki}} \; \overline{\Gamma_k} - \nabla\cdot \overline{\alpha_k}\left(\overline{\Phi^x_k} + \overline{\Phi^{Re}_k}\right) -$$

$$- \overline{s_k} - \overline{\alpha_k} \; \overline{\rho^x_k} \; \overline{\sigma^{x,\rho}_k} = \overline{\Delta s_k} \geq 0 \tag{5.1.7.9}$$

with

$$\overline{\Delta s_k} = \overline{s_{Tk}} + \overline{\alpha_k} \; \overline{\underline{\tau}^x_{sk}} : \nabla \overline{\mathbf{v}^{x,\rho}_k} + \overline{\alpha_k} \; \overline{D_{sk}} \,. \tag{5.1.7.10}$$

ii) Entropy equation

The entropy equation can be derived directly combining (5.1.7.9) and (5.1.7.10) or independently from the entropy equation for the continuum. The latter derivation is shown in detail to illustrate how the analytical expression of $\overline{\Delta s_k}$, given by equation (5.1.7.10), can be obtained independently of (5.1.7.7).

We multiply equation (2.3.55) by X_k and average:

$$\overline{X_k \frac{\partial}{\partial t}\left(\rho s\right)} + \overline{X_k \nabla\cdot(\rho s \mathbf{v})} = -\overline{\frac{X_k}{T}\left(\nabla\cdot\mathbf{q}\right)} + \overline{\frac{X_k}{T}\left(\underline{\tau} : \nabla \mathbf{v}\right)} + \overline{X_k \frac{\rho r}{T}} \,. \tag{5.1.7.11}$$

With the identities

$$X_k \frac{\partial}{\partial t}(\rho s) = \frac{\partial}{\partial t}(X_k \rho s) - \rho s \frac{\partial X_k}{\partial t} , \qquad (5.1.7.3)$$

$$X_k \nabla \cdot (\rho s \mathbf{v}) = \nabla \cdot (X_k \rho s \mathbf{v}) - \rho s \mathbf{v} \cdot \nabla X_k , \qquad (5.1.7.4)$$

$$\frac{X_k}{T}\left(\nabla \cdot \mathbf{q}\right) = \nabla \cdot \left(X_k \frac{\mathbf{q}}{T}\right) - \mathbf{q} \cdot \nabla\left(\frac{X_k}{T}\right) = \nabla \cdot \left(X_k \frac{\mathbf{q}}{T}\right) - \frac{\mathbf{q}}{T} \cdot \nabla X_k - X_k \mathbf{q} \cdot \nabla\left(\frac{1}{T}\right), \qquad [5.1.7.12]$$

and by means of equation (3.2.4.3) we obtain

$$\frac{\partial}{\partial t} \overline{X_k \rho s} + \nabla \cdot \overline{X_k \rho s\left(\overline{\mathbf{v}_k^{x,\rho}} + \mathbf{v}_k'\right)} = \overline{\rho s\left(\mathbf{v} - \mathbf{v}_i\right) \cdot \nabla X_k} +$$

[5.1.7.13]

$$+ \overline{\frac{X_k}{T}\left[\underline{\tau} : \left(\nabla \overline{\mathbf{v}_k^{x,\rho}} + \mathbf{v}_k'\right)\right]} - \nabla \cdot \overline{X_k \frac{\mathbf{q}}{T}} + \overline{\frac{\mathbf{q}}{T} \cdot \nabla X_k} + \overline{X_k \mathbf{q} \cdot \nabla\left(\frac{1}{T}\right)} + \overline{X_k \rho\, r/T} .$$

With the definitions

$$\overline{s_k^{x,\rho}} = \overline{X_k \rho s} \,/\, \left(\overline{\alpha_k}\; \overline{\rho_k^x}\right) , \qquad (4.1.6)$$

$$\overline{\Phi_k^{Re}} = -\overline{X_k \rho s' \mathbf{v}'} \,/\, \overline{\alpha_k} , \qquad (4.4.7)$$

$$\overline{s_{ki}}\; \overline{\Gamma_k} = \overline{\rho s\left(\mathbf{v} - \mathbf{v}_i\right) \cdot \nabla X_k} , \qquad (4.5.8)$$

$$\overline{\underline{\tau}_{sk}^x} = \frac{\overline{X_k \frac{\underline{\underline{\tau}}}{T}}}{\overline{\alpha_k}} , \qquad (4.3.8)$$

$$\overline{D_{sk}} = \left(\overline{X_k \frac{\underline{\underline{\tau}}}{T} : \nabla \mathbf{v}_k'}\right) / \overline{\alpha_k} , \qquad (4.7.8)$$

$$\overline{\Phi_k^x} = -\overline{X_k \mathbf{q}/T} \,/\, \overline{\alpha_k} , \qquad (4.3.7)$$

$$\overline{s_k} = \overline{\frac{\mathbf{q}}{T} \cdot \nabla X_k} , \qquad (4.6.7)$$

$$\overline{s_{Tk}} = \overline{X_k\, \mathbf{q} \cdot \nabla\left(\frac{1}{T}\right)} , \qquad (4.8.3)$$

$$\overline{\sigma_k^{x,\rho}} = \overline{X_k \, \rho \, r/T} \,/ \left(\overline{\alpha_k} \; \overline{\rho_k^x} \right), \qquad (4.2.4)$$

equation (5.1.7.5) becomes

$$\frac{\partial}{\partial t}\left(\overline{\alpha_k} \; \overline{\rho_k^x} \; \overline{s_k^{x,\rho}} \right) + \nabla \cdot \left(\overline{\alpha_k} \; \overline{\rho_k^x} \; \overline{\mathbf{v}_k^{x,\rho}} \; \overline{s_k^{x,\rho}} \right) - \overline{\alpha_k} \; \overline{\underline{\tau}_{sk}^x} : \nabla \, \overline{\mathbf{v}_k^{x,\rho}} - \overline{\alpha_k} \; \overline{D_{sk}} -$$

$$[5.1.7.14]$$

$$- \nabla \cdot \overline{\alpha_k} \left(\overline{\Phi_k^x} + \overline{\Phi_k^{Re}} \right) - \overline{s_{ki}} \; \overline{\Gamma_k} - \overline{s_k} - \overline{s_{Tk}} - \overline{\alpha_k} \; \overline{\rho_k^x} \; \overline{\sigma_k^{x,\rho}} = 0 \, .$$

Comparing (5.1.7.14) with (5.1.7.9) one derives again the analytical expression of $\overline{\Delta s_k}$ as given by equation (5.1.7.10).

5.2 Langrangian form of averaged three-dimensional conservation equations

5.2.1 Momentum equation

We consider the left hand side of equation (5.1.2.8) and decompose it according to the identity

$$\nabla \cdot (\mathbf{v}\,\mathbf{w}) = \mathbf{v} \cdot \nabla \, \mathbf{w} + \mathbf{w} \, (\nabla \cdot \mathbf{v}) \, .$$

Thus, we obtain

$$\frac{\partial}{\partial t}\left(\overline{\alpha_k} \; \overline{\rho_k^x} \; \overline{\mathbf{v}_k^{x,\rho}} \right) + \nabla \cdot \left(\overline{\alpha_k} \; \overline{\rho_k^x} \; \overline{\mathbf{v}_k^{x,\rho}} \; \overline{\mathbf{v}_k^{x,\rho}} \right)$$

$$[5.2.1.1]$$

$$= \left(\frac{\partial \alpha_k \rho_k^x}{\partial t} \right) \overline{\mathbf{v}^{x,\rho}} + \left(\nabla \cdot \overline{\alpha_k} \; \overline{\rho_k^x} \; \overline{\mathbf{v}_k^{x,\rho}} \right) \overline{\mathbf{v}_k^{x,\rho}} + \overline{\alpha_k} \; \overline{\rho_k^x} \; \frac{\partial \, \overline{\mathbf{v}_k^{x,\rho}}}{\partial t} + \overline{\alpha_k} \; \overline{\rho_k^x} \; \overline{\mathbf{v}_k^{x,\rho}} \cdot \nabla \, \overline{\mathbf{v}_k^{x,\rho}} \, .$$

The continuity equation (5.1.1.5) combined with equation (5.2.1.1) yields

$$\frac{\partial}{\partial t}\left(\overline{\alpha_k} \; \overline{\rho_k^x} \; \overline{\mathbf{v}_k^{x,\rho}} \right) + \nabla \cdot \left(\overline{\alpha_k} \; \overline{\rho_k^x} \; \overline{\mathbf{v}_k^{x,\rho}} \; \overline{\mathbf{v}_k^{x,\rho}} \right) =$$

$$= \overline{\mathbf{v}_k^{x,\rho}} \; \overline{\Gamma_k} + \overline{\alpha_k} \; \overline{\rho_k^x} \; \frac{\partial \, \overline{\mathbf{v}_k^{x,\rho}}}{\partial t} + \overline{\alpha_k} \; \overline{\rho_k^x} \; \overline{\mathbf{v}_k^{x,\rho}} \cdot \nabla \, \overline{\mathbf{v}_k^{x,\rho}} = \qquad [5.2.1.2]$$

$$= \overline{\mathbf{v}_k^{x,\rho}} \; \overline{\Gamma_k} + \overline{\alpha_k} \; \overline{\rho_k^x} \; \frac{D_k \, \overline{\mathbf{v}_k^{x,\rho}}}{Dt} \, ,$$

with the substantial derivative of $\overline{\mathbf{v}_k^{x,\rho}}$ given by

$$\frac{D_k \overline{\mathbf{v}_k^{x,\rho}}}{Dt} = \frac{\partial \overline{\mathbf{v}_k^{x,\rho}}}{\partial t} + \overline{\mathbf{v}_k^{x,\rho}} \cdot \nabla \overline{\mathbf{v}_k^{x,\rho}} \ .$$

Thus, we obtain the Langrangian form of the momentum equation:

$$\overline{\alpha_k}\ \overline{\rho_k^x}\ \frac{D_k \overline{\mathbf{v}_k^{x,\rho}}}{Dt} = \nabla \cdot \overline{\alpha_k} \left(\overline{\underline{\mathbf{T}}_k^x} + \overline{\underline{\mathbf{T}}_k^{Re}} \right) + \overline{\alpha_k}\ \overline{\rho_k^x}\ \mathbf{g} + \left(\overline{\mathbf{v}_{ki}} - \overline{\mathbf{v}_k^{x,\rho}} \right) \overline{\Gamma_k} + \overline{\mathbf{M}_k} \ , \qquad [5.2.1.3]$$

or, by means of equation (4.10.9),

$$\overline{\alpha_k}\ \overline{\rho_k^x}\ \frac{D_k \overline{\mathbf{v}_k^{x,\rho}}}{Dt} = \nabla \cdot \overline{\alpha_k} \left(\overline{\underline{\mathbf{T}}_k^x} + \overline{\underline{\mathbf{T}}_k^{Re}} \right) + \overline{\alpha_k}\ \overline{\rho_k^x}\ \mathbf{g} +$$

$$[5.2.1.4]$$

$$+ \left(\overline{\mathbf{v}_{ki}} - \overline{\mathbf{v}_k^{x,\rho}} \right) \overline{\Gamma_k} + \overline{p_{ki}}\ \nabla \overline{\alpha_k} - \overline{\underline{\tau}_{ki}} \cdot \nabla \overline{\alpha_k} + \overline{\mathbf{M}_k'} \ .$$

5.2.2 Total energy equation

The left hand side of equation (5.1.3.19) is written in the following form

$$\frac{\partial}{\partial t} \left(\overline{\alpha_k}\ \overline{\rho_k^x}\ \tilde{\overline{e_k}} \right) + \nabla \cdot \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{\mathbf{v}_k^{x,\rho}}\ \tilde{\overline{e_k}} =$$

$$[5.2.2.1]$$

$$= \left(\frac{\partial \overline{\alpha_k}\ \overline{\rho_k^x}}{\partial t} \right) \tilde{\overline{e_k}} + \left(\nabla \cdot \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{\mathbf{v}_k^{x,\rho}} \right) \tilde{\overline{e_k}} + \overline{\alpha_k}\ \overline{\rho_k^x}\ \frac{\partial \tilde{\overline{e_k}}}{\partial t} + \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{\mathbf{v}_k^{x,\rho}} \cdot \nabla \tilde{\overline{e_k}} \ ,$$

with the auxiliary definition

$$\tilde{\overline{e_k}} = \overline{u_k^{x,\rho}} + \frac{1}{2} \left(\overline{\upsilon_k^{x,\rho}} \right)^2 + \overline{u_k^{Re}} \ . \qquad [5.2.2.2]$$

The continuity equation (5.1.1.5) combined with equation (5.2.2.1) yields the Lagrangian form of the total energy equation (5.1.3.19)

$$\tilde{\overline{e_k}}\ \overline{\Gamma_k} + \overline{\alpha_k}\ \overline{\rho_k^x}\ \frac{D_k \tilde{\overline{e_k}}}{Dt} = \nabla \cdot \overline{\alpha_k} \left[\left(\overline{\underline{\mathbf{T}}_k^x} + \overline{\underline{\mathbf{T}}_k^{Re}} \right) \cdot \overline{\mathbf{v}_k^{x,\rho}} - \overline{\mathbf{q}_k^x} - \overline{\mathbf{q}_k^{Re}} \right] +$$

$$[5.2.2.3]$$

$$+ \left(\overline{u_{ki}} + \frac{1}{2}\ \overline{\upsilon_{ki}^2} \right) \overline{\Gamma_k} + \overline{E_k} + \overline{W_k} + \overline{\alpha_k}\ \overline{\rho_k^x} \left(\mathbf{g} \cdot \overline{\mathbf{v}_k^{x,\rho}} + \overline{r^{x,\rho}} \right) .$$

Equation (5.2.2.3) may be rearranged by the following procedure: In equation (4.11.8), the interfacial work W_k has been defined by

$$\overline{W_k} = \overline{W_k'} - \overline{p_{ki}} \frac{\partial \overline{\alpha_k}}{\partial t} + \overline{p\left(\mathbf{v}-\mathbf{v}_i\right)\cdot\nabla X_k} \qquad (4.11.8)$$

with the interfacial extra work $\overline{W_k'}$ defined by equation (4.7.7).
Furthermore, noting that the specific enthalpy is $h = u + p/\rho$, we may write

$$\overline{\rho\, h\left(\mathbf{v}-\mathbf{v}_i\right)\cdot\nabla X_k} = \overline{\rho\, u\left(\mathbf{v}-\mathbf{v}_i\right)\cdot\nabla X_k} + \overline{p\left(\mathbf{v}-\mathbf{v}_i\right)\cdot\nabla X_k} \qquad [5.2.2.4]$$

hence, with definitions (4.5.4) and (4.5.5),

$$\overline{p\left(\mathbf{v}-\mathbf{v}_i\right)\cdot\nabla X_k} = \overline{h_{ki}}\,\overline{\Gamma_k} - \overline{u_{ki}}\,\overline{\Gamma_k}\,. \qquad [5.2.2.5]$$

Thus, we may replace equation (5.2.2.3) by the following form

$$\overline{\alpha_k}\,\overline{\rho_k^x}\,\frac{D_k\,\tilde{e}_k}{Dt} = \nabla\cdot\,\overline{\alpha_k}\left[\left(\overline{\underline{T}_k^x} + \overline{\underline{T}_k^{Re}}\right)\cdot\,\overline{\mathbf{v}_k^{x,\rho}} - \overline{\mathbf{q}_k^x} - \overline{\mathbf{q}_k^{Re}}\right] +$$

$$[5.2.2.6]$$

$$+\left(\overline{h_{ki}} + \frac{1}{2}\,\overline{v_{ki}^2}\right)'\overline{\Gamma_k} - \tilde{e}_k\,\overline{\Gamma_k} + \overline{E_k} + \overline{W_k'} - \overline{p_{ki}}\,\frac{\partial\overline{\alpha_k}}{\partial t} + \overline{\alpha_k}\,\overline{\rho_k^x}\left(\mathbf{g}\cdot\overline{\mathbf{v}_k^{x,\rho}} + \overline{r^{x,\rho}}\right).$$

Drew and Wood [1] introduce an alternative form of equation (5.2.2.6) in the following way. By means of the identity

$$\frac{D_k}{Dt}\left[\frac{1}{2}\left(\overline{v_k^{x,\rho}}\right)^2\right] = \frac{1}{2}\,\frac{D_k}{Dt}\left[\overline{\mathbf{v}_k^{x,\rho}}\cdot\overline{\mathbf{v}_k^{x,\rho}}\right] = \overline{\mathbf{v}_k^{x,\rho}}\cdot\frac{D_k\,\overline{\mathbf{v}_k^{x,\rho}}}{Dt}$$

and the Lagrangian momentum equation (5.2.1.4), noting that

$$\tilde{e}_k = \overline{u_k^{x,\rho}} + \frac{1}{2}\left(\overline{v_k^{x,\rho}}\right)^2 + \overline{u_k^{Re}} \qquad (5.2.2.2)$$

we rearrange the left hand side of equation (5.2.2.6) to obtain

$$\overline{\alpha_k}\,\overline{\rho_k^x}\,\frac{D_k\,\tilde{e}_k}{Dt} = \overline{\alpha_k}\,\overline{\rho_k}\,\frac{D_k}{Dt}\left(\overline{u_k^{x,\rho}} + \overline{u_k^{Re}}\right) + \overline{\alpha_k}\,\overline{\rho_k^x}\,\overline{\mathbf{v}_k^{x,\rho}}\cdot\frac{D_k\,\overline{\mathbf{v}_k^{x,\rho}}}{Dt} =$$

$$= \overline{\alpha_k}\,\overline{\rho_k^x}\,\frac{D_k}{Dt}\left(\overline{u_k^{x,\rho}} + \overline{u_k^{Re}}\right) + \overline{\mathbf{v}_k^{x,\rho}}\cdot\left[\nabla\cdot\,\overline{\alpha_k}\left(\overline{\underline{T}_k^x} + \overline{\underline{T}_k^{Re}}\right) + \right. \qquad [5.2.2.7]$$

$$\left. + \overline{\alpha_k}\,\overline{\rho_k^x}\cdot\mathbf{g} + \left(\overline{\mathbf{v}_{ki}^m} - \overline{\mathbf{v}_k^{x,\rho}}\right)\overline{\Gamma_k} + \overline{p_{ki}}\,\nabla\overline{\alpha_k} - \overline{\underline{\tau}_{ki}}\cdot\nabla\overline{\alpha_k} + \overline{\mathbf{M}_k'}\right].$$

Thus, equation (5.2.2.6) yields

$$\overline{\alpha_k}\ \overline{\rho_k^x}\ \frac{D_k}{Dt}\left(\overline{u_k^{x,\rho}} + \overline{u_k^{Re}}\right) + \overline{\mathbf{v}_k^{x,\rho}} \cdot \Big[\nabla \cdot \overline{\alpha_k}\left(\overline{\underline{\mathbf{T}}_k^x} + \overline{\underline{\mathbf{T}}_k^{Re}}\right) + \overline{\alpha_k}\ \overline{\rho_k^x}\ \mathbf{g} +$$

$$+ \left(\overline{\mathbf{v}_{ki}} - \overline{\mathbf{v}_k^{x,\rho}}\right)\overline{\Gamma_k} + \overline{p_{ki}}\ \nabla\overline{\alpha_k} - \overline{\underline{\tau}_{ki}} \cdot \nabla\overline{\alpha_k} + \overline{\mathbf{M}_k'}\Big] =$$

$$= \nabla \cdot \overline{\alpha_k}\left[\left(\overline{\underline{\mathbf{T}}_k^x} + \overline{\underline{\mathbf{T}}_k^{Re}}\right) \cdot \overline{\mathbf{v}_k^{x,\rho}} - \overline{\mathbf{q}_k^x} - \overline{\mathbf{q}_k^{Re}}\right] + \qquad [5.2.2.8]$$

$$+ \left(\overline{h_{ki}} + \frac{1}{2}\ \overline{v_{ki}^2}\right)\overline{\Gamma_k} - \widetilde{\overline{e_k}}\ \overline{\Gamma_k} + \overline{E_k} + \overline{W_k'} -$$

$$- \overline{p_{ki}}\ \frac{\partial\overline{\alpha_k}}{\partial t} + \overline{\alpha_k}\ \overline{\rho_k^x}\left(\mathbf{g} \cdot \overline{\mathbf{v}_k^{x,\rho}} + \overline{r^{x,\rho}}\right).$$

Noting the tensor identity

$$\underline{\mathbf{T}} : \nabla\mathbf{v} = \nabla \cdot (\underline{\mathbf{T}} \cdot \mathbf{v}) - \mathbf{v} \cdot (\nabla \cdot \underline{\mathbf{T}}) \qquad (2.3.22)$$

and canceling the gravity terms, one obtains

$$\overline{\alpha_k}\ \overline{\rho_k^x}\ \frac{D_k}{Dt}\left(\overline{u_k^{x,\rho}} + \overline{u_k^{Re}}\right) = \overline{\alpha_k}\left(\overline{\underline{\mathbf{T}}_k^x} + \overline{\underline{\mathbf{T}}_k^{Re}}\right) : \nabla\ \overline{\mathbf{v}_k^{x,\rho}} -$$

$$- \overline{\mathbf{v}_k^{x,\rho}} \cdot \overline{p_{ki}}\ \nabla\overline{\alpha_k} + \overline{\mathbf{v}_k^{x,\rho}} \cdot \left(\overline{\underline{\tau}_{ki}} \cdot \nabla\overline{\alpha_k}\right) - \overline{\mathbf{v}_k} \cdot \overline{\mathbf{M}_k'} -$$

$$[5.2.2.9]$$

$$- \nabla \cdot \overline{\alpha_k}\left(\overline{\mathbf{q}_k^x} + \overline{\mathbf{q}_k^{Re}}\right) + \overline{E_k} + \overline{W_k'} - \overline{p_{ki}}\ \frac{\partial\overline{\alpha_k}}{\partial t} + \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{r_k} +$$

$$+ \left[- \overline{\mathbf{v}_k^{x,\rho}} \cdot \left(\overline{\mathbf{v}_{ki}} - \overline{\mathbf{v}_k^{x,\rho}}\right) + \overline{h_{ki}} + \frac{1}{2}\ \overline{v_{ki}^2} - \widetilde{\overline{e_k}}\right]\overline{\Gamma_k}.$$

Recalling the definition (5.2.2.2), the last term at the right side can be expanded as follows:

$$\left[- \overline{\mathbf{v}_k^{x,\rho}} \cdot \left(\overline{\mathbf{v}_{ki}} - \overline{\mathbf{v}_k^{x,\rho}}\right) + \overline{h_{ki}} + \frac{1}{2}\ \overline{v_{ki}^2} - \widetilde{\overline{e_k}}\right]\overline{\Gamma_k} =$$

$$= \left[- \overline{\mathbf{v}_k^{x,\rho}} \cdot \overline{\mathbf{v}_{ki}} + \left(\overline{v_k^{x,\rho}}\right)^2 + \overline{h_{ki}} + \frac{1}{2}\ \overline{v_{ki}^2} - \overline{u_k^{x,\rho}} - \frac{1}{2}\left(\overline{v_k^{x,\rho}}\right)^2 - \overline{u_k^{Re}}\right]\overline{\Gamma_k} = \qquad [5.2.2.10]$$

$$= \left[\overline{h_{ki}} - \overline{u_k^{x,\rho}} - \overline{u_k^{Re}} + \frac{1}{2}\left(\overline{v_{ki}^{x,\rho}}\right)^2 + \frac{1}{2}\ \overline{v_{ki}^2} - \overline{\mathbf{v}_k^{x,\rho}} \cdot \overline{\mathbf{v}_{ki}}\right]\overline{\Gamma_k}.$$

Introducing this expression into (5.2.2.9) one derives

$$\overline{\alpha_k}\ \overline{\rho^x_k}\ \frac{D_k}{Dt}\left(\overline{u^{x,\rho}_k} + \overline{u^{Re}_k}\right) =$$

$$= \overline{\alpha_k}\left(\overline{\underline{T}^x_k} + \overline{\underline{T}^{Re}_k}\right) : \nabla\ \overline{\mathbf{v}^{x,\rho}_k} - \nabla\cdot\ \overline{\alpha_k}\left(\overline{\mathbf{q}^x_k} + \overline{\mathbf{q}^{Re}_k}\right) +$$

$$+ \left\{\overline{h_{ki}} - \overline{u^{x,\rho}_k} - \overline{u^{Re}_k} + \frac{1}{2}\left[\overline{v^2_{ki}} + \left(\overline{v^{x,\rho}_k}\right)^2 - 2\,\overline{\mathbf{v}_{ki}}\cdot\overline{\mathbf{v}^{x,\rho}_k}\right]\right\}\overline{\Gamma_k} + \qquad [5.2.2.11]$$

$$+ \overline{\mathbf{E}_k} + \overline{W'_k} - \overline{p_{ki}}\left(\frac{\partial\overline{\alpha_k}}{\partial t} + \overline{\mathbf{v}^{x,\rho}_k}\cdot\nabla\overline{\alpha_k}\right) + \overline{\alpha_k}\ \overline{\rho^x_k}\ \overline{r^{x,\rho}_k} -$$

$$- \overline{\mathbf{M}'_k}\cdot\overline{\mathbf{v}^{x,\rho}_k} + \overline{\mathbf{v}^{x,\rho}_k}\cdot\left(\overline{\underline{\tau}_{ki}}\cdot\nabla\overline{\alpha_k}\right).$$

Using the identity

$$\left(\overline{h_{ki}} - \overline{u^{x,\rho}_k}\right)\overline{\Gamma_k} = \left(\overline{u_{ki}} - \overline{u^{x,\rho}_k}\right)\overline{\Gamma_k} + \overline{p\,(\mathbf{v}-\mathbf{v}_i)\cdot\nabla X_k}\ , \qquad [5.2.2.12]$$

and (4.10.9), (4.11.8), equation (5.2.2.11) can be written in more compact form in terms of $\overline{\mathbf{M}_k}$ and $\overline{\mathbf{W}_k}$ (instead of $\overline{\mathbf{M}_k}'$, $\overline{\mathbf{W}_k}'$):

$$\overline{\alpha_k}\ \overline{\rho^x_k}\ \frac{D_k}{Dt}\left(\overline{u^{x,\rho}_k} + \overline{u^{Re}_k}\right) = \overline{\alpha_k}\left(\overline{\underline{T}_k} + \overline{\underline{T}^{Re}_k}\right) : \nabla\ \overline{\mathbf{v}^{x,\rho}_k} - \nabla\cdot\ \overline{\alpha_k}\left(\overline{\mathbf{q}^x_k} + \overline{\mathbf{q}^{Re}_k}\right) +$$

$$+ \left\{\overline{u_{ki}} - \overline{u^{x,\rho}_k} - \overline{u^{Re}_k} + \frac{1}{2}\left[\overline{v^2_{ki}} + \left(\overline{v^{x,\rho}_k}\right)^2 - 2\,\overline{\mathbf{v}_{ki}}\cdot\overline{\mathbf{v}^{x,\rho}_k}\right]\right\}\overline{\Gamma_k} + \qquad [5.2.2.13]$$

$$+ \overline{W_k} + \overline{\mathbf{E}_k} - \overline{\mathbf{M}_k}\cdot\overline{\mathbf{v}^{x,\rho}_k} + \overline{\alpha_k}\ \overline{\rho^x_k}\ \overline{r^{x,\rho}_k}\ .$$

5.2.3 Mechanical energy equation

For the left hand side of equation (5.1.4.10) we write by means of the continuity equation (5.1.1.5)

$$\frac{\partial}{\partial t}\left\{\overline{\alpha_k}\ \overline{\rho^x_k}\left[\frac{1}{2}\left(\overline{v^{x,\rho}_k}\right)^2 + \overline{\mathrm{u}^{Re}_k}\right]\right\} + \nabla\cdot\left\{\overline{\alpha_k}\ \overline{\rho^x_k}\ \overline{\mathbf{v}^{x,\rho}_k}\left[\frac{1}{2}\left(\overline{v^{x,\rho}_k}\right)^2 + \overline{\mathrm{u}^{Re}_k}\right]\right\} =$$

$$= \overline{\alpha_k}\ \overline{\rho^x_k}\ \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\overline{v^{x,\rho}_k}\right)^2 + \overline{\mathrm{u}^{Re}_k}\right] + \overline{\alpha_k}\ \overline{\rho^x_k}\ \overline{\mathbf{v}^{x,\rho}_k}\cdot\nabla\left[\frac{1}{2}\left(\overline{v^{x,\rho}_k}\right)^2 + \overline{\mathrm{u}^{Re}_k}\right] + \qquad [5.2.3.1]$$

$$+ \left[\frac{1}{2}\left(\overline{v^{x,\rho}_k}\right)^2 + \overline{\mathrm{u}^{Re}_k}\right]\overline{\Gamma_k}\ .$$

Thus, we find for the Lagrangian form of the mechanical energy equation

$$\overline{\alpha_k}\ \overline{\rho_k^x}\ \frac{D}{Dt}\left[\frac{1}{2}\left(\overline{v_k^{x,\rho}}\right)^2 + \overline{u_k^{Re}}\right] =$$

$$= \nabla \cdot \overline{\alpha_k}\left[\left(\overline{\underline{T}_k^x} + \overline{\underline{T}_k^{Re}}\right)\cdot \overline{\mathbf{v}_k^{x,\rho}} - \overline{\mathbf{q}_k^{kin}} - \overline{\mathbf{q}_k^{p}} - \overline{\mathbf{q}_k^{\tau}}\right] +$$

$$+ \left[\frac{1}{2}\ \overline{v_{ki}^2} - \frac{1}{2}\left(\overline{v_k^{x,\rho}}\right)^2 - \overline{u_k^{Re}}\right]\overline{\Gamma_k} + \overline{W_k} - \qquad [5.2.3.2]$$

$$- \overline{\alpha_k}\ \overline{\underline{T}_k^x} : \nabla\overline{\mathbf{v}_k^{x,\rho}} + \overline{X_k\, p\left(\nabla\cdot\mathbf{v}_k'\right)} - \overline{\alpha_k}\ \overline{D_k} + \overline{\alpha_k}\ \overline{\rho_k^x}\ \mathbf{g}\cdot\overline{\mathbf{v}^{x,\rho}}\ .$$

5.2.4 Internal energy equation

The same procedure applied in section 5.2.3 leads to the following Lagrangian form of the internal energy equation (5.1.5.1):

$$\overline{\alpha_k}\ \overline{\rho_k^x}\ \frac{D}{Dt}\ \overline{u_k^{x,\rho}} = -\nabla\cdot\overline{\alpha_k}\left(\overline{\mathbf{q}_k^x} + \hat{\mathbf{q}}_k^{Re}\right) + \left(\overline{u_{ki}} - \overline{u_k^{x,\rho}}\right)\overline{\Gamma_k} + \qquad [5.2.4.1]$$

$$+ \overline{E_k} + \overline{\alpha_k}\ \overline{D_k} + \overline{\alpha_k}\ \overline{\underline{T}_k^x} : \nabla\overline{\mathbf{v}_k^{x,\rho}} - \overline{X_k\, p\ \nabla\cdot\mathbf{v}_k'} + \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{r_k^{x,\rho}}\ .$$

5.2.5 Enthalpy equation

The Lagrangian form of the enthalpy equation is deduced in the same way as before by application of the continuity equation (5.1.1.5) to equation (5.1.6.6). We obtain

$$\overline{\alpha_k}\ \overline{\rho_k^x}\ \frac{D}{Dt}\ \overline{h_k^{x,\rho}} = -\nabla\cdot\overline{\alpha_k}\left(\overline{\mathbf{q}_k^x} + \hat{\mathbf{q}}_k^{Re}\right) + \left(\overline{h_{ki}} - \overline{h_k^{x,\rho}}\right)\overline{\Gamma_k} + \overline{\alpha_k}\ \frac{D_k\overline{p_k^x}}{Dt} + \qquad [5.2.5.1]$$

$$+ \overline{X_k\frac{D_k p_k'}{Dt}} + \overline{E_k} + \overline{\alpha_k}\ \overline{\underline{T}_k^x} : \nabla\overline{\mathbf{v}_k^{x,\rho}} + \overline{\alpha_k}\ \overline{D_k} - \overline{X_k\, p\ \nabla\cdot\mathbf{v}_k'} + \overline{\alpha_k}\,\overline{\rho_k^x}\ \overline{r_k^{x,\rho}}\ .$$

5.2.6 Entropy inequality and entropy equation

i) Entropy inequality

$$\overline{\alpha_k}\ \overline{\rho_k^x}\ \frac{D}{Dt}\ \overline{s_k^{x,\rho}} - \nabla \cdot \overline{\alpha_k}\left(\overline{\Phi_k^x} + \overline{\Phi_k^{Re}}\right) - \left(\overline{s_{ki}} - \overline{s_k^{x,\rho}}\right)\overline{\Gamma_k} - \overline{s_k} - \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{\sigma_k^{x,\rho}} = \Delta s_k \geq 0 \quad [5.2.6.1]$$

with

$$\Delta s_k = s_{Tk} + \overline{\alpha_k}\ \overline{\underline{\tau}_{sk}^x} : \nabla \overline{v_k^{x,\rho}} + \overline{\alpha_k}\ \overline{D_{sk}} \geq 0 . \quad (5.1.7.10)$$

ii) Entropy equation

For the Lagrangian form of the entropy equation, we obtain finally from (5.1.7.14)

$$\overline{\alpha_k}\ \overline{\rho_k^x}\ \frac{D}{Dt}\ \overline{s_k^{x,\rho}} - \overline{\alpha_k}\ \overline{\underline{\tau}_{sk}^x} : \nabla \overline{v_k^{x,\rho}} - \overline{\alpha_k}\ \overline{D_{sk}} - \nabla \cdot \overline{\alpha_k}\left(\overline{\Phi_k^x} + \overline{\Phi_k^{Re}}\right) - \left(\overline{s_{ki}} - \overline{s_k^{x,\rho}}\right)\overline{\Gamma_k} - \quad [5.2.6.2]$$

$$- \overline{s_k} - \overline{s_{Tk}} - \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{\sigma_k^{x,\rho}} = 0 .$$

5.3 Averaged jump conditions

The general form of the averaged jump conditions is given by equation (3.4.2):

$$\overline{\left[\rho_1\psi_1\left(v_1 - v_i\right) - J_1\right] \cdot \nabla X_1} + \overline{\left[\rho_2\psi_2\left(v_2 - v_i\right) - J_2\right] \cdot \nabla X_2} = \quad (3.4.2)$$

$$= \overline{\left(n_1 \cdot \nabla X_1\right)m} = \overline{\left(n_k \cdot \nabla X_k\right)m} .$$

(Index k is not summed on the right hand side. The corresponding values of ψ_k, J_k and m are presented in Table I).

Thus, we derive the following averaged jump conditions:

(i) Mass

$$\overline{\Gamma_1} + \overline{\Gamma_2} = 0 \qquad [5.3.1]$$

with

$$\overline{\Gamma_k} = \overline{\rho\ (\mathbf{v}_k - \mathbf{v}_i) \cdot \nabla X_k}\ . \qquad (4.5.2)$$

(ii) Momentum

$$\overline{\mathbf{M}_1} + \overline{\mathbf{M}_2} + \overline{\mathbf{v}_{1i}}\ \overline{\Gamma_1} + \overline{\mathbf{v}_{2i}}\ \overline{\Gamma_2} = \overline{\mathbf{m}} \qquad [5.3.2]$$

with definitions

$$\overline{\mathbf{v}_{ki}}\ \overline{\Gamma_k} = \overline{\rho_k\ \mathbf{v}_k\,(\mathbf{v}_k - \mathbf{v}_i) \cdot \nabla X_k} \qquad (4.5.3)$$

$$\overline{\mathbf{M}_k} = \overline{-\underline{\mathbf{T}}_k \cdot \nabla X_k} \qquad (4.6.2)$$

and with

$$\overline{\mathbf{m}} = \overline{\left(\mathbf{n}_1 \cdot \nabla X_1\right) \mathbf{m}_i^{\sigma}} = \overline{\left(\mathbf{n}_2 \cdot \nabla X_2\right) \mathbf{m}_i^{\sigma}} =$$

$$[5.3.3]$$

$$= \overline{\left(\mathbf{n}_k \cdot \nabla X_k\right) \mathbf{m}_i^{\sigma}}\ .$$

(iii) Total energy

$$\overline{E_1} + \overline{W_1} + \overline{E_2} + \overline{W_2} + \left(\overline{u_{1i}} + \frac{1}{2}\ \overline{\upsilon_{1i}^2}\right) \overline{\Gamma_1} +$$

$$[5.3.4]$$

$$+ \left(\overline{u_{2i}} + \frac{1}{2}\ \overline{\upsilon_{2i}^2}\right) \overline{\Gamma_2} = \overline{\left(\mathbf{n}_k \cdot \nabla X_k\right)\left(e_i^{\sigma} + q_i\right)} = \bar{\varepsilon}\ ,$$

with the definitions

$$\overline{E_k} = \overline{\mathbf{q} \cdot \nabla X_k}\ , \qquad (4.6.6)$$

$$\overline{\mathbf{W}_k} = \overline{-\left(\underline{\mathbf{T}} \cdot \mathbf{v}\right) \cdot \nabla X_k}\ , \qquad (4.6.5)$$

$$\overline{u_{ki}}\ \overline{\Gamma_k} = \overline{\rho\ u\ (\mathbf{v}-\mathbf{v}_i)\cdot\nabla X_k}\ , \tag{4.5.5}$$

$$\frac{1}{2}\ \overline{v_{ki}^2}\ \overline{\Gamma_k} = \frac{1}{2}\ \overline{\rho\ v^2\ (\mathbf{v}-\mathbf{v}_i)\cdot\nabla X_k}\ . \tag{4.5.6}$$

(iv) Mechanical energy

$$\overline{W_1} + \overline{W_2} + \frac{1}{2}\ \overline{v_{1i}^2}\ \overline{\Gamma_1} + \frac{1}{2}\ \overline{v_{2i}^2}\ \overline{\Gamma_2} = \overline{\left(\mathbf{n}_k\cdot\nabla X_k\right)e_i^a}\ . \qquad [5.3.5]$$

(v) Internal energy

$$\overline{E_1} + \overline{E_2} + \overline{u_{1i}}\ \overline{\Gamma_1} + \overline{u_{2i}}\ \overline{\Gamma_2} = \overline{\left(\mathbf{n}_k\cdot\nabla X_k\right)q_i}\ . \qquad [5.3.6]$$

(vi) Enthalpy

With h = u + p/ρ, we rearrange equation (5.3.6) to obtain

$$\overline{E_1} + \overline{E_2} + \overline{h_{1i}}\ \overline{\Gamma_1} + \overline{h_{2i}}\ \overline{\Gamma_2} - \overline{p\ (\mathbf{v}_1-\mathbf{v}_i)\cdot\nabla X_1}\ -$$

[5.3.7]

$$-\ \overline{p\ (\mathbf{v}_2-\mathbf{v}_i)\cdot\nabla X_2} = \overline{\left(\mathbf{n}_k\cdot\nabla X_k\right)q_i}\ .$$

(vii) Entropy

We find

$$\overline{s_1} + \overline{s_2} + \overline{s_{1i}}\ \overline{\Gamma_1} + \overline{s_{2i}}\ \overline{\Gamma_2} = \overline{\left(\mathbf{n}_1\cdot\nabla X_1\right)q_{si}} = \overline{\left(\mathbf{n}_2\cdot\nabla X_2\right)q_{s2}} = \Delta s_i \geq 0 \qquad [5.3.8]$$

with the definitions

$$\overline{s_k} = \overline{\frac{\mathbf{q}}{T}\cdot\nabla X_k} \tag{4.6.7}$$

$$\overline{s_{ki}}\ \overline{\Gamma_k} = \overline{\rho\ s\ (\mathbf{v}-\mathbf{v}_i)\cdot\nabla X_k}\ . \tag{4.5.8}$$

Equation (5.3.8) yields a definition of Δs_i.

Eulerian form

Continuity equation

$$\frac{\partial}{\partial t}\left(\overline{\alpha_k}\ \overline{\rho_k^x}\right) + \nabla\cdot\left(\overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{\mathbf{v}_k^{x,\rho}}\right) = \overline{\Gamma_k}\ . \qquad (5.1.1.5)$$

Momentum equation

$$\frac{\partial}{\partial t}\left(\overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{\mathbf{v}_k^{x,\rho}}\right) + \nabla\cdot\left(\overline{\alpha_k}\ \overline{\rho_k}\ \overline{\mathbf{v}_k^{x,\rho}}\ \overline{\mathbf{v}_k^{x,\rho}}\right) = \nabla\cdot\ \overline{\alpha_k}\left(\overline{\underline{\mathbf{T}}_k^x} + \overline{\underline{\mathbf{T}}_k^{Re}}\right) + \qquad (5.1.2.8)$$

$$+\ \overline{\alpha_k}\ \overline{\rho_k^x}\ \mathbf{g} + \overline{\mathbf{v}_{ki}}\ \overline{\Gamma_k} + \overline{\mathbf{M}_k}\ ,$$

and

$$\frac{\partial}{\partial t}\left(\overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{\mathbf{v}_k^{x,\rho}}\right) + \nabla\cdot\left(\overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{\mathbf{v}_k^{x,\rho}}\ \overline{\mathbf{v}_k^{x,\rho}}\right) = \qquad (5.1.2.12)$$

$$= -\ \overline{\alpha_k}\nabla\ \overline{p_k^x} + \nabla\cdot\ \overline{\alpha_k}\left(\overline{\underline{\tau}_k^x} + \overline{\underline{\mathbf{T}}_k^{Re}}\right) + \overline{\alpha_k}\ \overline{\rho_k^x}\mathbf{g} + \overline{\mathbf{v}_{ki}}\ \overline{\Gamma_k} + \left(\overline{p_{ki}} - \overline{p_k^x}\right)\nabla\overline{\alpha_k} + \overline{\mathbf{M}_k^d}\ .$$

Total energy equation

$$\frac{\partial}{\partial t}\left\{\overline{\alpha_k}\ \overline{\rho_k^x}\left[\overline{u_k^{x,\rho}} + \frac{1}{2}\left(\overline{\upsilon_k^{x,\rho}}\right)^2 + \overline{\mathbf{u}_k^{Re}}\right]\right\} + \nabla\cdot\ \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{\mathbf{v}_\kappa^{x,\rho}}\left[\overline{u_k^{x,\rho}} + \frac{1}{2}\left(\overline{\upsilon_k^{x,\rho}}\right)^2 + \overline{\mathbf{u}_k^{Re}}\right] =$$

$$= \nabla\cdot\ \overline{\alpha_k}\left[\left(\overline{\underline{\mathbf{T}}_k^x} + \overline{\underline{\mathbf{T}}_k^{Re}}\right)\cdot\ \overline{\mathbf{v}_k^{x,\rho}} - \overline{\mathbf{q}_k^x} - \overline{\mathbf{q}_k^{Re}}\right] + \qquad (5.1.3.19)$$

$$+\left(\overline{u_{ki}} + \frac{1}{2}\ \overline{\upsilon_{ki}^2}\right)\overline{\Gamma_k} + \overline{\mathrm{E}_k} + \overline{W_k} + \overline{\alpha_k}\ \overline{\rho_k^x}\ \mathbf{g}\cdot\ \overline{\mathbf{v}_k^{x,\rho}} + \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{r_k^{x,\rho}}\ .$$

Mechanical energy equation

$$\frac{\partial}{\partial t}\ \overline{\alpha_k}\ \overline{\rho_k^x}\left[\frac{1}{2}\left(\overline{\upsilon_k^{x,\rho}}\right)^2 + \overline{\mathbf{u}_k^{Re}}\right] + \nabla\cdot\ \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{\mathbf{v}_k^{x,\rho}}\left[\frac{1}{2}\left(\overline{\upsilon_k^{x,\rho}}\right)^2 + \overline{\mathbf{u}_k^{Re}}\right] =$$

$$= \nabla\cdot\ \overline{\alpha_k}\left[\left(\overline{\underline{\mathbf{T}}_k^x} + \overline{\underline{\mathbf{T}}_k^{Re}}\right)\cdot\ \overline{\mathbf{v}_k^{x,\rho}} - \overline{\mathbf{q}_k^{kin}} - \overline{\mathbf{q}_k^p} - \overline{\mathbf{q}_k^\tau}\right] + \qquad (5.1.4.11)$$

$$+\ \frac{1}{2}\ \overline{\upsilon_{ki}^2}\ \overline{\Gamma_k} + \overline{W_k} - \overline{\alpha_k}\ \overline{\underline{\mathbf{T}}_k^x} : \nabla\overline{\mathbf{v}_k^{x,\rho}} + \overline{X_k\, p\left(\nabla\cdot\mathbf{v}_k'\right)} - \overline{\alpha_k}\ \overline{D_k} + \overline{\alpha_k}\ \overline{\rho_k^x}\ \mathbf{g}\cdot\ \overline{\mathbf{v}_k^{x,\rho}}\ .$$

TABLE VI - Summary of averaged conservation equations in detailed form

Internal energy equation

$$\frac{\partial}{\partial t}\left(\overline{\alpha_k}\;\overline{\rho_k^x}\;\overline{u_k^{x,\rho}}\right)+\nabla\cdot\left(\overline{\alpha_k}\;\overline{\rho_k^x}\;\overline{\mathbf{v}_k^{x,\rho}}\;\overline{u_k^{x,\rho}}\right)=$$

$$=-\nabla\cdot\overline{\alpha_k}\left(\overline{\mathbf{q}_k^x}+\widehat{\mathbf{q}}_k^{Re}\right)+\overline{u_{ki}}\;\overline{\Gamma_k}+\overline{E_k}+\overline{\alpha_k}\;\overline{D_k}+ \qquad (5.1.5.1)$$

$$+\;\overline{\alpha_k}\;\overline{\underline{\mathbf{T}}_k^x}:\nabla\overline{\mathbf{v}_k^{x,\rho}}-\overline{X_k\,p\;\nabla\cdot\mathbf{v}_k'}+\overline{\alpha_k}\;\overline{\rho_k^x}\;\overline{r_k^{x,\rho}}\;.$$

Enthalpy equation

$$\frac{\partial}{\partial t}\left(\overline{\alpha_k}\;\overline{\rho_k^x}\;\overline{h_k^{x,\rho}}\right)+\nabla\cdot\left(\overline{\alpha_k}\;\overline{\rho_k^x}\;\overline{h_k^{x,\rho}}\;\overline{\mathbf{v}_k^{x,\rho}}\right)=$$

$$=-\nabla\cdot\overline{\alpha_k}\left(\overline{\mathbf{q}_k^x}+\widehat{\mathbf{q}}_k^{Re}\right)+\overline{h_{ki}}\;\overline{\Gamma_k}+\overline{\alpha_k}\left(\frac{\partial\overline{p_k^x}}{\partial t}+\overline{\mathbf{v}_k^{x,\rho}}\cdot\nabla\overline{p_k^x}\right)+ \qquad (5.1.6.6)$$

$$+\overline{X_k\left(\frac{\partial p_k'}{\partial t}+\mathbf{v}_k\cdot\nabla p_k'\right)}+\overline{\mathbf{E}_k}+\overline{\alpha_k}\,\overline{\underline{\mathbf{T}}_k^x}:\nabla\overline{\mathbf{v}_k^{x,\rho}}+\overline{\alpha_k}\;\overline{D_k}-\overline{X_k\,p\,\nabla\cdot\mathbf{v}_k'}+\overline{\alpha_k}\,\overline{\rho_k^x}\;\overline{r_k^{x,\rho}}\,.$$

Entropy inequality

$$\frac{\partial}{\partial t}\left(\overline{\alpha_k}\;\overline{\rho_k^x}\;\overline{s_k^{x,\rho}}\right)+\nabla\cdot\left(\overline{\alpha_k}\;\overline{\rho_k^x}\;\overline{s_k^{x,\rho}}\;\overline{\mathbf{v}_k^{x,\rho}}\right)-\overline{s_{ki}}\;\overline{\Gamma_k}-\nabla\cdot\overline{\alpha_k}\left(\overline{\Phi_k^x}+\overline{\Phi_k^{Re}}\right)- \qquad (5.1.7.9)$$

$$-\;\overline{s_k}-\overline{\alpha_k}\;\overline{\rho_k^x}\;\overline{\sigma_k^{x,\rho}}=\overline{\Delta s_k}\geq 0$$

with

$$\overline{\Delta s_k}=\overline{s_{Tk}}+\overline{\alpha_k}\;\overline{\underline{\tau}_{sk}^x}:\nabla\overline{\mathbf{v}_k^{x,\rho}}+\overline{\alpha_k}\;\overline{D_{sk}}\;. \qquad (5.1.7.10)$$

Entropy equation

$$\frac{\partial}{\partial t}\left(\overline{\alpha_k}\;\overline{\rho_k^x}\;\overline{s_k^{x,\rho}}\right)+\nabla\cdot\left(\overline{\alpha_k}\;\overline{\rho_k^x}\;\overline{\mathbf{v}_k^{x,\rho}}\;\overline{s_k^{x,\rho}}\right)-\overline{\alpha_k}\;\overline{\underline{\tau}_{sk}^x}:\nabla\overline{\mathbf{v}_k^{x,\rho}}-\overline{\alpha_k}\;\overline{D_{sk}}- \qquad (5.1.7.14)$$

$$-\;\nabla\cdot\overline{\alpha_k}\left(\overline{\Phi_k^x}+\overline{\Phi_k^{Re}}\right)-\overline{s_{ki}}\;\overline{\Gamma_k}-\overline{s_k}-\overline{s_{Tk}}-\overline{\alpha_k}\;\overline{\rho_k^x}\;\overline{\sigma_k^{x,\rho}}=0\;.$$

TABLE VI - continued

Lagrangian form

Momentum equation

$$\overline{\alpha}_k \; \overline{\rho}^x_k \; \frac{D_k \overline{\mathbf{v}^{x,\rho}_k}}{Dt} = \nabla \cdot \; \overline{\alpha}_k \left(\overline{\underline{\mathbf{T}}^x_k} + \overline{\underline{\mathbf{T}}^{Re}_k} \right) + \overline{\alpha}_k \; \overline{\rho}^x_k \; \mathbf{g} \; + \qquad (5.2.1.4)$$

$$+ \left(\overline{\mathbf{v}_{ki}} - \overline{\mathbf{v}^{x,\rho}_k} \right) \overline{\Gamma}_k + \overline{p_{ki}} \; \nabla \overline{\alpha}_k - \overline{\underline{\tau}_{ki}} \cdot \nabla \overline{\alpha}_k + \overline{\mathbf{M}'_k} \; .$$

Total energy equation

a) With explicit term $v^2/2$

$$\overline{\alpha}_k \; \overline{\rho}^x_k \; \frac{D_k}{Dt} \left[\overline{u^{x,\rho}_k} + \frac{\left(\overline{\upsilon^{x,\rho}_k} \right)^2}{2} + \overline{u^{Re}_k} \right] =$$

$$= \nabla \cdot \; \overline{\alpha}_k \left[\left(\overline{\underline{\mathbf{T}}^x_k} + \overline{\underline{\mathbf{T}}^{Re}_k} \right) \cdot \overline{\mathbf{v}^{x,\rho}_k} - \overline{\mathbf{q}^x_k} - \overline{\mathbf{q}^{Re}_k} \right] + \left(\overline{h_{ki}} + \frac{\overline{\upsilon^2_{ki}}}{2} \right) \overline{\Gamma}_k \; - \qquad (5.2.2.6)$$

$$- \left[\overline{u^{x,\rho}_k} + \frac{\left(\overline{\upsilon^{x,\rho}_k} \right)^2}{2} + \overline{u^{Re}_k} \right] \overline{\Gamma}_k + \overline{E_k} + \overline{W'_k} - \overline{p_{ki}} \; \frac{\partial \overline{\alpha}_k}{\partial t} \; +$$

$$+ \; \overline{\alpha}_k \; \overline{\rho}^x_k \; \mathbf{g} \cdot \overline{\mathbf{v}^{x,\rho}_k} + \overline{\alpha}_k \; \overline{\rho}^x_k \; \overline{r^{x,\rho}_k} \; .$$

b) Alternative form

b1) In terms of $\overline{\mathbf{M}'_k}$ and $\overline{\mathbf{W}'_k}$

$$\overline{\alpha}_k \; \overline{\rho}^x_k \; \frac{D_k}{Dt} \left(\overline{u^{x,\rho}_k} + \overline{u^{Re}_k} \right) =$$

$$= \overline{\alpha}_k \left(\overline{\underline{\mathbf{T}}^x_k} + \overline{\underline{\mathbf{T}}^{Re}_k} \right) : \nabla \; \overline{\mathbf{v}^{x,\rho}_k} - \nabla \cdot \; \overline{\alpha}_k \left(\overline{\mathbf{q}^x_k} + \overline{\mathbf{q}^{Re}_k} \right) + \qquad (5.2.2.11)$$

$$+ \left\{ \left(\overline{h_{ki}} - \overline{u^{x,\rho}_k} - \overline{u^{Re}_k} \right) + \frac{1}{2} \left[\overline{\upsilon^2_{ki}} + \left(\overline{\upsilon^{x,\rho}_k} \right)^2 - 2 \, \overline{\mathbf{v}_{ki}} \cdot \overline{\mathbf{v}^{x,\rho}_k} \right] \right\} \overline{\Gamma}_k \; +$$

$$+ \overline{E}_k + \overline{W'_k} - \overline{p_{ki}} \left(\frac{\partial \overline{\alpha}_k}{\partial t} + \overline{\mathbf{v}^{x,\rho}_k} \cdot \nabla \overline{\alpha}_k \right) + \overline{\alpha}_k \; \overline{\rho}^x_k \; \overline{r^{x,\rho}_k} - \overline{\mathbf{M}'_k} \cdot \overline{\mathbf{v}^{x,\rho}_k} + \overline{\mathbf{v}^{x,\rho}_k} \cdot \left(\overline{\underline{\tau}_{ki}} \cdot \nabla \overline{\alpha}_k \right)$$

TABLE VI - continued

b2) In terms of M_k and W_k

$$\overline{\alpha_k}\ \overline{\rho_k^x}\ \frac{D_k}{Dt}\left(\overline{u_k^{x,\rho}} + \overline{u_k^{Re}}\right) = \overline{\alpha_k}\left(\overline{\underline{T}_k} + \overline{\underline{T}_k^{Re}}\right) : \nabla\ \overline{\mathbf{v}_k^{x,\rho}} - \nabla \cdot \overline{\alpha_k}\left(\overline{\mathbf{q}_k^x} + \overline{\mathbf{q}_k^{Re}}\right) +$$

$$+ \left\{\overline{u_{ki}} - \overline{u_k^{x,\rho}} - \overline{u_k^{Re}} + \frac{1}{2}\left[\overline{v_{ki}^2} + \left(\overline{v_k^{x,\rho}}\right)^2 - 2\,\overline{\mathbf{v}_{ki}} \cdot \overline{\mathbf{v}_k^{x,\rho}}\right]\right\}\overline{\Gamma_k} + \qquad (5.2.2.13)$$

$$+ \overline{W_k} + \overline{E_k} - \overline{M_k} \cdot \overline{\mathbf{v}_k^{x,\rho}} + \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{r_k^{x,\rho}} .$$

Mechanical energy equation

$$\overline{\alpha_k}\ \overline{\rho_k^x}\ \frac{D}{Dt}\left[\frac{1}{2}\left(\overline{v_k^{x,\rho}}\right)^2 + \overline{u_k^{Re}}\right] =$$

$$= \nabla \cdot \overline{\alpha_k}\left[\left(\overline{\underline{T}_k^x} + \overline{\underline{T}_k^{Re}}\right) \cdot \overline{\mathbf{v}_k^{x,\rho}} - \overline{\mathbf{q}_k^{kin}} - \overline{\mathbf{q}_k^{p}} - \overline{\mathbf{q}_k^{\tau}}\right] +$$

$$(5.2.3.2)$$

$$+\left[\frac{1}{2}\ \overline{v_{ki}^2} - \frac{1}{2}\left(\overline{v_k^{x,\rho}}\right)^2 - \overline{u_k^{Re}}\right]\overline{\Gamma_k} + \overline{W_k} -$$

$$- \overline{\alpha_k}\ \overline{\underline{T}_k^x} : \nabla\,\overline{\mathbf{v}_k^{x,\rho}} + \overline{X_k\, p\left(\nabla \cdot \mathbf{v}_k'\right)} - \overline{\alpha_k}\ \overline{D_k} + \overline{\alpha_k}\ \overline{\rho_k^x}\ \mathbf{g} \cdot \overline{\mathbf{v}^{x,\rho}} .$$

Internal energy equation

$$\overline{\alpha_k}\ \overline{\rho_k^x}\ \frac{D}{Dt}\ \overline{u_k^{x,\rho}} = -\nabla \cdot \overline{\alpha_k}\left(\overline{\mathbf{q}_k^x} + \hat{\mathbf{q}}_k^{Re}\right) + \left(\overline{u_{ki}} - \overline{u_k^{x,\rho}}\right)\overline{\Gamma_k} +$$

$$(5.2.4.1)$$

$$+ \overline{E_k} + \overline{\alpha_k}\ \overline{D_k} + \overline{\alpha_k}\ \overline{\underline{T}_k^x} : \nabla\,\overline{\mathbf{v}_k^{x,\rho}} - \overline{X_k\, p\ \nabla \cdot \mathbf{v}_k'} + \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{r_k^{x,\rho}} .$$

TABLE VI - continued

Enthalpy equation

$$\overline{\alpha_k}\ \overline{\rho_k^x}\ \frac{D}{Dt}\ \overline{h_k^{x,\rho}} = -\nabla\cdot\ \overline{\alpha_k}\left(\overline{\mathbf{q}_k^x} + \hat{\mathbf{q}}_k^{Re}\right) + \left(\overline{h_{ki}} - \overline{h_k^{x,\rho}}\right)\overline{\Gamma_k} + \overline{\alpha_k}\ \frac{D_k\ \overline{p_k^x}}{Dt} + \tag{5.2.5.1}$$

$$+ X_k \overline{\frac{D_k p_k'}{Dt}} + \overline{E_k} + \overline{\alpha_k}\ \overline{\underline{T}_k^x} : \nabla\ \overline{\mathbf{v}_k^{x,\rho}} + \overline{\alpha_k}\ \overline{D_k} - \overline{X_k\ p\ \nabla\cdot\mathbf{v}_k'} + \overline{\alpha_k}\,\overline{\rho_k^x}\ \overline{r_k^{x,\rho}}\ .$$

Entropy inequality

$$\overline{\alpha_k}\ \overline{\rho_k^x}\ \frac{D}{Dt}\ \overline{s_k^{x,\rho}} - \nabla\cdot\ \overline{\alpha_k}\left(\overline{\boldsymbol{\phi}_k^x} + \overline{\boldsymbol{\phi}_k^{Re}}\right) - \left(\overline{s_{ki}} - \overline{s_k^{x,\rho}}\right)\overline{\Gamma_k} - \overline{s_k} - \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{\sigma_k^{x,\rho}} = \overline{\Delta s_k} \geq 0 \tag{5.2.6.1}$$

with

$$\overline{\Delta s_k} = \overline{s_{Tk}} + \overline{\alpha_k}\ \overline{\underline{\tau}_{sk}^x} : \nabla\ \overline{\mathbf{v}_k^{x,\rho}} + \overline{\alpha_k}\ \overline{D_{sk}} \geq 0 \tag{5.1.7.10}$$

Entropy equation

$$\overline{\alpha_k}\ \overline{\rho_k^x}\ \frac{D}{Dt}\ \overline{s_k^{x,\rho}} - \overline{\alpha_k}\ \overline{\underline{\tau}_{sk}^x} : \nabla\ \overline{\mathbf{v}_k^{x,\rho}} - \overline{\alpha_k}\ \overline{D_{sk}} - \nabla\cdot\ \overline{\alpha_k}\left(\overline{\boldsymbol{\phi}_k^x} + \overline{\boldsymbol{\phi}_k^{Re}}\right) - \left(\overline{s_{ki}} - \overline{s_k^{x,\rho}}\right)\overline{\Gamma_k} -$$

$$- \overline{s_k} - \overline{s_{Tk}} - \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{\sigma_k^{x,\rho}} = 0\ . \tag{5.2.6.2}$$

TABLE VI - continued

6. Conservation equations for the mixture

Purpose of this section is to derive a system of governing equations for the fluid mixture from the equations written for the two phases separately, which were derived in the previous section. To this purpose it is necessary to define first the physical quantities of interest for the mixture.

6.1 Definition of averaged variables for the fluid mixture

We classify the definitions of averaged variables for the mixture in three classes, as follows:

6.1.1 Class (i) - Volume weighted averages

These are simply defined by

$$f_m = \overline{\alpha_1}\ \overline{f_1^x} + \overline{\alpha_2}\ \overline{f_2^x} = \sum_1^2{}_k\ \overline{X_k f}\ . \qquad [6.1.1.1]$$

Applying this definition to the set of physical quantities

$$f \equiv \overline{f_k^x}\ \left\{ \overline{\rho_k^x}\ ,\ \overline{p_k^x}\ ,\ \overline{D_k}\ ,\ \overline{D_{sk}} \right\}$$

one obtains:

Mixture density ($f = \rho_k^x$)

$$\rho_m = \overline{\alpha_1}\ \overline{\rho_1^x} + \overline{\alpha_2}\ \overline{\rho_2^x} \qquad \left(\frac{kg}{m^3}\right) \qquad [6.1.1.2]$$

Mixture pressure ($f = p_k^x$)

$$p_m = \overline{\alpha_1}\ \overline{p_1^x} + \overline{\alpha_2}\ \overline{p_2^x} \qquad \left(\frac{N}{m^2}\right) \qquad [6.1.1.3]$$

Turbulent energy dissipation of mixture ($f = \overline{D}_k$)

$$D_m = \overline{\alpha_1}\ \overline{D_1} + \overline{\alpha_2}\ \overline{D_2} \qquad \left(\frac{W}{m^3}\right) \qquad [6.1.1.4]$$

Turbulent entropy source in the mixture ($f = \overline{D}_{sk}$)

$$D_m^{st} = \overline{\alpha_1}\ \overline{D_{s1}} + \overline{\alpha_2}\ \overline{D_{s2}} \qquad \left(\frac{W}{m^3 K}\right)\ . \qquad [6.1.1.5]$$

6.1.2 Class (ii) - Mass weighted averages

The mean value of a phase velocity $\overline{\mathrm{v}_k^{x,\rho}}$ can be considered as sum of the mixture velocity v_m (yet to be defined) and a (positive or negative) shift $\tilde{\mathrm{v}}_k$ with respect to v_m; hence

$$\tilde{\mathrm{v}}_k = \overline{\mathrm{v}_k^{x,\rho}} - \mathrm{v}_m \; . \qquad [6.1.2.1]$$

We call $\tilde{\mathrm{v}}_k$ "phase velocity difference" with respect to the mean value of the mixture v_m. Similarly we write for physical scalar quantities

$$\tilde{u}_k = \overline{u_k^{x,\rho}} - u_m \qquad [6.1.2.2]$$

$$\tilde{h}_k = \overline{h_k^{x,\rho}} - h_m \qquad \textit{etc.} \qquad [6.1.2.3]$$

Let us first define v_m by means of

$$\rho_m \mathrm{v}_m = \overline{\alpha_1}\; \overline{\rho_1^x}\; \overline{\mathrm{v}_1^{x,\rho}} + \overline{\alpha_2}\; \overline{\rho_2^x}\; \overline{\mathrm{v}_2^{x,\rho}} \; . \qquad [6.1.2.4]$$

Using (6.1.2.1) and (6.1.1.2) this definition implies:

$$\overline{\alpha_1}\; \overline{\rho_1^x}\; \overline{\mathrm{v}_1^{x,\rho}} + \overline{\alpha_2}\; \overline{\rho_2^x}\; \overline{\mathrm{v}_2^{x,\rho}} = \left(\overline{\alpha_1}\; \overline{\rho_1^x} + \overline{\alpha_2}\; \overline{\rho_2^x} \right) \mathrm{v}_m =$$

$$= \overline{\alpha_1}\; \overline{\rho_1^x} \left(\overline{\mathrm{v}_1^{x,\rho}} - \tilde{\mathrm{v}}_1 \right) + \overline{\alpha_2}\; \overline{\rho_2^x} \left(\overline{\mathrm{v}_2^{x,\rho}} - \tilde{\mathrm{v}}_2 \right), \qquad [6.1.2.5]$$

hence:

$$\overline{\alpha_1}\; \overline{\rho_1^x}\; \tilde{\mathrm{v}}_1 + \overline{\alpha_2}\; \overline{\rho_2^x}\; \tilde{\mathrm{v}}_2 = 0 \; . \qquad [6.1.2.6]$$

Similarly, for any physical quantity f, one has:

$$\overline{\alpha_1}\; \overline{\rho_1^x}\; \tilde{f}_1 + \overline{\alpha_2}\; \overline{\rho_2^x}\; \tilde{f}_2 = 0 \qquad [6.1.2.7]$$

Definitions of mass weighted averages are given by the formula

$$f_m = \frac{1}{\rho_m} \left(\overline{\alpha_1}\; \overline{\rho_1^x}\; \overline{f_1} + \overline{\alpha_2}\; \overline{\rho_2^x}\; \overline{f_2} \right) \qquad [6.1.2.8]$$

applied to the set of physical quantities:

$$f \equiv \overline{f_k} = \left\{ \overline{\mathrm{v}_k^{x,\rho}},\ \overline{u_k^{x,\rho}},\ \overline{h_k^{x,\rho}},\ \overline{s_k^{x,\rho}},\ \overline{\sigma_k^{x,\rho}},\ \frac{1}{2}\left(\overline{\mathrm{v}_k^{x,\rho}}\right)^2,\ \overline{\underline{\tau}_k^x},\ \overline{r_k^{x,\rho}},\ \overline{u_k^{Re}},\ \overline{\underline{\mathrm{T}}_k^x},\ \overline{\underline{\mathrm{T}}_k^{Re}},\ \overline{\mathrm{q}_k^x},\ \hat{\mathrm{q}}_k^{Re},\ \overline{\Phi_k^x},\ \overline{\Phi_k^{Re}} \right\} .$$

For $f = \mathrm{v}$ one derives (6.1.2.4), and similarly:

Internal energy of mixture

$$\left(f = \overline{u_k^{x,\rho}}\right) \quad u_m = \frac{\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{u_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{u_2^{x,\rho}}}{\rho_m} \quad \left(\frac{J}{kg}\right) \qquad [6.1.2.9]$$

Enthalpy of mixture

$$\left(f = \overline{h_k^{x,\rho}}\right) \quad h_m = \frac{\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{h_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{h_2^{x,\rho}}}{\rho_m} \quad \left(\frac{J}{kg}\right) \qquad [6.1.2.10]$$

Entropy of mixture

$$\left(f = \overline{s_k^{x,\rho}}\right) \quad s_m = \frac{\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{s_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{s_2^{x,\rho}}}{\rho_m} \quad \left(\frac{J}{kg\ K}\right) \qquad [6.1.2.11]$$

Entropy source of mixture

$$\left(f = \overline{\sigma_k^{x,\rho}}\right) \quad \sigma_m = \frac{\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\sigma_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\sigma_2^{x,\rho}}}{\rho_m} \quad \left(\frac{W}{kg\ K}\right) \qquad [6.1.2.12]$$

Kinetic energy of mixture

$$\left(f = \frac{\left(\overline{v_k^{x,\rho}}\right)^2}{2}\right) \quad \frac{v_m^2}{2} = \frac{\overline{\alpha_1}\ \overline{\rho_1^x}\left(\overline{v_k^{x,\rho}}\right)^2 + \overline{\alpha_2}\ \overline{\rho_2^x}\left(\overline{v_k^{x,\rho}}\right)^2}{2\rho_m} \quad \left(\frac{J}{kg}\right) \qquad [6.1.2.13]$$

Shear stress of mixture

$$\left(f = \overline{\underline{\tau}_k^x}\right) \quad \underline{\tau}_m = \frac{\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\underline{\tau}_1^x} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\underline{\tau}_2^x}}{\rho_m} \quad \left(\frac{N}{m^2}\right) \qquad [6.1.2.14]$$

Energy source

$$\left(f = \overline{r_k^{x,\rho}}\right) \quad r_m = \frac{\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{r_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{r_1^{x,\rho}}}{\rho_m} \quad \left(\frac{W}{m^3}\right) \qquad [6.1.2.15]$$

Turbulent kinetic energy

$$\left(f = \overline{u_k^{Re}}\right) \quad u_m^{Re} = \frac{\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{u_1^{Re}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{u_2^{Re}}}{\rho_m} \quad \left(\frac{J}{m^3}\right) \qquad [6.1.2.16]$$

Stress term of mixture

$$\left(f=\overline{\underline{T}_k^x}\right) \quad \underline{T}_m^x = \frac{\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\underline{T}_1^x} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\underline{T}_2^x}}{\rho_m} \quad \left(\frac{N}{m^2}\right) \qquad [6.1.2.17]$$

Reynolds stress of mixture

$$\left(f=\overline{\underline{T}_k^{Re}}\right) \quad \underline{T}_m^{Re} = \frac{\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\underline{T}_1^{Re}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\underline{T}_2^{Re}}}{\rho_m} \quad \left(\frac{N}{m^2}\right) \qquad [6.1.2.18]$$

Energy flux of mixture

$$\left(f=\overline{\mathbf{q}_k^x}\right) \quad \mathbf{q}_m^x = \frac{\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\mathbf{q}_1} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\mathbf{q}_2^x}}{\rho_m} \quad \left(\frac{W}{m^2}\right) \qquad [6.1.2.19]$$

Turbulent enthalpy flux of mixture

$$\left(f=\hat{\hat{\mathbf{q}}}_k^{Re}\right) \quad \hat{\hat{\mathbf{q}}}_m^{Re} = \frac{\overline{\alpha_1}\ \overline{\rho_1^x}\ \hat{\hat{\mathbf{q}}}_1^{Re} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \hat{\hat{\mathbf{q}}}_2^{Re}}{\rho_m} \quad \left(\frac{W}{m^2}\right) \qquad [6.1.2.20]$$

Entropy flux of mixture

$$\left(f=\overline{\Phi_k^x}\right) \quad \Phi_m = \frac{\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\Phi_1^x} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\Phi_2^x}}{\rho_m} \quad \left(W/m^2\,K\right) \qquad [6.1.2.21]$$

Turbulent entropy flux of mixture

$$\left(f=\overline{\Phi_k^{Re}}\right) \quad \Phi_m^{Re} = \frac{\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\Phi_1^{Re}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\Phi_2^{Re}}}{\rho_m} \quad \left(\frac{W}{m^2 K}\right) \qquad [6.1.2.22]$$

Multiplying scalarly equation (6.1.2.4) by the gravitational acceleration g one has

$$\rho_m\ \mathbf{v}_m \cdot \mathbf{g} = \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\mathbf{v}_1^{x,\rho}} \cdot \mathbf{g} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\mathbf{v}_2^{x,\rho}} \cdot \mathbf{g} \ . \qquad [6.1.2.23]$$

6.1.3 Class (iii) - Combined definitions: volume weighted and mass flux weighted averages

In the following we use, for a physical quantity F, the following definition of mass flux weighted average:

$$\overline{F_k} = \frac{\overline{X_k \rho \, v \, F}}{\overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{v_k^{x,\rho}}} \qquad [6.1.3.1]$$

<u>1st case</u>

Let f denote a scalar or vector or tensor quantity. Using the definition (6.1.2.1) **for the** phase velocities and (6.1.2.2) for a physical quantity $\overline{F}_k = F_m + \tilde{F}_k$ one has identically:

$$\sum_1^2{}_k \left(\overline{X_k f} + \overline{X_k \rho \, v \, F} \right) = \sum_1^2{}_k \left(\overline{\alpha_k}\ \overline{f_k^x} + \overline{\alpha_k}\ \overline{\rho_k}\ \overline{v_k^{x,\rho}}\ \overline{F_k} \right) =$$

$$= \sum_1^2{}_k \left[\overline{\alpha_k}\ \overline{f_k^x} + \overline{\alpha_k}\ \overline{\rho_k} \left(v_m + \tilde{v}_k \right) \left(F_m + \tilde{F}_k \right) \right] = \qquad [6.1.3.2]$$

$$= \left(\overline{\alpha_1}\ \overline{f_1^x} + \overline{\alpha_2}\ \overline{f_2^x} \right) + \left(\overline{\alpha_1}\ \overline{\rho_1^x} + \overline{\alpha_2}\,\overline{\rho_2^x} \right) v_m F_m + v_m \left(\overline{\alpha_1}\,\overline{\rho_1^x}\ \tilde{F}_1 + \overline{\alpha_2}\ \overline{\rho_2^x}\ \tilde{F}_2 \right) +$$

$$+ \left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \tilde{v}_1 + \overline{\alpha_2}\ \overline{\rho_2^x}\ \tilde{v}_2 \right) F_m + \overline{\alpha_1}\ \overline{\rho_1^x}\ \tilde{v}_1 \tilde{F}_1 + \overline{\alpha_2}\ \overline{\rho_2^x}\ \tilde{v}_2 \tilde{F}_2 .$$

The third and fourth terms of the right hand side are equal to zero, because of (6.1.2.7) and (6.1.2.6), respectively. Using the previous definitions (6.1.1.1) formally extended to any vector entity, and (6.1.1.2) for the mixture density, we can write

$$\sum_1^2{}_k \ \overline{X_k f} = f_m = \overline{\alpha_1}\ \overline{f_1^x} + \overline{\alpha_2}\ \overline{f_2^x} , \qquad [6.1.1.1]$$

$$\sum_1^2{}_k \left(\overline{X_k \rho \, v \, F} \right) = \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{v_1^{x,\rho}}\ \overline{F_1} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{v_2^{x,\rho}}\ \overline{F_2} =$$

[6.1.3.3]

$$= \rho_m \ v_m F_m + \overline{\alpha_1}\ \overline{\rho_1^x}\ \tilde{v}_1 \tilde{F}_1 + \overline{\alpha_2}\ \overline{\rho_2^x}\ \tilde{v}_2 \tilde{F}_2$$

Hence, from the latter:

$$\rho_m \ v_m F_m = \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{v_1^{x,\rho}}\ \overline{F_1} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{v_2^{x,\rho}}\ \overline{F_2} - \overline{\alpha_1}\,\overline{\rho_1^x}\ \tilde{v}_1 \tilde{F}_1 - \overline{\alpha_2}\ \overline{\rho_2^x}\ \tilde{v}_2 \tilde{F}_2 . \qquad [6.1.3.4]$$

Summing (6.1.1.1) and (6.1.3.4) one derives the following definition

$$J_m = f_m + \rho_m \ v_m F_m = \overline{\alpha_1} \ \overline{f_1^x} + \overline{\alpha_2} \ \overline{f_2^x} + \qquad [6.1.3.5]$$

$$+ \overline{\alpha_1} \ \overline{\rho_1^x} \ \overline{v_1^{x,\rho}} \ \overline{F_1} + \overline{\alpha_2} \ \overline{\rho_2^x} \ \overline{v_2^{x,\rho}} \ \overline{F_2} - \left(\overline{\alpha_1} \ \overline{\rho_1^x} \ \widetilde{v}_1 \widetilde{F}_1 + \overline{\alpha_2} \ \overline{\rho_2^x} \ \widetilde{v}_2 \widetilde{F}_2 \right).$$

Let us take for example $f_k = 0$, $F_k = \overline{v_k^{x,\rho}}$. Equation (6.1.3.2) yields

$$\sum_{1}^{2}{}_k \ \overline{\alpha_k} \ \overline{\rho_k^x} \ \overline{v_k^{x,\rho}} \ \overline{v_k^{x,\rho}} = \sum_k \overline{\alpha_k} \ \overline{\rho_k^x} \left(v_m + \widetilde{v}_k \right)\left(v_m + \widetilde{v}_k \right) =$$

$$= \sum_k \left(\overline{\alpha_k} \ \overline{\rho_k^x} \ v_m v_m + \overline{\alpha_k} \ \overline{\rho_k^x} \ v_m \widetilde{v}_k + \overline{\alpha_k} \ \overline{\rho_k^x} \ \widetilde{v}_k \ v_m + \overline{\alpha_k} \ \overline{\rho_k^x} \ \widetilde{v}_k \ \widetilde{v}_k \right).$$

Hence, using (6.1.2.7)

$$\rho_m \ v_m \ v_m = \overline{\alpha_1} \ \overline{\rho_1^x} \ \overline{v_1^{x,\rho}} \ \overline{v_1^{x,\rho}} + \overline{\alpha_2} \ \overline{\rho_2^x} \ \overline{v_2^{x,\rho}} \ \overline{v_2^{x,\rho}} - \qquad [6.1.3.6]$$

$$- \left(\overline{\alpha_1} \ \overline{\rho_1^x} \ \widetilde{v}_1 \ \widetilde{v}_1 + \overline{\alpha_2} \ \overline{\rho_2^x} \ \widetilde{v}_2 \ \widetilde{v}_2 \right).$$

The left hand side represents a mean stress tensor for the mixture while the first and second terms at the right side represent an equivalent definition of stress tensor within each phase.

<u>2nd case</u>

The above definition holds formally also when the physical quantities involved do not denote mean values but turbulent fluctuations. In this case (6.1.2.1) is replaced by

$$\widetilde{v}'_k = v'_k - v'_m , \qquad [6.1.3.7]$$

and the definition of average becomes:

$$J'_m = f_m + \rho_m \ v'_m \ F_m = \overline{\alpha_1} \ \overline{f_1^x} + \overline{\alpha_2} \ \overline{f_2^x} + \qquad [6.1.3.8]$$

$$+ \overline{\alpha_1} \ \overline{\rho_1^x \ v'_1} \ \overline{F_1} + \overline{\alpha_2} \ \overline{\rho_2^x \ v'_2} \ \overline{F_2} - \overline{\alpha_1} \ \overline{\rho_1^x \widetilde{v}'_1} \ \widetilde{F}_1 - \overline{\alpha_2} \ \overline{\rho_2^x \ \widetilde{v}'_2} \widetilde{F}_2 ,$$

with the identity (similar to (6.1.2.6))

$$\overline{\alpha_1} \ \overline{\rho_1^x \widetilde{v}'_1} + \overline{\alpha_2} \ \overline{\rho_2^x \widetilde{v}'_2} = 0 . \qquad [6.1.3.9]$$

3rd case

As a third case let us assume that, for both quantities involved (v and F), turbulent fluctuations are considered, with

$$\tilde{v}'_k = v'_k - v'_m \quad , \qquad (6.1.3.7)$$

$$\tilde{F}'_k = F'_k - F'_m . \qquad [6.1.3.10]$$

Thus one has:

$$\sum_{1}^{2} {}_k \left(\overline{X_k f} + \overline{X_k \rho \, v' F'} \right) = \overline{\alpha_1} \; \overline{f_1^x} + \overline{\alpha_2} \; \overline{f_2^x} + \overline{X_1 \rho_1 v'_1 F'_1} + \overline{X_2 \rho_2 v'_2 F'_2} =$$

$$= \overline{\alpha_1} \; \overline{f_1^x} + \overline{\alpha_2} \; \overline{f_2^x} + \overline{X_1 \rho_1 \left(v'_m + \tilde{v}'_1\right) \left(F'_m + \tilde{F}'_1\right)} + \overline{X_2 \rho_2 \left(v'_m + \tilde{v}'_2\right) \left(F'_m + \tilde{F}'_2\right)} =$$

$$[6.1.3.11]$$

$$= \overline{\alpha_1} \; \overline{f_1^x} + \overline{\alpha_2} \; \overline{f_2^x} + \overline{X_1 \rho_1 v'_m F'_m} + \overline{X_1 \rho_1 \tilde{v}'_1 \tilde{F}'_1} + \overline{X_2 \rho_2 v'_m F'_m} + \overline{X_2 \rho_2 \tilde{v}'_2 \tilde{F}'_2} +$$

$$+ \overline{v'_m \left(X_1 \rho_1 \tilde{F}'_1 + X_2 \rho_2 \tilde{F}'_2\right)} + \overline{F'_m \left(X_1 \rho_1 \tilde{v}'_1 + X_2 \rho_2 \tilde{v}'_2\right)} .$$

The last two terms at the right side are equal to zero because (in analogy with (6.1.2.6) and (6.1.2.7))

$$\overline{\left(X_1 \rho_1 \tilde{v}'_1 + X_2 \rho_2 \tilde{v}'_2\right)} = \overline{\alpha_1} \; \overline{\rho_1^x} \; \overline{\tilde{v}'_1} + \overline{\alpha_2} \; \overline{\rho_2^x} \; \overline{\tilde{v}'_2} = 0 \qquad [6.1.3.12]$$

$$\overline{\left(X_1 \rho_1 \tilde{F}'_1 + X_2 \rho_2 \tilde{F}'_2\right)} = \overline{\alpha_1} \; \overline{\rho_1^x} \; \overline{\tilde{F}'_1} + \overline{\alpha_2} \; \overline{\rho_2^x} \; \overline{\tilde{F}'_2} = 0 . \qquad [6.1.3.13]$$

Defining

$$\overline{F_k^{Re}} = \frac{\overline{X_k \rho_k v'_k F'_k}}{\overline{\alpha_k}} \quad , \qquad [6.1.3.14]$$

$$F_m^{Re} = \overline{\left(X_1 \rho_1 + X_2 \rho_2\right) v'_m F'_m} \qquad [6.1.3.15]$$

one has from (6.1.3.11)

$$\alpha_1 \; \overline{f_1^x} + \alpha_2 \; \overline{f_2^x} + \overline{\alpha_1} \; \overline{F_1^{Re}} + \overline{\alpha_2} \; \overline{F_2^{Re}} =$$

$$= f_m + F_m^{Re} + \overline{X_1 \rho_1 \tilde{v}'_1 \tilde{F}'_1} + \overline{X_1 \rho_1 \tilde{v}'_2 \tilde{F}'_2} = \qquad [6.1.3.16]$$

$$= f_m + F_m^{Re} + \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\tilde{v}'_1 \tilde{F}'_1} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\tilde{v}'_2 \tilde{F}'_2} ,$$

with the assumption

$$\overline{X_k\, \rho_k\, \tilde{v}'_k \tilde{F}'_k} = \overline{\alpha_k}\ \overline{\rho_k^x}\ \overline{\tilde{v}'_k \tilde{F}'_k} . \qquad [6.1.3.17]$$

Thus, from (6.1.3.16)

$$J_m^{Re} = f_m + F_m^{Re} = \overline{\alpha_1}\ \overline{f_1^x} + \overline{\alpha_2}\ \overline{f_2^x} + \overline{\alpha_1}\ \overline{F_1^{Re}} + \overline{\alpha_2}\ \overline{F_2^{Re}} - \qquad [6.1.3.18]$$

$$- \left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\tilde{v}'_1 \tilde{F}'_1} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\tilde{v}'_2 \tilde{F}'_2} \right) .$$

Remark:
The definitions of volume weighted and mass flux weighted averages (6.1.3.5), (6.1.3.9), (6.1.3.18) for the mixture imply an important consequence. The averages for the mixture are not given by the sum of the contributions of the single phases. Instead, from this sum two terms must be subtracted, which depend on the "phase velocity differences" of the single phases.

6.2 Derivation of conservation equations for the mixture

In this section the conservation equations for the mixture are derived from the averaged conservation equations for the single phases (obtained in section 5) by adding the separate contributions of the two phases.

6.2.1 - Continuity equation

Using equation (5.1.1.5) for $k = 1, 2$, adding, and considering the jump condition (5.3.1) one derives

$$\frac{\partial}{\partial t}\left(\overline{\alpha_1}\ \overline{\rho_1^x} + \overline{\alpha_2}\ \overline{\rho_2^x} \right) + \nabla \cdot \left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{v_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{v_2^{x,\rho}} \right) = \overline{\Gamma_1} + \overline{\Gamma_2} = 0 . \qquad [6.2.1.1]$$

With the definitions (6.1.1.2) and (6.1.2.4) one obtains

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m\, v_m) = 0 . \qquad [6.2.1.2]$$

6.2.2 - Momentum equation

1st Form

From equation (5.1.2.8) one derives, upon summation for both phases:

$$\frac{\partial}{\partial t}\left(\overline{\alpha_1}\,\overline{\rho_1^x}\,\overline{v_1^{x,\rho}} + \overline{\alpha_2}\,\overline{\rho_2^x}\,\overline{v_2^{x,\rho}}\right) + \nabla\cdot\left(\overline{\alpha_1}\,\overline{\rho_1^x}\,\overline{v_1^{x,\rho}}\,\overline{v_1^{x,\rho}} + \overline{\alpha_2}\,\overline{\rho_2^x}\,\overline{v_2^{x,\rho}}\,\overline{v_2^{x,\rho}}\right) =$$

$$= \nabla\cdot\left[\overline{\alpha_1}\left(\overline{\underline{T}_1^x} + \overline{\underline{T}_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\underline{T}_2^x} + \overline{\underline{T}_2^{Re}}\right)\right] + \left(\overline{\alpha_1}\,\overline{\rho_1^x} + \overline{\alpha_2}\,\overline{\rho_2^x}\right)g + \qquad [6.2.2.1]$$

$$+ \overline{M_1} + \overline{v_{1i}}\,\overline{\Gamma_1} + \overline{M_2} + \overline{v_{2i}}\,\overline{\Gamma_2}\ .$$

Letting $\overline{v_k^{x,\rho}} = v_m + \tilde{v}_k$ in the divergence term at the left side of equation (6.2.2.1), using the definitions of ρ_m (6.1.1.2), v_m (6.1.2.4) and the jump condition (5.3.2) yields

$$\frac{\partial}{\partial t}(\rho_m v_m) + \nabla\cdot\left[\left(\overline{\alpha_1}\,\overline{\rho_1^x} + \overline{\alpha_2}\,\overline{\rho_2^x}\right)v_m v_m + 2v_m\left(\overline{\alpha_1}\,\overline{\rho_1^x}\,\tilde{v}_1 + \overline{\alpha_2}\,\overline{\rho_2^x}\,\tilde{v}_2\right)\right] =$$

$$= \nabla\cdot\left[\overline{\alpha_1}\left(\overline{\underline{T}_1^x} + \overline{\underline{T}_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\underline{T}_2^x} + \overline{\underline{T}_2^{Re}}\right)\right] - \qquad [6.2.2.2]$$

$$- \nabla\cdot\left(\overline{\alpha_1}\,\overline{\rho_1^x}\,\tilde{v}_1\,\tilde{v}_1 + \overline{\alpha_2}\,\overline{\rho_2^x}\,\tilde{v}_2\,\tilde{v}_2\right) + \rho_m\,g + \overline{m}\ .$$

Using equations (6.1.1.2) and (6.1.2.6) the previous equation can be written

$$\frac{\partial}{\partial t}(\rho_m v_m) + \nabla\cdot\rho_m v_m v_m = \nabla\cdot\underline{T}_m + \rho_m\,g + \overline{m}\ , \qquad [6.2.2.3]$$

with

$$\underline{T}_m = \overline{\alpha_1}\left(\overline{\underline{T}_1^x} + \overline{\underline{T}_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\underline{T}_2^x} + \overline{\underline{T}_2^{Re}}\right) - \left(\overline{\alpha_1}\,\overline{\rho_1^x}\,\tilde{v}_1\,\tilde{v}_1 + \overline{\alpha_2}\,\overline{\rho_2^x}\,\tilde{v}_2\,\tilde{v}_2\right). \qquad [6.2.2.4]$$

<u>2nd Form</u>

An equivalent formulation can be obtained from equation (5.1.2.12) as follows:

$$\frac{\partial}{\partial t}\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{v_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{v_2^{x,\rho}}\right) + \nabla\cdot\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{v_1^{x,\rho}}\ \overline{v_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{v_2^{x,\rho}}\ \overline{v_2^{x,\rho}}\right) =$$

$$= \nabla\cdot\left(\overline{\alpha_1}\ \overline{\underline{\tau}_1^x} + \overline{\alpha_2}\ \overline{\underline{\tau}_2^x}\right) + \nabla\cdot\left(\overline{\alpha_1}\ \overline{\underline{T}_1^{Re}} + \overline{\alpha_2}\ \overline{\underline{T}_2^{Re}}\right) + \left(\overline{\alpha_1}\ \overline{\rho_1^x} + \overline{\alpha_2}\ \overline{\rho_2^x}\right) g +$$

$$+ \overline{v_{1i}}\ \overline{\Gamma_1} + \overline{v_{2i}}\ \overline{\Gamma_2} - \left(\overline{\alpha_1}\,\nabla\,\overline{p_1^x} + \overline{\alpha_2}\,\nabla\,\overline{p_2^x}\right) + \qquad [6.2.2.5]$$

$$+ \left(\overline{p_{1i}^x} - \overline{p_1^x}\right)\nabla\,\overline{\alpha_1} + \left(\overline{p_{2i}^x} - \overline{p_2^x}\right)\nabla\,\overline{\alpha_2} + \overline{M_1^d} + \overline{M_2^d}\,.$$

Using again $\overline{v_k^{x,\rho}} = v_m + \tilde{v}_k$ in the divergence term at the left side one has:

$$\frac{\partial}{\partial t}(\rho_m v_m) + \nabla\cdot(\rho_m v_m v_m) = -\nabla\cdot\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \tilde{v}_1\tilde{v}_1 + \overline{\alpha_2}\ \overline{\rho_2^x}\ \tilde{v}_2\tilde{v}_2\right) +$$

$$+ \nabla\cdot\left[\alpha_1\left(\overline{\underline{\tau}_1^x} + \overline{\underline{T}_1^{Re}}\right) + \alpha_2\left(\overline{\underline{\tau}_2^x} + \overline{\underline{T}_2^{Re}}\right)\right] + \rho_m\, g +$$

$$+ \overline{v_{1i}}\ \overline{\Gamma_1} + \overline{v_{2i}}\ \overline{\Gamma_2} - \left(\overline{\alpha_1}\,\nabla\,\overline{p_1^x} + \overline{\alpha_2}\,\nabla\,\overline{p_2^x}\right) + \qquad [6.2.2.6]$$

$$+ \left(\overline{p_{1i}^x} - \overline{p_1^x}\right)\nabla\,\overline{\alpha_1} + \left(\overline{p_{2i}^x} - \overline{p_2^x}\right)\nabla\,\overline{\alpha_2} + \overline{M_1^d} + \overline{M_2^d}\,.$$

<u>3rd or alternative form</u>

In the following we call "alternative form" of the conservation equations the expression obtained without expanding the divergence term at the left hand side. This alternative form is most suitable to derive the conservation equations in terms of slip velocity (see section 6.3). Starting again from equation (5.1.2.12) one obtains:

$$\frac{\partial}{\partial t}(\rho_m \mathbf{v}_m) + \nabla\cdot\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\mathbf{v}_1^{x,\rho}}\ \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\mathbf{v}_2^{x,\rho}}\ \overline{\mathbf{v}_2^{x,\rho}}\right) =$$

$$= \nabla\cdot\left(\overline{\alpha_1}\ \overline{\underline{\tau}_1^x} + \overline{\alpha_2}\ \overline{\underline{\tau}_2^x}\right) + \nabla\cdot\left(\overline{\alpha_1}\ \overline{\underline{\mathbf{T}}_1^{Re}} + \overline{\alpha_2}\ \overline{\underline{\mathbf{T}}_2^{Re}}\right) + \rho_m\ \mathbf{g}\ +$$

$$+\ \overline{\mathbf{v}_{1i}}\ \overline{\Gamma_1} + \overline{\mathbf{v}_{2i}}\ \overline{\Gamma_2} - \left(\overline{\alpha_1}\ \nabla\ \overline{p_1^x} + \overline{\alpha_2}\ \nabla\ \overline{p_2^x}\right) + \qquad [6.2.2.7]$$

$$+\left(\overline{p_{1i}^x} - \overline{p_1^x}\right)\nabla\ \overline{\alpha_1} + \left(\overline{p_{2i}^x} - \overline{p_2^x}\right)\nabla\ \overline{\alpha_2} + \overline{\mathbf{M}_1^d} + \overline{\mathbf{M}_2^d}\ .$$

This equation is more suitable to derive the momentum equation for the mixture in terms of the slip velocity ($\mathbf{v}_{Sl} = \overline{\mathbf{v}_1^{x,\rho}} - \overline{\mathbf{v}_2^{x,\rho}}$) as shown in section 6.3.

6.2.3 Total Energy equation

1st form

From equation (5.1.3.19) one derives

$$\frac{\partial}{\partial t}\left\{\overline{\alpha_1}\ \overline{\rho_1^x}\left[\overline{u_1^{x,\rho}} + \frac{\left(\overline{\upsilon_1^{x,\rho}}\right)^2}{2} + \overline{u_1^{Re}}\right] + \overline{\alpha_2}\ \overline{\rho_2^x}\left[\overline{u_2^{x,\rho}} + \frac{\left(\overline{\upsilon_2^{x,\rho}}\right)^2}{2} + \overline{u_2^{Re}}\right]\right\} +$$

$$+\nabla\cdot\left\{\overline{\alpha_1}\,\overline{\rho_1^x}\,\overline{\mathbf{v}_1^{x,\rho}}\left[\overline{u_1^{x,\rho}} + \frac{1}{2}\left(\overline{\upsilon_1^{x,\rho}}\right)^2 + \overline{u_1^{Re}}\right] + \overline{\alpha_2}\,\overline{\rho_2^x}\,\overline{\mathbf{v}_2^{x,\rho}}\left[\overline{u_2^{x,\rho}} + \frac{\left(\overline{\upsilon_2^{x,\rho}}\right)^2}{2} + \overline{u_2^{Re}}\right]\right\} =$$

$$= \nabla\cdot\left[\overline{\alpha_1}\left(\overline{\underline{\mathbf{T}}_1^x} + \overline{\underline{\mathbf{T}}_1^{Re}}\right)\cdot\overline{\mathbf{v}_1^{x,\rho}} + \alpha_2\left(\overline{\underline{\mathbf{T}}_2^x} + \overline{\underline{\mathbf{T}}_2^{Re}}\right)\cdot\overline{\mathbf{v}_2^{x,\rho}}\ -\right.$$

$$\left. -\left(\overline{\alpha_1}\ \overline{\mathbf{q}_1^x} + \overline{\alpha_2}\ \overline{\mathbf{q}_2^x}\right) - \left(\overline{\alpha_1}\ \overline{\mathbf{q}_1^{Re}} + \overline{\alpha_2}\ \overline{\mathbf{q}_2^{Re}}\right)\right] + \qquad [6.2.3.1]$$

$$+\left(\overline{u_{1i}} + \frac{1}{2}\overline{\upsilon_{1i}^2}\right)\overline{\Gamma_1} + \left(\overline{u_{2i}} + \frac{1}{2}\overline{\upsilon_{2i}^2}\right)\overline{\Gamma_2} + \overline{E_1} + \overline{E_2} + \overline{W_1} + \overline{W_2}\ +$$

$$+\ \overline{\alpha_1}\ \overline{\rho_1^x}\ \mathbf{g}\cdot\overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \mathbf{g}\cdot\overline{\mathbf{v}_2^{x,\rho}} + \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{r_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{r_2^{x,\rho}}\ .$$

Introducing in the above equation

$$\bar{\varepsilon} = \left(\overline{u_{1i}} + \frac{1}{2} \overline{v_{1i}^2} \right) \Gamma_1 + \left(\overline{u_{2i}} + \frac{1}{2} \overline{v_{2i}^2} \right) \Gamma_2 + \overline{E_1} + \overline{E_2} + \overline{W_1} + \overline{W_2} , \tag{5.3.4}$$

$$\rho_m \mathbf{v}_m \cdot \mathbf{g} = \overline{\alpha_1}\, \overline{\rho_1^x}\, \mathbf{g} \cdot \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\, \overline{\rho_2^x}\, \mathbf{g} \cdot \overline{\mathbf{v}_2^{x,\rho}} , \tag{6.1.2.23}$$

$$\rho_m r_m = \overline{\alpha_1}\, \overline{\rho_1^x}\, \overline{r_1^{x,\rho}} + \overline{\alpha_2}\, \overline{\rho_2^x}\, \overline{r_2^{x,\rho}} , \tag{6.1.2.15}$$

$$\rho_m u_m = \overline{\alpha_1}\, \overline{\rho_1^x}\, \overline{u_1^{x,\rho}} + \overline{\alpha_2}\, \overline{\rho_2^x}\, \overline{u_2^{x,\rho}} , \tag{6.1.2.9}$$

$$\rho_m u_m^{Re} = \overline{\alpha_1}\, \overline{\rho_1^x}\, \overline{u_1^{Re}} + \overline{\alpha_2}\, \overline{\rho_2^x}\, \overline{u_2^{Re}} , \tag{6.1.2.16}$$

$$\rho_m v_m^2 = \overline{\alpha_1}\, \overline{\rho_1^x} \left(\overline{v_1^{x,\rho}} \right)^2 + \overline{\alpha_2}\, \overline{\rho_2^x} \left(\overline{v_2^{x,\rho}} \right)^2 , \tag{6.1.2.13}$$

and using

$$\left(\overline{v_k^{x,\rho}} \right)^2 = (v_m + \tilde{v}_k)^2 = (\mathbf{v}_m + \tilde{\mathbf{v}}_k) \cdot (\mathbf{v}_m + \tilde{\mathbf{v}}_k) = v_m^2 + 2 v_m \tilde{v}_k + \tilde{v}_k^2 , \qquad [6.2.3.2]$$

one derives

$$\frac{\partial}{\partial t} \left(\rho_m u_m + \rho_m \frac{1}{2} v_m^2 + \rho_m u_m^{Re} \right) +$$

$$+ \nabla \cdot \left\{ \overline{\alpha_1}\, \overline{\rho_1^x} \left(\mathbf{v}_m + \tilde{\mathbf{v}}_1 \right) \left[u_m + \tilde{u}_1 + \frac{1}{2} v_m^2 + \frac{1}{2} \tilde{v}_1^2 + v_m \tilde{v}_1 + \overline{u_1^{Re}} \right] + \right.$$

$$\left. + \overline{\alpha_2}\, \overline{\rho_2^x} \left(\mathbf{v}_m + \tilde{\mathbf{v}}_2 \right) \left[u_m + \tilde{u}_2 + \frac{1}{2} v_m^2 + \frac{1}{2} \tilde{v}_2^2 + v_m \tilde{v}_2 + \overline{u_2^{Re}} \right] \right\} = \qquad [6.2.3.3]$$

$$= \nabla \cdot \left[\overline{\alpha_1} \left(\overline{\underline{\mathbf{T}}_1^x} + \overline{\underline{\mathbf{T}}_1^{Re}} \right) \cdot \left(\mathbf{v}_m + \tilde{\mathbf{v}}_1 \right) + \alpha_2 \left(\overline{\underline{\mathbf{T}}_2^x} + \overline{\underline{\mathbf{T}}_2^{Re}} \right) \cdot \left(\mathbf{v}_m + \tilde{\mathbf{v}}_2 \right) \right] -$$

$$- \nabla \cdot \left(\overline{\alpha_1}\, \overline{\mathbf{q}_1^x} + \overline{\alpha_2}\, \overline{\mathbf{q}_2^x} \right) - \nabla \cdot \left(\overline{\alpha_1}\, \overline{\mathbf{q}_1^{Re}} + \overline{\alpha_2}\, \overline{\mathbf{q}_2^{Re}} \right) + \rho_m \mathbf{v}_m \cdot \mathbf{g} + \rho_m r_m + \bar{\varepsilon} .$$

Expanding the terms at the left side of the previous equation one derives:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\, \rho_m \left(u_m + \frac{1}{2} v_m^2 + u_m^{Re} \right) + \\
&+ \nabla \cdot \Bigg[\rho_m \mathbf{v}_m \left(u_m + \frac{v_m^2}{2} \right) + \mathbf{v}_m \left(\overline{\alpha}_1 \overline{\rho_1^x}\, \tilde{u}_1 + \overline{\alpha}_2 \overline{\rho_2^x}\, \tilde{u}_2 \right) + \mathbf{v}_m \left(\overline{\alpha}_1 \overline{\rho_1^x}\, \frac{1}{2} \tilde{v}_1^2 + \overline{\alpha}_2 \overline{\rho_2^x}\, \frac{1}{2} \tilde{v}_2^2 \right) + \\
&+ \rho_m u_m^{Re} \mathbf{v}_m + \mathbf{v}_m \left(\overline{\alpha}_1 \overline{\rho_1^x}\, v_m \tilde{v}_1 + \overline{\alpha}_2 \overline{\rho_2^x}\, v_m \tilde{v}_2 \right) + \qquad [6.2.3.4] \\
&+ \left(\overline{\alpha}_1 \overline{\rho_1^x}\, \tilde{\mathbf{v}}_1 + \overline{\alpha}_2 \overline{\rho_2^x}\, \tilde{\mathbf{v}}_2 \right) \left(u_m + \frac{1}{2} v_m^2 \right) + \overline{\alpha}_1 \overline{\rho_1^x}\, \mathbf{v}_1 v_m \tilde{v}_1 + \overline{\alpha}_2 \overline{\rho_2^x}\, \mathbf{v}_2 v_m \tilde{v}_2 \Big) + \\
&+ \left(\overline{\alpha}_1 \overline{\rho_1^x}\, \mathbf{v}_1 \tilde{u}_1 + \overline{\alpha}_2 \overline{\rho_2^x}\, \mathbf{v}_2 \tilde{u}_2 \right) + \left(\overline{\alpha}_1 \overline{\rho_1^x}\, \tilde{\mathbf{v}}_1 \frac{1}{2} \tilde{v}_1^2 + \overline{\alpha}_2 \overline{\rho_2^x}\, \tilde{\mathbf{v}}_2 \frac{1}{2} \tilde{v}_2^2 \right) + \\
&\left(\overline{\alpha}_1 \overline{\rho_1^x}\, \tilde{\mathbf{v}}_1 \overline{u_1^{Re}} + \overline{\alpha}_2 \overline{\rho_2^x}\, \tilde{\mathbf{v}}_2 \overline{u_2^{Re}} \right) \Bigg] = \\
&= \nabla \cdot \left\{ \left[\overline{\alpha}_1 \left(\overline{\underline{\mathbf{T}}_1^x} + \overline{\underline{\mathbf{T}}_1^{Re}} \right) + \overline{\alpha}_2 \left(\overline{\underline{\mathbf{T}}_2^x} + \overline{\underline{\mathbf{T}}_2^{Re}} \right) \right] \cdot \mathbf{v}_m \right\} + \\
&+ \nabla \cdot \left\{ \overline{\alpha}_1 \left(\overline{\underline{\mathbf{T}}_1^x} + \overline{\underline{\mathbf{T}}_1^{Re}} \right) \cdot \tilde{\mathbf{v}}_1 + \overline{\alpha}_2 \left(\overline{\underline{\mathbf{T}}_2^x} + \overline{\underline{\mathbf{T}}_2^{Re}} \right) \cdot \tilde{\mathbf{v}}_2 \right\} - \\
&- \nabla \cdot \left[\overline{\alpha}_1 \left(\overline{\mathbf{q}_1^x} + \overline{\mathbf{q}_1^{Re}} \right) + \alpha_2 \left(\overline{\mathbf{q}_2^x} + \overline{\mathbf{q}_2^{Re}} \right) \right] + \rho_m \mathbf{v}_m \cdot \mathbf{g} + \rho_m r_m + \overline{\varepsilon} .
\end{aligned}
$$

The second, fifth and sixth terms at the left hand side within the brackets are equal to zero because of (6.1.2.7) and (6.1.2.6).

Let us define:

$$
\begin{aligned}
\mathbf{q}_m = {} & \overline{\alpha}_1 \overline{\mathbf{q}_1^x} + \overline{\alpha}_2 \overline{\mathbf{q}_2^x} + \overline{\alpha}_1 \overline{\mathbf{q}_1^{Re}} + \overline{\alpha}_2 \overline{\mathbf{q}_2^{Re}} - \\
& - \overline{\alpha}_1 \left(\overline{\underline{\mathbf{T}}_1^x} + \overline{\underline{\mathbf{T}}_1^{Re}} \right) \cdot \tilde{\mathbf{v}}_1 - \overline{\alpha}_2 \left(\overline{\underline{\mathbf{T}}_2^x} + \overline{\underline{\mathbf{T}}_2^{Re}} \right) \cdot \tilde{\mathbf{v}}_2 + \qquad [6.2.3.5] \\
& + \overline{\alpha}_1 \overline{\rho_1^x}\, \tilde{\mathbf{v}}_1 \left(u_1 + \frac{v_1^2}{2} \right) + \overline{\alpha}_2 \overline{\rho_2^x}\, \tilde{\mathbf{v}}_2 \left(u_2 + \frac{v_2^2}{2} \right) .
\end{aligned}
$$

Thus, one obtains:

$$\frac{\partial}{\partial t}\left[\rho_m\left(u_m+\frac{v_m^2}{2}+u_m^{Re}\right)\right]+\nabla\cdot\left[\rho_m\mathbf{v}_m\left(u_m+\frac{v_m^2}{2}+u_m^{Re}\right)\right]+$$

$$+\nabla\cdot\left[\frac{1}{2}\mathbf{v}_m\left(\overline{\alpha_1}\,\overline{\rho_1^x}\,\widetilde{v_1^2}+\overline{\alpha_2}\,\overline{\rho_2^x}\,\widetilde{v_2^2}\right)+v_m\left(\overline{\alpha_1}\,\overline{\rho_1^x}\,\tilde{\mathbf{v}}_1 v_1+\overline{\alpha_2}\,\overline{\rho_2^x}\,\tilde{\mathbf{v}}_2 v_2\right)+\right.$$

$$\left.+\overline{\alpha_1}\,\overline{\rho_1^x}\,\tilde{\mathbf{v}}_1\,\overline{u_1^{Re}}+\overline{\alpha_2}\,\overline{\rho_2^x}\,\tilde{\mathbf{v}}_2\,\overline{u_2^{Re}}\right]= \qquad [6.2.3.6]$$

$$=-\nabla\cdot\mathbf{q}_m+\rho_m\mathbf{v}_m\cdot\mathbf{g}+\rho_m r_m+\bar{\varepsilon}+\nabla\cdot\left\{\left[\overline{\alpha_1}\left(\overline{\underline{\mathbf{T}}_1^x}+\overline{\underline{\mathbf{T}}_1^{Re}}\right)+\overline{\alpha_2}\left(\overline{\underline{\mathbf{T}}_2^x}+\overline{\underline{\mathbf{T}}_2^{Re}}\right)\right]\cdot\mathbf{v}_m\right\}.$$

<u>Alternative form</u>

We start again from equation

$$\frac{\partial}{\partial t}\left\{\overline{\alpha_1}\,\overline{\rho_1^x}\left[\overline{u_1^{x,\rho}}+\frac{\left(\overline{v_1^{x,\rho}}\right)^2}{2}+\overline{u_1^{Re}}\right]+\overline{\alpha_2}\,\overline{\rho_2^x}\left[\overline{u_2^{x,\rho}}+\frac{\left(\overline{v_2^{x,\rho}}\right)^2}{2}+\overline{u_2^{Re}}\right]\right\}+$$

$$+\nabla\cdot\left\{\overline{\alpha_1}\,\overline{\rho_1^x}\,\overline{\mathbf{v}_1^{x,\rho}}\left[\overline{u_1^{x,\rho}}+\frac{1}{2}\left(\overline{v_1^{x,\rho}}\right)^2+\overline{u_1^{Re}}\right]+\overline{\alpha_2}\,\overline{\rho_2^x}\,\overline{\mathbf{v}_2^{x,\rho}}\left[\overline{u_2^{x,\rho}}+\frac{1}{2}\left(\overline{v_2^{x,\rho}}\right)^2+\overline{u_2^{Re}}\right]\right\}=$$

$$=\nabla\cdot\left[\overline{\alpha_1}\left(\overline{\underline{\mathbf{T}}_1^x}+\overline{\underline{\mathbf{T}}_1^{Re}}\right)\cdot\overline{\mathbf{v}_1^{x,\rho}}+\alpha_2\left(\overline{\underline{\mathbf{T}}_2^x}+\overline{\underline{\mathbf{T}}_2^{Re}}\right)\cdot\overline{\mathbf{v}_2^{x,\rho}}-\right. \qquad (6.2.3.1)$$

$$\left.-\left(\overline{\alpha_1}\,\overline{\mathbf{q}_1^x}+\overline{\alpha_2}\,\overline{\mathbf{q}_2^x}\right)-\left(\overline{\alpha_1}\,\overline{\mathbf{q}_1^{Re}}+\overline{\alpha_2}\,\overline{\mathbf{q}_2^{Re}}\right)\right]+$$

$$+\left(\overline{u_{1i}}+\frac{1}{2}\overline{v_{1i}^2}\right)\overline{\Gamma_1}+\left(\overline{u_{2i}}+\frac{1}{2}\overline{v_{2i}^2}\right)\overline{\Gamma_2}+\overline{E_1}+\overline{E_2}+\overline{W_1}+\overline{W_2}+$$

$$+\overline{\alpha_1}\,\overline{\rho_1^x}\,\mathbf{g}\cdot\overline{\mathbf{v}_1^{x,\rho}}+\overline{\alpha_2}\,\overline{\rho_2^x}\,\mathbf{g}\cdot\overline{\mathbf{v}_2^{x,\rho}}+\overline{\alpha_1}\,\overline{\rho_1^x}\,\overline{r_1^{x,\rho}}+\overline{\alpha_2}\,\overline{\rho_2^x}\,\overline{r_2^{x,\rho}}\,.$$

Introducing in the above equation

$$\bar{\varepsilon}=\left(\overline{u_{1i}}+\frac{1}{2}\overline{v_{1i}^2}\right)\overline{\Gamma_1}+\left(\overline{u_{2i}}+\frac{1}{2}\overline{v_{2i}^2}\right)\overline{\Gamma_2}+\overline{E_1}+\overline{E_2}+\overline{W_1}+\overline{W_2}\,, \qquad (5.3.4)$$

$$\rho_m\mathbf{v}_m\cdot\mathbf{g}=\overline{\alpha_1}\,\overline{\rho_1^x}\,\mathbf{g}\cdot\overline{\mathbf{v}_1^{x,\rho}}+\overline{\alpha_2}\,\overline{\rho_2^x}\,\mathbf{g}\cdot\overline{\mathbf{v}_2^{x,\rho}}\,, \qquad (6.1.2.23)$$

$$\rho_m\, r_m = \overline{\alpha_1}\;\overline{\rho_1^x}\;\overline{r_1^{x,\rho}} + \overline{\alpha_2}\;\overline{\rho_2^x}\;\overline{r_2^{x,\rho}}\,, \tag{6.1.2.15}$$

$$\rho_m\, u_m = \overline{\alpha_1}\;\overline{\rho_1^x}\;\overline{u_1^{x,\rho}} + \overline{\alpha_2}\;\overline{\rho_2^x}\;\overline{u_2^{x,\rho}}\,, \tag{6.1.2.9}$$

$$\rho_m\, u_m^{Re} = \overline{\alpha_1}\;\overline{\rho_1^x}\;\overline{u_1^{Re}} + \overline{\alpha_2}\;\overline{\rho_2^x}\;\overline{u_2^{Re}}\,, \tag{6.1.2.16}$$

$$\rho_m\, v_m^2 = \overline{\alpha_1}\;\overline{\rho_1^x}\left(\overline{v_1^{x,\rho}}\right)^2 + \overline{\alpha_2}\;\overline{\rho_2^x}\left(\overline{v_2^{x,\rho}}\right)^2, \tag{6.1.2.13}$$

and using

$$\left(\overline{v_k^{x,\rho}}\right)^2 = \left(v_m + \tilde{v}_k\right)^2 = \left(\mathbf{v}_m + \tilde{\mathbf{v}}_k\right)\cdot\left(\mathbf{v}_m + \tilde{\mathbf{v}}_k\right) = v_m^2 + 2v_m\tilde{v}_k + \tilde{v}_k^2 \tag{6.2.3.2}$$

one derives:

$$\frac{\partial}{\partial t}\left[\rho_m\, u_m + \rho_m \frac{1}{2} v_m^2\right] + \frac{\partial}{\partial t}\left(\rho_m\, u_m^{Re}\right) +$$

$$+\nabla\cdot\left\{\overline{\alpha_1}\,\overline{\rho_1^x}\,\overline{\mathbf{v}_1^{x,\rho}}\left[\overline{u_1^{x,\rho}} + \frac{1}{2}\left(\overline{v_1^{x,\rho}}\right)^2 + \overline{u_1^{Re}}\right] + \overline{\alpha_2}\,\overline{\rho_2^x}\,\overline{\mathbf{v}_2^{x,\rho}}\left[\overline{u_2^{x,\rho}} + \frac{1}{2}\left(\overline{v_2^{x,\rho}}\right)^2 + \overline{u_2^{Re}}\right]\right\} =$$

$$= \nabla\cdot\left[\overline{\alpha_1}\left(\overline{\underline{\mathbf{T}}_1^x} + \overline{\underline{\mathbf{T}}_1^{Re}}\right)\cdot\left(\mathbf{v}_m + \tilde{\mathbf{v}}_1\right) + \alpha_2\left(\overline{\underline{\mathbf{T}}_2^x} + \overline{\underline{\mathbf{T}}_2^{Re}}\right)\cdot\left(\mathbf{v}_m + \tilde{\mathbf{v}}_2\right)\right] - \tag{6.2.3.7}$$

$$-\nabla\cdot\left(\overline{\alpha_1}\;\overline{\mathbf{q}_1^x} + \overline{\alpha_2}\;\overline{\mathbf{q}_2^x}\right) - \nabla\cdot\left(\overline{\alpha_1}\;\overline{\mathbf{q}_1^{Re}} + \overline{\alpha_2}\;\overline{\mathbf{q}_2^{Re}}\right) + \rho_m\,\mathbf{v}_m\cdot\mathbf{g} + \rho_m\, r_m + \bar{\varepsilon}\,.$$

Expanding the first term at the left side of the previous equation one derives:

$$\frac{\partial}{\partial t}\,\rho_m\left(u_m + \frac{1}{2}v_m^2\right) + \frac{\partial}{\partial t}\left(\rho_m\, u_m^{Re}\right) +$$

$$+\nabla\cdot\left\{\overline{\alpha_1}\,\overline{\rho_1^x}\,\overline{\mathbf{v}_1^{x,\rho}}\left[\overline{u_1^{x,\rho}} + \frac{1}{2}\left(\overline{v_1^{x,\rho}}\right)^2 + \overline{u_1^{Re}}\right] + \overline{\alpha_2}\,\overline{\rho_2^x}\,\overline{\mathbf{v}_2^{x,\rho}}\left[\overline{u_2^{x,\rho}} + \frac{1}{2}\left(\overline{v_2^{x,\rho}}\right)^2 + \overline{u_2^{Re}}\right]\right\} =$$

$$= \nabla\cdot\left\{\left[\overline{\alpha_1}\left(\overline{\underline{\mathbf{T}}_1^x} + \overline{\underline{\mathbf{T}}_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\underline{\mathbf{T}}_2^x} + \overline{\underline{\mathbf{T}}_2^{Re}}\right)\right]\cdot\mathbf{v}_m\right\} + \tag{6.2.3.8}$$

$$+\nabla\cdot\left[\overline{\alpha_1}\left(\overline{\underline{\mathbf{T}}_1^x} + \overline{\underline{\mathbf{T}}_1^{Re}}\right)\cdot\tilde{\mathbf{v}}_1 + \overline{\alpha_2}\left(\overline{\underline{\mathbf{T}}_2^x} + \overline{\underline{\mathbf{T}}_2^{Re}}\right)\cdot\tilde{\mathbf{v}}_2\right] -$$

$$-\nabla\cdot\left[\overline{\alpha_1}\left(\overline{\mathbf{q}_1^x} + \overline{\mathbf{q}_1^{Re}}\right) + \alpha_2\left(\overline{\mathbf{q}_2^x} + \overline{\mathbf{q}_2^{Re}}\right)\right] + \rho_m\,\mathbf{v}_m\cdot\mathbf{g} + \rho_m\, r_m + \bar{\varepsilon}\,.$$

The second, fifth and sixth terms at the left hand side within the bracket are equal to zero because of (6.1.2.7) and (6.1.2.6).

Let us define:

$$\tilde{q}_m = \overline{\alpha_1}\,\overline{q_1^x} + \overline{\alpha_2}\,\overline{q_2^x} + \overline{\alpha_1}\,\overline{q_1^{Re}} + \overline{\alpha_2}\,\overline{q_2^{Re}} - \overline{\alpha_1}\left(\overline{\underline{T}_1^x} + \overline{\underline{T}_1^{Re}}\right)\cdot\tilde{v}_1 - \overline{\alpha_2}\left(\overline{\underline{T}_2^x} + \overline{\underline{T}_2^{Re}}\right)\cdot\tilde{v}_2 \,. \qquad [6.2.3.9]$$

Thus, one obtains:

$$\frac{\partial}{\partial t}\left[\rho_m\left(u_m + \frac{v_m^2}{2} + u_m^{Re}\right)\right] + \nabla\cdot\left\{\overline{\alpha_1}\,\overline{\rho_1^x}\,\overline{v_1^{x,\rho}}\left[\overline{u_1^{x,\rho}} + \frac{1}{2}\left(\overline{v_1^{x,\rho}}\right)^2 + \overline{u_1^{Re}}\right] + \overline{\alpha_2}\,\overline{\rho_2^x}\,\overline{v_2^{x,\rho}}\left[\overline{u_2^{x,\rho}} + \frac{1}{2}\left(\overline{v_2^{x,\rho}}\right)^2 + \overline{u_2^{Re}}\right]\right\} =$$

$$= -\nabla\cdot\tilde{q}_m + \rho_m\,v_m\cdot g + \rho_m\,r_m + \bar{\varepsilon} + \nabla\cdot\left\{\left[\overline{\alpha_1}\left(\overline{\underline{T}_1^x} + \overline{\underline{T}_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\underline{T}_2^x} + \overline{\underline{T}_2^{Re}}\right)\right]\cdot v_m\right\}. \qquad [6.2.3.10]$$

6.2.4 Mechanical energy equation

1st Form

From equation (5.1.4.10) one derives

$$\frac{\partial}{\partial t}\left\{\overline{\alpha_1}\,\overline{\rho_1^x}\left[\frac{1}{2}\left(\overline{v_1^{x,\rho}}\right)^2 + \overline{u_1^{Re}}\right] + \overline{\alpha_2}\,\overline{\rho_2^x}\left[\frac{1}{2}\left(\overline{v_2^{x,\rho}}\right)^2 + \overline{u_2^{Re}}\right]\right\} +$$

$$+ \nabla\cdot\left\{\overline{\alpha_1}\,\overline{\rho_1^x}\,\overline{v_1^{x,\rho}}\left[\frac{1}{2}\left(\overline{v_1^{x,\rho}}\right)^2 + \overline{u_1^{Re}}\right] + \overline{\alpha_2}\,\overline{\rho_2^x}\,\overline{v_2^{x,\rho}}\left[\frac{1}{2}\left(\overline{v_2^{x,\rho}}\right)^2 + \overline{u_2^{Re}}\right]\right\} =$$

$$= \nabla\cdot\left[\overline{\alpha_1}\left(\overline{\underline{T}_1^x} + \overline{\underline{T}_1^{Re}}\right)\cdot\overline{v_1^{x,\rho}} + \alpha_2\left(\overline{\underline{T}_2^x} + \overline{\underline{T}_2^{Re}}\right)\cdot\overline{v_2^{x,\rho}}\right] -$$

$$- \nabla\cdot\left(\overline{q_1^{kin}} + \overline{q_1^p} + \overline{q_1^\tau} + \overline{q_2^{kin}} + \overline{q_2^p} + \overline{q_2^\tau}\right) + \qquad [6.2.4.1]$$

$$+ \frac{1}{2} \overline{\upsilon_{1i}^{2}} \; \overline{\Gamma_1} + \frac{1}{2} \overline{\upsilon_{2i}^{2}} \; \overline{\Gamma_2} + \overline{W_1} + \overline{W_2} -$$

$$- \overline{\alpha_1} \; \overline{\underline{T}_1^x} : \nabla \overline{\mathbf{v}_1^{x,\rho}} - \alpha_2 \; \overline{\underline{T}_2^x} : \nabla \overline{\mathbf{v}_2^{x,\rho}} + \overline{X_1 \, p \left(\nabla \cdot \mathbf{v}_1'\right)} +$$

$$+ \overline{X_2 \, p \left(\nabla \cdot \mathbf{v}_2'\right)} - \left(\overline{\alpha_1} \; \overline{D_1} + \overline{\alpha_2} \; \overline{D_2}\right) + \overline{\alpha_1} \; \overline{\rho_1^x} \; \mathbf{g} \cdot \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2} \; \overline{\rho_2^x} \; \mathbf{g} \cdot \overline{\mathbf{v}_2^{x,\rho}} \; .$$

Now, we use the definitions

$$\rho_m \, \mathrm{u}_m^{Re} = \overline{\alpha_1} \; \overline{\rho_1^x} \; \overline{\mathrm{u}_1^{Re}} + \overline{\alpha_2} \; \overline{\rho_2^x} \; \overline{\mathrm{u}_2^{Re}} \; , \qquad (6.1.2.16)$$

$$\mathbf{g} \cdot \rho_m \, \mathbf{v}_m = \mathbf{g} \cdot \left(\overline{\alpha_1} \; \overline{\rho_1^x} \; \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2} \; \overline{\rho_2^x} \; \overline{\mathbf{v}_2^{x,\rho}} \right) , \qquad (6.1.2.23)$$

$$D_m = \overline{\alpha_1} \; \overline{D_1} + \overline{\alpha_2} \; \overline{D_2} \; , \qquad (6.1.1.4)$$

and the jump condition for mechanical energy

$$\overline{W_1} + \overline{W_2} + \frac{1}{2} \overline{\upsilon_{1i}^{2}} \; \overline{\Gamma_1} + \frac{1}{2} \overline{\upsilon_{2i}^{2}} \; \overline{\Gamma_2} = \overline{\left(\mathbf{n}_k \cdot \nabla X_k\right) e_i^{\sigma}} \; , \; (k = 1 \; or \; 2). \qquad (5.3.5)$$

Hence from equation (6.2.4.1) one derives

$$\frac{\partial}{\partial t} \left(\rho_m \, \mathrm{u}_m^{Re} + \rho_m \frac{\upsilon_m^2}{2}\right) +$$

$$+ \nabla \cdot \Bigg[\overline{\alpha_1} \, \overline{\rho_1^x} \, \overline{\mathbf{v}_1^{x,\rho}} \left(\frac{\upsilon_m^2}{2} + \upsilon_m \tilde{\upsilon}_1 + \frac{\tilde{\upsilon}_1^2}{2} \right) + \overline{\alpha_1} \; \overline{\rho_1^x} \; \mathbf{v}_m \; \overline{\mathrm{u}_1^{Re}} + \overline{\alpha_1} \; \overline{\rho_1^x} \, \tilde{\mathbf{v}}_1 \; \overline{\mathrm{u}_1^{Re}} +$$

$$+ \overline{\alpha_2} \, \overline{\rho_2^x} \, \overline{\mathbf{v}_2^{x,\rho}} \left(\frac{\upsilon_m^2}{2} + \upsilon_m \tilde{\upsilon}_2 + \frac{\tilde{\upsilon}_2^2}{2} \right) + \overline{\alpha_2} \; \overline{\rho_2^x} \; \mathbf{v}_m \; \overline{\mathrm{u}_2^{Re}} + \overline{\alpha_2} \; \overline{\rho_2^x} \, \tilde{\mathbf{v}}_2 \; \overline{\mathrm{u}_2^{Re}} \Bigg] =$$

$$= \nabla \cdot \left[\overline{\alpha_1} \left(\overline{\underline{T}_1^x} + \overline{\underline{T}_1^{Re}} \right) \cdot \left(\mathbf{v}_m + \tilde{\mathbf{v}}_1\right) + \alpha_2 \left(\overline{\underline{T}_2^x} + \overline{\underline{T}_2^{Re}} \right) \cdot \left(\mathbf{v}_m + \tilde{\mathbf{v}}_2\right) \right] - \qquad [6.2.4.2]$$

$$- \nabla \cdot \left(\overline{\mathbf{q}_1^{kin}} + \overline{\mathbf{q}_2^{kin}} + \overline{\mathbf{q}_1^{p}} + \overline{\mathbf{q}_2^{p}} + \overline{\mathbf{q}_1^{\tau}} + \overline{\mathbf{q}_2^{\tau}} \right) + \overline{\left(\mathbf{n}_k \cdot \nabla X_k\right) e_i^{\sigma}} -$$

$$- \overline{\alpha_1} \; \overline{\underline{T}_1^x} : \nabla \overline{\mathbf{v}_1^{x,\rho}} - \alpha_2 \overline{\underline{T}_2^x} : \nabla \overline{\mathbf{v}_2^{x,\rho}} + \overline{X_1 \, p \left(\nabla \cdot \mathbf{v}_1'\right)} + \overline{X_2 \, p \left(\nabla \cdot \mathbf{v}_2'\right)} - D_m + \mathbf{g} \cdot \rho_m \, \mathbf{v}_m .$$

Furthermore, we use the definition

$$\rho_m \mathbf{v}_m = \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\mathbf{v}_2^{x,\rho}} \tag{6.1.2.4}$$

and obtain

$$\frac{\partial}{\partial t}\left(\rho_m \mathrm{u}_m^{Re} + \rho_m \frac{\upsilon_m^2}{2}\right) + \nabla\cdot\left(\rho_m \frac{\upsilon_m^2}{2}\mathbf{v}_m\right) +$$

$$+ \nabla\cdot\left(\overline{\alpha_1}\,\overline{\rho_1^x}\,\overline{\mathbf{v}_1^{x,\rho}}\,\upsilon_m \tilde{\upsilon}_1 + \overline{\alpha_1}\,\overline{\rho_1^x}\,\overline{\mathbf{v}_1^{x,\rho}}\,\frac{\tilde{\upsilon}_1^2}{2} + \overline{\alpha_2}\,\overline{\rho_2^x}\,\overline{\mathbf{v}_2^{x,\rho}}\,\upsilon_m \tilde{\upsilon}_2 + \overline{\alpha_2}\,\overline{\rho_2^x}\,\overline{\mathbf{v}_2^{x,\rho}}\,\frac{\tilde{\upsilon}_2^2}{2}\right) +$$

$$+ \nabla\cdot\left(\rho_m u_m^{Re}\mathbf{v}_m\right) + \nabla\cdot\left(\overline{\alpha_1}\,\overline{\rho_1^x}\,\tilde{\mathbf{v}}_1\,\overline{u_1^{Re}} + \overline{\alpha_2}\,\overline{\rho_2^x}\,\tilde{\mathbf{v}}_2\,\overline{u_2^{Re}}\right) =$$

$$= \nabla\cdot\left\{\left[\overline{\alpha_1}\left(\overline{\underline{\mathbf{T}}_1^x} + \overline{\underline{\mathbf{T}}_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\underline{\mathbf{T}}_2^x} + \overline{\underline{\mathbf{T}}_2^{Re}}\right)\right]\cdot\mathbf{v}_m\right\} + \tag{6.2.4.3}$$

$$+ \nabla\cdot\left[\overline{\alpha_1}\left(\overline{\underline{\mathbf{T}}_1^x} + \overline{\underline{\mathbf{T}}_1^{Re}}\right)\cdot\tilde{\mathbf{v}}_1 + \overline{\alpha_2}\left(\overline{\underline{\mathbf{T}}_2^x} + \overline{\underline{\mathbf{T}}_2^{Re}}\right)\cdot\tilde{\mathbf{v}}_2\right] -$$

$$- \nabla\cdot\left(\overline{\mathbf{q}_1^{kin}} + \overline{\mathbf{q}_2^{kin}} + \overline{\mathbf{q}_1^{p}} + \overline{\mathbf{q}_2^{p}} + \overline{\mathbf{q}_1^{\tau}} + \overline{\mathbf{q}_2^{\tau}}\right) + \overline{\left(\mathbf{n}_k\cdot\nabla X_k\right)e_i^\sigma} -$$

$$- \overline{\alpha_1}\,\overline{\underline{\mathbf{T}}_1^x} : \nabla\,\overline{\mathbf{v}_1^{x,\rho}} - \alpha_2\overline{\underline{\mathbf{T}}_2^x} : \nabla\,\overline{\mathbf{v}_2^{x,\rho}} + \overline{X_1\,p\left(\nabla\cdot\mathbf{v}_1'\right)} + \overline{X_2\,p\left(\nabla\cdot\mathbf{v}_2'\right)} - D_m + \mathbf{g}\cdot\rho_m\mathbf{v}_m .$$

<u>Alternative form</u>

We start again from equation

$$\frac{\partial}{\partial t}\left\{\overline{\alpha_1}\ \overline{\rho_1^x}\left[\frac{1}{2}\left(\overline{\upsilon_1^{x,\rho}}\right)^2 + \overline{\mathrm{u}_1^{Re}}\right] + \overline{\alpha_2}\ \overline{\rho_2^x}\left[\frac{1}{2}\left(\overline{\upsilon_2^{x,\rho}}\right)^2 + \overline{\mathrm{u}_2^{Re}}\right]\right\} +$$

$$+ \nabla\cdot\left\{\overline{\alpha_1}\,\overline{\rho_1^x}\,\overline{\mathbf{v}_1^{x,\rho}}\left[\frac{1}{2}\left(\overline{\upsilon_1^{x,\rho}}\right)^2 + \overline{\mathrm{u}_1^{Re}}\right] + \overline{\alpha_2}\,\overline{\rho_2^x}\,\overline{\mathbf{v}_2^{x,\rho}}\left[\frac{1}{2}\left(\overline{\upsilon_2^{x,\rho}}\right)^2 + \overline{\mathrm{u}_2^{Re}}\right]\right\} =$$

$$= \nabla\cdot\left[\overline{\alpha_1}\left(\overline{\underline{\mathbf{T}}_1^x} + \overline{\underline{\mathbf{T}}_1^{Re}}\right)\cdot\overline{\mathbf{v}_1^{x,\rho}} + \alpha_2\left(\overline{\underline{\mathbf{T}}_2^x} + \overline{\underline{\mathbf{T}}_2^{Re}}\right)\cdot\overline{\mathbf{v}_2^{x,\rho}}\right] -$$

$$- \nabla\cdot\left(\overline{\mathbf{q}_1^{kin}} + \overline{\mathbf{q}_1^{p}} + \overline{\mathbf{q}_1^{\tau}} + \overline{\mathbf{q}_2^{kin}} + \overline{\mathbf{q}_2^{p}} + \overline{\mathbf{q}_2^{\tau}}\right) + \tag{6.2.4.1}$$

$$+ \frac{1}{2}\overline{\upsilon_{1i}^2}\,\overline{\Gamma_1} + \frac{1}{2}\overline{\upsilon_{2i}^2}\,\overline{\Gamma_2} + \overline{W_1} + \overline{W_2} -$$

$$- \overline{\alpha_1}\ \overline{\underline{T}_1^x} : \nabla \overline{v_1^{x,\rho}} - \alpha_2\ \overline{\underline{T}_2^x} : \nabla \overline{v_2^{x,\rho}} + \overline{X_1\, p \left(\nabla \cdot v_1'\right)} +$$

$$+ \overline{X_2\, p \left(\nabla \cdot v_2'\right)} - \left(\overline{\alpha_1}\ \overline{D_1} + \overline{\alpha_2}\ \overline{D_2} \right) + \overline{\alpha_1}\ \overline{\rho_1^x}\ g \cdot \overline{v_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ g \cdot \overline{v_2^{x,\rho}} .$$

Using the definitions

$$\rho_m\, u_m^{Re} = \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{u_1^{Re}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{u_2^{Re}} , \tag{6.1.2.16}$$

$$g \cdot \rho_m\, v_m = g \cdot \left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{v_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{v_2^{x,\rho}} \right) , \tag{6.1.2.23}$$

$$D_m = \overline{\alpha_1}\ \overline{D_1} + \overline{\alpha_2}\ \overline{D_2} , \tag{6.1.1.4}$$

and the jump condition for mechanical energy

$$\overline{W_1} + \overline{W_2} + \frac{1}{2}\ \overline{\upsilon_{1i}^2}\ \overline{\Gamma_1} + \frac{1}{2}\ \overline{\upsilon_{2i}^2}\ \overline{\Gamma_2} = \overline{\left(n_k \cdot \nabla X_k\right) e_i^{\sigma}} , \ (k = 1\ or\ 2) \tag{5.3.5}$$

one derives

$$\frac{\partial}{\partial t} \left(\rho_m\, u_m^{Re} + \rho_m \frac{\upsilon_m^2}{2} \right) +$$

$$+ \nabla \cdot \left\{ \overline{\alpha_1}\, \overline{\rho_1^x}\, \overline{v_1^{x,\rho}} \left[\frac{1}{2} \left(\overline{\upsilon_1^{x,\rho}} \right)^2 + \overline{u_1^{Re}} \right] + \overline{\alpha_2}\, \overline{\rho_2^x}\, \overline{v_2^{x,\rho}} \left[\frac{1}{2} \left(\overline{\upsilon_2^{x,\rho}} \right)^2 + \overline{u_2^{Re}} \right] \right\} =$$

$$= \nabla \cdot \left\{ \left[\overline{\alpha_1} \left(\overline{\underline{T}_1^x} + \overline{\underline{T}_1^{Re}} \right) + \overline{\alpha_2} \left(\overline{\underline{T}_2^x} + \overline{\underline{T}_2^{Re}} \right) \right] \cdot v_m \right\} +$$

$$+ \nabla \cdot \left[\overline{\alpha_1} \left(\overline{\underline{T}_1^x} + \overline{\underline{T}_1^{Re}} \right) \cdot \tilde{v}_1 + \overline{\alpha_2} \left(\overline{\underline{T}_2^x} + \overline{\underline{T}_2^{Re}} \right) \cdot \tilde{v}_2 \right] - \tag{6.2.4.4}$$

$$- \nabla \cdot \left(\overline{q_1^{kin}} + \overline{q_2^{kin}} + \overline{q_1^p} + \overline{q_2^p} + \overline{q_1^{\tau}} + \overline{q_2^{\tau}} \right) + \overline{\left(n_k \cdot \nabla X_k\right) e_i^{\sigma}} -$$

$$- \overline{\alpha_1}\ \overline{\underline{T}_1^x} : \nabla \overline{v_1^{x,\rho}} - \alpha_2\ \overline{\underline{T}_2^x} : \nabla \overline{v_2^{x,\rho}} +$$

$$+ \overline{X_1\, p \left(\nabla \cdot v_1'\right)} + \overline{X_2\, p \left(\nabla \cdot v_2'\right)} - D_m + g \cdot \rho_m\, v_m .$$

6.2.5 Internal energy equation

1st Form

From equation (5.1.5.1) one derives, summing the contribution of both phases:

$$\frac{\partial}{\partial t}\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{u_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{u_2^{x,\rho}}\right) + \nabla\cdot\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\mathbf{v}_1^{x,\rho}}\ \overline{u_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\mathbf{v}_2^{x,\rho}}\ \overline{u_2^{x,\rho}}\right) =$$

$$= -\nabla\cdot\left[\overline{\alpha_1}\left(\overline{\mathbf{q}_1^x} + \hat{\mathbf{q}}_1^{Re}\right) + \overline{\alpha_2}\left(\overline{\mathbf{q}_2^x} + \hat{\mathbf{q}}_2^{Re}\right)\right] + \overline{u_{1i}}\ \overline{\Gamma_1} + \overline{u_{2i}}\ \overline{\Gamma_2} + \overline{E_1} + \overline{E_2} + \qquad [6.2.5.1]$$

$$+ \overline{\alpha_1}\ \overline{D_1} + \overline{\alpha_2}\ \overline{D_2} + \overline{\alpha_1}\ \overline{\underline{\mathbf{T}}_1^x} : \nabla\overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\underline{\mathbf{T}}_2^x} : \nabla\overline{\mathbf{v}_2^{x,\rho}} -$$

$$- \overline{X_1\, p\, \nabla\cdot\mathbf{v}_1'} - \overline{X_2\, p\, \nabla\cdot\mathbf{v}_2'} + \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{r_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{r_2^{x,\rho}}\ .$$

With the usual procedure one derives

$$\frac{\partial}{\partial t}(\rho_m u_m) + \nabla\cdot\left[\overline{\alpha_1}\ \overline{\rho_1^x}\left(\mathbf{v}_m + \tilde{\mathbf{v}}_1\right)\left(u_m + \tilde{u}_1\right) + \overline{\alpha_2}\ \overline{\rho_2^x}\left(\mathbf{v}_m + \tilde{\mathbf{v}}_2\right)\left(u_m + \tilde{u}_2\right)\right] =$$

$$= -\nabla\cdot\left[\overline{\alpha_1}\left(\overline{\mathbf{q}_1^x} + \hat{\mathbf{q}}_1^{Re}\right) + \overline{\alpha_2}\left(\overline{\mathbf{q}_2^x} + \hat{\mathbf{q}}_2^{Re}\right)\right] + \bar{\varepsilon} - \overline{W_1} - \overline{W_2} - \frac{1}{2}\ \overline{\upsilon_{1i}^2}\ \overline{\Gamma_1} - \frac{1}{2}\ \overline{\upsilon_{2i}^2}\ \overline{\Gamma_2} + \qquad [6.2.5.2]$$

$$+ D_m + \overline{\alpha_1}\ \overline{\underline{\mathbf{T}}_1^x} : \nabla\left(\mathbf{v}_m + \tilde{\mathbf{v}}_1\right) + \overline{\alpha_2}\ \overline{\underline{\mathbf{T}}_2^x} : \nabla\left(\mathbf{v}_m + \tilde{\mathbf{v}}_2\right) - \overline{X_1\, p\, \nabla\cdot\mathbf{v}_1'} - \overline{X_2\, p\, \nabla\cdot\mathbf{v}_2'} + \rho_m r_m\ .$$

Expanding the terms and rearranging one derives

$$\frac{\partial}{\partial t}(\rho_m u_m) + \nabla\cdot\left[\mathbf{v}_m u_m\left(\overline{\alpha_1}\ \overline{\rho_1^x} + \overline{\alpha_2}\ \overline{\rho_2^x}\right) + \mathbf{v}_m\left(\overline{\alpha_1}\ \overline{\rho_1^x}\,\tilde{u}_1 + \overline{\alpha_2}\ \overline{\rho_2^x}\,\tilde{u}_2\right) +\right.$$

$$\left. + u_m\left(\overline{\alpha_1}\ \overline{\rho_1^x}\,\tilde{\mathbf{v}}_1 + \overline{\alpha_2}\ \overline{\rho_2^x}\,\tilde{\mathbf{v}}_2\right) + \overline{\alpha_1}\ \overline{\rho_1^x}\,\tilde{\mathbf{v}}_1\,\tilde{u}_1 + \overline{\alpha_2}\ \overline{\rho_2^x}\,\tilde{\mathbf{v}}_2\,\tilde{u}_2\right] =$$

$$= -\nabla\cdot\left[\overline{\alpha_1}\left(\overline{\mathbf{q}_1^x} + \hat{\mathbf{q}}_1^{Re}\right) + \overline{\alpha_2}\left(\overline{\mathbf{q}_2^x} + \hat{\mathbf{q}}_2^{Re}\right)\right] + \qquad [6.2.5.3]$$

$$+ \bar{\varepsilon} - \overline{W_1} - \overline{W_2} - \frac{1}{2}\ \overline{\upsilon_{1i}^2}\ \overline{\Gamma_1} - \frac{1}{2}\ \overline{\upsilon_{2i}^2}\ \overline{\Gamma_2} + D_m +$$

$$+ \left(\overline{\alpha_1}\ \overline{\underline{\mathbf{T}}_1^x} + \overline{\alpha_2}\ \overline{\underline{\mathbf{T}}_2^x}\right) : \nabla\mathbf{v}_m + \alpha_1\ \overline{\underline{\mathbf{T}}_1^x} : \nabla\tilde{\mathbf{v}}_1 + \alpha_2\ \overline{\underline{\mathbf{T}}_2^x} : \nabla\tilde{\mathbf{v}}_2 -$$

$$- \overline{X_1 p \nabla \cdot \mathbf{v}_1'} - \overline{X_2 p \nabla \cdot \mathbf{v}_2'} + \rho_m r_m .$$

With the definition:

$$\hat{q}_m = \overline{\alpha_1}\left(\overline{q_1^x} + \hat{q}_1^{Re}\right) + \overline{\alpha_2}\left(\overline{q_2^x} + \hat{q}_2^{Re}\right) + \overline{\alpha_1}\,\overline{\rho_1^x}\,\tilde{\mathbf{v}}_1\tilde{u}_1 + \overline{\alpha_2}\,\overline{\rho_2^x}\,\tilde{\mathbf{v}}_2\tilde{u}_2 \qquad [6.2.5.4]$$

we obtain:

$$\frac{\partial(\rho_m u_m)}{\partial t} + \nabla\cdot(\rho_m u_m \mathbf{v}_m) = -\nabla\cdot\hat{\mathbf{q}}_m + D_m + \rho_m r_m +$$

$$+ \bar{\varepsilon} - \overline{W_1} - \overline{W_2} - \frac{1}{2}\overline{v_{1i}^2}\,\overline{\Gamma_1} - \frac{1}{2}\overline{v_{2i}^2}\,\overline{\Gamma_2} + \qquad [6.2.5.5]$$

$$+ \left(\overline{\alpha_1}\,\overline{\underline{\mathbf{T}}_1^x} + \overline{\alpha_2}\,\overline{\underline{\mathbf{T}}_2^x}\right) : \nabla\mathbf{v}_m + \alpha_1\overline{\underline{\mathbf{T}}_1^x} : \nabla\tilde{\mathbf{v}}_1 + \alpha_2\overline{\underline{\mathbf{T}}_2^x} : \nabla\tilde{\mathbf{v}}_2 - \overline{X_1 p \nabla\cdot\mathbf{v}_1'} - \overline{X_2 p \nabla\cdot\mathbf{v}_2'} .$$

Using the identity $\underline{\mathbf{I}} : \nabla\mathbf{v} = \nabla\cdot\mathbf{v}$ (see equation (2.3.31)) we have

$$-\overline{\alpha_1}\,\overline{\underline{\mathbf{T}}_1^x} : \nabla\tilde{\mathbf{v}}_1 = \overline{\alpha_1}\left(-\overline{p_1^x}\,\underline{\mathbf{I}} + \overline{\underline{\tau}_1^x}\right) : \nabla\tilde{\mathbf{v}}_1 = -\overline{\alpha_1}\,\overline{p_1^x}\,\nabla\cdot\tilde{\mathbf{v}}_1 + \overline{\alpha_1}\,\overline{\underline{\tau}_1^x} : \nabla\tilde{\mathbf{v}}_1 \qquad [6.2.5.6]$$

and similarly for the second phase. Introducing into equation (6.2.5.5) and defining

$$W_m = \overline{-X_1 p\left(\nabla\cdot\mathbf{v}_1'\right)} - \overline{X_2 p\left(\nabla\cdot\mathbf{v}_2'\right)} - \overline{\alpha_1}\,\overline{p_1^x}\,\nabla\cdot\tilde{\mathbf{v}}_1' - \overline{\alpha_2}\,\overline{p_2^x}\,\nabla\cdot\tilde{\mathbf{v}}_2' \qquad [6.2.5.7]$$

one obtains:

$$\frac{\partial(\rho_m u_m)}{\partial t} + \nabla\cdot(\rho_m u_m \mathbf{v}_m) = -\nabla\cdot\hat{\mathbf{q}}_m + D_m + \rho_m r_m +$$

$$+ \bar{\varepsilon} - \overline{W_1} - \overline{W_2} - \frac{1}{2}\overline{v_{1i}^2}\,\overline{\Gamma_1} - \frac{1}{2}\overline{v_{2i}^2}\,\overline{\Gamma_2} -$$

$$[6.2.5.8]$$

$$- \overline{\alpha_1}\,\overline{p_1^x}\,\nabla\cdot\mathbf{v}_m + \overline{\alpha_1}\,\overline{\underline{\tau}_1^x} : \nabla\mathbf{v}_m - \overline{\alpha_2}\,\overline{p_2^x}\,\nabla\cdot\mathbf{v}_m + \overline{\alpha_2}\,\overline{\underline{\tau}_2^x} : \nabla\mathbf{v}_m +$$

$$+ \overline{\alpha_1}\,\overline{\underline{\tau}_1^x} : \nabla\tilde{\mathbf{v}}_1 + \overline{\alpha_2}\,\overline{\underline{\tau}_2^x} : \nabla\tilde{\mathbf{v}}_2 + W_m .$$

Using

$$\overline{W_1} + \overline{W_2} = \overline{-\left(\underline{\mathbf{T}}\cdot\mathbf{v}\right)\cdot\left(\nabla X_1 + \nabla X_2\right)} , \qquad (4.6.5)$$

$$p_m = \overline{\alpha_1}\,\overline{p_1^x} + \overline{\alpha_2}\,\overline{p_2^x} , \qquad (6.1.1.2)$$

one derives:

$$\frac{\partial (\rho_m u_m)}{\partial t} + \nabla \cdot (\rho_m u_m \mathbf{v}_m) = -\nabla \cdot \widehat{\mathbf{q}}_m + D_m + \rho_m r_m + \bar{\varepsilon} - p_m \nabla \cdot \mathbf{v}_m + W_m -$$

$$- \frac{1}{2} \overline{v_{1i}^2}\ \overline{\Gamma_1} - \frac{1}{2} \overline{v_{2i}^2}\ \overline{\Gamma_2} + \overline{\left(\underline{\mathbf{T}} \cdot \mathbf{v}\right) \cdot \left(\nabla X_1 + \nabla X_2\right)} + \qquad [6.2.5.9]$$

$$+ \overline{\alpha_1}\ \overline{\underline{\tau}_1^x} : \left(\nabla \mathbf{v}_m + \nabla \tilde{\mathbf{v}}_1\right) + \overline{\alpha_2}\ \overline{\underline{\tau}_2^x} : \left(\nabla \mathbf{v}_m + \nabla \tilde{\mathbf{v}}_2\right).$$

Alternative form

Starting again from equation

$$\frac{\partial}{\partial t}\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{u_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{u_2^{x,\rho}}\right) + \nabla \cdot \left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\mathbf{v}_1^{x,\rho}}\ \overline{u_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\mathbf{v}_2^{x,\rho}}\ \overline{u_2^{x,\rho}}\right) =$$

$$= -\nabla \cdot \left[\overline{\alpha_1}\left(\overline{\mathbf{q}_1^x} + \widehat{\mathbf{q}}_1^{Re}\right) + \overline{\alpha_2}\left(\overline{\mathbf{q}_2^x} + \widehat{\mathbf{q}}_2^{Re}\right)\right] + \overline{u_{1i}}\ \overline{\Gamma_1} + \overline{u_{2i}}\ \overline{\Gamma_2} + \overline{E_1} + \overline{E_2} + \qquad (6.2.5.1)$$

$$+ \overline{\alpha_1}\ \overline{D_1} + \overline{\alpha_2}\ \overline{D_2} + \overline{\alpha_1}\ \overline{\underline{\mathbf{T}}_1^x} : \nabla \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\underline{\mathbf{T}}_2^x} : \nabla \overline{\mathbf{v}_2^{x,\rho}} -$$

$$- \overline{X_1 p \nabla \cdot \mathbf{v}_1'} - \overline{X_2 p \nabla \cdot \mathbf{v}_2'} + \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{r_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{r_2^{x,\rho}},$$

one derives with the usual procedure

$$\frac{\partial}{\partial t}(\rho_m u_m) + \nabla \cdot \left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\mathbf{v}_1^{x,\rho}}\ \overline{u_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\mathbf{v}_2^{x,\rho}}\ \overline{u_2^{x,\rho}}\right) =$$

$$= -\nabla \cdot \left[\overline{\alpha_1}\left(\overline{\mathbf{q}_1^x} + \widehat{\mathbf{q}}_1^{Re}\right) + \overline{\alpha_2}\left(\overline{\mathbf{q}_2^x} + \widehat{\mathbf{q}}_2^{Re}\right)\right] + \bar{\varepsilon} - \overline{W_1} - \overline{W_2} - \frac{1}{2} \overline{v_{1i}^2}\ \overline{\Gamma_1} - \frac{1}{2} \overline{v_{2i}^2}\ \overline{\Gamma_2} + \qquad [6.2.5.10]$$

$$+ D_m + \overline{\alpha_1}\ \overline{\underline{\mathbf{T}}_1^x} : \nabla\left(\mathbf{v}_m + \tilde{\mathbf{v}}_1\right) + \overline{\alpha_2}\ \overline{\underline{\mathbf{T}}_2^x} : \nabla\left(\mathbf{v}_m + \tilde{\mathbf{v}}_2\right) - \overline{X_1 p \nabla \cdot \mathbf{v}_1'} - \overline{X_2 p \nabla \cdot \mathbf{v}_2'} + \rho_m r_m .$$

Using the identity $\underline{\mathbf{I}} : \nabla \mathbf{v} = \nabla \cdot \mathbf{v}$ (see equation (2.3.31)) we have

$$-\overline{\alpha_1}\ \overline{\underline{\mathbf{T}}_1^x} : \nabla \tilde{\mathbf{v}}_1 = \overline{\alpha_1}\left(-\overline{p_1^x}\,\underline{\mathbf{I}} + \overline{\underline{\tau}_1^x}\right) : \nabla \tilde{\mathbf{v}}_1 = -\overline{\alpha_1}\ \overline{p_1^x}\ \nabla \cdot \tilde{\mathbf{v}}_1 + \overline{\alpha_1}\ \overline{\underline{\tau}_1^x} : \nabla \tilde{\mathbf{v}}_1 \qquad (6.2.5.6)$$

and similarly for the second phase. Introducing into equation (6.2.5.10) and defining

$$W_m = \overline{-X_1 p \left(\nabla \cdot \mathbf{v}_1'\right)} - \overline{X_2 p \left(\nabla \cdot \mathbf{v}_2'\right)} - \overline{\alpha_1}\ \overline{p_1^x}\ \nabla \cdot \tilde{\mathbf{v}}_1' - \overline{\alpha_2}\ \overline{p_2^x}\ \nabla \cdot \tilde{\mathbf{v}}_2' \qquad (6.2.5.7)$$

one obtains:

$$\frac{\partial (\rho_m u_m)}{\partial t} + \nabla \cdot \left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\mathbf{v}_1^{x,\rho}}\ \overline{u_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\mathbf{v}_2^{x,\rho}}\ \overline{u_2^{x,\rho}} \right) =$$

$$= - \nabla \cdot \left[\overline{\alpha_1} \left(\overline{\mathbf{q}_1^x} + \hat{\mathbf{q}}_1^{Re} \right) + \overline{\alpha_2} \left(\overline{\mathbf{q}_2^x} + \hat{\mathbf{q}}_2^{Re} \right) \right] + D_m + \rho_m r_m +$$

$$+ \bar{\varepsilon} - \overline{W_1} - \overline{W_2} - \frac{1}{2}\ \overline{\upsilon_{1i}^2}\ \overline{\Gamma_1} - \frac{1}{2}\ \overline{\upsilon_{2i}^2}\ \overline{\Gamma_2} - \qquad [6.2.5.11]$$

$$- \overline{\alpha_1}\ \overline{p_1^x}\ \nabla \cdot \mathbf{v}_m + \overline{\alpha_1}\ \overline{\underline{\tau}_1^x} : \nabla \mathbf{v}_m - \overline{\alpha_2}\ \overline{p_2^x}\ \nabla \cdot \mathbf{v}_m + \overline{\alpha_2}\ \overline{\underline{\tau}_2^x} : \nabla \mathbf{v}_m +$$

$$+ \overline{\alpha_1}\ \overline{\underline{\tau}_1^x} : \nabla \tilde{\mathbf{v}}_1 + \overline{\alpha_2}\ \overline{\underline{\tau}_2^x} : \nabla \tilde{\mathbf{v}}_2 + W_m .$$

Using

$$\overline{W_1} + \overline{W_2} = - \overline{(\underline{\mathbf{T}} \cdot \mathbf{v}) \cdot (\nabla X_1 + \nabla X_2)} , \qquad (4.6.5)$$

$$\rho_m = \overline{\alpha_1}\ \overline{\rho_1^x} + \overline{\alpha_2}\ \overline{\rho_2^x} , \qquad (6.1.1.2)$$

one derives:

$$\frac{\partial (\rho_m u_m)}{\partial t} + \nabla \cdot \left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\mathbf{v}_1^{x,\rho}}\ \overline{u_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\mathbf{v}_2^{x,\rho}}\ \overline{u_2^{x,\rho}} \right) =$$

$$= - \nabla \cdot \left[\overline{\alpha_1} \left(\overline{\mathbf{q}_1^x} + \hat{\mathbf{q}}_1^{Re} \right) + \overline{\alpha_2} \left(\overline{\mathbf{q}_2^x} + \hat{\mathbf{q}}_2^{Re} \right) \right] + D_m + \rho_m r_m + \bar{\varepsilon} - p_m \nabla \cdot \mathbf{v}_m + W_m -$$

[6.2.5.12]

$$- \frac{1}{2}\ \overline{\upsilon_{1i}^2}\ \overline{\Gamma_1} - \frac{1}{2}\ \overline{\upsilon_{2i}^2}\ \overline{\Gamma_2} + \overline{\left(\underline{\mathbf{T}} \cdot \mathbf{v}\right) \cdot \left(\nabla X_1 + \nabla X_2\right)} +$$

$$+ \overline{\alpha_1}\ \overline{\underline{\tau}_1^x} : \left(\nabla \mathbf{v}_m + \nabla \tilde{\mathbf{v}}_1 \right) + \overline{\alpha_2}\ \overline{\underline{\tau}_2^x} : \left(\nabla \mathbf{v}_m + \nabla \tilde{\mathbf{v}}_2 \right) .$$

6.2.6 Enthalpy equation

From equation (5.1.6.6) one derives

$$\frac{\partial}{\partial t}\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{h_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{h_2^{x,\rho}}\right) + \nabla\cdot\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{h_1^{x,\rho}}\ \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{h_2^{x,\rho}}\ \overline{\mathbf{v}_2^{x,\rho}}\right) =$$

$$= -\nabla\cdot\left[\overline{\alpha_1}\left(\overline{q_1^x} + \widehat{q}_1^{Re}\right) + \overline{\alpha_2}\left(\overline{q_2^x} + \widehat{q}_2^{Re}\right)\right] + \overline{h_{1i}}\ \overline{\Gamma_1} + \overline{h_{2i}}\ \overline{\Gamma_2} +$$

$$+ \overline{\alpha_1}\ \frac{\partial \overline{p_1^x}}{\partial t} + \overline{\alpha_2}\ \frac{\partial \overline{p_2^x}}{\partial t} + \overline{\alpha_1}\ \overline{\mathbf{v}_1^{x,\rho}}\cdot\nabla\overline{p_1^x} + \overline{\alpha_2}\ \overline{\mathbf{v}_2^{x,\rho}}\cdot\nabla\overline{p_2^x} + \tag{6.2.6.1}$$

$$+ \overline{X_1\left(\frac{\partial p_1'}{\partial t} + \mathbf{v}_1'\cdot\nabla p_1'\right)} + \overline{X_2\left(\frac{\partial p_2'}{\partial t} + \mathbf{v}_2'\cdot\nabla p_2'\right)} + \overline{E_1} + \overline{E_2} +$$

$$+ \overline{\alpha_1}\ \overline{\underline{\mathbf{T}}_1^x} : \nabla\overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\underline{\mathbf{T}}_2^x} : \nabla\overline{\mathbf{v}_2^{x,\rho}} + \overline{\alpha_1}\ \overline{D_1} + \overline{\alpha_2}\ \overline{D_2} -$$

$$- \overline{X_1 p_1 \nabla\cdot\mathbf{v}_1'} - \overline{X_2 p_2 \nabla\cdot\mathbf{v}_2'} + \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{r_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{r_2^{x,\rho}}\ .$$

Now, we use the definitions

$$\rho_m h_m = \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{h_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\,\overline{h_1^{x,\rho}}\ , \tag{6.1.2.10}$$

$$D_m = \overline{\alpha_1}\ \overline{D_1} + \overline{\alpha_2}\ \overline{D_2}\ , \tag{6.1.1.4}$$

$$\rho_m r_m = \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{r_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{r_2^{x,\rho}}\ , \tag{6.1.2.15}$$

and the jump condition for total energy

$$\overline{E_1} + \overline{W_1} + \overline{E_2} + \overline{W_2} + \left(\overline{u_{1i}} + \frac{1}{2}\ \overline{v_{1i}^2}\right)\overline{\Gamma_1} + \left(\overline{u_{2i}} + \frac{1}{2}\ \overline{v_{2i}^2}\right)\overline{\Gamma_2} = \bar{\varepsilon} \tag{5.3.4}$$

with

$$\overline{W_1} + \overline{W_2} = -\overline{\left(\underline{\mathbf{T}}_1\cdot\mathbf{v}_1\right)\cdot\nabla X_1} - \overline{\left(\underline{\mathbf{T}}_2\cdot\mathbf{v}_2\right)\cdot\nabla X_2} \tag{4.6.5}$$

and obtain

$$\frac{\partial}{\partial t}(\rho_m h_m) + \nabla\cdot\left[\overline{\alpha_1}\ \overline{\rho_1^x}\left(h_m + \widetilde{h}_1\right)\left(\mathbf{v}_m + \widetilde{\mathbf{v}}_1\right) + \overline{\alpha_2}\ \overline{\rho_2^x}\left(h_m + \widetilde{h}_2\right)\left(\mathbf{v}_m + \widetilde{\mathbf{v}}_2\right)\right] =$$

$$= -\nabla\cdot\left[\overline{\alpha_1}\left(\overline{\mathbf{q}_1^x} + \widehat{\mathbf{q}}_1^{Re}\right) + \overline{\alpha_2}\left(\overline{\mathbf{q}_2^x} + \widehat{\mathbf{q}}_2^{Re}\right)\right] + \overline{h_{1i}}\ \overline{\Gamma_1} + \overline{h_{2i}}\ \overline{\Gamma_2} +$$

$$+ \overline{\alpha_1} \frac{\partial p_m}{\partial t} + \overline{\alpha_1} \frac{\partial \widetilde{p}_1}{\partial t} + \overline{\alpha_2} \frac{\partial p_m}{\partial t} + \overline{\alpha_2} \frac{\partial \widetilde{p}_2}{\partial t} +$$

$$+ \alpha_1 \overline{\mathbf{v}_1^{x,\rho}} \cdot \nabla p_m + \alpha_1 \overline{\mathbf{v}_1^{x,\rho}} \cdot \nabla \widetilde{p}_1 + \alpha_2 \overline{\mathbf{v}_2^{x,\rho}} \cdot \nabla p_m + \alpha_2 \overline{\mathbf{v}_2^{x,\rho}} \cdot \nabla \widetilde{p}_2 + \qquad [6.2.6.2]$$

$$+ \overline{X_1 \left(\frac{\partial p_1'}{\partial t} + \mathbf{v}_1 \cdot \nabla p_1' \right)} + \overline{X_2 \left(\frac{\partial p_2'}{\partial t} + \mathbf{v}_2 \cdot \nabla p_2' \right)} + \bar{\varepsilon}$$

$$+ \overline{\left(\underline{\mathbf{T}} \cdot \mathbf{v}\right)_1 \cdot \nabla X_1} + \overline{\left(\underline{\mathbf{T}} \cdot \mathbf{v}\right)_2 \cdot \nabla X_2} - \left(u_{1i} + \frac{1}{2} \overline{v_{1i}^2} \right) \overline{\Gamma_1} - \left(u_{2i} + \frac{1}{2} \overline{v_{2i}^2} \right) \overline{\Gamma_2} +$$

$$+ \overline{\alpha_1} \overline{\underline{\mathbf{T}}_1^x} : \nabla \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2} \overline{\underline{\mathbf{T}}_2^x} : \nabla \overline{\mathbf{v}_2^{x,\rho}} + D_m - \overline{X_1 p_1 \nabla \cdot \mathbf{v}_1'} - \overline{X_2 p_2 \nabla \cdot \mathbf{v}_2'} + \rho_m r_m .$$

Furthermore, we introduce the definitions

$$\hat{\hat{\mathbf{q}}}_m = \overline{\alpha_1} \left(\overline{\mathbf{q}_1^x} + \hat{\hat{\mathbf{q}}}_1^{Re} \right) + \overline{\alpha_2} \left(\overline{\mathbf{q}_2^x} + \hat{\hat{\mathbf{q}}}_2^{Re} \right) + \overline{\alpha_1}\, \overline{\rho_1^x}\, \widetilde{h}_1 \widetilde{\mathbf{v}}_1 + \overline{\alpha_2}\, \overline{\rho_2^x}\, \widetilde{h}_2 \widetilde{\mathbf{v}}_2 , \qquad [6.2.6.3]$$

$$\widehat{\mathrm{W}}_m = \overline{X_1 \left(\frac{\partial p_1'}{\partial t} + \mathbf{v}_1' \cdot \nabla p_1' \right)} + \overline{X_2 \left(\frac{\partial p_2'}{\partial t} + \mathbf{v}_2' \cdot \nabla p_2' \right)} ,$$

$$+ \overline{\alpha_1} \left(\frac{\partial \widetilde{p}_1}{\partial t} + \overline{\mathbf{v}_1^{x,\rho}} \cdot \nabla \widetilde{p}_1 \right) + \overline{\alpha_2} \left(\frac{\partial \widetilde{p}_2}{\partial t} + \overline{\mathbf{v}_2^{x,\rho}} \cdot \nabla \widetilde{p}_2 \right) , \qquad [6.2.6.4]$$

and obtain, rearranging equation (6.2.6.2),

$$\frac{\partial}{\partial t} (\rho_m h_m) + \nabla \cdot (\rho_m h_m \mathbf{v}_m) =$$

$$= - \nabla \cdot \hat{\hat{\mathbf{q}}}_m + \overline{h_{1i}}\, \overline{\Gamma_1} + \overline{h_{2i}}\, \overline{\Gamma_2} + \frac{\partial p_m}{\partial t} + \left(\overline{\alpha_1}\, \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\, \overline{\mathbf{v}_2^{x,\rho}} \right) \cdot \nabla p_m +$$

$$+ \bar{\varepsilon} - \left(\bar{u}_{1i} + \frac{1}{2} \overline{v_{1i}^2} \right) \overline{\Gamma_1} - \left(\bar{u}_{2i} + \frac{1}{2} \overline{v_{2i}^2} \right) \overline{\Gamma_2} + \qquad [6.2.6.5]$$

$$+ \overline{\left(\underline{T} \cdot v\right)_1 \cdot \nabla X_1} + \overline{\left(\underline{T} \cdot v\right)_2 \cdot \nabla X_2} + \overline{\alpha_1}\ \overline{\underline{T}_1^x} : \nabla \overline{v_1^{x,\rho}} +$$

$$+ \overline{\alpha_2}\ \overline{\underline{T}_2^x} : \nabla \overline{v_2^{x,\rho}} + D_m - \overline{X_1 p_1 \nabla \cdot v_1'} - \overline{X_2 p_2 \nabla \cdot v_2'} + \rho_m r_m + \widehat{W}_m .$$

Finally, we define

$$\widehat{v}_m = \overline{\alpha_1}\ \overline{v_1^{x,\rho}} + \overline{\alpha_2}\ \overline{v_2^{x,\rho}} , \qquad [6.2.6.6]$$

$$\widehat{\widehat{W}}_m = \widehat{W}_m - \overline{X_1 p_1 \nabla \cdot v_1'} - \overline{X_2 p_2 \nabla \cdot v_2'} , \qquad [6.2.6.7]$$

and use

$$\overline{u_{ki}} = \overline{h_{ki}} - \frac{\overline{p_k^x}}{\overline{\overline{\rho_k^x}}} , \qquad [6.2.6.8]$$

thus obtaining the final form of the mixture enthalpy equation

$$\frac{\partial}{\partial t}(\rho_m h_m) + \nabla \cdot (\rho_m h_m v_m) =$$

$$= - \nabla \cdot \widehat{\widehat{q}}_m + \frac{\partial p_m}{\partial t} + \widehat{v}_m \cdot \nabla p_m + \bar{\varepsilon} + D_m + \rho_m r_m + \widehat{\widehat{W}}_m -$$

$$- \left(\frac{1}{2} \overline{v_{1i}^2} - \frac{\overline{p_1^x}}{\overline{\overline{\rho_1^x}}} \right) \overline{\Gamma_1} - \left(\frac{1}{2} \overline{v_{2i}^2} - \frac{\overline{p_2^x}}{\overline{\overline{\rho_2^x}}} \right) \overline{\Gamma_2} + \qquad [6.2.6.9]$$

$$+ \overline{\left(\underline{T} \cdot v\right)_1 \cdot \nabla X_1} + \overline{\left(\underline{T} \cdot v\right)_2 \cdot \nabla X_2} + \overline{\alpha_1}\ \overline{\underline{T}_1^x} : \nabla \overline{v_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\underline{T}_2^x} : \nabla \overline{v_2^{x,\rho}} .$$

<u>Alternative formulation</u>

Starting again from equation (5.1.6.6) we obtain:

$$\frac{\partial}{\partial t}\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{h_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{h_2^{x,\rho}} \right) + \nabla \cdot \left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{h_1^{x,\rho}}\ \overline{v_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{h_2^{x,\rho}}\ \overline{v_2^{x,\rho}} \right) =$$

$$= -\nabla \cdot \left[\overline{\alpha_1} \left(\overline{q_1^x} + \widehat{\widehat{q}}_1^{Re} \right) + \overline{\alpha_2} \left(\overline{q_2^x} + \widehat{\widehat{q}}_2^{Re} \right) \right] + \overline{h_{1i}}\ \overline{\Gamma_1} + \overline{h_{2i}}\ \overline{\Gamma_2} +$$

$$+ \overline{\alpha_1}\ \frac{\partial\ \overline{p_1^x}}{\partial t} + \overline{\alpha_2}\ \frac{\partial\ \overline{p_2^x}}{\partial t} + \overline{\alpha_1}\ \overline{\mathbf{v}_1^{x,\rho}} \cdot \overline{\nabla p_1^x} + \overline{\alpha_2}\ \overline{\mathbf{v}_2^{x,\rho}} \cdot \overline{\nabla p_2^x} +$$

$$+ \overline{X_1 \left(\frac{\partial p_1'}{\partial t} + \mathbf{v}_1' \cdot \nabla p_1' \right)} + \overline{X_2 \left(\frac{\partial p_2'}{\partial t} + \mathbf{v}_2' \cdot \nabla p_2' \right)} + \overline{E_1} + \overline{E_2} + \qquad (6.2.6.1)$$

$$+ \overline{\alpha_1}\ \overline{\underline{\mathbf{T}}_1^x} : \nabla \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\underline{\mathbf{T}}_2^x} : \nabla \overline{\mathbf{v}_2^{x,\rho}} + \overline{\alpha_1}\ \overline{D_1} + \overline{\alpha_2}\ \overline{D_2} -$$

$$- \overline{X_1\, p_1\, \nabla \cdot \mathbf{v}_1'} - \overline{X_2\, p_2\, \nabla \cdot \mathbf{v}_2'} + \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{r_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{r_2^{x,\rho}} .$$

With the same definitions

$$\rho_m\, h_m = \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{h_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\, \overline{h_2^{x,\rho}} , \qquad (6.1.2.10)$$

$$D_m = \overline{\alpha_1}\ \overline{D_1} + \overline{\alpha_2}\ \overline{D_2} , \qquad (6.1.1.4)$$

$$\rho_m\, r_m = \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{r_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{r_2^{x,\rho}} , \qquad (6.1.2.15)$$

and the jump condition for total energy

$$\overline{E_1} + \overline{W_1} + \overline{E_2} + \overline{W_2} + \left(\overline{u_{1i}} + \frac{1}{2}\ \overline{v_{1i}^2} \right) \overline{\Gamma_1} + \left(\overline{u_{2i}} + \frac{1}{2}\ \overline{v_{2i}^2} \right) \overline{\Gamma_2} = \bar{\varepsilon} \qquad (5.3.4)$$

with

$$\overline{W_1} + \overline{W_2} = - \overline{\left(\underline{\mathbf{T}}_1 \cdot \mathbf{v}_1 \right) \cdot \nabla X_1} - \overline{\left(\underline{\mathbf{T}}_2 \cdot \mathbf{v}_2 \right) \cdot \nabla X_2} , \qquad (4.6.5)$$

but without expanding the divergence term at the left hand side, we obtain

$$\frac{\partial}{\partial t} (\rho_m\, h_m) + \nabla \cdot \left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{h_1^{x,\rho}}\ \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{h_2^{x,\rho}}\ \overline{\mathbf{v}_2^{x,\rho}} \right) =$$

$$= - \nabla \cdot \left[\overline{\alpha_1} \left(\overline{\mathbf{q}_1^x} + \widehat{\mathbf{q}}_1^{Re} \right) + \overline{\alpha_2} \left(\overline{\mathbf{q}_2^x} + \widehat{\mathbf{q}}_2^{Re} \right) \right] + \overline{h_{1i}}\ \overline{\Gamma_1} + \overline{h_{2i}}\ \overline{\Gamma_2} +$$

$$+ \overline{\alpha_1}\ \frac{\partial p_m}{\partial t} + \overline{\alpha_1}\ \frac{\partial \widetilde{p}_1}{\partial t} + \overline{\alpha_2}\ \frac{\partial p_m}{\partial t} + \overline{\alpha_2}\ \frac{\partial \widetilde{p}_2}{\partial t} +$$

$$+ \alpha_1\ \overline{\mathbf{v}_1^{x,\rho}} \cdot \nabla p_m + \alpha_1\ \overline{\mathbf{v}_1^{x,\rho}} \cdot \nabla \widetilde{p}_1 + \alpha_2\ \overline{\mathbf{v}_2^{x,\rho}} \cdot \nabla p_m + \alpha_2\ \overline{\mathbf{v}_2^{x,\rho}} \cdot \nabla \widetilde{p}_2 + \qquad [6.2.6.10]$$

$$+ \overline{X_1 \left(\frac{\partial p_1'}{\partial t} + v_1 \cdot \nabla p_1' \right)} + \overline{X_2 \left(\frac{\partial p_2'}{\partial t} + v_2 \cdot \nabla p_2' \right)} +$$

$$+ \bar{\varepsilon} + \overline{\left(\underline{T} \cdot v\right)_1 \cdot \nabla X_1} + \overline{\left(\underline{T} \cdot v\right)_2 \cdot \nabla X_2} - \left(u_{1i} + \frac{1}{2} \overline{v_{1i}^2} \right) \overline{\Gamma}_1 - \left(u_{2i} + \frac{1}{2} \overline{v_{2i}^2} \right) \overline{\Gamma}_2 +$$

$$+ \overline{\alpha}_1 \overline{\underline{T}_1^x} : \nabla \overline{v_1^{x,\rho}} + \overline{\alpha}_2 \overline{\underline{T}_2^x} : \nabla \overline{v_2^{x,\rho}} + D_m - \overline{X_1 p_1 \nabla \cdot v_1'} - \overline{X_2 p_2 \nabla \cdot v_2'} + \rho_m r_m \, .$$

Furthermore with the definition

$$\widehat{W}_m = \overline{X_1 \left(\frac{\partial p_1'}{\partial t} + v_1' \cdot \nabla p_1' \right)} + \overline{X_2 \left(\frac{\partial p_2'}{\partial t} + v_2' \cdot \nabla p_2' \right)} +$$

$$+ \overline{\alpha}_1 \left(\frac{\partial \tilde{p}_1}{\partial t} + \overline{v_1^{x,\rho}} \cdot \nabla \tilde{p}_1 \right) + \overline{\alpha}_2 \left(\frac{\partial \tilde{p}_2}{\partial t} + \overline{v_2^{x,\rho}} \cdot \nabla \tilde{p}_2 \right) , \qquad (6.2.6.4)$$

we obtain, rearranging equation (6.2.6.10),

$$\frac{\partial}{\partial t} (\rho_m h_m) + \nabla \cdot \left[\overline{\alpha}_1 \overline{\rho_1^x} \, \overline{h_1^{x,\rho}} \, \overline{v_1^{x,\rho}} + \overline{\alpha}_2 \overline{\rho_2^x} \, \overline{h_2^{x,\rho}} \, \overline{v_2^{x,\rho}} \right] =$$

$$= - \nabla \cdot \left[\overline{\alpha}_1 \left(\overline{q_1^x} + \widehat{\widehat{q}}_1^{Re} \right) + \overline{\alpha}_2 \left(\overline{q_2^x} + \widehat{\widehat{q}}_2^{Re} \right) \right] +$$

$$+ \overline{h}_{1i} \overline{\Gamma}_1 + \overline{h}_{2i} \overline{\Gamma}_2 + \frac{\partial p_m}{\partial t} + \left(\overline{\alpha}_1 \overline{v_1^{x,\rho}} + \overline{\alpha}_2 \overline{v_2^{x,\rho}} \right) \cdot \nabla p_m +$$

$$+ \bar{\varepsilon} - \left(\overline{u}_{1i} + \frac{1}{2} \overline{v_{1i}^2} \right) \overline{\Gamma}_1 - \left(\overline{u}_{2i} + \frac{1}{2} \overline{v_{2i}^2} \right) \overline{\Gamma}_2 + \qquad [6.2.6.11]$$

$$+ \overline{\left(\underline{T} \cdot v\right)_1 \cdot \nabla X_1} + \overline{\left(\underline{T} \cdot v\right)_2 \cdot \nabla X_2} + \overline{\alpha}_1 \overline{\underline{T}_1^x} : \nabla \overline{v_1^{x,\rho}} +$$

$$+ \overline{\alpha}_2 \overline{\underline{T}_2^x} : \nabla \overline{v_2^{x,\rho}} + D_m - \overline{X_1 p_1 \nabla \cdot v_1'} - \overline{X_2 p_2 \nabla \cdot v_2'} + \rho_m r_m + \widehat{W}_m .$$

Finally using again

$$\hat{v}_m = \overline{\alpha}_1 \overline{v_1^{x,\rho}} + \overline{\alpha}_2 \overline{v_2^{x,\rho}} , \qquad (6.2.6.6)$$

$$\widehat{\widehat{W}}_m = \widehat{W}_m - \overline{X_1 p_1 \nabla \cdot v_1'} - \overline{X_2 p_2 \nabla \cdot v_2'} , \qquad (6.2.6.7)$$

$$\overline{u_{ki}} = \overline{h_{ki}} - \frac{\overline{p_k^x}}{\overline{\overline{\rho_k^x}}} , \qquad (6.2.6.8)$$

we obtain the alternative form of the mixture enthalpy equation

$$\frac{\partial}{\partial t}(\rho_m h_m) + \nabla \cdot \left(\overline{\alpha_1}\, \overline{\overline{\rho_1^x}}\, \overline{\overline{h_1^{x,\rho}}}\, \overline{\overline{\mathbf{v}_1^{x,\rho}}} + \overline{\alpha_2}\, \overline{\overline{\rho_2^x}}\, \overline{\overline{h_2^{x,\rho}}}\, \overline{\overline{\mathbf{v}_2^{x,\rho}}} \right) =$$

$$= - \nabla \cdot \left[\overline{\alpha_1} \left(\overline{\mathbf{q}_1^x} + \widehat{\widehat{\mathbf{q}}}_1^{Re} \right) + \overline{\alpha_2} \left(\overline{\mathbf{q}_2^x} + \widehat{\widehat{\mathbf{q}}}_2^{Re} \right) \right] +$$

$$+ \frac{\partial p_m}{\partial t} + \widehat{\mathbf{v}}_m \cdot \nabla p_m + \bar{\varepsilon} + D_m + \rho_m r_m + \widehat{\widehat{W}}_m - \qquad [6.2.6.12]$$

$$- \left(\frac{1}{2}\, \overline{v_{1i}^2} - \frac{\overline{p_1^x}}{\overline{\overline{\rho_1^x}}} \right) \overline{\Gamma_1} - \left(\frac{1}{2}\, \overline{v_{2i}^2} - \frac{\overline{p_2^x}}{\overline{\overline{\rho_2^x}}} \right) \overline{\Gamma_2} +$$

$$+ \overline{\left(\underline{\mathbf{T}} \cdot \mathbf{v}\right)_1 \cdot \nabla X_1} + \overline{\left(\underline{\mathbf{T}} \cdot \mathbf{v}\right)_2 \cdot \nabla X_2} + \overline{\alpha_1}\, \overline{\underline{\mathbf{T}}_1^x} : \nabla \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\, \overline{\underline{\mathbf{T}}_2^x} : \nabla \overline{\mathbf{v}_2^{x,\rho}} .$$

6.2.7 Entropy inequality and entropy equation

1st Form

i) Entropy inequality

From equation (5.1.7.9) one obtains

$$\frac{\partial}{\partial t}\left(\overline{\alpha_1}\, \overline{\rho_1^x}\, \overline{s_1^{x,\rho}} + \overline{\alpha_2}\, \overline{\rho_2^x}\, \overline{s_2^{x,\rho}} \right) + \nabla \cdot \left(\overline{\alpha_1}\, \overline{\rho_1^x}\, \overline{s_1^{x,\rho}}\, \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\, \overline{\rho_2^x}\, \overline{s_2^{x,\rho}}\, \overline{\mathbf{v}_2^{x,\rho}} \right) -$$

$$- \overline{s_{1i}}\, \overline{\Gamma_1} - \overline{s_{2i}}\, \overline{\Gamma_2} - \nabla \cdot \left[\overline{\alpha_1} \left(\overline{\boldsymbol{\phi}_1^x} + \overline{\boldsymbol{\phi}_1^{Re}} \right) + \overline{\alpha_2} \left(\overline{\boldsymbol{\phi}_2^x} + \overline{\boldsymbol{\phi}_2^{Re}} \right) \right] - \overline{s_1} - \overline{s_2} - \qquad [6.2.7.1]$$

$$- \overline{\alpha_1}\, \overline{\rho_1^x}\, \overline{\sigma_1^{x,\rho}} - \overline{\alpha_2}\, \overline{\rho_2^x}\, \overline{\sigma_2^{x,\rho}} = \overline{\Delta s_1} + \overline{\Delta s_2} \geq 0$$

with (from equation (5.1.7.10))

$$\overline{\Delta s_1} + \overline{\Delta s_2} = \overline{s_{T1}} + \overline{s_{T2}} + \overline{\alpha_1}\, \overline{\underline{\tau}_{s1}^x} : \nabla \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\, \overline{\underline{\tau}_{s2}^x} : \nabla \overline{\mathbf{v}_2^{x,\rho}} + \overline{\alpha_1}\, \overline{D_{s1}} + \overline{\alpha_2}\, \overline{D_{s2}} \geq 0 \qquad (6.2.7.2)$$

We use the following definitions in equation (6.2.7.1):

$$\rho_m\, s_m = \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{s_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{s_2^{x,\rho}} \ , \qquad (6.1.2.11)$$

$$\rho_m\, \sigma_m = \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\sigma_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\sigma_2^{x,\rho}} \ , \qquad (6.1.2.12)$$

$$\overline{s_{1i}}\ \overline{\Gamma_1} + \overline{s_{2i}}\ \overline{\Gamma_2} + \overline{s_1} + \overline{s_2} = \Delta s_i \ \geq 0 \ , \qquad (5.3.8)$$

$$\overline{\mathbf{v}_k^{x,\rho}} = \mathbf{v}_m + \tilde{\mathbf{v}}_k \quad (k = 1, 2) \ , \qquad (6.1.2.1)$$

$$\Delta s_m = \overline{\Delta s_1} + \overline{\Delta s_2} \ , \qquad [6.2.7.3]$$

thus obtaining

$$\frac{\partial}{\partial t}(\rho_m\, s_m) + \nabla\cdot(\rho_m\, s_m\, \mathbf{v}_m) + \nabla\cdot\left(\overline{\alpha_1}\ \overline{\rho_1}\ \overline{s_1^{x,\rho}}\,\tilde{\mathbf{v}}_1 + \overline{\alpha_2}\ \overline{\rho_2}\ \overline{s_2^{x,\rho}}\,\tilde{\mathbf{v}}_2\right) -$$

$$- \nabla\cdot\left[\overline{\alpha_1}\left(\overline{\boldsymbol{\Phi}_1^x} + \overline{\boldsymbol{\Phi}_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\boldsymbol{\Phi}_2^x} + \overline{\boldsymbol{\Phi}_2^{Re}}\right)\right] - \rho_m\, \sigma_m = \Delta s_m + \Delta s_i \ \geq 0 \qquad [6.2.7.4]$$

In this equation the contribution to entropy increase within the phases (Δs_m) is separated from the entropy increase at the phase interface (Δs_i).

In equation (6.2.7.2) we use the definition

$$D_m^{st} = \overline{\alpha_1}\ \overline{D_{s1}} + \overline{\alpha_2}\ \overline{D_{s2}} \qquad (6.1.1.5)$$

and the following new ones:

$$s_{Tm} = \overline{s_{T1}} + \overline{s_{T2}} = \overline{X_1\,\mathbf{q}\cdot\nabla\left(\frac{1}{T}\right)} + \overline{X_2\,\mathbf{q}\cdot\nabla\left(\frac{1}{T}\right)} \ , \qquad [6.2.7.5]$$

$$D_m^s = \overline{\alpha_1}\ \overline{\underline{\tau}_{s1}^x} : \nabla\,\overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\underline{\tau}_{s2}^x} : \nabla\,\overline{\mathbf{v}_2^{x,\rho}} \ . \qquad [6.2.7.6]$$

s_{Tm} and D^s_m are the mean entropy sources due to temperature gradient and shear stresses, respectively. Thus equation (6.2.7.2) becomes:

$$\Delta s_m = s_{Tm} + D_m^s + D_m^{st} \ \geq 0 \ . \qquad [6.2.7.7]$$

ii) Entropy equation

The entropy equation can be obtained combining (6.2.7.4) with (6.2.7.7) or from equation (5.1.7.14) as follows:

$$\frac{\partial}{\partial t}\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{s_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{s_2^{x,\rho}}\right) + \nabla\cdot\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{s_1^{x,\rho}}\ \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{s_2^{x,\rho}}\ \overline{\mathbf{v}_2^{x,\rho}}\right) -$$

$$- \overline{\alpha_1}\ \overline{\underline{\tau}_{s1}^x} : \nabla\overline{\mathbf{v}_1^{x,\rho}} - \overline{\alpha_2}\ \overline{\underline{\tau}_{s2}^x} : \nabla\overline{\mathbf{v}_2^{x,\rho}} - \overline{\alpha_1}\ \overline{D_{s1}} - \overline{\alpha_2}\ \overline{D_{s2}} -$$

$$- \nabla\cdot\left[\overline{\alpha_1}\left(\overline{\mathbf{\Phi}_1^x} + \overline{\mathbf{\Phi}_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\mathbf{\Phi}_2^x} + \overline{\mathbf{\Phi}_2^{Re}}\right)\right] - \qquad [6.2.7.8]$$

$$- \overline{s_{1i}}\ \overline{\Gamma_1} - \overline{s_{2i}}\ \overline{\Gamma_2} - \overline{s_1} - \overline{s_2} - \overline{s_{T1}} - \overline{s_{T2}} - \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\sigma_1^{x,\rho}} - \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\sigma_2^{x,\rho}} = 0 .$$

Using again the definitions

$$\rho_m\ s_m = \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{s_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{s_2^{x,\rho}} , \qquad (6.1.2.11)$$

$$\rho_m\ \sigma_m = \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\sigma_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\sigma_2^{x,\rho}} , \qquad (6.1.2.12)$$

$$\overline{s_{1i}}\ \overline{\Gamma_1} + \overline{s_{2i}}\ \overline{\Gamma_2} + \overline{s_1} + \overline{s_2} = \Delta s_i \geq 0 , \qquad (5.3.8)$$

$$D_m^{st} = \overline{\alpha_1}\ \overline{D_{s1}} + \overline{\alpha_2}\ \overline{D_{s2}} , \qquad (6.1.1.5)$$

$$\overline{\mathbf{v}_k^{x,\rho}} = \mathbf{v}_m + \check{\tilde{\mathbf{v}}}_k \quad \left(k = 1,2\right) , \qquad (6.1.2.1)$$

$$s_{Tm} = \overline{s_{T1}} + \overline{s_{T2}} = \overline{X_1\,\mathbf{q}\cdot\nabla\left(\frac{1}{T}\right)} + \overline{X_2\,\mathbf{q}\cdot\nabla\left(\frac{1}{T}\right)} , \qquad (6.2.7.5)$$

$$D_m^s = \overline{\alpha_1}\ \overline{\underline{\tau}_{s1}^x} : \nabla\overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\underline{\tau}_{s2}^x} : \nabla\overline{\mathbf{v}_2^{x,\rho}} , \qquad (6.2.7.6)$$

one derives from equation (6.2.7.8)

$$\frac{\partial}{\partial t}(\rho_m\ s_m) + \nabla\cdot(\rho_m\ s_m\ \mathbf{v}_m) + \nabla\cdot\left(\overline{\alpha_1}\ \overline{\rho_1}\ \overline{s_1^{x,\rho}}\ \check{\tilde{\mathbf{v}}}_1 + \overline{\alpha_2}\ \overline{\rho_2}\ \overline{s_2^{x,\rho}}\ \check{\tilde{\mathbf{v}}}_2\right) - D_m^s - D_m^{st} -$$

$$- \nabla\cdot\left[\overline{\alpha_1}\left(\overline{\mathbf{\Phi}_1^x} + \overline{\mathbf{\Phi}_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\mathbf{\Phi}_2^x} + \overline{\mathbf{\Phi}_2^{Re}}\right)\right] - s_{Tm} - \rho_m\ \sigma_m = \Delta s_i \geq 0. \qquad [6.2.7.9]$$

With the definition (not to be confused with (6.1.2.21))

$$\tilde{\Phi}_m = \overline{\alpha_1}\left(\overline{\Phi_1^x} + \overline{\Phi_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\Phi_2^x} + \overline{\Phi_2^{Re}}\right) - \left(\overline{\alpha_1}\ \overline{\rho_1}\ \overline{s_1^{x,\rho}}\ \check{v}_1 + \overline{\alpha_2}\ \overline{\rho_2}\ \overline{s_2^{x,\rho}}\ \check{v}_2\right) \quad [6.2.7.10]$$

one derives:

$$\frac{\partial}{\partial t}(\rho_m s_m) + \nabla\cdot(\rho_m s_m v_m) - \nabla\cdot\tilde{\Phi}_m - D_m^s - D_m^{st} - s_{Tm} - \rho_m \sigma_m = \Delta s_i \geq 0 . \quad [6.2.7.11]$$

For irreversible processes the term Δs_i (5.3.8) is positive. For both reversible and irreversible processes we have therefore the entropy equation:

$$\frac{\partial}{\partial t}(\rho_m s_m) + \nabla\cdot(\rho_m s_m v_m) - \nabla\cdot\tilde{\Phi}_m - D_m^s - D_m^{st} - s_{Tm} - \rho_m \sigma_m - \Delta s_i = 0 . \quad [6.2.7.12]$$

<u>Alternative formulation</u>

i) <u>Entropy inequality</u>

Starting again from equation (6.2.7.1), but without expanding the second term, one obtains:

$$\frac{\partial}{\partial t}(\rho_m s_m) + \nabla\cdot\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{s_1^{x,\rho}}\ \overline{v_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{s_2^{x,\rho}}\ \overline{v_2^{x,\rho}}\right) -$$

$$- \nabla\cdot\left[\overline{\alpha_1}\left(\overline{\Phi_1^x} + \overline{\Phi_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\Phi_2^x} + \overline{\Phi_2^{Re}}\right)\right] - \rho_m \sigma_m = \Delta s_m + \Delta s_i \geq 0 \quad [6.2.7.13]$$

with Δs_m and Δs_i given by (6.2.7.7) and (5.3.8), respectively, as before.

ii) <u>Entropy equation</u>

The entropy equation can be obtained combining (6.2.7.13) with (6.2.7.7) and (5.3.8) or again from equation (5.1.7.14) as follows:

$$\frac{\partial}{\partial t}\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{s_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{s_2^{x,\rho}}\right) + \nabla\cdot\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{s_1^{x,\rho}}\ \overline{v_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{s_2^{x,\rho}}\ \overline{v_2^{x,\rho}}\right) =$$

$$= \overline{\alpha_1}\ \overline{\underline{\tau}_{s1}^x} : \nabla \overline{v_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\underline{\tau}_{s2}^x} : \nabla \overline{v_2^{x,\rho}} + \overline{\alpha_1}\ \overline{D_{s1}} + \overline{\alpha_2}\ \overline{D_{s2}} +$$

$$+ \nabla\cdot\left[\overline{\alpha_1}\left(\overline{\Phi_1^x} + \overline{\Phi_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\Phi_2^x} + \overline{\Phi_2^{Re}}\right)\right] + \quad (6.2.7.8)$$

$$+ \overline{s_{1i}}\ \overline{\Gamma_1} + \overline{s_{2i}}\ \overline{\Gamma_2} + \overline{s_1} + \overline{s_2} + \overline{s_{T1}} + \overline{s_{T2}} + \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\sigma_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\sigma_2^{x,\rho}} .$$

Using the definitions

$$\overline{\rho_m \; s_m} = \overline{\alpha_1}\;\overline{\rho_1^x}\;\overline{s_1^{x,\rho}} + \overline{\alpha_2}\;\overline{\rho_2^x}\;\overline{s_1^{x,\rho}} \quad , \qquad (6.1.2.11)$$

$$\rho_m \; \sigma_m = \overline{\alpha_1}\;\overline{\rho_1^x}\;\overline{\sigma_1^{x,\rho}} + \overline{\alpha_2}\;\overline{\rho_2^x}\;\overline{\sigma_2^{x,\rho}} \quad , \qquad (6.1.2.12)$$

$$\overline{s_{1i}}\;\overline{\Gamma_1} + \overline{s_{2i}}\;\overline{\Gamma_2} + \overline{s_1} + \overline{s_2} = \Delta s_i \geq 0 \quad , \qquad (5.3.8)$$

$$D_m^{st} = \overline{\alpha_1}\;\overline{D_{s1}} + \overline{\alpha_2}\;\overline{D_{s2}} \quad , \qquad (6.1.1.5)$$

$$\overline{\mathbf{v}_k^{x,\rho}} = \mathbf{v}_m + \tilde{\mathbf{v}}_k \quad \left(k = 1, 2\right), \qquad (6.1.2.1)$$

$$s_{Tm} = \overline{s_{T1}} + \overline{s_{T2}} = \overline{X_1\,\mathbf{q}\cdot\nabla\left(\frac{1}{T}\right)} + \overline{X_2\,\mathbf{q}\cdot\nabla\left(\frac{1}{T}\right)} \quad , \qquad (6.2.7.5)$$

$$D_m^s = \overline{\alpha_1}\;\overline{\underline{\tau}_{s1}^x} : \nabla\,\overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\;\overline{\underline{\tau}_{s2}^x} : \nabla\,\overline{\mathbf{v}_2^{x,\rho}} \quad , \qquad (6.2.7.6)$$

one derives from equation (6.2.7.8)

$$\frac{\partial}{\partial t}(\rho_m\, s_m) + \nabla\cdot\left(\overline{\alpha_1}\;\overline{\rho_1^x}\;\overline{s_1^{x,\rho}}\;\overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\;\overline{\rho_2^x}\;\overline{s_2^{x,\rho}}\;\overline{\mathbf{v}_2^{x,\rho}}\right) - D_m^s - D_m^{st} -$$

$$- \nabla\cdot\left[\overline{\alpha_1}\left(\overline{\mathbf{\Phi}_1^x} + \overline{\mathbf{\Phi}_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\mathbf{\Phi}_2^x} + \overline{\mathbf{\Phi}_2^{Re}}\right)\right] - s_{Tm} - \rho_m\,\sigma_m = \Delta s_i \geq 0 \;. \qquad [6.2.7.14]$$

For irreversible processes the term Δs_i (5.3.8) is positive. For both reversible and irreversible processes we have therefore the entropy equation:

$$\frac{\partial}{\partial t}(\rho_m\, s_m) + \nabla\cdot\left(\overline{\alpha_1}\;\overline{\rho_1^x}\;\overline{s_1^{x,\rho}}\;\overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\;\overline{\rho_2^x}\;\overline{s_2^{x,\rho}}\;\overline{\mathbf{v}_2^{x,\rho}}\right) -$$

$$- \nabla\cdot\left[\overline{\alpha_1}\left(\overline{\mathbf{\Phi}_1^x} + \overline{\mathbf{\Phi}_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\mathbf{\Phi}_2^x} + \overline{\mathbf{\Phi}_2^{Re}}\right)\right] - D_m^s - D_m^{st} - s_{Tm} - \rho_m\sigma_m - \Delta s_i = 0. \qquad [6.2.7.15]$$

6.3 Conservation equation for the mixture in terms of slip velocity

6.3.1 General relationships

The conservation equations for the mixture in terms of the slip velocity v_{Sl} and of the slip quantity f_{Sl}, defined by:

$$v_{Sl} = \overline{v_2^{x,\rho}} - \overline{v_1^{x,\rho}}, \qquad [6.3.1.1]$$

$$f_{Sl} = \overline{f_2} - \overline{f_1}, \qquad [6.3.1.2]$$

are often applied in computer codes and therefore of practical importance. In definition (6.3.1.2) f is a tensor or vector or scalar physical quantity. We derive new conservation equations in terms of these slip quantities by making use of two general relationships which are first proved.

For every tensor or vector or scalar physical quantity f for which the mass weighted average definition

$$f_m = \frac{1}{\rho_m}\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{f_1} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{f_2}\right) \qquad (6.1.2.8)$$

holds, the following two identities also hold:

$$\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{f_1}\ \overline{v_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{f_2}\ \overline{v_2^{x,\rho}} = \rho_m f_m v_m + \frac{\overline{\alpha_1}\ \overline{\alpha_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x}}{\rho_m}\left(\overline{f_2} - \overline{f_1}\right)v_{Sl} \qquad [6.3.1.3]$$

$$\overline{\alpha_1}\ \overline{f_1} + \overline{\alpha_2}\ \overline{f_2} = f_m + \frac{1}{\rho_m}\ \overline{\alpha_1}\ \overline{\alpha_2}\left(\overline{\rho_1^x} - \overline{\rho_2^x}\right)\left(\overline{f_2} - \overline{f_1}\right). \qquad [6.3.1.4]$$

Proof of identity (6.3.1.3)

$$\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{f_1}\ \overline{v_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{f_2}\ \overline{v_2^{x,\rho}} =$$

$$= \frac{1}{\rho_m}\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{f_1}\ \overline{v_1^{x,\rho}}\ \rho_m + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{f_2}\ \overline{v_2^{x,\rho}}\ \rho_m\right) =$$

$$= \frac{1}{\rho_m}\left[\overline{v_1^{x,\rho}}\ \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{f_1}\left(\overline{\alpha_1}\ \overline{\rho_1^x} + \overline{\alpha_2}\ \overline{\rho_2^x}\right) + \overline{v_2^{x,\rho}}\ \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{f_2}\left(\overline{\alpha_1}\ \overline{\rho_1^x} + \overline{\alpha_2}\ \overline{\rho_2^x}\right)\right] =$$

$$= \frac{1}{\rho_m}\left\{\overline{v_1^{x,\rho}}\left[\left(\overline{\alpha_1}\ \overline{\rho_1^x}\right)^2 \overline{f_1} + \overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{f_1}\right] + \right.$$

$$+ \overline{v_2^{x,\rho}} \left[\overline{a_1}\ \overline{\rho_1^x}\ \overline{a_2}\ \overline{\rho_2^x}\ \overline{f_2} + \left(\overline{a_2}\ \overline{\rho_2^x} \right)^2 \overline{f_2} \right] \Big\} .$$

Adding and subtracting the terms

$$\pm\ \overline{a_1}\ \overline{a_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x}\ \overline{f_2}\ \overline{v_1^{x,\rho}}$$

$$\pm\ \overline{a_1}\ \overline{a_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x}\ \overline{f_1}\ \overline{v_2^{x,\rho}}$$

one derives:

$$\overline{a_1}\ \overline{\rho_1^x}\ \overline{f_1}\ \overline{v_1^{x,\rho}} + \overline{a_2}\ \overline{\rho_2^x}\ \overline{f_2}\ \overline{v_2^{x,\rho}} =$$

$$= \frac{1}{\rho_m} \left[\left(\overline{a_1}\ \overline{\rho_1^x} \right)^2 \overline{f_1}\ \overline{v_1^{x,\rho}} + \overline{a_1}\ \overline{a_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x}\ \overline{f_1}\ \overline{v_2^{x,\rho}} - \right.$$

$$- \overline{a_1}\ \overline{a_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x}\ \overline{f_1} \left(\overline{v_2^{x,\rho}} - \overline{v_1^{x,\rho}} \right) + \overline{a_1}\ \overline{a_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x}\ \overline{f_2}\ \overline{v_1^{x,\rho}} +$$

$$\left. + \overline{a_1}\ \overline{a_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x}\ \overline{f_2} \left(\overline{v_2^{x,\rho}} - \overline{v_1^{x,\rho}} \right) + \left(\overline{a_2}\ \overline{\rho_2^x} \right)^2 \overline{f_2}\ \overline{v_2^{x,\rho}} \right] =$$

$$= \frac{1}{\rho_m} \left[\overline{a_1}\ \overline{\rho_1^x}\ \overline{f_1} \left(\overline{a_1}\ \overline{\rho_1^x}\ \overline{v_1^{x,\rho}} + \overline{a_2}\ \overline{\rho_2^x}\ \overline{v_2^{x,\rho}} \right) + \right.$$

$$+ \overline{a_1}\ \overline{a_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x} \left(\overline{f_2} - \overline{f_1} \right) \left(\overline{v_2^{x,\rho}} - \overline{v_1^{x,\rho}} \right) +$$

$$\left. + \overline{a_2}\ \overline{\rho_2^x}\ \overline{f_2} \left(\overline{a_1}\ \overline{\rho_1^x}\ \overline{v_1^{x,\rho}} + \overline{a_2}\ \overline{\rho_2^x}\ \overline{v_2^{x,\rho}} \right) \right] .$$

Hence using

$$\rho_m\ v_m = \overline{a_1}\ \overline{\rho_1^x}\ \overline{v_1^{x,\rho}} + \overline{a_2}\ \overline{\rho_2^x}\ \overline{v_2^{x,\rho}} \qquad (6.1.2.4)$$

one obtains

$$\overline{a_1}\ \overline{\rho_1^x}\ \overline{f_1}\ \overline{v_1^{x,\rho}} + \overline{a_2}\ \overline{\rho_2^x}\ \overline{f_2}\ \overline{v_2^{x,\rho}} =$$

$$= \frac{1}{\rho_m} \left[\overline{a_1}\ \overline{\rho_1^x}\ \overline{f_1}\ \rho_m\ v_m + \overline{a_1}\ \overline{a_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x} \left(\overline{f_2} - \overline{f_1} \right) \left(\overline{v_2^{x,\rho}} - \overline{v_1^{x,\rho}} \right) + \right.$$

$$\left. + \overline{a_2}\ \overline{\rho_2^x}\ \overline{f_2}\ \rho_m\ v_m \right] =$$

$$= v_m \left(\overline{a_1}\ \overline{\rho_1^x}\ \overline{f_1} + \overline{a_2}\ \overline{\rho_2^x}\ \overline{f_2} \right) + \frac{\overline{a_1}\ \overline{a_2}\ \overline{\rho_1^x}\ \overline{\rho_1^x}}{\rho_m} \left(\overline{f_2} - \overline{f_1} \right) \left(\overline{v_2^{x,\rho}} - \overline{v_1^{x,\rho}} \right) =$$

$$= \rho_m \, f_m \, \mathrm{v}_m \; + \; \frac{\overline{\alpha_1}\;\overline{\alpha_2}\;\overline{\rho_1^x}\;\overline{\rho_2^x}}{\rho_m} \left(\overline{f_2} - \overline{f_1}\right)\left(\overline{\mathrm{v}_2^{x,\rho}} - \overline{\mathrm{v}_1^{x,\rho}}\right), \quad (q.e.d.)$$

having used definition (6.1.2.8) at last.

Proof of identity (6.3.1.4)

$$\overline{\alpha_1}\;\overline{f_1} + \overline{\alpha_2}\;\overline{f_2} = \frac{1}{\rho_m}\left[\overline{\alpha_1}\;\overline{f_1}\left(\overline{\alpha_1}\;\overline{\rho_1^x} + \overline{\alpha_2}\,\overline{\rho_2^x}\right) + \overline{\alpha_2}\;\overline{f_2}\left(\overline{\alpha_1}\,\overline{\rho_1^x} + \overline{\alpha_2}\,\overline{\rho_2^x}\right)\right] =$$

$$= \frac{1}{\rho_m}\left[\left(\overline{\alpha_1}\right)^2\overline{\rho_1^x}\;\overline{f_1} + \overline{\alpha_1}\;\overline{\alpha_2}\;\overline{f_1}\;\overline{\rho_2^x} + \overline{\alpha_2}\;\overline{f_2}\;\overline{\alpha_1}\;\overline{\rho_1^x} + \left(\overline{\alpha_2}\right)^2\overline{f_2}\;\overline{\rho_2^x}\right].$$

Adding and subtracting the terms

$$\pm\; \overline{\alpha_1}\;\overline{\alpha_2}\;\overline{\rho_1^x}\;\overline{f_1}\,,$$

$$\pm\; \overline{\alpha_1}\;\overline{\alpha_2}\;\overline{\rho_2^x}\;\overline{f_2}\,,$$

one derives:

$$\overline{\alpha_1}\;\overline{f_1} + \overline{\alpha_2}\;\overline{f_2} = \frac{1}{\rho_m}\left[\left(\overline{\alpha_1}\right)^2\overline{\rho_1^x}\;\overline{f_1} + \overline{\alpha_1}\;\overline{\alpha_2}\;\overline{\rho_1^x}\;\overline{f_1} - \overline{\alpha_1}\;\overline{\alpha_2}\;\overline{f_1}\left(\overline{\rho_1^x} - \overline{\rho_2^x}\right) + \right.$$

$$\left. + \;\overline{\alpha_1}\;\overline{\alpha_2}\;\overline{\rho_2^x}\;\overline{f_2} + \overline{\alpha_1}\;\overline{\alpha_2}\;\overline{f_2}\left(\overline{\rho_1^x} - \overline{\rho_2^x}\right) + \left(\overline{\alpha_2}\right)^2\overline{f_2}\;\overline{\rho_2^x}\right] =$$

$$= \frac{1}{\rho_m}\left[\overline{\alpha_1}\,\overline{\rho_1^x}\;\overline{f_1}\left(\overline{\alpha_1} + \overline{\alpha_2}\right) + \overline{\alpha_1}\,\overline{\alpha_2}\left(\overline{\rho_1^x} - \overline{\rho_2^x}\right)\left(\overline{f_2} - \overline{f_1}\right) + \overline{\alpha_2}\,\overline{\rho_2^x}\,\overline{f_2}\left(\overline{\alpha_1} + \overline{\alpha_2}\right)\right] =$$

$$= f_m \; + \; \frac{\overline{\alpha_1}\;\overline{\alpha_2}}{\rho_m}\left(\overline{\rho_1^x} - \overline{\rho_2^x}\right)\left(\overline{f_2} - \overline{f_1}\right) \quad (q.e.d.)\,,$$

having used $\alpha_1 + \alpha_2 = 1$.

Using the two above identities we can reformulate the conservation equations for the mixture in terms of the slip quantities (6.3.1.1) and (6.3.1.2).

We derive the new equations in two cases:

i) For two phases and two unmiscible components (see Section 6.3.2);
ii) For a single component with two phases in thermodynamic equilibrium,

when the concept of thermodynamic equilibrium can be introduced (see Section 6.3.3).

An example of case i) is a flow of air-water mixture (as far as the solubility of air in water can be neglected); an example of case ii) is the flow of a boiling liquid.

6.3.2 Mixture of two unmiscible components

The starting points for the equations derived in this section are the alternative forms obtained in section 6.2.

i) Momentum equation

Let us recall the alternative form of the mixture equation

$$\frac{\partial}{\partial t}(\rho_m \mathbf{v}_m) + \nabla\cdot\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\mathbf{v}_1^{x,\rho}}\ \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\mathbf{v}_2^{x,\rho}}\ \overline{\upsilon_2^{x,\rho}}\right) =$$

$$= \nabla\cdot\left(\overline{\alpha_1}\ \overline{\underline{\tau}_1^x} + \overline{\alpha_2}\ \overline{\underline{\tau}_2^x}\right) + \nabla\cdot\left(\overline{\alpha_1}\ \overline{\underline{\mathbf{T}}_1^{Re}} + \overline{\alpha_2}\ \overline{\underline{\mathbf{T}}_2^{Re}}\right) + \rho_m \mathbf{g} +$$

$$+ \overline{\mathbf{v}_{1i}}\ \overline{\Gamma_1} + \overline{\mathbf{v}_{2i}}\ \overline{\Gamma_2} - \left(\overline{\alpha_1}\ \nabla\ \overline{p_1^x} + \overline{\alpha_2}\ \nabla\ \overline{p_2^x}\right) + \qquad (6.2.2.7)$$

$$+ \left(\overline{p_{1i}^x} - \overline{p_1^x}\right)\nabla\ \overline{\alpha_1} + \left(\overline{p_{2i}^x} - \overline{p_2^x}\right)\nabla\ \overline{\alpha_2} + \overline{\mathbf{M}_1^d} + \overline{\mathbf{M}_2^d}.$$

Using identities (6.3.1.3) and (6.3.1.4) we can write

$$\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\mathbf{v}_1^{x,\rho}}\ \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{\mathbf{v}_2^{x,\rho}}\ \overline{\mathbf{v}_2^{x,\rho}} = \rho_m \mathbf{v}_m \mathbf{v}_m + \frac{\overline{\alpha_1}\ \overline{\alpha_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x}}{\rho_m}\mathbf{v}_{Sl}\mathbf{v}_{Sl}, \qquad [6.3.2.1]$$

$$\overline{\alpha_1}\ \overline{\underline{\tau}_1^x} + \overline{\alpha_2}\ \overline{\underline{\tau}_2^x} = \underline{\tau}_m + \frac{1}{\rho_m}\ \overline{\alpha_1}\ \overline{\alpha_2}\left(\overline{\rho_1^x} - \overline{\rho_2^x}\right)\left(\overline{\underline{\tau}_2^x} - \overline{\underline{\tau}_1^x}\right), \qquad [6.3.2.2]$$

$$\overline{\alpha_1}\ \overline{\underline{\mathbf{T}}_1^{Re}} + \overline{\alpha_2}\ \overline{\underline{\mathbf{T}}_2^{Re}} = \underline{\mathbf{T}}_m^{Re} + \frac{1}{\rho_m}\ \overline{\alpha_1}\ \overline{\alpha_2}\left(\overline{\rho_1^x} - \overline{\rho_2^x}\right)\left(\overline{\underline{\mathbf{T}}_2^{Re}} - \overline{\underline{\mathbf{T}}_1^{Re}}\right). \qquad [6.3.2.3]$$

Introducing in equation (6.2.2.7) one derives

$$\frac{\partial}{\partial t}(\rho_m \mathbf{v}_m) + \nabla\cdot\left(\rho_m \mathbf{v}_m \mathbf{v}_m + \frac{\overline{\alpha_1}\ \overline{\alpha_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x}}{\rho_m}\mathbf{v}_{Sl}\mathbf{v}_{Sl}\right) =$$

$$= \nabla \cdot \left\{ \underline{\tau}_m + \underline{T}_m^{Re} + \frac{1}{\rho_m} \overline{\alpha_1}\, \overline{\alpha_2} \left(\overline{\rho_1^x} - \overline{\rho_2^x} \right) \left[\left(\overline{\underline{\tau}_2^x} - \overline{\underline{\tau}_1^x} \right) + \left(\overline{\underline{T}_2^{Re}} - \overline{\underline{T}_1^{Re}} \right) \right] \right\} + \rho_m \mathbf{g} +$$

[6.3.2.4]

$$+ \overline{\mathbf{v}_{1i}}\, \overline{\Gamma_1} + \overline{\mathbf{v}_{2i}}\, \overline{\Gamma_2} - \left(\overline{\alpha_1} \nabla \overline{p_1^x} + \overline{\alpha_2} \nabla \overline{p_2^x} \right) +$$

$$+ \left(\overline{p_{1i}^x} - \overline{p_1^x} \right) \nabla \overline{\alpha_1} + \left(\overline{p_{2i}^x} - \overline{p_2^x} \right) \nabla \overline{\alpha_2} + \overline{\mathbf{M}_1^d} + \overline{\mathbf{M}_2^d} \, .$$

ii) Total energy equation

Let us recall the alternative form

$$\frac{\partial}{\partial t} \left[\rho_m \left(u_m + \frac{v_m^2}{\rho_m} + u_m^{Re} \right) \right] +$$

$$+ \nabla \cdot \left\{ \overline{\alpha_1} \overline{\rho_1^x}\, \overline{\mathbf{v}_1^{x,\rho}} \left[\overline{u_1^{x,\rho}} + \frac{1}{2} \left(\overline{\mathbf{v}_1^{x,\rho}} \right)^2 + \overline{u_1^{Re}} \right] + \overline{\alpha_2}\, \overline{\rho_2^x}\, \overline{\mathbf{v}_2^{x,\rho}} \left[\overline{u_2^{x,\rho}} + \frac{1}{2} \left(\overline{\mathbf{v}_2^{x,\rho}} \right)^2 + \overline{u_2^{Re}} \right] \right\} =$$

(6.2.3.10)

$$= - \nabla \cdot \tilde{\mathbf{q}}_m + \rho_m \mathbf{v}_m \cdot g + \rho_m r_m + \bar{\varepsilon} +$$

$$+ \nabla \cdot \left\{ \left[\overline{\alpha_1} \left(\overline{\underline{T}_1^x} - \overline{\underline{T}_1^{Re}} \right) + \overline{\alpha_2} \left(\overline{\underline{T}_2^x} - \overline{\underline{T}_2^{Re}} \right) \right] \cdot \mathbf{v}_m \right\} .$$

Using the identities

$$\overline{\alpha_1}\, \overline{\rho_1^x}\, \overline{u_1^{x,\rho}}\, \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\, \overline{\rho_2^x}\, \overline{u_2^{x,\rho}}\, \overline{\mathbf{v}_2^{x,\rho}} = \rho_m u_m \mathbf{v}_m + \frac{\overline{\alpha_1}\, \overline{\alpha_2}\, \overline{\rho_1^x}\, \overline{\rho_2^x}}{\rho_m} \left(\overline{u_2^{x,\rho}} - \overline{u_1^{x,\rho}} \right) \mathbf{v}_{Sl} , \qquad [6.3.2.5]$$

$$\overline{\alpha_1}\, \overline{\rho_1^x} \left(\overline{\mathbf{v}_1^{x,\rho}} \right)^2 \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\, \overline{\rho_2^x} \left(\overline{\mathbf{v}_2^{x,\rho}} \right)^2 \overline{\mathbf{v}_2^{x,\rho}} = \rho_m v_m^2 \mathbf{v}_m + \frac{\overline{\alpha_1}\, \overline{\alpha_2}\, \overline{\rho_1^x}\, \overline{\rho_2^x}}{\rho_m} \left[\left(\overline{\mathbf{v}_2^{x,\rho}} \right)^2 - \left(\overline{\mathbf{v}_1^{x,\rho}} \right)^2 \right] \mathbf{v}_{Sl} ,$$

[6.3.2.6]

$$\overline{\alpha_1}\, \overline{\rho_1^x}\, \overline{u_1^{Re}}\, \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\, \overline{\rho_2^x}\, \overline{u_2^{Re}}\, \overline{\mathbf{v}_2^{x,\rho}} = \rho_m u_m^{Re} \mathbf{v}_m + \frac{\overline{\alpha_1}\, \overline{\alpha_2}\, \overline{\rho_1^x}\, \overline{\rho_2^x}}{\rho_m} \left(\overline{u_2^{Re}} - \overline{u_1^{Re}} \right) \mathbf{v}_{Sl} , \qquad [6.3.2.7]$$

$$\overline{\alpha_1}\, \overline{\underline{T}_1^x} + \overline{\alpha_2}\, \overline{\underline{T}_2^x} = \underline{T}_m + \frac{1}{\rho_m} \overline{\alpha_1}\, \overline{\alpha_2} \left(\overline{\rho_1^x} - \overline{\rho_2^x} \right) \left(\overline{\underline{T}_2^x} - \overline{\underline{T}_1^x} \right) , \qquad [6.3.2.8]$$

$$\overline{\alpha_1}\, \overline{\underline{T}_1^{Re}} + \overline{\alpha_2}\, \overline{\underline{T}_2^{Re}} = \underline{T}_m^{Re} + \frac{1}{\rho_m} \overline{\alpha_1}\, \overline{\alpha_2} \left(\overline{\rho_1^x} - \overline{\rho_2^x} \right) \left(\overline{\underline{T}_2^{Re}} - \overline{\underline{T}_1^{Re}} \right) , \qquad (6.3.2.3)$$

equation (6.2.3.10) becomes:

$$\frac{\partial}{\partial t}\left[\rho_m\left(u_m+\frac{v_m^2}{2}+u_m^{Re}\right)\right]+\nabla\cdot\left[\rho_m\left(u_m+\frac{v_m^2}{2}+u_m^{Re}\right)\mathbf{v}_m\right]+$$

$$\nabla\cdot\left\{\frac{1}{\rho_m}\left(\overline{\alpha_1}\,\overline{\alpha_2}\,\overline{\rho_1^x}\,\overline{\rho_2^x}\right)\left[\overline{u_2^{x,\rho}}-\overline{u_1^{x,\rho}}+\frac{\left(\overline{\mathbf{v}_2^{x,\rho}}\right)^2}{2}-\frac{\left(\overline{\mathbf{v}_1^{x,\rho}}\right)^2}{2}+\overline{u_2^{Re}}-\overline{u_1^{Re}}\right]\mathbf{v}_{Sl}\right\}= \qquad [6.3.2.9]$$

$$=-\nabla\cdot\tilde{\mathbf{q}}_m+\rho_m\,\mathbf{v}_m\cdot g+\rho_m\,r_m+\bar{\varepsilon}+$$

$$+\nabla\cdot\left\{\left[\underline{\mathbf{T}}_m+\underline{\mathbf{T}}_m^{Re}+\frac{1}{\rho_m}\overline{\alpha_1}\,\overline{\alpha_2}\,\overline{\rho_1^x}\,\overline{\rho_2^x}\left(\overline{\underline{\mathbf{T}}_2^x}-\overline{\underline{\mathbf{T}}_1^x}+\overline{\underline{\mathbf{T}}_2^{Re}}-\overline{\underline{\mathbf{T}}_1^{Re}}\right)\right]\cdot\mathbf{v}_m\right\}.$$

iii) Mechanical energy equation

Let us recall the alternative form

$$\frac{\partial}{\partial t}\left(\rho_m\,\mathrm{u}_m^{Re}+\rho_m\frac{v_m^2}{2}\right)+$$

$$+\nabla\cdot\left\{\overline{\alpha_1}\,\overline{\rho_1^x}\,\overline{\mathbf{v}_1^{x,\rho}}\left[\frac{1}{2}\left(\overline{\mathbf{v}_1^{x,\rho}}\right)^2+\overline{u_1^{Re}}\right]+\overline{\alpha_2}\,\overline{\rho_2^x}\,\overline{\mathbf{v}_2^{x,\rho}}\left[\frac{1}{2}\left(\overline{\mathbf{v}_2^{x,\rho}}\right)^2+\overline{u_2^{Re}}\right]\right\}=$$

$$=\nabla\cdot\left\{\left[\overline{\alpha_1}\left(\overline{\underline{\mathbf{T}}_1^x}+\overline{\underline{\mathbf{T}}_1^{Re}}\right)+\overline{\alpha_2}\left(\overline{\underline{\mathbf{T}}_2^x}+\overline{\underline{\mathbf{T}}_2^{Re}}\right)\right]\cdot\mathbf{v}_m\right\}+$$

$$+\nabla\cdot\left[\overline{\alpha_1}\left(\overline{\underline{\mathbf{T}}_1^x}+\overline{\underline{\mathbf{T}}_1^{Re}}\right)\cdot\mathbf{v}_1'+\overline{\alpha_2}\left(\overline{\underline{\mathbf{T}}_2^x}+\overline{\underline{\mathbf{T}}_2^{Re}}\right)\cdot\mathbf{v}_2'\right]- \qquad (6.2.4.4)$$

$$-\nabla\cdot\left(\overline{\mathbf{q}_1^{kin}}+\overline{\mathbf{q}_2^{kin}}+\overline{\mathbf{q}_1^p}+\overline{\mathbf{q}_2^p}+\overline{\mathbf{q}_1^\iota}+\overline{\mathbf{q}_2^\iota}\right)+\overline{\left(\mathbf{n}_k\cdot\nabla X_k\right)e_i^\sigma}-$$

$$-\overline{\alpha_1}\,\overline{\underline{\mathbf{T}}_1^x}:\nabla\,\overline{\mathbf{v}_1^{x,\rho}}-\alpha_2\,\overline{\underline{\mathbf{T}}_2^x}:\nabla\,\overline{\mathbf{v}_2^{x,\rho}}+$$

$$+\overline{X_1\,p\left(\nabla\cdot\mathbf{v}_1'\right)}+\overline{X_2\,p\left(\nabla\cdot\mathbf{v}_2'\right)}-D_m+\mathbf{g}\cdot\rho_m\,\mathbf{v}_m.$$

Using the identities

$$\overline{\alpha_1}\,\overline{\rho_1^x}\,\overline{u_1^{x,\rho}}\,\overline{\mathbf{v}_1^{x,\rho}}+\overline{\alpha_2}\,\overline{\rho_2^x}\,\overline{u_2^{x,\rho}}\,\overline{\mathbf{v}_2^{x,\rho}}=\rho_m\,u_m\,\mathbf{v}_m+\frac{\overline{\alpha_1}\,\overline{\alpha_2}\,\overline{\rho_1^x}\,\overline{\rho_2^x}}{\rho_m}\left(\overline{u_2^{x,\rho}}-\overline{u_1^{x,\rho}}\right)\mathbf{v}_{Sl}\,, \qquad (6.3.2.5)$$

$$\overline{\alpha}_1 \overline{\rho}_1^x \left(\overline{\mathbf{v}_1^{x,\rho}} \right)^2 \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha}_2 \overline{\rho}_2^x \left(\overline{\mathbf{v}_2^{x,\rho}} \right)^2 \overline{\mathbf{v}_2^{x,\rho}} = \rho_m \upsilon_m^2 \mathbf{v}_m + \frac{\overline{\alpha}_1 \overline{\alpha}_2 \overline{\rho}_1^x \overline{\rho}_2^x}{\rho_m} \left[\left(\overline{\mathbf{v}_2^{x,\rho}} \right)^2 - \left(\overline{\mathbf{v}_1^{x,\rho}} \right)^2 \right] \mathbf{v}_{Sl} \,, \tag{6.3.2.6}$$

$$\overline{\alpha}_1 \overline{\underline{\mathbf{T}}_1^{Re}} + \overline{\alpha}_2 \overline{\underline{\mathbf{T}}_2^{Re}} = \underline{\mathbf{T}}_m^{Re} + \frac{1}{\rho_m} \overline{\alpha}_1 \overline{\alpha}_2 \left(\overline{\rho}_1 - \overline{\rho}_2^x \right) \left(\overline{\underline{\mathbf{T}}_2^{Re}} - \overline{\underline{\mathbf{T}}_1^{Re}} \right), \tag{6.3.2.3}$$

$$\overline{\alpha}_1 \overline{\underline{\mathbf{T}}_1^x} + \overline{\alpha}_2 \overline{\underline{\mathbf{T}}_2^x} = \underline{\mathbf{T}}_m + \frac{1}{\rho_m} \overline{\alpha}_1 \overline{\alpha}_2 \left(\overline{\rho}_1^x - \overline{\rho}_2^x \right) \left(\overline{\underline{\mathbf{T}}_2^x} - \overline{\underline{\mathbf{T}}_1^x} \right), \tag{6.3.2.8}$$

one obtains

$$\frac{\partial}{\partial t} \left(\rho_m u_m^{Re} + \rho_m \frac{\upsilon_m^2}{2} \right) + \frac{\partial}{\partial t} \left(\overline{\alpha}_1 \overline{\rho}_1^x \frac{\upsilon_1^2}{2} + \overline{\alpha}_2 \overline{\rho}_2^x \frac{\upsilon_2^2}{2} \right) +$$

$$+ \nabla \cdot \rho_m \left(u_m + \frac{\upsilon_m^2}{2} \right) \mathbf{v}_m + \nabla \cdot \left\{ \frac{\overline{\alpha}_1 \overline{\alpha}_2 \overline{\rho}_1^x \overline{\rho}_2^x}{\rho_m} \left[\overline{u_2^{x,\rho}} - \overline{u_1^{x,\rho}} + \frac{\left(\overline{\mathbf{v}_2^{x,\rho}} \right)^2}{2} - \frac{\left(\overline{\mathbf{v}_1^{x,\rho}} \right)^2}{2} \right] \mathbf{v}_{Sl} \right\} =$$

$$= \nabla \cdot \left\{ \left[\underline{\mathbf{T}}_m + \underline{\mathbf{T}}_m^{Re} + \frac{1}{\rho_m} \overline{\alpha}_1 \overline{\alpha}_2 \left(\overline{\rho}_1^x - \overline{\rho}_2^x \right) \left(\overline{\underline{\mathbf{T}}_2^x} - \overline{\underline{\mathbf{T}}_1^x} + \overline{\underline{\mathbf{T}}_2^{Re}} - \overline{\underline{\mathbf{T}}_1^{Re}} \right) \right] \cdot \mathbf{v}_m \right\} +$$

$$+ \nabla \cdot \left[\overline{\alpha}_1 \left(\overline{\underline{\mathbf{T}}_1^x} + \overline{\underline{\mathbf{T}}_1^{Re}} \right) \cdot \mathbf{v}_1 + \overline{\alpha}_2 \left(\overline{\underline{\mathbf{T}}_2^x} + \overline{\underline{\mathbf{T}}_2^{Re}} \right) \cdot \mathbf{v}_2 \right] - \tag{6.3.2.10}$$

$$- \nabla \cdot \left(\overline{\mathbf{q}_1^{kin}} + \overline{\mathbf{q}_2^{kin}} + \overline{\mathbf{q}_1^p} + \overline{\mathbf{q}_2^p} + \overline{\mathbf{q}_1^\iota} + \overline{\mathbf{q}_2^\iota} \right) + \overline{\left(\mathbf{n}_k \cdot \nabla X_k \right) e_i^\sigma} -$$

$$- \overline{\alpha}_1 \overline{\underline{\mathbf{T}}_1^x} : \nabla \overline{\mathbf{v}_1^{x,\rho}} - \alpha_2 \overline{\underline{\mathbf{T}}_2^x} : \nabla \overline{\mathbf{v}_2^{x,\rho}} +$$

$$+ \overline{X_1 \, p \left(\nabla \cdot \mathbf{v}_1' \right)} + \overline{X_2 \, p \left(\nabla \cdot \mathbf{v}_2' \right)} - D_m + \mathbf{g} \cdot \rho_m \mathbf{v}_m .$$

iv) Internal energy equation

Let us recall the alternative form

$$\frac{\partial (\rho_m u_m)}{\partial t} + \nabla \cdot \left(\overline{\alpha}_1 \overline{\rho}_1^x \overline{\mathbf{v}_1^{x,\rho}} \overline{u_1^{x,\rho}} + \overline{\alpha}_2 \overline{\rho}_2^x \overline{\mathbf{v}_2^{x,\rho}} \overline{u_2^{x,\rho}} \right) =$$

$$= -\nabla \cdot \left[\overline{\alpha}_1 \left(\overline{\mathbf{q}_1^x} + \hat{\mathbf{q}}_1^{Re} \right) + \overline{\alpha}_2 \left(\overline{\mathbf{q}_2^x} + \hat{\mathbf{q}}_2^{Re} \right) \right] + D_m + \rho_m r_m + \bar{\varepsilon} - p_m \nabla \cdot \mathbf{v}_m + W_m -$$

$$- \frac{1}{2} \overline{\upsilon_{1i}^2} \, \overline{\Gamma}_1 - \frac{1}{2} \overline{\upsilon_{2i}^2} \, \overline{\Gamma}_2 + \overline{\left(\underline{\mathbf{T}} \cdot \mathbf{v} \right) \cdot \left(\nabla X_1 + \nabla X_2 \right)} + \tag{6.2.5.12}$$

$$+ \ \overline{\alpha_1} \ \overline{\underline{\tau}_1^x} : \left(\nabla \mathbf{v}_m + \nabla \tilde{\mathbf{v}}_1 \right) + \overline{\alpha_2} \ \overline{\underline{\tau}_2^x} : \left(\nabla \mathbf{v}_m + \nabla \tilde{\mathbf{v}}_2 \right).$$

Using identity (6.3.2.5) for $u_k^{x,\rho}$ and

$$\overline{\alpha_1} \ \overline{\mathbf{q}_1^x} + \overline{\alpha_2} \ \overline{\mathbf{q}_2^x} = \mathbf{q}_m + \frac{1}{\rho_m} \overline{\alpha_1} \ \overline{\alpha_2} \left(\overline{\rho_1^x} - \overline{\rho_2^x} \right) \left(\overline{\mathbf{q}_2^x} - \overline{\mathbf{q}_1^x} \right), \qquad [6.3.2.11]$$

$$\overline{\alpha_1} \ \hat{\mathbf{q}}_1^{Re} + \overline{\alpha_2} \ \hat{\mathbf{q}}_2^{Re} = \hat{\mathbf{q}}_m^{Re} + \frac{1}{\rho_m} \overline{\alpha_1} \ \overline{\alpha_2} \left(\overline{\rho_1^x} - \overline{\rho_2^x} \right) \left(\hat{\mathbf{q}}_2^{Re} - \hat{\mathbf{q}}_1^{Re} \right), \qquad [6.3.2.12]$$

one derives

$$\frac{\partial}{\partial t} (\rho_m u_m) + \nabla \cdot (\rho_m u_m \mathbf{v}_m) + \nabla \cdot \left[\frac{\overline{\alpha_1} \ \overline{\alpha_2} \ \overline{\rho_1^x} \ \overline{\rho_2^x}}{\rho_m} \left(\overline{u_2^{x,\rho}} - \overline{u_1^{x,\rho}} \right) \mathbf{v}_{Sl} \right] =$$

$$= - \nabla \cdot \left[\mathbf{q}_m + \hat{\mathbf{q}}_m^{Re} + \frac{1}{\rho_m} \overline{\alpha_1} \ \overline{\alpha_2} \left(\overline{\rho_1^x} - \overline{\rho_2^x} \right) \left(\overline{\mathbf{q}_2^x} - \overline{\mathbf{q}_1^x} + \hat{\mathbf{q}}_2^{Re} - \hat{\mathbf{q}}_1^{Re} \right) \right] +$$

$$+ D_m + \rho_m r_m + \bar{\varepsilon} - p_m \nabla \cdot \mathbf{v}_m + W_m - \qquad [6.3.2.13]$$

$$- \frac{1}{2} \overline{v_{1i}^2} \ \overline{\Gamma_1} - \frac{1}{2} \overline{v_{2i}^2} \ \overline{\Gamma_2} + \overline{\left(\underline{\mathbf{T}} \cdot \mathbf{v} \right) \cdot \left(\nabla X_1 + \nabla X_2 \right)} +$$

$$+ \ \overline{\alpha_1} \ \overline{\underline{\tau}_1^x} : \left(\nabla \mathbf{v}_m + \nabla \tilde{\mathbf{v}}_1 \right) + \overline{\alpha_2} \ \overline{\underline{\tau}_2^x} : \left(\nabla \mathbf{v}_m + \nabla \tilde{\mathbf{v}}_2 \right).$$

v) Enthalpy equation

Let us recall the alternative form

$$\frac{\partial}{\partial t} (\rho_m h_m) + \nabla \cdot \left(\overline{\alpha_1} \ \overline{\rho_1^x} \ \overline{h_1^{x,\rho}} \ \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2} \ \overline{\rho_2^x} \ \overline{h_2^{x,\rho}} \ \overline{\mathbf{v}_2^{x,\rho}} \right) =$$

$$= - \nabla \cdot \left[\overline{\alpha_1} \left(\overline{\mathbf{q}_1^x} + \hat{\hat{q}}_1^{Re} \right) + \overline{\alpha_2} \left(\overline{\mathbf{q}_2^x} + \hat{\hat{\mathbf{q}}}_2^{Re} \right) \right] +$$

$$+ \frac{\partial p_m}{\partial t} + \hat{\mathbf{v}}_m \cdot \nabla p_m + \bar{\varepsilon} + D_m + \rho_m r_m + \hat{\hat{W}}_m -$$

$$-\left(\frac{1}{2}\,\overline{v_{1i}^2}-\frac{\overline{p_1^x}}{\overline{\overline{\rho_1^x}}}\right)\overline{\Gamma_1}-\left(\frac{1}{2}\,\overline{v_{2i}^2}-\frac{\overline{p_2^x}}{\overline{\overline{\rho_2^x}}}\right)\overline{\Gamma_2}+ \qquad (6.2.6.12)$$

$$+\overline{\left(\underline{\mathbf{T}}\cdot\mathbf{v}\right)_1\cdot\nabla X_1}+\overline{\left(\underline{\mathbf{T}}\cdot\mathbf{v}\right)_2\cdot\nabla X_2}+\overline{\alpha_1}\,\overline{\underline{\mathbf{T}}_1^x}:\nabla\,\overline{\mathbf{v}_1^{x,\rho}}+\overline{\alpha_2}\,\overline{\underline{\mathbf{T}}_2^x}:\nabla\,\overline{\mathbf{v}_2^{x,\rho}}\;.$$

Using identities

$$\overline{\alpha_1}\;\overline{\mathbf{q}_1^x}+\overline{\alpha_2}\;\overline{\mathbf{q}_2^x}=\mathbf{q}_m+\frac{1}{\rho_m}\,\overline{\alpha_1}\;\overline{\alpha_2}\left(\overline{\rho_1^x}-\overline{\rho_2^x}\right)\left(\overline{\mathbf{q}_2^x}-\overline{\mathbf{q}_1^x}\right), \qquad (6.3.2.11)$$

$$\overline{\alpha_1}\;\widehat{\widehat{\mathbf{q}}}_1^{Re}+\overline{\alpha_2}\;\widehat{\widehat{\mathbf{q}}}_2^{Re}=\widehat{\widehat{\mathbf{q}}}_m^{Re}+\frac{1}{\rho_m}\,\overline{\alpha_1}\;\overline{\alpha_2}\left(\overline{\rho_1^x}-\overline{\rho_2^x}\right)\left(\widehat{\widehat{\mathbf{q}}}_2^{Re}-\widehat{\widehat{\mathbf{q}}}_1^{Re}\right), \qquad [6.3.2.14]$$

$$\overline{\alpha_1}\;\overline{\rho_1^x}\;\overline{h_1^{x,\rho}}\;\overline{\mathbf{v}_1^{x,\rho}}+\overline{\alpha_2}\;\overline{\rho_2^x}\;\overline{h_2^{x,\rho}}\;\overline{\mathbf{v}_2^{x,\rho}}=\rho_m h_m \mathbf{v}_m+\frac{\overline{\alpha_1}\;\overline{\alpha_2}\;\overline{\rho_1^x}\;\overline{\rho_2^x}}{\rho_m}\left(\overline{h_2^{x,\rho}}-\overline{h_1^{x,\rho}}\right)\mathbf{v}_{Sl}\,, \qquad [6.3.2.15]$$

$$\widehat{\mathbf{v}}_m=\overline{\alpha_1}\;\overline{\mathbf{v}_1^{x,\rho}}+\overline{\alpha_2}\;\overline{\mathbf{v}_2^{x,\rho}}=\mathbf{v}_m+\frac{\overline{\alpha_1}\;\overline{\alpha_2}}{\rho_m}\left(\overline{\rho_1^x}-\overline{\rho_2^x}\right)\mathbf{v}_{Sl}\,, \qquad [6.3.2.16]$$

one obtains

$$\frac{\partial}{\partial t}\left(\rho_m h_m\right)+\nabla\cdot\left(\rho_m h_m \mathbf{v}_m\right)+\nabla\cdot\left[\frac{\overline{\alpha_1}\;\overline{\alpha_2}\;\overline{\rho_1^x}\;\overline{\rho_2^x}}{\rho_m}\left(\overline{h_2^{x,\rho}}-\overline{h_1^{x,\rho}}\right)\mathbf{v}_{Sl}\right]=$$

$$=-\nabla\cdot\left[\mathbf{q}_m+\widehat{\widehat{\mathbf{q}}}_m^{Re}+\frac{1}{\rho_m}\,\overline{\alpha_1}\;\overline{\alpha_2}\left(\overline{\rho_1^x}-\overline{\rho_2^x}\right)\left(\overline{\mathbf{q}_2^x}-\overline{\mathbf{q}_1^x}+\widehat{\widehat{\mathbf{q}}}_2^{Re}-\widehat{\widehat{\mathbf{q}}}_1^{Re}\right)\right]+$$

$$+\frac{\partial p_m}{\partial t}+\mathbf{v}_m\cdot\nabla p_m+\frac{\overline{\alpha_1}\;\overline{\alpha_2}}{\rho_m}\left(\overline{\rho_1^x}-\overline{\rho_2^x}\right)\mathbf{v}_{Sl}\cdot\nabla p_m+\bar{\varepsilon}+D_m+\rho_m r_m+\widehat{\widehat{W}}_m- \qquad [6.3.2.17]$$

$$-\left(\frac{1}{2}\,\overline{v_{1i}^2}-\frac{\overline{p_1^x}}{\overline{\overline{\rho_1^x}}}\right)\overline{\Gamma_1}-\left(\frac{1}{2}\,\overline{v_{2i}^2}-\frac{\overline{p_2^x}}{\overline{\overline{\rho_2^x}}}\right)\overline{\Gamma_2}+$$

$$+\overline{\left(\underline{\mathbf{T}}\cdot\mathbf{v}\right)_1\cdot\nabla X_1}+\overline{\left(\underline{\mathbf{T}}\cdot\mathbf{v}\right)_2\cdot\nabla X_2}+\overline{\alpha_1}\,\overline{\underline{\mathbf{T}}_1^x}:\nabla\,\overline{\mathbf{v}_1^{x,\rho}}+\overline{\alpha_2}\,\overline{\underline{\mathbf{T}}_2^x}:\nabla\,\overline{\mathbf{v}_2^{x,\rho}}\;.$$

vi) Entropy inequality and entropy equation

a) Entropy inequality

We start from the alternative form of the entropy inequality:

$$\frac{\partial(\rho_m s_m)}{\partial t} + \nabla\cdot\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{s_1^{x,\rho}}\ \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{s_2^{x,\rho}}\ \overline{\mathbf{v}_2^{x,\rho}}\right) -$$

$$-\nabla\cdot\left[\overline{\alpha_1}\left(\overline{\Phi_1^x} + \overline{\Phi_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\Phi_2^x} + \overline{\Phi_2^{Re}}\right)\right] - \rho_m\sigma_m = \Delta s_m + \Delta s_i \geq 0 . \qquad (6.2.7.13)$$

Using the identities

$$\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{s_1^{x,\rho}}\ \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{s_2^{x,\rho}}\ \overline{\mathbf{v}_2^{x,\rho}} = \rho_m s_m \mathbf{v}_m + \frac{\overline{\alpha_1}\ \overline{\alpha_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x}}{\rho_m}\left(\overline{s_2^{x,\rho}} - \overline{s_1^{x,\rho}}\right)\mathbf{v}_{Sl} , \qquad [6.3.2.18]$$

$$\overline{\alpha_1}\ \overline{\Phi_1^x} + \overline{\alpha_2}\ \overline{\Phi_2^x} = \Phi_m + \frac{\overline{\alpha_1}\ \overline{\alpha_2}}{\rho_m}\left(\overline{\rho_1^x} - \overline{\rho_2^x}\right)\left(\overline{\Phi_2^x} - \overline{\Phi_1^x}\right), \qquad [6.3.2.19]$$

$$\overline{\alpha_1}\ \overline{\Phi_1^{Re}} + \overline{\alpha_2}\ \overline{\Phi_2^{Re}} = \Phi_m^{Re} + \frac{\overline{\alpha_1}\ \overline{\alpha_2}}{\rho_m}\left(\overline{\rho_1^x} - \overline{\rho_2^x}\right)\left(\overline{\Phi_2^{Re}} - \overline{\Phi_1^{Re}}\right), \qquad [6.3.2.20]$$

one obtains:

$$\frac{\partial(\rho_m s_m)}{\partial t} + \nabla\cdot(\rho_m s_m \mathbf{v}_m) + \nabla\cdot\left[\frac{\overline{\alpha_1}\ \overline{\alpha_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x}}{\rho_m}\left(\overline{s_2^{x,\rho}} - \overline{s_1^{x,\rho}}\right)\mathbf{v}_{Sl}\right] -$$

$$-\nabla\cdot\left[\Phi_m + \Phi_m^{Re} + \frac{\overline{\alpha_1}\ \overline{\alpha_2}}{\rho_m}\left(\overline{\rho_1^x} - \overline{\rho_2^x}\right)\left(\overline{\Phi_2^x} - \overline{\Phi_1^x} + \overline{\Phi_2^{Re}} - \overline{\Phi_1^{Re}}\right)\right] - \qquad [6.3.2.21]$$

$$-\rho_m\sigma_m = \Delta s_m + \Delta s_i \geq 0 .$$

b) Entropy equation

Let us recall the alternative form

$$\frac{\partial}{\partial t}(\rho_m s_m) + \nabla\cdot\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{s_1^{x,\rho}}\ \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{s_2^{x,\rho}}\ \overline{\mathbf{v}_2^{x,\rho}}\right) -$$

(6.2.7.15)

$$-\nabla\cdot\left[\overline{\alpha_1}\left(\overline{\Phi_1^x} + \overline{\Phi_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\Phi_2^x} + \overline{\Phi_2^{Re}}\right)\right] - D_m^s - D_m^{st} - s_{Tm} - \rho_m\,\sigma_m - \Delta s_i = 0\,.$$

Using again the identities (6.3.2.18) - (6.3.2.20) one obtains:

$$\frac{\partial}{\partial t}(\rho_m s_m) + \nabla\cdot(\rho_m s_m \mathbf{v}_m) + \nabla\left[\cdot\ \frac{\overline{\alpha_1}\ \overline{\alpha_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x}}{\rho_m}\left(\overline{s_2^{x,\rho}} - \overline{s_1^{x,\rho}}\right)\mathbf{v}_{Sl}\right] -$$

$$-\nabla\cdot\left[\Phi_m + \Phi_m^{Re} + \frac{\overline{\alpha_1}\ \overline{\alpha_2}}{\rho_m}\left(\overline{\rho_1^x} - \overline{\rho_2^x}\right)\left(\overline{\Phi_2^x} - \overline{\Phi_1^x} + \overline{\Phi_2^{Re}} - \overline{\Phi_1^{Re}}\right)\right] - \qquad [6.3.2.22]$$

$$- D_m^s - D_m^{st} - s_{Tm} - \rho_m\,\sigma_m - \Delta s_i = 0\,.$$

6.3.3 Mixture of two phases of a single component in thermodynamical equilibrium

When two phases of a single component are considered, the thermodynamic quality of the mixture can be defined by

$$h_m = x\ \overline{h_2^{x,\rho}} + (1-x)\ \overline{h_1^{x,\rho}} \qquad [6.3.3.1a]$$

or

$$x = \frac{h_m - \overline{h_1^{x,\rho}}}{\overline{h_2^{x,\rho}} - \overline{h_1^{x,\rho}}}\,. \qquad [6.3.3.1b]$$

The relationship between volume fractions of the phases and the thermodynamic quality is given by

$$\frac{1-x}{x} = \frac{\overline{\alpha_1}}{\overline{\alpha_2}}\ \frac{\overline{\rho_1^x}}{\overline{\rho_2^x}} \qquad [6.3.3.2]$$

which yields also

$$x = \frac{\overline{\alpha_2}\ \overline{\rho_2^x}}{\overline{\alpha_1}\ \overline{\rho_1^x} + \overline{\alpha_2}\ \overline{\rho_2^x}} = \frac{\overline{\alpha_2}\ \overline{\rho_2^x}}{\rho_m}, \qquad [6.3.3.3]$$

$$1 - x = \frac{\overline{\alpha_1}\ \overline{\rho_1^x}}{\rho_m}. \qquad [6.3.3.4]$$

Multiplying the last two equations member by member one derives:

$$\frac{\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{\alpha_2}\ \overline{\rho_2^x}}{\rho_m} = x\ (1 - x)\ \rho_m \qquad [6.3.3.5]$$

and

$$\frac{\overline{\alpha_1}\ \overline{\alpha_2}}{\rho_m} = x\ (1 - x)\ \frac{\rho_m}{\overline{\rho_1^x}\ \overline{\rho_2^x}}. \qquad [6.3.3.6]$$

Equations (6.3.3.5) and (6.3.3.6) are used to express the conservation equations for the mixture in terms of the thermodynamic quality. The results are obtained straightforwardly from the equations of section (6.3.2) and are as follows:

i) Momentum equation

$$\frac{\partial}{\partial t}(\rho_m \mathbf{v}_m) + \nabla \cdot \left(\rho_m \mathbf{v}_m \mathbf{v}_m + x\ (1 - x)\ \rho_m \mathbf{v}_{Sl} \mathbf{v}_{Sl}\right) =$$

$$= \nabla \cdot \left[\underline{\tau}_m + \underline{\mathbf{T}}_m^{Re} + x\ (1 - x)\ \frac{\rho_m}{\overline{\rho_1^x}\ \overline{\rho_2^x}} \left(\overline{\rho_1^x} - \overline{\rho_2^x}\right)\left(\overline{\underline{\tau}_2^x} - \overline{\underline{\tau}_1^x} + \overline{\underline{\mathbf{T}}_2^{Re}} - \overline{\underline{\mathbf{T}}_1^{Re}}\right)\right] +$$

$$+ \rho_m \mathbf{g} + \overline{\mathbf{v}_{1i}}\ \overline{\Gamma_1} + \overline{\mathbf{v}_{2i}}\ \overline{\Gamma_2} - \left(\overline{\alpha_1}\ \nabla\ \overline{p_1^x} + \overline{\alpha_2}\ \nabla\ \overline{p_2^x}\right) + \qquad [6.3.3.7]$$

$$+ \left(\overline{p_{1i}^x} - \overline{p_1^x}\right) \nabla\ \overline{\alpha_1} + \left(\overline{p_{2i}^x} - \overline{p_2^x}\right) \nabla\ \overline{\alpha_2} + \overline{\mathbf{M}_1^d} + \overline{\mathbf{M}_2^d}.$$

ii) Total energy equation

$$\frac{\partial}{\partial t}\left[\rho_m \left(u_m + \frac{v_m^2}{2} + u_m^{Re}\right)\right] + \nabla \cdot \left[\rho_m \left(u_m + \frac{v_m^2}{2} + u_m^{Re}\right) \mathbf{v}_m\right] +$$

$$+ \nabla \cdot \left\{ x\,(1-x)\,\rho_m \left[\overline{u_2^{x,\rho}} - \overline{u_1^{x,\rho}} + \frac{\left(\overline{\mathbf{v}_2^{x,\rho}}\right)^2}{2} - \frac{\left(\overline{\mathbf{v}_2^{x,\rho}}\right)^2}{2} + u_2^{x,\rho} - u_1^{x,\rho} \right] \mathbf{v}_{Sl} \right\} =$$

[6.3.3.8]

$$= - \nabla \cdot \tilde{\mathbf{q}}_m + \rho_m\, \mathbf{v}_m \cdot \mathbf{g} + \rho_m\, r_m + \bar{\varepsilon} +$$

$$+ \nabla \cdot \left\{ \left[\underline{\mathbf{T}}_m + \underline{\mathbf{T}}_m^{Re} + x\,(1-x)\, \frac{\rho_m}{\overline{\overline{\rho_1^x\, \rho_2^x}}} \left(\overline{\rho_1^x} - \overline{\rho_2^x} \right) \left(\overline{\underline{\mathbf{T}}_2^x} - \overline{\underline{\mathbf{T}}_1^x} + \overline{\underline{\mathbf{T}}_2^{Re}} - \overline{\underline{\mathbf{T}}_1^{Re}} \right) \right] \cdot \mathbf{v}_m \right\}.$$

iii) Mechanical energy equation

$$\frac{\partial}{\partial t} \left(\rho_m\, u_m^{Re} + \rho_m \frac{v_m^2}{2} \right) + \nabla \cdot \left[\rho_m \left(u_m + \frac{v_m^2}{2} \right) \mathbf{v}_m \right] +$$

$$+ \nabla \cdot \left\{ x\,(1-x)\,\rho_m \left[\overline{u_2^{x,\rho}} - \overline{u_1^{x,\rho}} + \frac{\left(\overline{\mathbf{v}_2^{x,\rho}}\right)^2}{2} - \frac{\left(\overline{\mathbf{v}_1^{x,\rho}}\right)^2}{2} \right] \mathbf{v}_{Sl} \right\} =$$

$$= \nabla \cdot \left\{ \left[\underline{\mathbf{T}}_m + \underline{\mathbf{T}}_m^{Re} + x\,(1-x)\, \frac{\rho_m}{\overline{\overline{\rho_1^x\, \rho_2^x}}} \left(\overline{\rho_1^x} - \overline{\rho_2^x} \right) \left(\overline{\underline{\mathbf{T}}_2^x} - \overline{\underline{\mathbf{T}}_1^x} + \overline{\underline{\mathbf{T}}_2^{Re}} - \overline{\underline{\mathbf{T}}_1^{Re}} \right) \right] \cdot \mathbf{v}_m \right\} +$$

$$+ \nabla \cdot \left[\overline{\alpha}_1 \left(\overline{\underline{\mathbf{T}}_1^x} + \overline{\underline{\mathbf{T}}_1^{Re}} \right) \cdot \tilde{\mathbf{v}}_1 + \overline{\alpha}_2 \left(\overline{\underline{\mathbf{T}}_2^x} + \overline{\underline{\mathbf{T}}_2^{Re}} \right) \cdot \tilde{\mathbf{v}}_2 \right] - \qquad [6.3.3.9]$$

$$- \nabla \cdot \left(\overline{\mathbf{q}_1^{kin}} + \overline{\mathbf{q}_2^{kin}} + \overline{\mathbf{q}_1^p} + \overline{\mathbf{q}_2^p} + \overline{\mathbf{q}_1^\tau} + \overline{\mathbf{q}_2^\tau} \right) + \overline{\left(\mathbf{n}_k \cdot \nabla X_k \right) e_i^\sigma} -$$

$$- \overline{\alpha}_1\, \overline{\underline{\mathbf{T}}_1^x} : \nabla\, \overline{\mathbf{v}_1^{x,\rho}} - \alpha_2\, \overline{\underline{\mathbf{T}}_2^x} : \nabla\, \overline{\mathbf{v}_2^{x,\rho}} +$$

$$+ \overline{X_1\, p \left(\nabla \cdot \mathbf{v}_1' \right)} + \overline{X_2\, p \left(\nabla \cdot \mathbf{v}_2' \right)} - D_m + \mathbf{g} \cdot \rho_m\, \mathbf{v}_m .$$

iv) Internal energy equation

$$\frac{\partial}{\partial t} \left(\rho_m\, u_m \right) + \nabla \cdot \left(\rho_m\, u_m\, \mathbf{v}_m \right) + \nabla \cdot \left[x\,(1-x)\,\rho_m \left(\overline{u_2^{x,\rho}} - \overline{u_1^{x,\rho}} \right) \mathbf{v}_{Sl} \right] =$$

$$= - \nabla \cdot \left[\mathbf{q}_m + \hat{\mathbf{q}}_m^{Re} + x\,(1-x)\, \frac{\rho_m}{\overline{\overline{\rho_1^x\, \rho_2^x}}} \left(\overline{\rho_1^x} - \overline{\rho_2^x} \right) \left(\overline{\mathbf{q}_2^x} - \overline{\mathbf{q}_1^x} + \hat{\mathbf{q}}_2^{Re} - \hat{\mathbf{q}}_1^{Re} \right) \right] +$$

$$+ D_m + \rho_m r_m + \bar{\varepsilon} - p_m \nabla \cdot \mathbf{v}_m + W_m - \qquad [6.3.3.10]$$

$$- \frac{1}{2} \overline{v_{1i}^2} \, \overline{\Gamma_1} - \frac{1}{2} \overline{v_{2i}^2} \, \overline{\Gamma_2} + \overline{\left(\underline{\mathbf{T}} \cdot \mathbf{v}\right) \cdot \left(\nabla X_1 + \nabla X_2\right)} +$$

$$+ \overline{\alpha_1} \, \overline{\underline{\tau}_1^x} : \left(\nabla \mathbf{v}_m + \nabla \check{\mathbf{v}}_1\right) + \overline{\alpha_2} \, \overline{\underline{\tau}_2^x} : \left(\nabla \mathbf{v}_m + \nabla \check{\mathbf{v}}_2\right).$$

v) <u>Enthalpy equation</u>

$$\frac{\partial}{\partial t} (\rho_m h_m) + \nabla \cdot (\rho_m h_m \mathbf{v}_m) + \nabla \cdot \left[x(1-x) \rho_m \left(\overline{h_2^{x,\rho}} - \overline{h_1^{x,\rho}} \right) \mathbf{v}_{Sl} \right] =$$

$$- \nabla \cdot \left[\mathbf{q}_m + \hat{\hat{\mathbf{q}}}_m^{Re} + x(1-x) \frac{\rho_m}{\overline{\overline{\rho_1^x \rho_2^x}}} \left(\overline{\rho_1^x} - \overline{\rho_2^x} \right) \left(\overline{\mathbf{q}_2^x} - \overline{\mathbf{q}_1^x} + \hat{\hat{\mathbf{q}}}_2^{Re} - \hat{\hat{\mathbf{q}}}_1^{Re} \right) \right] +$$

$$+ \frac{\partial p_m}{\partial t} + \mathbf{v}_m \cdot \nabla p_m + x(1-x) \frac{\rho_m}{\overline{\overline{\rho_1^x \rho_2^x}}} \left(\overline{\rho_1^x} - \overline{\rho_2^x} \right) \mathbf{v}_{Sl} \cdot \nabla p_m + \qquad [6.3.3.11]$$

$$+ \bar{\varepsilon} + D_m + \rho_m r_m + \hat{\hat{W}}_m - \left(\frac{1}{2} \overline{v_{1i}^2} - \frac{\overline{p_1^x}}{\overline{\overline{\rho_1^x}}} \right) \overline{\Gamma_1} - \left(\frac{1}{2} \overline{v_{2i}^2} - \frac{\overline{p_2^x}}{\overline{\overline{\rho_2^x}}} \right) \overline{\Gamma_2} +$$

$$+ \overline{\left(\underline{\mathbf{T}} \cdot \mathbf{v}\right)_1 \cdot \nabla X_1} + \overline{\left(\underline{\mathbf{T}} \cdot \mathbf{v}\right)_2 \cdot \nabla X_2} + \overline{\alpha_1} \, \overline{\underline{\mathbf{T}}_1^x} : \nabla \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2} \, \overline{\underline{\mathbf{T}}_2^x} : \nabla \overline{\mathbf{v}_2^{x,\rho}} .$$

vi) <u>Entropy inequality and entropy equation</u>

a) <u>Entropy inequality</u>

From the inequality (6.3.2.21) one obtains

$$\frac{\partial}{\partial t} (\rho_m s_m) + \nabla \cdot (\rho_m s_m \mathbf{v}_m) + \nabla \cdot \left[x(1-x) \rho_m \left(\overline{s_2^{x,\rho}} - \overline{s_1^{x,\rho}} \right) \mathbf{v}_{Sl} \right] -$$

$$- \nabla \cdot \left[\mathbf{\Phi}_m + \mathbf{\Phi}_m^{Re} + x(1-x) \frac{\rho_m}{\overline{\overline{\rho_1^x \rho_2^x}}} \left(\overline{\rho_1^x} - \overline{\rho_2^x} \right) \left(\overline{\mathbf{\Phi}_2^x} - \overline{\mathbf{\Phi}_1^x} + \overline{\mathbf{\Phi}_2^{Re}} - \overline{\mathbf{\Phi}_1^{Re}} \right) \right] - \qquad [6.3.3.12]$$

$$- \rho_m \sigma_m = \Delta s_m + \Delta s_i \geq 0 .$$

b) Entropy equation

From equation (6.3.2.22) one obtains

$$\frac{\partial}{\partial t}(\rho_m s_m) + \nabla\cdot(\rho_m s_m \mathbf{v}_m) + \nabla\cdot\left[x(1-x)\rho_m\left(\overline{s_2^{x,\rho}} - \overline{s_1^{x,\rho}}\right)\mathbf{v}_{Sl}\right] -$$

$$- \nabla\cdot\left[\Phi_m + \Phi_m^{Re} + x(1-x)\frac{\rho_m}{\overline{\overline{\rho_1^x}}\,\overline{\overline{\rho_2^x}}}\left(\overline{\rho_1^x} - \overline{\rho_2^x}\right)\left(\overline{\Phi_2^x} - \overline{\Phi_1^x} + \overline{\Phi_2^{Re}} - \overline{\Phi_1^{Re}}\right)\right] - \qquad [6.3.3.13]$$

$$- D_m^s - D_m^{st} - s_{Tm} - \rho_m\,\sigma_m - \Delta s_i = 0\,.$$

Part A: Averages and further definitions

Volume weighted averages

$$f_m = \overline{\alpha_1}\ \overline{f_1^x} + \overline{\alpha_2}\ \overline{f_2^x} \tag{6.1.1.1}$$

defined for the following set of physical quantities:

$$f = \left\{ \overline{\rho_k^x},\ \overline{p_k^x},\ \overline{D_k},\ \overline{D_{sk}} \right\},$$

yielding the set of averages

$$\left\{ \rho_m,\ p_m,\ D_m,\ D_m^{st} \right\}. \tag{6.1.1.1}$$

$$\overline{f_k^{x,\rho}} = f_m + \tilde{f}_k \tag{6.1.2.1 ff}$$

$$\overline{\alpha_1}\ \overline{\rho_1^x}\ \tilde{f}_1 + \overline{\alpha_2}\ \overline{\rho_2^x}\ \tilde{f}_2 = 0 \tag{6.1.2.7}$$

Mass weighted averages

$$f_m = \frac{1}{\rho_m}\left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{f_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{f_2^{x,\rho}} \right) \tag{6.1.2.8}$$

defined for the following set of physical quantities:

$$f = \left\{ \overline{\mathbf{v}_k^{x,\rho}},\ \overline{u_k^{x,\rho}},\ \overline{h_k^{x,\rho}},\ \overline{s_k^{x,\rho}},\ \overline{\sigma_k^{x,\rho}},\ \frac{1}{2}\left(\overline{\mathbf{v}_k^{x,\rho}}\right)^2,\ \overline{\underline{\tau}_k^x},\ \overline{r_k^{x,\rho}},\ \overline{u_k^{Re}},\ \overline{\underline{\mathbf{T}}_k^x},\ \overline{\underline{\mathbf{T}}_k^{Re}},\ \overline{\mathbf{q}_k^x},\ \hat{\mathbf{q}}_k^{Re},\ \overline{\Phi_k^x},\ \overline{\Phi_k^{Re}} \right\},$$

yielding the set of averages:

$$\left\{ \mathbf{v}_m,\ u_m,\ h_m,\ s_m,\ \sigma_m,\ \frac{v_m^2}{2},\ \underline{\tau}_m,\ r_m,\ u_k^{Re},\ \underline{\mathbf{T}}_m^x,\ \underline{\mathbf{T}}_m^{Re},\ \mathbf{q}_m^x,\ \hat{\mathbf{q}}_m^{Re},\ \Phi_m,\ \Phi_m^{Re} \right\}.$$

TABLE VII - Summary of main equations obtained in section 6

Volume and mass flux weighted averages

$$\overline{F_k} = \frac{\overline{x_k \, \rho \, \mathrm{v} \, F}}{\overline{\alpha_k} \; \overline{\rho_k^x} \; \overline{\overline{\mathrm{v}_k^{x,\rho}}}} \tag{6.1.3.1}$$

$$J_m = f_m + \rho_m \, \mathrm{v}_m \, F_m = \overline{\alpha_1} \; \overline{f_1^x} + \overline{\alpha_2} \; \overline{f_2^x} +$$

$$+ \overline{\alpha_1} \; \overline{\rho_1^x} \; \overline{v_1^{x,\rho}} \; \overline{F_1} + \overline{\alpha_2} \; \overline{\rho_2^x} \; \overline{v_2^{x,\rho}} \; \overline{F_2} - \left(\overline{\alpha_1} \; \overline{\rho_1} \; \tilde{\mathrm{v}}_1 \; \tilde{F}_1 + \overline{\alpha_2} \; \overline{\rho_2^x} \, \tilde{\mathrm{v}}_2 \; \tilde{F}_2 \right). \tag{6.1.3.5}$$

Further definitions

$$J'_m = f_m + \rho_m \, \mathrm{v}'_m \, F_m = \overline{\alpha_1} \; \overline{f_1^x} + \overline{\alpha_2} \; \overline{f_2^x} +$$

$$+ \overline{\alpha_1} \; \overline{\rho_1^x} \; \mathrm{v}'_1 \; \overline{F_1} + \overline{\alpha_2} \; \overline{\rho_2^x} \; \mathrm{v}'_2 \; \overline{F_2} - \left(\overline{\alpha_1} \; \overline{\rho_1^x} \; \tilde{\mathrm{v}}'_1 \; \tilde{F}_1 + \overline{\alpha_2} \; \overline{\rho_2^x} \; \tilde{\mathrm{v}}'_2 \; \tilde{F}_2 \right). \tag{6.1.3.8}$$

$$\overline{F_k^{Re}} = \frac{\overline{X_k \, \rho_k \, \mathrm{v}'_k \, F'_k}}{\overline{\alpha_k}} . \tag{6.1.3.14}$$

$$F_m^{Re} = \overline{\left(X_1 \, \rho_1 + X_2 \, \rho_2 \right) \mathrm{v}'_m \; F'_m} . \tag{6.1.3.15}$$

$$J_m^{Re} = f_m + F_m^{Re} = \overline{\alpha_1} \; \overline{f_1^x} + \overline{\alpha_2} \; \overline{f_2^x} + \overline{\alpha_1} \; \overline{F_1^{Re}} + \overline{\alpha_2} \; \overline{F_2^{Re}} -$$

$$- \left(\overline{\alpha_1} \; \overline{\rho_1^x} \; \overline{\mathrm{v}'_1 \, F'_1} + \overline{\alpha_2} \; \overline{\rho_2^x} \; \overline{\mathrm{v}'_2 \, F'_2} \right). \tag{6.1.3.18}$$

$$\underline{\mathrm{T}}_m = \overline{\alpha_1} \left(\overline{\underline{\mathrm{T}}_1^x} + \overline{\underline{\mathrm{T}}_1^{Re}} \right) + \overline{\alpha_2} \left(\overline{\underline{\mathrm{T}}_2^x} + \overline{\underline{\mathrm{T}}_2^{Re}} \right) - \left(\overline{\alpha_1} \; \overline{\rho_1^x} \; \tilde{\mathrm{v}}_1 \; \tilde{\mathrm{v}}_1 + \overline{\alpha_2} \; \overline{\rho_2^x} \; \tilde{\mathrm{v}}_2 \, \tilde{\mathrm{v}}_2 \right). \tag{6.2.2.4}$$

$$\mathbf{q}_m = \overline{\alpha_1} \; \overline{\mathbf{q}_1^x} + \overline{\alpha_2} \; \overline{\mathbf{q}_2^x} + \overline{\alpha_1} \; \overline{\mathbf{q}_1^{Re}} + \overline{\alpha_2} \; \overline{\mathbf{q}_2^{Re}} -$$

$$- \overline{\alpha_1} \left(\overline{\underline{\mathrm{T}}_1^x} + \overline{\underline{\mathrm{T}}_1^{Re}} \right) \cdot \tilde{\mathrm{v}}_1 - \overline{\alpha_2} \left(\overline{\underline{\mathrm{T}}_2^x} + \overline{\underline{\mathrm{T}}_2^{Re}} \right) \cdot \tilde{\mathrm{v}}_2 + \tag{6.2.3.5}$$

$$+ \overline{\alpha_1} \; \overline{\rho_1^x} \, \tilde{\mathrm{v}}_1 \left(\tilde{u}_1 + \frac{\tilde{v}_1^2}{2} \right) + \overline{\alpha_2} \; \overline{\rho_2^x} \; \tilde{\mathrm{v}}_2 \left(\tilde{u}_2 + \frac{\tilde{v}_2^2}{2} \right).$$

TABLE VII - continued.

$$\widetilde{\mathbf{q}}_m = \overline{\alpha_1}\ \overline{\mathbf{q}_1^x} + \overline{\alpha_2}\ \overline{\mathbf{q}_2^x} + \overline{\alpha_1}\ \overline{\mathbf{q}_1^{Re}} + \overline{\alpha_2}\ \overline{\mathbf{q}_2^{Re}}$$

$$- \overline{\alpha_1}\left(\overline{\underline{\mathbf{T}}_1^x} + \overline{\underline{\mathbf{T}}_1^{Re}}\right)\cdot\widetilde{\mathbf{v}}_1 - \overline{\alpha_2}\left(\overline{\underline{\mathbf{T}}_2^x} + \overline{\underline{\mathbf{T}}_2^{Re}}\right)\cdot\widetilde{\mathbf{v}}_2 \ . \qquad (6.2.3.9)$$

$$\widehat{\mathbf{q}}_m = \overline{\alpha_1}\left(\overline{\mathbf{q}_1^x} + \widehat{\mathbf{q}}_1^{Re}\right) + \overline{\alpha_2}\left(\overline{\mathbf{q}_2^x} + \widehat{\mathbf{q}}_2^{Re}\right) + \overline{\alpha_1}\ \overline{\rho_1^x}\ \widetilde{\mathbf{v}}_1\ \widetilde{u}_1 + \overline{\alpha_2}\ \overline{\rho_2^x}\ \widetilde{\mathbf{v}}_2\ \widetilde{u}_2 \ . \qquad (6.2.5.4)$$

$$W_m = \overline{-X_1\, p\left(\nabla\cdot\mathbf{v}_1'\right)} - \overline{X_2\, p\left(\nabla\cdot\mathbf{v}_2'\right)} - \overline{\alpha_1}\ \overline{p_1^x}\ \nabla\cdot\widetilde{\mathbf{v}}_1' - \overline{\alpha_2}\ \overline{p_2^x}\ \nabla\cdot\widetilde{\mathbf{v}}_2' \ . \qquad (6.2.5.7)$$

$$\widehat{\widehat{\mathbf{q}}}_m = \overline{\alpha_1}\left(\overline{\mathbf{q}_1^x} + \widehat{\widehat{\mathbf{q}}}_1^{Re}\right) + \overline{\alpha_2}\left(\overline{\mathbf{q}_2^x} + \widehat{\widehat{\mathbf{q}}}_2^{Re}\right) + \overline{\alpha_1}\ \overline{\rho_1^x}\ \widetilde{h}_1\,\widetilde{\mathbf{v}}_1 + \overline{\alpha_2}\ \overline{\rho_2^x}\ \widetilde{h}_2\,\widetilde{\mathbf{v}}_2 \ . \qquad (6.2.6.3)$$

$$\widehat{W}_m = \overline{X_1\left(\frac{\partial p_1'}{\partial t} + \mathbf{v}_1'\cdot\nabla p_1'\right)} + \overline{X_2\left(\frac{\partial p_2'}{\partial t} + \mathbf{v}_2'\cdot\nabla p_2'\right)} +$$

$$+ \overline{\alpha_1}\left(\frac{\partial \widetilde{p}_1}{\partial t} + \overline{\mathbf{v}_1^{x,\rho}}\cdot\nabla\widetilde{p}_1\right) + \overline{\alpha_2}\left(\frac{\partial \widetilde{p}_2}{\partial t} + \overline{\mathbf{v}_2^{x,\rho}}\cdot\nabla\widetilde{p}_2\right) . \qquad (6.2.6.4)$$

$$\widehat{\mathbf{v}}_m = \overline{\alpha_1}\ \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\mathbf{v}_2^{x,\rho}} \ . \qquad (6.2.6.6)$$

$$\widehat{\widehat{W}}_m = \widehat{W}_m - \overline{X_1\, p_1\, \nabla\cdot\mathbf{v}_1'} - \overline{X_2\, p_2\, \nabla\cdot\mathbf{v}_2'} \ . \qquad (6.2.6.7)$$

$$D_m^s = \overline{\alpha_1}\ \overline{\underline{\tau}_{s1}^x} : \nabla\,\overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\underline{\tau}_{s2}^x} : \nabla\,\overline{\mathbf{v}_2^{x,\rho}} \ . \qquad (6.2.7.6)$$

$$s_{Tm} = \overline{s_{T1}} + \overline{s_{T2}} = \overline{X_1\,\mathbf{q}\cdot\nabla\left(\frac{1}{T}\right)} + \overline{X_2\,\mathbf{q}\cdot\nabla\left(\frac{1}{T}\right)} \ . \qquad (6.2.7.5)$$

$$\widetilde{\Phi}_m = \overline{\alpha_1}\left(\overline{\Phi_1^x} + \overline{\Phi_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\Phi_2^x} + \overline{\Phi_2^{Re}}\right) - \left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{s_1^{x,\rho}}\ \widetilde{\mathbf{v}}_1 + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{s_2^{x,\rho}}\ \widetilde{\mathbf{v}}_2\right) . \qquad (6.2.7.10)$$

TABLE VII - continued.

Part B: Summary of conservation equations for the mixture

Continuity equation

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{v}_m) = 0 . \tag{6.2.1.2}$$

Momentum equation

1st form

$$\frac{\partial}{\partial t}(\rho_m \mathbf{v}_m) + \nabla \cdot (\rho_m \mathbf{v}_m \mathbf{v}_m) = \nabla \cdot \underline{\mathbf{T}}_m + \rho_m \mathbf{g} + \overline{\mathbf{m}} . \tag{6.2.2.3}$$

2nd form

$$\begin{aligned}\frac{\partial}{\partial t}(\rho_m \mathbf{v}_m) + \nabla \cdot (\rho_m \mathbf{v}_m \mathbf{v}_m) = & - \nabla \cdot \left(\overline{\alpha_1}\, \overline{\rho_1^x}\, \tilde{\mathbf{v}}_1 \tilde{\mathbf{v}}_1 + \overline{\alpha_2}\, \overline{\rho_2^x}\, \tilde{\mathbf{v}}_2 \tilde{\mathbf{v}}_2 \right) + \\ & + \nabla \cdot \left[\alpha_1 \left(\overline{\underline{\tau}_1^x} + \overline{\underline{\mathbf{T}}_1^{Re}} \right) + \alpha_2 \left(\overline{\underline{\tau}_2^x} + \overline{\underline{\mathbf{T}}_2^{Re}} \right) \right] + \rho_m \mathbf{g} + \\ & + \overline{\mathbf{v}_{1i}}\, \overline{\Gamma_1} + \overline{\mathbf{v}_{2i}}\, \overline{\Gamma_2} - \left(\overline{\alpha_1}\, \nabla \overline{p_1^x} + \overline{\alpha_2}\, \nabla \overline{p_2^x} \right) + \\ & + \left(\overline{p_{1i}^x} - \overline{p_1^x} \right) \nabla \overline{\alpha_1} + \left(\overline{p_{2i}^x} - \overline{p_2^x} \right) \nabla \overline{\alpha_2} + \overline{\mathbf{M}_1^d} + \overline{\mathbf{M}_2^d} . \end{aligned} \tag{6.2.2.6}$$

3rd or alternative form

$$\begin{aligned}\frac{\partial}{\partial t}(\rho_m \mathbf{v}_m) + \nabla \cdot \left(\overline{\alpha_1}\, \overline{\rho_1^x}\, \overline{\mathbf{v}_1^{x,\rho}}\, \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\, \overline{\rho_2^x}\, \overline{\mathbf{v}_2^{x,\rho}}\, \overline{\mathbf{v}_2^{x,\rho}} \right) = & \\ = \nabla \cdot \left(\overline{\alpha_1}\, \overline{\underline{\tau}_1^x} + \overline{\alpha_2}\, \overline{\underline{\tau}_2^x} \right) + \nabla \cdot \left(\overline{\alpha_1}\, \overline{\underline{\mathbf{T}}_1^{Re}} + \overline{\alpha_2}\, \overline{\underline{\mathbf{T}}_2^{Re}} \right) + \rho_m \mathbf{g} & + \\ + \overline{\mathbf{v}_{1i}}\, \overline{\Gamma_1} + \overline{\mathbf{v}_{2i}}\, \overline{\Gamma_2} - \left(\overline{\alpha_1}\, \nabla \overline{p_1^x} + \overline{\alpha_2}\, \nabla \overline{p_2^x} \right) & + \\ + \left(\overline{p_{1i}^x} - \overline{p_1^x} \right) \nabla \overline{\alpha_1} + \left(\overline{p_{2i}^x} - \overline{p_2^x} \right) \nabla \overline{\alpha_2} + \overline{\mathbf{M}_1^d} + \overline{\mathbf{M}_2^d} & . \end{aligned} \tag{6.2.2.7}$$

TABLE VII - continued.

Total energy equation

1st form

$$\frac{\partial}{\partial t}\left[\rho_m\left(u_m+\frac{\upsilon_m^2}{2}+u_m^{Re}\right)\right]+\nabla\cdot\left[\rho_m\mathbf{v}_m\left(u_m+\frac{\upsilon_m^2}{2}+u_m^{Re}\right)\right]+$$

$$+\nabla\cdot\left[\frac{1}{2}\mathbf{v}_m\left(\overline{\alpha}_1\overline{\rho_1^x}\,\tilde{\upsilon}_1^2+\overline{\alpha}_2\overline{\rho_2^x}\,\tilde{\upsilon}_2^2\right)+\upsilon_m\left(\overline{\alpha}_1\overline{\rho_1^x}\,\tilde{\mathbf{v}}_1\tilde{\upsilon}_1+\overline{\alpha}_2\overline{\rho_2^x}\,\tilde{\mathbf{v}}_2\tilde{\upsilon}_2\right)+\right.$$

$$\left.+\overline{\alpha}_1\overline{\rho_1^x}\,\tilde{\mathbf{v}}_1\overline{u_1^{Re}}+\overline{\alpha}_2\overline{\rho_2^x}\,\tilde{\mathbf{v}}_2\overline{u_2^{Re}}\right]= \qquad (6.2.3.6)$$

$$=-\nabla\cdot\mathbf{q}_m+\rho_m\mathbf{v}_m\cdot\mathbf{g}+\rho_m r_m+\bar{\varepsilon}+\nabla\cdot\left\{\left[\overline{\alpha}_1\left(\overline{\underline{\mathbf{T}}_1^x}+\overline{\underline{\mathbf{T}}_1^{Re}}\right)+\overline{\alpha}_2\left(\overline{\underline{\mathbf{T}}_2^x}+\overline{\underline{\mathbf{T}}_2^{Re}}\right)\right]\cdot\mathbf{v}_m\right\}.$$

Alternative form

$$\frac{\partial}{\partial t}\left[\rho_m\left(u_m+\frac{\upsilon_m^2}{2}+u_m^{Re}\right)\right]+$$

$$+\nabla\cdot\left\{\overline{\alpha}_1\overline{\rho_1^x}\,\overline{\mathbf{v}_1^{x,\rho}}\left[\overline{u_1^{x,\rho}}+\frac{1}{2}\left(\overline{\upsilon_1^{x,\rho}}\right)^2+\overline{u_1^{Re}}\right]+\overline{\alpha}_2\overline{\rho_2^x}\,\overline{\mathbf{v}_2^{x,\rho}}\left[\overline{u_2^{x,\rho}}+\frac{1}{2}\left(\overline{\upsilon_2^{x,\rho}}\right)^2+\overline{u_2^{Re}}\right]\right\}= \qquad (6.2.3.10)$$

$$=-\nabla\cdot\mathbf{q}_m+\rho_m\mathbf{v}_m\cdot\mathbf{g}+\rho_m r_m+\bar{\varepsilon}+$$

$$+\nabla\cdot\left\{\left[\overline{\alpha}_1\left(\overline{\underline{\mathbf{T}}_1^x}+\overline{\underline{\mathbf{T}}_1^{Re}}\right)+\overline{\alpha}_2\left(\overline{\underline{\mathbf{T}}_2^x}+\overline{\underline{\mathbf{T}}_2^{Re}}\right)\right]\cdot\mathbf{v}_m\right\}.$$

Mechanical energy equation

1st form

$$\frac{\partial}{\partial t}\left(\rho_m u_m^{Re}+\rho_m\frac{\upsilon_m^2}{2}\right)+\nabla\cdot\left(\rho_m\frac{\upsilon_m^2}{2}\mathbf{v}_m\right)+$$

$$+\nabla\cdot\left(\overline{\alpha}_1\overline{\rho_1^x}\,\overline{\mathbf{v}_1^{x,\rho}}\upsilon_m\tilde{\upsilon}_1+\overline{\alpha}_1\overline{\rho_1^x}\,\overline{\mathbf{v}_1^{x,\rho}}\frac{\tilde{\upsilon}_1^2}{2}+\overline{\alpha}_2\overline{\rho_2^x}\,\overline{\mathbf{v}_2^{x,\rho}}\upsilon_m\tilde{\upsilon}_2+\overline{\alpha}_2\overline{\rho_2^x}\,\overline{\mathbf{v}_2^{x,\rho}}\frac{\tilde{\upsilon}_2^2}{2}\right)+ \qquad (6.2.4.3)$$

$$+\nabla\cdot\left(\rho_m u_m^{Re}\mathbf{v}_m\right)+\nabla\cdot\left(\overline{\alpha}_1\overline{\rho_1^x}\,\tilde{\mathbf{v}}_1\overline{u_1^{Re}}+\overline{\alpha}_2\overline{\rho_2^x}\,\tilde{\mathbf{v}}_2\overline{u_2^{Re}}\right)=$$

$$=\nabla\cdot\left\{\left[\overline{\alpha}_1\left(\overline{\underline{\mathbf{T}}_1^x}+\overline{\underline{\mathbf{T}}_1^{Re}}\right)+\alpha_2\left(\overline{\underline{\mathbf{T}}_2^x}+\overline{\underline{\mathbf{T}}_2^{Re}}\right)\right]\cdot\mathbf{v}_m\right\}+$$

TABLE VII - continued.

$$+ \nabla \cdot \left| \overline{\alpha_1} \left(\overline{\underline{T}_1^x} + \overline{\underline{T}_1^{Re}} \right) \cdot \tilde{\mathbf{v}}_1 + \alpha_2 \left(\overline{\underline{T}_2^x} + \overline{\underline{T}_2^{Re}} \right) \cdot \tilde{\mathbf{v}}_2 \right| -$$

$$- \nabla \cdot \left(\overline{\mathbf{q}_1^{kin}} + \overline{\mathbf{q}_2^{kin}} + \overline{\mathbf{q}_1^p} + \overline{\mathbf{q}_2^p} + \overline{\mathbf{q}_1^\iota} + \overline{\mathbf{q}_2^\iota} \right) + \overline{\left(\mathbf{n}_k \cdot \nabla X_k \right) e_i^\sigma} -$$

$$- \overline{\alpha_1} \; \overline{\underline{T}_1^x} : \nabla \overline{\mathbf{v}_1^{x,\rho}} - \alpha_2 \overline{\underline{T}_2^x} : \nabla \overline{\mathbf{v}_2^{x,\rho}} +$$

$$+ \overline{X_1 \, p \left(\nabla \cdot \mathbf{v}_1' \right)} + \overline{X_2 \, p \left(\nabla \cdot \mathbf{v}_2' \right)} - D_m + \mathbf{g} \cdot \rho_m \mathbf{v}_m .$$

<u>Alternative form</u>

$$\frac{\partial}{\partial t} \left(\rho_m u_m^{Re} + \rho_m \frac{v_m^2}{2} \right) +$$

$$+ \nabla \cdot \left\{ \overline{\alpha_1} \overline{\rho_1^x} \overline{\mathbf{v}_1^{x,\rho}} \left| \frac{1}{2} \left(\overline{v_1^{x,\rho}} \right)^2 + \overline{u_1^{Re}} \right| + \overline{\alpha_2} \overline{\rho_2^x} \overline{\mathbf{v}_2^{x,\rho}} \left[\frac{1}{2} \left(\overline{v_2^{x,\rho}} \right)^2 + \overline{u_2^{Re}} \right] \right\} =$$

$$= \nabla \cdot \left\{ \left| \overline{\alpha_1} \left(\overline{\underline{T}_1^x} + \overline{\underline{T}_1^{Re}} \right) + \overline{\alpha_2} \left(\overline{\underline{T}_2^x} + \overline{\underline{T}_2^{Re}} \right) \right| \cdot \mathbf{v}_m \right\} +$$

$$+ \nabla \cdot \left| \overline{\alpha_1} \left(\overline{\underline{T}_1^x} + \overline{\underline{T}_1^{Re}} \right) \cdot \tilde{\mathbf{v}}_1 + \overline{\alpha_2} \left(\overline{\underline{T}_2^x} + \overline{\underline{T}_2^{Re}} \right) \cdot \tilde{\mathbf{v}}_2 \right| - \qquad (6.2.4.4)$$

$$- \nabla \cdot \left(\overline{\mathbf{q}_1^{kin}} + \overline{\mathbf{q}_2^{kin}} + \overline{\mathbf{q}_1^p} + \overline{\mathbf{q}_2^p} + \overline{\mathbf{q}_1^\iota} + \overline{\mathbf{q}_2^\iota} \right) + \overline{\left(\mathbf{n}_k \cdot \nabla X_k \right) e_i^\sigma} -$$

$$- \overline{\alpha_1} \; \overline{\underline{T}_1^x} : \nabla \overline{\mathbf{v}_1^{x,\rho}} - \alpha_2 \overline{\underline{T}_2^x} : \nabla \overline{\mathbf{v}_2^{x,\rho}} +$$

$$+ \overline{X_1 \, p \left(\nabla \cdot \mathbf{v}_1' \right)} + \overline{X_2 \, p \left(\nabla \cdot \mathbf{v}_2' \right)} - D_m + \mathbf{g} \cdot \rho_m \mathbf{v}_m .$$

Internal energy equation

<u>1st form</u>

$$\frac{\partial (\rho_m u_m)}{\partial t} + \nabla \cdot (\rho_m u_m \mathbf{v}_m) = - \nabla \cdot \hat{\mathbf{q}}_m + D_m + \rho_m r_m + \bar{\varepsilon} - p_m \nabla \cdot \mathbf{v}_m + W_m -$$

TABLE VII - continued.

$$- \frac{1}{2} \overline{v_{1i}^2} \; \overline{\Gamma_1} - \frac{1}{2} \overline{v_{2i}^2} \; \overline{\Gamma_2} + \overline{\left(\underline{T} \cdot v\right) \cdot \left(\nabla X_1 + \nabla X_2\right)} + \tag{6.2.5.9}$$

$$+ \overline{\alpha_1} \; \overline{\underline{\tau}_1^x} : \left(\nabla v_m + \nabla \tilde{v}_1\right) + \overline{\alpha_2} \; \overline{\underline{\tau}_2^x} : \left(\nabla v_m + \nabla \tilde{v}_2\right).$$

Alternative form

$$\frac{\partial (\rho_m u_m)}{\partial t} + \nabla \cdot \left(\overline{\alpha_1} \; \overline{\rho_1^x} \; \overline{v_1^{x,\rho}} \; \overline{u_1^{x,\rho}} + \overline{\alpha_2} \; \overline{\rho_2^x} \; \overline{v_2^{x,\rho}} \; \overline{u_2^{x,\rho}}\right) =$$

$$= - \nabla \cdot \left[\overline{\alpha_1}\left(\overline{q_1^x} + \hat{q}_1^{Re}\right) + \overline{\alpha_2}\left(\overline{q_2^x} + \hat{q}_2^{Re}\right)\right] +$$

$$+ D_m + \rho_m r_m + \bar{\varepsilon} - p_m \nabla \cdot v_m + W_m - \tag{6.2.5.12}$$

$$- \frac{1}{2} \overline{v_{1i}^2} \; \overline{\Gamma_1} - \frac{1}{2} \overline{v_{2i}^2} \; \overline{\Gamma_2} + \overline{\left(\underline{T} \cdot v\right) \cdot \left(\nabla X_1 + \nabla X_2\right)} +$$

$$+ \overline{\alpha_1} \; \overline{\underline{\tau}_1^x} : \left(\nabla v_m + \nabla \tilde{v}_1\right) + \overline{\alpha_2} \; \overline{\underline{\tau}_2^x} : \left(\nabla v_m + \nabla \tilde{v}_2\right).$$

Enthalpy equation

1st form

$$\frac{\partial}{\partial t} (\rho_m h_m) + \nabla \cdot (\rho_m h_m v_m) =$$

$$= - \nabla \cdot \hat{\hat{q}}_m + \frac{\partial p_m}{\partial t} + v_m \cdot \nabla p_m + \bar{\varepsilon} + D_m + \rho_m r_m + \hat{\hat{W}}_m -$$

$$- \left(\frac{1}{2} \overline{v_{1i}^2} - \frac{\overline{p_1^x}}{\overline{\rho_1^x}}\right) \overline{\Gamma_1} - \left(\frac{1}{2} \overline{v_{2i}^2} - \frac{\overline{p_2^x}}{\overline{\rho_2^x}}\right) \overline{\Gamma_2} + \tag{6.2.6.9}$$

$$+ \overline{\left(\underline{T} \cdot v\right)_1 \cdot \nabla X_1} + \overline{\left(\underline{T} \cdot v\right)_2 \cdot \nabla X_2} + \overline{\alpha_1} \; \overline{\underline{T}_1^x} : \nabla \overline{v_1^{x,\rho}} + \overline{\alpha_2} \; \overline{\underline{T}_2^x} : \nabla \overline{v_2^{x,\rho}} .$$

TABLE VII - continued.

Alternative form

$$\frac{\partial}{\partial t}\left(\rho_m h_m\right) + \nabla\cdot\left(\overline{\alpha_1}\;\overline{\rho_1^x}\;\overline{h_1^{x,\rho}}\;\overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\;\overline{\rho_2^x}\;\overline{h_2^{x,\rho}}\;\overline{\mathbf{v}_2^{x,\rho}}\right) =$$

$$= -\nabla\cdot\left[\overline{\alpha_1}\left(\overline{\mathbf{q}_1^x} + \widehat{\mathbf{q}}_1^{Re}\right) + \overline{\alpha_2}\left(\overline{\mathbf{q}_2^x} + \widehat{\mathbf{q}}_2^{Re}\right)\right] +$$

$$+ \frac{\partial p_m}{\partial t} + \widehat{\mathbf{v}}_m\cdot\nabla p_m + \bar{\varepsilon} + D_m + \rho_m r_m + \widehat{W}_m - \tag{6.2.6.12}$$

$$-\left(\frac{1}{2}\,\overline{v_{1i}^2} - \frac{\overline{p_1^x}}{\overline{\overline{\rho_1^x}}}\right)\overline{\Gamma}_1 - \left(\frac{1}{2}\,\overline{v_{2i}^2} - \frac{\overline{p_2^x}}{\overline{\overline{\rho_2^x}}}\right)\overline{\Gamma}_2 +$$

$$+ \overline{\left(\underline{\mathbf{T}}\cdot\mathbf{v}\right)_1\cdot\nabla X_1} + \overline{\left(\underline{\mathbf{T}}\cdot\mathbf{v}\right)_2\cdot\nabla X_2} + \overline{\alpha_1}\;\overline{\underline{\mathbf{T}}_1^x} : \nabla\,\overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\;\overline{\underline{\mathbf{T}}_2^x} : \nabla\,\overline{\mathbf{v}_2^{x,\rho}} \,.$$

Entropy inequality

1st form

$$\frac{\partial}{\partial t}\left(\rho_m s_m\right) + \nabla\cdot\left(\rho_m s_m \mathbf{v}_m\right) + \nabla\cdot\left(\overline{\alpha_1}\;\overline{\rho_1}\;\overline{s_1^{x,\rho}\tilde{\mathbf{v}}_1} + \overline{\alpha_2}\;\overline{\rho_2}\;\overline{s_2^{x,\rho}\tilde{\mathbf{v}}_2}\right) -$$

$$-\nabla\cdot\left[\overline{\alpha_1}\left(\overline{\boldsymbol{\Phi}_1^x} + \overline{\boldsymbol{\Phi}_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\boldsymbol{\Phi}_2^x} + \overline{\boldsymbol{\Phi}_2^{Re}}\right)\right] - \rho_m\sigma_m = \Delta s_m + \Delta s_i \geq 0 \tag{6.2.7.4}$$

with

$$\Delta s_m = s_{Tm} + D_m^s + D_m^{st} \geq 0 \,. \tag{6.2.7.7}$$

Alternative form

$$\frac{\partial}{\partial t}\left(\rho_m s_m\right) + \nabla\cdot\left(\overline{\alpha_1}\;\overline{\rho_1^x}\;\overline{s_1^{x,\rho}}\;\overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\;\overline{\rho_2^x}\;\overline{s_2^{x,\rho}}\;\overline{\mathbf{v}_2^{x,\rho}}\right) -$$

$$-\nabla\cdot\left[\overline{\alpha_1}\left(\overline{\boldsymbol{\Phi}_1^x} + \overline{\boldsymbol{\Phi}_1^{Re}}\right) + \overline{\alpha_2}\left(\overline{\boldsymbol{\Phi}_2^x} + \overline{\boldsymbol{\Phi}_2^{Re}}\right)\right] - \rho_m\sigma_m = \Delta s_m + \Delta s_i \geq 0 \tag{6.2.7.13}$$

Entropy equation

1st form

$$\frac{\partial}{\partial t}(\rho_m s_m) + \nabla \cdot (\rho_m s_m \mathbf{v}_m) - \nabla \cdot \tilde{\boldsymbol{\Phi}}_m - D^s_m - D^{st}_m - s_{Tm} - \rho_m \sigma_m - \Delta s_i = 0 \qquad (6.2.7.12)$$

Alternative form

$$\frac{\partial}{\partial t}(\rho_m s_m) + \nabla \cdot \left(\overline{\alpha_1}\ \overline{\rho_1^x}\ \overline{s_1^{x,\rho}}\ \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\rho_2^x}\ \overline{s_2^{x,\rho}}\ \overline{\mathbf{v}_2^{x,\rho}} \right) -$$

$$- \nabla \cdot \left[\overline{\alpha_1} \left(\overline{\boldsymbol{\Phi}_1^x} + \overline{\boldsymbol{\Phi}_1^{Re}} \right) + \overline{\alpha_2} \left(\overline{\boldsymbol{\Phi}_2^x} + \overline{\boldsymbol{\Phi}_2^{Re}} \right) \right] - \qquad (6.2.7.15)$$

$$- D^s_m - D^{st}_m - s_{Tm} - \rho_m \sigma_m - \Delta s_i = 0 .$$

TABLE VII - continued.

Part C: Conservations equations for the mixture in terms of slip velocity (Two phases, two components)

Momentum equation

$$\frac{\partial}{\partial t}(\rho_m \mathbf{v}_m) + \nabla \cdot \left(\rho_m \mathbf{v}_m \mathbf{v}_m + \frac{\overline{\alpha_1}\,\overline{\alpha_2}\,\overline{\rho_1^x}\,\overline{\rho_2^x}}{\rho_m}\mathbf{v}_{Sl}\mathbf{v}_{Sl}\right) =$$

$$= \nabla \cdot \left[\underline{\tau}_m + \underline{T}_m^{Re} + \frac{1}{\rho_m}\overline{\alpha_1}\,\overline{\alpha_2}\left(\overline{\rho_1^x} - \overline{\rho_2^x}\right)\left(\overline{\underline{\tau}_2^x} - \overline{\underline{\tau}_1^x} + \overline{\underline{T}_2^{Re}} - \overline{\underline{T}_1^{Re}}\right)\right] + \rho_m \mathbf{g} +$$

$$+ \overline{\mathbf{v}_{1i}}\;\overline{\Gamma_1} + \overline{\mathbf{v}_{2i}}\;\overline{\Gamma_2} - \left(\overline{\alpha_1}\nabla\overline{p_1^x} + \overline{\alpha_2}\nabla\overline{p_2^x}\right) + \qquad (6.3.2.4)$$

$$+ \left(\overline{p_{1i}^x} - \overline{p_1^x}\right)\nabla\overline{\alpha_1} + \left(\overline{p_{2i}^x} - \overline{p_2^x}\right)\nabla\overline{\alpha_2} + \overline{\mathbf{M}_1^d} + \overline{\mathbf{M}_2^d}\ .$$

Total energy equation

$$\frac{\partial}{\partial t}\left[\rho_m\left(u_m + \frac{v_m^2}{2} + u_m^{Re}\right)\right] + \nabla\cdot\left[\rho_m\left(u_m + \frac{v_m^2}{2} + u_m^{Re}\right)\mathbf{v}_m\right] +$$

$$\nabla\cdot\left\{\frac{1}{\rho_m}\left(\overline{\alpha_1}\,\overline{\alpha_2}\,\overline{\rho_1^x}\,\overline{\rho_2^x}\right)\left[\overline{u_2^{x,\rho}} - \overline{u_1^{x,\rho}} + \frac{\left(\overline{\mathbf{v}_2^{x,\rho}}\right)^2}{2} - \frac{\left(\overline{\mathbf{v}_1^{x,\rho}}\right)^2}{2} + u_2^{Re} - u_1^{Re}\right]\mathbf{v}_{Sl}\right\} = \qquad (6.3.2.9)$$

$$= -\nabla\cdot\tilde{\mathbf{q}}_m + \rho_m \mathbf{v}_m\cdot g + \rho_m r_m + \bar{\varepsilon} +$$

$$+ \nabla\cdot\left\{\left[\underline{T}_m + \underline{T}_m^{Re} + \frac{1}{\rho_m}\overline{\alpha_1}\,\overline{\alpha_2}\,\overline{\rho_1^x}\,\overline{\rho_2^x}\left(\overline{\underline{T}_2^x} - \overline{\underline{T}_1^x} + \overline{\underline{T}_2^{Re}} - \overline{\underline{T}_1^{Re}}\right)\right]\cdot\mathbf{v}_m\right\}.$$

Mechanical energy equation

$$\frac{\partial}{\partial t}\left(\rho_m u_m^{Re} + \rho_m\frac{v_m^2}{2}\right) +$$

$$+ \nabla\cdot\rho_m\left(u_m + \frac{v_m^2}{2}\right)\mathbf{v}_m + \nabla\cdot\left\{\frac{\overline{\alpha_1}\,\overline{\alpha_2}\,\overline{\rho_1^x}\,\overline{\rho_2^x}}{\rho_m}\left[\overline{u_2^{x,\rho}} - \overline{u_1^{x,\rho}} + \frac{\left(\overline{\mathbf{v}_2^{x,\rho}}\right)^2}{2} - \frac{\left(\overline{\mathbf{v}_1^{x,\rho}}\right)^2}{2}\right]\mathbf{v}_{Sl}\right\} =$$

TABLE VII - continued.

$$= \nabla \cdot \left\{ \left[\underline{T}_m + \underline{T}_m^{Re} + \frac{1}{\rho_m} \overline{\alpha_1}\ \overline{\alpha_2} \left(\overline{\rho_1^x} - \overline{\rho_2^x} \right) \left(\overline{\underline{T}_2^x} - \overline{\underline{T}_1^x} + \overline{\underline{T}_2^{Re}} - \overline{\underline{T}_1^{Re}} \right) \right] \cdot \mathbf{v}_m \right\} +$$

$$+ \nabla \cdot \left[\overline{\alpha_1} \left(\overline{\underline{T}_1^x} + \overline{\underline{T}_1^{Re}} \right) \cdot \tilde{\mathbf{v}}_1 + \overline{\alpha_2} \left(\overline{\underline{T}_2^x} + \overline{\underline{T}_2^{Re}} \right) \cdot \tilde{\mathbf{v}}_2 \right] - \qquad (6.3.2.10)$$

$$- \nabla \cdot \left(\overline{\mathbf{q}_1^{kin}} + \overline{\mathbf{q}_2^{kin}} + \overline{\mathbf{q}_1^p} + \overline{\mathbf{q}_2^p} + \overline{\mathbf{q}_1^\tau} + \overline{\mathbf{q}_2^\tau} \right) + \overline{\left(\mathbf{n}_k \cdot \nabla X_k \right) e_i^\sigma} -$$

$$- \overline{\alpha_1}\ \overline{\underline{T}_1^x} : \nabla\, \overline{\mathbf{v}_1^{x,\rho}} - \alpha_2 \overline{\underline{T}_2^x} : \nabla\, \overline{\mathbf{v}_2^{x,\rho}} +$$

$$+ \overline{X_1\, p \left(\nabla \cdot \mathbf{v}_1' \right)} + \overline{X_2\, p \left(\nabla \cdot \mathbf{v}_2' \right)} - D_m + \mathbf{g} \cdot \rho_m \mathbf{v}_m .$$

Internal energy equation

$$\frac{\partial}{\partial t} (\rho_m u_m) + \nabla \cdot (\rho_m u_m \mathbf{v}_m) + \nabla \cdot \left[\frac{\overline{\alpha_1}\ \overline{\alpha_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x}}{\rho_m} \left(\overline{u_2^{x,\rho}} - \overline{u_1^{x,\rho}} \right) \mathbf{v}_{Sl} \right] =$$

$$= - \nabla \cdot \left[\mathbf{q}_m + \hat{\mathbf{q}}_m^{Re} + \frac{1}{\rho_m} \overline{\alpha_1}\ \overline{\alpha_2} \left(\overline{\rho_1^x} - \overline{\rho_2^x} \right) \left(\overline{\mathbf{q}_2^x} - \overline{\mathbf{q}_1^x} + \hat{\mathbf{q}}_2^{Re} - \hat{\mathbf{q}}_1^{Re} \right) \right] +$$

$$+ D_m + \rho_m r_m + \bar{\varepsilon} - p_m \nabla \cdot \mathbf{v}_m + W_m - \qquad (6.3.2.13)$$

$$- \frac{1}{2} \overline{v_{1i}^2}\ \overline{\Gamma_1} - \frac{1}{2} \overline{v_{2i}^2}\ \overline{\Gamma_2} + \overline{\left(\underline{T} \cdot \mathbf{v} \right) \cdot \left(\nabla X_1 + \nabla X_2 \right)} +$$

$$+ \overline{\alpha_1}\ \overline{\underline{\tau}_1^x} : \left(\nabla \mathbf{v}_m + \nabla \tilde{\mathbf{v}}_1 \right) + \overline{\alpha_2}\ \overline{\underline{\tau}_2^x} : \left(\nabla \mathbf{v}_m + \nabla \tilde{\mathbf{v}}_2 \right) .$$

Enthalpy equation

$$\frac{\partial}{\partial t} (\rho_m h_m) + \nabla \cdot (\rho_m h_m \mathbf{v}_m) + \nabla \cdot \left[\frac{\overline{\alpha_1}\ \overline{\alpha_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x}}{\rho_m} \left(\overline{h_2^{x,\rho}} - \overline{h_1^{x,\rho}} \right) \mathbf{v}_{Sl} \right] =$$

$$= - \nabla \cdot \left[\mathbf{q}_m + \hat{\mathbf{q}}_m^{Re} + \frac{1}{\rho_m} \overline{\alpha_1}\ \overline{\alpha_2} \left(\overline{\rho_1^x} - \overline{\rho_2^x} \right) \left(\overline{\mathbf{q}_2^x} - \overline{\mathbf{q}_1^x} + \hat{\mathbf{q}}_2^{Re} - \hat{\mathbf{q}}_1^{Re} \right) \right] +$$

TABLE VII - continued.

$$+\frac{\partial p_m}{\partial t} + \mathbf{v}_m \cdot \nabla p_m + \frac{\overline{\alpha_1}\ \overline{\alpha_2}}{\rho_m}\left(\overline{\rho_1^x} - \overline{\rho_2^x}\right)\mathbf{v}_{Sl} \cdot \nabla p_m + \bar{\varepsilon} + D_m + \rho_m r_m + \widehat{W}_m - \qquad (6.3.2.17)$$

$$-\left(\frac{1}{2}\ \overline{v_{1i}^2} - \frac{\overline{p_1^x}}{\overline{\overline{\rho_1^x}}}\right)\overline{\Gamma_1} - \left(\frac{1}{2}\ \overline{v_{2i}^2} - \frac{\overline{p_2^x}}{\overline{\overline{\rho_2^x}}}\right)\overline{\Gamma_2} +$$

$$+ \overline{\left(\underline{\mathbf{T}} \cdot \mathbf{v}\right)_1 \cdot \nabla X_1} + \overline{\left(\underline{\mathbf{T}} \cdot \mathbf{v}\right)_2 \cdot \nabla X_2} + \overline{\alpha_1}\ \overline{\underline{\mathbf{T}}_1^x} : \nabla \overline{\mathbf{v}_1^{x,\rho}} + \overline{\alpha_2}\ \overline{\underline{\mathbf{T}}_2^x} : \nabla \overline{\mathbf{v}_2^{x,\rho}} .$$

Entropy inequality

$$\frac{\partial(\rho_m s_m)}{\partial t} + \nabla \cdot (\rho_m s_m \mathbf{v}_m) + \nabla \cdot \left[\frac{\overline{\alpha_1}\ \overline{\alpha_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x}}{\rho_m}\left(\overline{s_2^{x,\rho}} - \overline{s_1^{x,\rho}}\right)\mathbf{v}_{Sl}\right] -$$

$$-\nabla \cdot \left[\boldsymbol{\Phi}_m + \boldsymbol{\Phi}_m^{Re} + \frac{\overline{\alpha_1}\ \overline{\alpha_2}}{\rho_m}\left(\overline{\rho_1^x} - \overline{\rho_2^x}\right)\left(\overline{\boldsymbol{\Phi}_2^x} - \overline{\boldsymbol{\Phi}_1^x} + \overline{\boldsymbol{\Phi}_2^{Re}} - \overline{\boldsymbol{\Phi}_1^{Re}}\right)\right] - \qquad (6.3.2.21)$$

$$-\rho_m \sigma_m = \Delta s_m + \Delta s_i \geq 0 .$$

Entropy equation

$$\frac{\partial}{\partial t}(\rho_m s_m) + \nabla \cdot (\rho_m s_m \mathbf{v}_m) + \nabla \cdot \left[\frac{\overline{\alpha_1}\ \overline{\alpha_2}\ \overline{\rho_1^x}\ \overline{\rho_2^x}}{\rho_m}\left(\overline{s_2^{x,\rho}} - \overline{s_1^{x,\rho}}\right)\mathbf{v}_{Sl}\right] -$$

$$-\nabla \cdot \left[\boldsymbol{\Phi}_m + \boldsymbol{\Phi}_m^{Re} + \frac{\overline{\alpha_1}\ \overline{\alpha_2}}{\rho_m}\left(\overline{\rho_1^x} - \overline{\rho_2^x}\right)\left(\overline{\boldsymbol{\Phi}_2^x} - \overline{\boldsymbol{\Phi}_1^x} + \overline{\boldsymbol{\Phi}_2^{Re}} - \overline{\boldsymbol{\Phi}_1^{Re}}\right)\right] - \qquad (6.3.2.22)$$

$$-D_m^s - D_m^{st} - s_{Tm} - \rho_m \sigma_m - \Delta s_i = 0 .$$

TABLE VII - continued.

$$\nabla \cdot (\rho \mathbf{v} \mathbf{v}) = \sum_k \sum_i \frac{\partial}{\partial x_i} \rho \, v_i \, v_k \, \boldsymbol{\delta}_k \qquad (2.3.4)$$

$$\nabla \cdot \underline{\mathbf{T}} = \sum_k \sum_i \frac{\partial}{\partial x_i} T_{ik} \, \boldsymbol{\delta}_k \qquad (2.3.6)$$

$$\underline{\mathbf{T}} \cdot \mathbf{v} = \sum_i \sum_j T_{ij} \, v_j \, \boldsymbol{\delta}_k \qquad (2.3.12)$$

$$\nabla \cdot (\underline{\mathbf{T}} \cdot \mathbf{v}) = \sum_i \sum_k \frac{\partial}{\partial x_i} T_{ik} \, v_k \qquad (2.3.13)$$

$$\mathbf{v} \cdot (\nabla \cdot \underline{\mathbf{T}}) = \nabla \cdot (\underline{\mathbf{T}} \cdot \mathbf{v}) - (\underline{\mathbf{T}} : \nabla \mathbf{v}) = \sum_i \sum_k v_i \frac{\partial T_{ik}}{\partial x_k} \qquad (2.3.22)$$

$$\underline{\mathbf{T}} : \nabla \mathbf{v} = \sum_i \sum_j T_{ij} \frac{\partial v_i}{\partial x_j} = \qquad (2.3.30)$$

$$= -p \, (\nabla \cdot \mathbf{v}) + \underline{\boldsymbol{\tau}} : \nabla \mathbf{v} \qquad (2.3.31)$$

$$-p \underline{\mathbf{I}} : \nabla \mathbf{v} = -p \, (\nabla \cdot \mathbf{v})$$ from (2.3.31) and (5.1.4.10)

$$\nabla \cdot p\mathbf{v} = p \, (\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla p \qquad (2.3.34)$$

$$\underline{\mathbf{I}} \cdot \nabla X_k = \nabla X_k \qquad (4.6.8)$$

$$p \underline{\mathbf{I}} \cdot \nabla X_k = p \nabla X_k \qquad (4.6.9)$$

$$p \underline{\mathbf{I}} \cdot \mathbf{v} = p \, \mathbf{v} \qquad (4.6.10)$$

$$X_k \, (\nabla \cdot \rho \mathbf{v} \mathbf{v}) = \nabla \cdot (X_k \, \rho \mathbf{v} \mathbf{v}) - \rho \mathbf{v} \mathbf{v} \cdot \nabla X_k \qquad (5.1.2.3)$$

$$\nabla \cdot p \underline{\mathbf{I}} = \nabla p \qquad (5.1.2.13)$$

$$X_k \, (\nabla \cdot \underline{\mathbf{T}}) = \nabla \cdot (X_k \underline{\mathbf{T}}) - \underline{\mathbf{T}} \cdot \nabla X_k \qquad (5.1.2.4)$$

$$(\mathbf{w} \cdot \mathbf{v}) \mathbf{v} = \mathbf{w} \cdot \mathbf{v} \mathbf{v} \qquad (5.1.3.20)$$

TABLE VIII: Summary of vector and tensor relations

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