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Abstract:

The problem is treated of a two dissimilar materials joint with arbitrary edge angles under thermal loading. Emphasis is placed on the investigation of the regular stress term, which is independent of the distance r, in the stress field near the singular point. A method of calculating the regular stress term $\sigma_{ij0}(\theta)$ analytically for arbitrary angles θ_1 and θ_2 is presented. Numerical results of σ_{ij0} for special wedges are included for all possible Dundurs parameters. It is shown that under thermal loading conditions the regular stress term is very important to the stress distribution in the near field of the singularity point.

Eine Methode zur Bestimmung von regulärem Spannungsterm für Verbund mit beliebigen Winkeln bei thermischer Belastung

Zusammenfassung:

Das Problem von Zweistoffverbunden mit beliebigen Winkeln bei thermischer Belastung wird behandelt. Der Schwerpunkt liegt auf der Untersuchung des regulären Spannungsterms im Spannungsfeld in der Nähe der Singularitätsstelle. Der reguläre Spannungsterm ist unabhängig vom Abstand r. Eine analytische Methode zur Bestimmung des regulären Spannungsterms $\sigma_{ij0}(\theta)$ für beliebige Winkel θ_1 und θ_2 wird angegeben. Numerische Ergebnisse von σ_{ij0} werden für spezielle Winkel θ_1, θ_2 als Funktion der Dundurs Parameter dargestellt. Es wird gezeigt, daß bei thermischer Belastung der reguläre Spannungsterm einen wesentlichen Beitrag zur Spannungsverteilung in der Nähe der Singularitätsstelle liefert.

Contents

Notation

E	Young's modulus
$f_{ijk}(\theta)$	Angular function
G	Shear modulus
K_k	Stress intensity factor
L	Characteristic length
r , θ	Polar coordinates
ΔT	Temperature difference
u , v	Displacement in polar coordinate system
x , y	Rectangular coordinates
α	1) Coefficient of thermal expansion
	2) Dundurs parameter
β	Dundurs parameter
ε_{ij}	Strain tensor
ε_r , ε_θ , $\gamma_{r\theta}$	Strain component in polar coodinates
ν	Poisson's ratio
ω_k	Stress exponents
Φ	Stress function
σ_{ij}	Stress tensor
σ_r , σ_θ , $\sigma_{r\theta}$	Stress component in polar coodinates
σ_x , σ_y , σ_{xy}	Stress component in rectangular coodinates
$\sigma_{ij0}(\theta)$	Regular stress term
σ_0	$\sigma_0 = \sigma_{\theta 0}$ for $\theta = 0$
θ_1 , θ_2	Angles of the two free edge

1. Introduction

If two different materials are bound together thermal stresses are developing during cooling or heating due to the different thermal expansions. Very high stresses occur near the edge of the interface of the compound and singularities of the stresses appear at point A in Fig.1. These stresses in the near field of the singularity point can be described analytically by

$$\sigma_{ij}(r,\theta) = \sum_{k=1}^{N} \frac{K_k}{(r/L)^{\omega_k}} f_{ijk}(\theta) + \sigma_{ij0}(\theta) \tag{1}$$

where r and θ are polar coordinates as shown in Fig.1 and L is a characteristic length of the compound. The stress exponents ω_k and the angular functions f_{ijk} can be calculated as a function of the elastic constants E_1, E_2, ν_1, ν_2 and the geometrical angles θ_1 and θ_2 [1] [2]. The stress intensity factors K_k and the regular stress term $\sigma_{ij0}(\theta)$ are proportional to the temperature difference ΔT and dependent on the elastic constants, the thermal expansion coefficients, and the geometrical angles.

Here the stress intensity factor K_k is different from that in fracture of mechanics. According to the definition in Eq.(1), K_k has the dimension of the stress and is independent of the size of the compound. They can be calculated by applying the results from the numerical method, e.g. the finite element method [3].

For the special geometry of $\theta_1 = 90°$, $\theta_2 = -90°$ the regular stress term $\sigma_{ij0}(\theta)$ was calculated analytically [2], [4]. In this paper a method of calculating $\sigma_{ij0}(\theta)$ analytically for arbitrary angles θ_1 and θ_2 will be presented.

2. Fundamental relations

For a two-dimensional problem the stresses can be analysed by means of the stress function. The following stress function was used for the problem shown in Fig.1.

$$\Phi_j(r,\theta) = \sum_{n=0}^{\infty} r^{(2-\omega_n)}\{A_{jn}\sin(\omega_n\theta) + B_{jn}\cos(\omega_n\theta)$$

$$+ C_{jn}\sin[(2-\omega_n)\theta] + D_{jn}\cos[(2-\omega_n)\theta]\} \tag{2}$$

The subscript j denotes the two materials (j = 1, 2).

The stresses are obtained from Eq.(2) :

$$\sigma_r = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} \tag{3a}$$

$$\sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2} \tag{3b}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right) \tag{3c}$$

Introducing Eq.(2) in Eq.(3) yields

$$\sigma_{jr}(r,\theta) = \sum_n r^{-\omega_n}(1-\omega_n)\{A_{jn}(2+\omega_n)\sigma + B_{jn}(2+\omega_n)\&co.$$

$$- C_{jn}(2-\omega_n)\&si2. - D_{jn}(2-\omega_n)\&co2.\} \tag{4a}$$

$$\sigma_{j\theta}(r,\theta) = \sum_n r^{-\omega_n}(2-\omega_n)(1-\omega_n)\{A_{jn}\sigma + B_{jn}\&co.$$

$$+ C_{jn}\&si2. + D_{jn}\&co2.\} \tag{4b}$$

$$\tau_{jr\theta}(r,\theta) = -\sum_n r^{-\omega_n}(1-\omega_n)\{A_{jn}\omega_n\&co. - B_{jn}\omega_n\sigma$$

$$+ C_{jn}(2-\omega_n)\&co2. - D_{jn}(2-\omega_n)\&si2.\} \tag{4c}$$

$$(j = 1, 2)$$

For plane stress the relations between the stresses and strains are given by

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) + \alpha\Delta T \tag{5a}$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) + \alpha\Delta T \tag{5b}$$

$$\gamma_{r\theta} = \frac{1}{G}\tau_{r\theta} \tag{5c}$$

with the shear modulus G, Young's modulus E, Poisson's ratio ν.

The components of the strain are related to the displacements by

$$\varepsilon_r = \frac{\partial u}{\partial r} \tag{6a}$$

$$\varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{6b}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial \upsilon}{\partial r} - \frac{\upsilon}{r} \tag{6c}$$

The displacements u_j and υ_j are obtained from Eqs.(4), (5), and (6) after vanishing of the rigid body displacements as

$$u_j(r, \theta) = \sum_n \frac{r^{(1-\omega_n)}}{E_j} \{A_{jn}[2(1-v_j) + \omega_n(1+v_j)]\sigma$$

$$+ B_{jn}[2(1-v_j) + \omega_n(1+v_j)]\&co.$$

$$-C_{jn}(1+v_j)(2-\omega_n)\&si2.$$

$$-D_{jn}(1+v_j)(2-\omega_n)\&co2.\} + r \bullet \alpha \bullet \Delta T \tag{7a}$$

$$\upsilon_j(r, \theta) = \sum_n \frac{r^{(1-\omega_n)}}{E_j} \{A_{jn}[2(1-v_j) + (2-\omega_n)(1+v_j)]\&co.$$

$$- B_{jn}[2(1-v_j) + (2-\omega_n)(1+v_j)]\sigma$$

$$-C_{jn}(1+v_j)(2-\omega_n)\&co2.$$

$$+D_{jn}(1+v_j)(2-\omega_n)\&si2.\} \tag{7b}$$

In order to determine the unknown coefficients A_{jn}, B_{jn}, C_{jn}, D_{jn}, ω_n, the following boundary conditions were used.

For $\theta = \theta_1$: $\sigma_{1\theta}(r, \theta_1) = 0$ $\tau_{1r\theta}(r, \theta_1) = 0$

For $\theta = \theta_2$: $\sigma_{2\theta}(r, \theta_2) = 0$ $\tau_{2r\theta}(r, \theta_2) = 0$

For $\theta = 0°$: $\sigma_{1\theta}(r, 0) = \sigma_{2\theta}(r, 0)$ $\tau_{1r\theta}(r, 0) = \tau_{2r\theta}(r, 0)$

$u_1(r, 0) = u_2(r, 0)$ $\upsilon_1(r, 0) = \upsilon_2(r, 0)$

These 8 relations lead to the following 8 equations :

$$\sum_n r^{-\omega_n}(2-\omega_n)(1-\omega_n)\{A_{1n}\&s/1. + B_{1n}\&c/1.$$

$$+ C_{1n}\&s2/1. + D_{1n}\&c2/1.\} = 0 \tag{8a}$$

$$\sum_n r^{-\omega_n}(\omega_n - 1)\{A_{1n}\omega_n \&c/1. - B_{1n}\omega_n \&s/1.$$

$$+ C_{1n}(2 - \omega_n)\&c2/1. - D_{1n}(2 - \omega_n)\&s2/1.\} = 0 \tag{8b}$$

$$\sum_n r^{-\omega_n}(2 - \omega_n)(1 - \omega_n)\{A_{2n}\&s/2. + B_{2n}\&c/2.$$

$$+ C_{2n}\&s2/2. + D_{2n}\&c2/2.\} = 0 \tag{8c}$$

$$\sum_n r^{-\omega_n}(\omega_n - 1)\{A_{2n}\omega_n \&c/2. - B_{2n}\omega_n \&s/2.$$

$$+ C_{2n}(2 - \omega_n)\&c2/2. - D_{2n}(2 - \omega_n)\&s2/2.\} = 0 \tag{8d}$$

$$\sum_n r^{-\omega_n}\{(2 - \omega_n)(1 - \omega_n)(B_{1n} + D_{1n}) - (2 - \omega_n)(1 - \omega_n)(B_{2n} + D_{2n})\} = 0 \tag{8e}$$

$$\sum_n r^{-\omega_n}(1 - \omega_n)\{A_{1n}\omega_n + C_{1n}(2 - \omega_n) - A_{2n}\omega_n - C_{2n}(2 - \omega_n)\} = 0 \tag{8f}$$

$$\sum_n r^{1-\omega_n}\{B_{1n}\mu[2(1 - \nu_1) + \omega_n(1 + \nu_1)] - D_{1n}\mu(1 + \nu_1)(2 - \omega_n)$$

$$- B_{2n}[2(1 - \nu_2) + \omega_n(1 + \nu_2)] + D_{2n}(1 + \nu_2)(2 - \omega_n)\}$$

$$= r\Delta T \bullet E_2(\alpha_2 - \alpha_1) \tag{8g}$$

$$\sum_n r^{1-\omega_n}\{A_{1n}\mu[2(1 - \nu_1) + (2 - \omega_n)(1 + \nu_1)] - C_{1n}\mu(1 + \nu_1)(2 - \omega_n)$$

$$- A_{2n}[2(1 - \nu_2) + (2 - \omega_n)(1 + \nu_2)] + C_{2n}(1 + \nu_2)(2 - \omega_n)\} = 0 \tag{8h}$$

where $\mu = E_2/E_1$, α_1, α_2 are the thermal expansion coefficients.

To satisfy Eq.(8g) one of the values ω_n - denoted ω_0 - has to be zero. This case will be treated in section 3. As r in Eq.(8) is arbitrary, if $\omega_n \neq 0$, we obtain

$$A_{1n}\&s/1. + B_{1n}\&c/1. + C_{1n}\&s2/1. + D_{1n}\&c2/1. = 0$$

$$A_{1n}\omega_n \&c/1. - B_{1n}\omega_n \&s/1.$$

$$+ C_{1n}(2 - \omega_n)\&c2/1. - D_{1n}(2 - \omega_n)\&s2/1. = 0$$

$$A_{2n}\&s/2. + B_{2n}\&c/2. + C_{2n}\&s2/2. + D_{2n}\&c2/2. = 0$$

$$A_{2n}\omega_n\&c/2. - B_{2n}\omega_n\&s/2.$$
$$+ C_{2n}(2-\omega_n)\&c2/2. - D_{2n}(2-\omega_n)\&s2/2. = 0$$

$$B_{1n} + D_{1n} - B_{2n} - D_{2n} = 0$$

$$A_{1n}\omega_n + C_{1n}(2-\omega_n) - A_{2n}\omega_n - C_{2n}(2-\omega_n) = 0$$

$$\mu[2(1-\nu_1) + \omega_n(1+\nu_1)]B_{1n} - \mu(1+\nu_1)(2-\omega_n)D_{1n}$$
$$- [2(1-\nu_2) + \omega_n(1+\nu_2)]B_{2n} + (1+\nu_2)(2-\omega_n)D_{2n} = 0$$

$$\mu[2(1-\nu_1) + (2-\omega_n)(1+\nu_1)]A_{1n} - \mu(1+\nu_1)(2-\omega_n)C_{1n}$$
$$- [2(1-\nu_2) + (2-\omega_n)(1+\nu_2)]A_{2n} + (1+\nu_2)(2-\omega_n)C_{2n} = 0 \qquad (9)$$

$$(n = 1,2,...)$$

Equation (9) can be rewritten

$$[A_n]\{X_n\} = \{0\} \qquad (10)$$

$[A_n]$ is the coefficient matrix in Eq.(9) and $\{X_n\} = \{A_{1n,}B_{1n,}C_{1n,}D_{1n,}A_{2n,}B_{2n,}C_{2n,}D_{2n}\}^T$. To obtain a non-trivial solution of $\{X_n\}$ in Eq.(10) $|A_n| = 0$ has to be satisfied. Its solutions are the eigenvalues of the problem. The displacements must be finite at r = 0, so that only the solution with $\omega_n \leq 1$ need to be considered. It was shown by Bogy [1] that dependent on the material properties and the angles θ_1 and θ_2, there may be one or two positive values of ω_n which lead to stress singularities. Also complex values of ω_n are possible. This will not be discussed in this paper.

3. Solution for $\omega_0 = 0$

The solution $\omega_0 = 0$ is a trivial solution of Eq.(8). For $\omega_0 = 0$ by application of normal solving method a solution of Eq.(8) exists only for special values of θ_1 and θ_2. These are $\theta_1 - \theta_2 = \pi$ or $\theta_1 - \theta_2 = 2\pi$ or $\theta_1 = \pi$ and θ_2 arbitrary or $\theta_2 = \pi$ and θ_1 arbitrary.

A general solution can be obtained by using a limit process, i.e. in Eq.(8) putting $\omega_0 \neq 0$, but $\omega_0 \to 0$. Then Eq.(8) can be written

$$\lim_{\omega_0 \to 0}\{A_{10}\sin(\omega_0\theta_1) + B_{10}\cos(\omega_0\theta_1)$$
$$+ C_{10}\sin[(2-\omega_0)\theta_1] + D_{10}\cos[(2-\omega_0)\theta_1]\} = 0$$

$$\lim_{\omega_0 \to 0}\{A_{10}\omega_0 \cos(\omega_0\theta_1) - B_{10}\omega_0 \sin(\omega_0\theta_1)$$

$$+C_{10}(2-\omega_0)\cos[(2-\omega_0)\theta_1] - D_{10}(2-\omega_0)\sin[(2-\omega_0)\theta_1]\} = 0$$

$$\lim_{\omega_0 \to 0}\{A_{20}\sin(\omega_0\theta_2) + B_{20}\cos(\omega_0\theta_2)$$

$$+C_{20}\sin[(2-\omega_0)\theta_2] + D_{20}\cos[(2-\omega_0)\theta_2]\} = 0$$

$$\lim_{\omega_0 \to 0}\{A_{20}\omega_0 \cos(\omega_0\theta_2) - B_{20}\omega_0 \sin(\omega_0\theta_2)$$

$$+ C_{20}(2-\omega_0)\cos[(2-\omega_0)\theta_2] - D_{20}(2-\omega_0)\sin[(2-\omega_0)\theta_2]\} = 0$$

$$B_{10} + D_{10} - B_{20} - D_{20} = 0$$

$$\lim_{\omega_0 \to 0}\{A_{10}\omega_0 + C_{10}(2-\omega_0) - A_{20}\omega_0 - C_{20}(2-\omega_0)\} = 0$$

$$\lim_{\omega_0 \to 0}\{\mu[2(1-\nu_1) + \omega_0(1+\nu_1)]B_{10} - \mu(1+\nu_1)(2-\omega_0)D_{10}$$

$$-[2(1-\nu_2) + \omega_0(1+\nu_2)]B_{20} + (1+\nu_2)(2-\omega_0)D_{20}\} = E_2\Delta T(\alpha_2 - \alpha_1)$$

$$\lim_{\omega_0 \to 0}\{\mu[2(1-\nu_1) + (2-\omega_0)(1+\nu_1)]A_{10} - \mu(1+\nu_1)(2-\omega_0)C_{10}$$

$$-[2(1-\nu_2) + (2-\omega_0)(1+\nu_2)]A_{20} + (1+\nu_2)(2-\omega_0)C_{20}\} = 0 \qquad (11)$$

These relations can be written

$$A_{10}\sin(\omega_0\theta_1) + B_{10}\cos(\omega_0\theta_1)$$

$$+ C_{10}\sin[(2-\omega_0)\theta_1] + D_{10}\cos[(2-\omega_0)\theta_1] = \delta$$

$$A_{10}\omega_0\cos(\omega_0\theta_1) - B_{10}\omega_0\sin(\omega_0\theta_1)$$

$$+C_{10}(2-\omega_0)\cos[(2-\omega_0)\theta_1] - D_{10}(2-\omega_0)\sin[(2-\omega_0)\theta_1] = \delta$$

$$A_{20}\sin(\omega_0\theta_2) + B_{20}\cos(\omega_0\theta_2)$$

$$+C_{20}\sin[(2-\omega_0)\theta_2] + D_{20}\cos[(2-\omega_0)\theta_2] = \delta$$

$$A_{20}\omega_0\cos(\omega_0\theta_2) - B_{20}\omega_0\sin(\omega_0\theta_2)$$

$$+ C_{20}(2-\omega_0)\cos[(2-\omega_0)\theta_2] - D_{20}(2-\omega_0)\sin[(2-\omega_0)\theta_2] = \delta$$

$$B_{10} + D_{10} - B_{20} - D_{20} = 0$$

$$A_{10}\omega_0 + C_{10}(2-\omega_0) - A_{20}\omega_0 - C_{20}(2-\omega_0) = \delta$$

$$\mu[2(1-\nu_1)+\omega_0(1+\nu_1)]B_{10}-\mu(1+\nu_1)(2-\omega_0)D_{10}$$

$$-[2(1-\nu_2)+\omega_0(1+\nu_2)]B_{20}+(1+\nu_2)(2-\omega_0)D_{20}=E_2\Delta T(\alpha_2-\alpha_1)+\delta$$

$$\mu[2(1-\nu_1)+(2-\omega_0)(1+\nu_1)]A_{10}-\mu(1+\nu_1)(2-\omega_0)C_{10}$$

$$-[2(1-\nu_2)+(2-\omega_0)(1+\nu_2)]A_{20}+(1+\nu_2)(2-\omega_0)C_{20}=\delta \tag{12}$$

where ω_0 and δ are infinitesimal.

Equation (12) can be written in matrix form

$$[A_0]\{X_0\}=\{B_0\} \tag{13}$$

where $[A_0]$ is the coefficient matrix in Eq.(12), $\{X_0\} = \{A_{10},B_{10},C_{10},D_{10},A_{20},B_{20},C_{20},D_{20}\}^T$, and $\{B_0\}=\{\delta \quad \delta \quad \delta \quad \delta \quad 0 \quad \delta \quad E_2\Delta T(\alpha_2-\alpha_1)+\delta \quad \delta\}^T$.

The coefficients $\{X_0\}$ can be solved by application of the Cramer's principle, e.g.

$$A_{10}=\frac{|\Delta_{A_{10}}|}{|A_0|}$$

$$B_{10}=\frac{|\Delta_{B_{10}}|}{|A_0|} \tag{14}$$

and corresponding relations for the other coefficients. $|\Delta_{A_{10}}|$ is the determinant obtained by replacing the first column of $|A_0|$ by $\{B_0\}$, $|\Delta_{B_{10}}|$ by replacing the second column ..., and so on.

Then the stress terms corresponding to $\omega_0=0$ are given by

$$\sigma_{jr0}(\theta)=\lim_{\substack{\omega_0\to 0\\ \delta\to 0}}\{r^{-\omega_0}(1-\omega_0)[A_{j0}(2+\omega_0)\sin(\omega_0\theta)+B_{j0}(2+\omega_0)\cos(\omega_0\theta)$$

$$-C_{j0}(2-\omega_0)\sin[(2-\omega_0)\theta]-D_{j0}(2-\omega_0)\cos[(2-\omega_0)\theta]]\} \tag{15a}$$

$$\sigma_{j\theta 0}(\theta)=\lim_{\substack{\omega_0\to 0\\ \delta\to 0}}\{r^{-\omega_0}(2-\omega_0)(1-\omega_0)[A_{j0}\sin(\omega_0\theta)+B_{j0}\cos(\omega_0\theta)$$

$$+C_{j0}\sin[(2-\omega_0)\theta]+D_{j0}\cos[(2-\omega_0)\theta]]\} \tag{15b}$$

$$\tau_{jr\theta 0}(\theta)=\lim_{\substack{\omega_0\to 0\\ \delta\to 0}}\{-r^{-\omega_0}(1-\omega_0)[A_{j0}\omega_0\cos(\omega_0\theta)-B_{j0}\omega_0\sin(\omega_0\theta)$$

$$+C_{j0}(2-\omega_0)\cos[(2-\omega_0)\theta]-D_{j0}(2-\omega_0)\sin[(2-\omega_0)\theta]]\} \tag{15c}$$

(j = 1, 2)

In Eq.(14), if $\omega_0 = 0$ and $\delta = 0$, there is

$$|A_0| = |\Delta_{B_{10}}| = |\Delta_{C_{10}}| = |\Delta_{D_{10}}| = |\Delta_{B_{20}}| = |\Delta_{C_{20}}| = |\Delta_{D_{20}}| = 0 \tag{16a}$$

$$|\Delta_{A_{10}}| = -128E_2(\alpha_2 - \alpha_1)\Delta T \sin(\theta_1)\sin(\theta_2)[\cos(\theta_1)\sin(\theta_2) - \cos(\theta_2)\sin(\theta_1)] \tag{16b}$$

$$|\Delta_{A_{20}}| = -128E_2 k(\alpha_2 - \alpha_1)\Delta T \sin(\theta_1)\sin(\theta_2)[\cos(\theta_1)\sin(\theta_2) - \cos(\theta_2)\sin(\theta_1)] \tag{16c}$$

However in Eq.(15) the coefficient A_{j0} is always combined with $\sin(\omega_0\theta)$ or $\omega_0\cos(\omega_0\theta)$. For arbitrary angles θ, if $\omega_0 = 0$ and $\delta = 0$, there are

$$|\Delta_{A_{j0}}| \sin(\omega_0\theta) = 0 \tag{17a}$$

and

$$|\Delta_{A_{j0}}| \omega_0 \cos(\omega_0\theta) = 0 \tag{17b}$$

Consequently, the l'Hospital principle can be used here and it holds

$$\overline{A}_{j0} = \lim_{\substack{\omega_0 \to 0 \\ \delta \to 0}} A_{j0}\omega_0\cos(\omega_0\theta)$$

$$= \lim_{\substack{\omega_0 \to 0 \\ \delta \to 0}} \frac{|\Delta_{A_{j0}}|\omega_0\cos(\omega_0\theta)}{|A_0|} = \frac{|\Delta_{A_{j0}}|_{\substack{\omega_0 = 0 \\ \delta = 0}}}{\frac{\partial |A_0|}{\partial \omega_0}\Big|_{\substack{\omega_0 = 0 \\ \delta = 0}}} \equiv \frac{A_j^*}{Z} \tag{18a}$$

$$\lim_{\substack{\omega_0 \to 0 \\ \delta \to 0}} A_{j0}\sin(\omega_0\theta) = \lim_{\substack{\omega_0 \to 0 \\ \delta \to 0}} \frac{|\Delta_{A_{j0}}|\sin(\omega_0\theta)}{|A_0|}$$

$$= \frac{|\Delta_{A_{j0}}|_{\substack{\omega_0 = 0 \\ \delta = 0}}}{\frac{\partial |A_0|}{\partial \omega_0}\Big|_{\substack{\omega_0 = 0 \\ \delta = 0}}}\theta = \overline{A}_{j0}\theta \tag{18b}$$

$$\overline{B}_{10} = \lim_{\substack{\omega_0 \to 0 \\ \delta \to 0}} \frac{|\Delta_{B_{10}}|}{|A_0|}$$

$$= \frac{\frac{\partial |\Delta_{B_{10}}|}{\partial \omega_0}\Big|_{\substack{\omega_0 = 0 \\ \delta = 0}}}{\frac{\partial |A_0|}{\partial \omega_0}\Big|_{\substack{\omega_0 = 0 \\ \delta = 0}}} \equiv \frac{B_1^*}{Z} \tag{18c}$$

and corresponding relations for the other coefficients. Finally, the regular stress terms $\sigma_{ij0}(\theta)$ can be calculated analytically by

$$\sigma_{jr0}(\theta) = 2\{\overline{A}_{j0}\theta + \overline{B}_{j0} - \overline{C}_{j0}\sin(2\theta) - \overline{D}_{j0}\cos(2\theta)\} \tag{19a}$$

$$\sigma_{j\theta 0}(\theta) = 2\{\overline{A}_{j0}\theta + \overline{B}_{j0} + \overline{C}_{j0}\sin(2\theta) + \overline{D}_{j0}\cos(2\theta)\} \tag{19b}$$

$$\tau_{jr\theta 0}(\theta) = -2\{\frac{1}{2}\overline{A}_{j0} + \overline{C}_{j0}\cos(2\theta) - \overline{D}_{j0}\sin(2\theta)\} \tag{19c}$$

The displacements according to $\omega_0 = 0$ are

$$u_{j0}(r, \theta) = \frac{2r}{E_j}[\overline{A}_{j0}\theta(1-\nu_j) + \overline{B}_{j0}(1-\nu_j) - \overline{C}_{j0}(1+\nu_j)\sin(2\theta)$$

$$-\overline{D}_{j0}(1+\nu_j)\cos(2\theta)] \tag{19d}$$

$$v_{j0}(r, \theta) = \frac{2r}{E_j}[-\overline{C}_{j0}(1+\nu_j)\cos(2\theta) + \overline{D}_{j0}(1+\nu_j)\sin(2\theta)] + H_{j0}r - \frac{2\overline{A}_{j0}}{E_j}r\ln(r) \tag{19e}$$

$$(j = 1, 2)$$

where H_{j0} is the unknown constant.

The coefficients $\overline{A}_{j0}$, $\overline{B}_{j0}$, $\overline{C}_{j0}$, $\overline{D}_{j0}$, can be calculated analytically by means of the following equations

$$Z = \alpha^2[F_{4s} - (\theta_1 + \theta_2)(F_{12} + F_{21}) - 3F_{4c} + 2F_{1p} - 1]$$

$$+ 2\alpha\beta[(F_{21} - \sin(2\theta_2))\theta_2 + (F_{12} - \sin(2\theta_1))\theta_1 + 2F_{4c} - 2F_{1p} + 2]$$

$$+ 2\alpha[F_{21}\theta_2 - F_{12}\theta_1 + F_{1n}]$$

$$+ 2\beta[(F_{12} - \sin(2\theta_1))\theta_1 - (F_{21} - \sin(2\theta_2))\theta_2]$$

$$+ [-F_{4s} + (\theta_1 - \theta_2)(F_{21} - F_{12}) - F_{4c} + 1] \tag{20a}$$

$$A_1^* = \frac{1}{2}q(\alpha + 1)\{(F_{21} - \sin(2\theta_2)) - (F_{12} - \sin(2\theta_1))\} \tag{20b}$$

$$B_1^* = \frac{1}{4}q\{\alpha[F_{4s} - 2(F_{21} - F_{12} + \sin(2\theta_1))\theta_1 - 2\sin(2\theta_2)\theta_2 + F_{4c} - 2F_{1p} - F_{1n} + 3]$$

$$+ [F_{4s} - 2(F_{21} - F_{12} + \sin(2\theta_1))\theta_1 + 2\sin(2\theta_2)\theta_2 + F_{4c} + F_{1n} - 1]\} \tag{20c}$$

$$C_1^* = \frac{1}{4}q\{\alpha[2F_{4s}(\theta_1 + \theta_2) + (F_{21} - \sin(2\theta_2)) + 3(F_{12} - \sin(2\theta_1))]$$

$$+ [2F_{4s}(\theta_1 - \theta_2) + (F_{21} - \sin(2\theta_2)) - (F_{12} - \sin(2\theta_1))]\} \tag{20d}$$

$$D_1^* = -\frac{1}{4} q\{\alpha[F_{4s} - 2F_{21}(\theta_1 + \theta_2) - 3F_{4c} + 2F_{1p} - F_{1n} - 1]$$

$$+ [F_{4s} + 2F_{21}(\theta_2 - \theta_1) + F_{4c} + F_{1n} - 1]\} \qquad (20e)$$

$$A_2^* = -\frac{1}{2} q(\alpha - 1)\{(F_{21} - \sin(2\theta_2)) - (F_{12} - \sin(2\theta_1))\} \qquad (20f)$$

$$B_2^* = \frac{1}{4} q\{\alpha[F_{4s} + 2(F_{21} - F_{12} - \sin(2\theta_2))\theta_2 - 2\sin(2\theta_1)\theta_1 + F_{4c} - 2F_{1p} - F_{1n} + 3]$$

$$+ [-F_{4s} - 2(F_{21} - F_{12} - \sin(2\theta_2))\theta_2 - 2\sin(2\theta_1)\theta_1 - F_{4c} + F_{1n} + 1]\} \qquad (20g)$$

$$C_2^* = \frac{1}{4} q\{\alpha[2F_{4s}(\theta_1 + \theta_2) + (F_{12} - \sin(2\theta_1)) + 3(F_{21} - \sin(2\theta_2))]$$

$$+ [2F_{4s}(\theta_1 - \theta_2) + (F_{21} - \sin(2\theta_2)) - (F_{12} - \sin(2\theta_1))]\} \qquad (20h)$$

$$D_2^* = -\frac{1}{4} q\{\alpha[F_{4s} - 2F_{12}(\theta_1 + \theta_2) - 3F_{4c} + 2F_{1p} + F_{1n} - 1]$$

$$+ [-F_{4s} + 2F_{12}(\theta_2 - \theta_1) - F_{4c} + F_{1n} + 1]\} \qquad (20i)$$

where α, β are the Dundurs parameter

$$\alpha = \frac{m_2 - km_1}{m_2 + km_1}$$

$$\beta = \frac{(m_2 - 2) - k(m_1 - 2)}{m_2 + km_1}$$

with

$$k = \frac{G_2}{G_1}$$

$$m = \begin{cases} \dfrac{4}{(1+\nu)} & \text{for plane stress} \\ 4(1-\nu) & \text{for plane strain} \end{cases}$$

and

$$q = \frac{1+\alpha}{2} E_2' \Delta T \, \Delta\alpha$$

with

$$E_2' = \begin{cases} E_2 & \text{for plane stress} \\ \dfrac{E_2}{(1-\nu_2^2)} & \text{for plane strain} \end{cases}$$

$$\Delta\alpha = \begin{cases} \alpha_1 - \alpha_2 & \text{for plane stress} \\ \alpha_1(1+\nu_1) - \alpha_2(1+\nu_2) & \text{for plane strain} \end{cases}$$

The parameters F_{4s}, F_{4c}, F_{12}, etc. are

$$F_{1n} = \cos(2\theta_2) - \cos(2\theta_1)$$

$$F_{1p} = \cos(2\theta_2) + \cos(2\theta_1)$$

$$F_{4c} = \cos(2\theta_2)\cos(2\theta_1)$$

$$F_{4s} = \sin(2\theta_2)\sin(2\theta_1)$$

$$F_{12} = \sin(2\theta_1)\cos(2\theta_2)$$

$$F_{21} = \sin(2\theta_2)\cos(2\theta_1)$$

If $\overline{B}_{j0} + \overline{D}_{j0} \neq 0$ a characteristic stress can be defined as

$$\sigma_0 = \sigma_{\theta 0}\,|_{\theta=0} = 2(\overline{B}_{j0} + \overline{D}_{j0})$$

Then the regular stress terms can be rewritten

$$\sigma_{jr0}(\theta) = \sigma_0\{\overline{A}_{j0}\theta + \overline{B}_{j0} - \overline{C}_{j0}\sin(2\theta) - \overline{D}_{j0}\cos(2\theta)\}/(\overline{B}_{10} + \overline{D}_{10}) \tag{21a}$$

$$\sigma_{j\theta 0}(\theta) = \sigma_0\{\overline{A}_{j0}\theta + \overline{B}_{j0} + \overline{C}_{j0}\sin(2\theta) + \overline{D}_{j0}\cos(2\theta)\}/(\overline{B}_{10} + \overline{D}_{10}) \tag{21b}$$

$$\tau_{jr\theta 0}(\theta) = -\sigma_0\{\frac{1}{2}\overline{A}_{j0} + \overline{C}_{j0}\cos(2\theta) - \overline{D}_{j0}\sin(2\theta)\}/(\overline{B}_{10} + \overline{D}_{10}) \tag{21c}$$

Generally, the regular stress term can be written

$$\sigma_{ij0}(r,\theta) = \sigma_0 f_{ij0}(\theta) \tag{22}$$

4. The regular stress term for special cases

4.1 $\theta_1 - \theta_2 = \pi$

In this case the coefficients are

$$Z = \{\alpha^2\&c1.[-\sin(2\theta_1)(2\theta_1 - \pi) - 2\&c1. + 2]$$

$$+ \alpha\beta(\&c1. - 1)[\sin(2\theta_1)(2\theta_1 - \pi) + 2(\&c1. - 1)]$$

$$- \alpha\sin(2\theta_1)\&c1.\pi + \beta\sin(2\theta_1)\pi(\&c1. - 1)\} \tag{23a}$$

$$A_1^* = 0 \tag{23b}$$

$$B_1^* = -\frac{1}{4} q\{\alpha[2(\&c1. - 1) + \sin(2\theta_1)(2\theta_1 - \pi)] + \pi \sin(2\theta_1)\} \tag{23c}$$

$$C_1^* = \frac{1}{4} q \sin(2\theta_1)\{\alpha[2(\&c1. - 1) + \sin(2\theta_1)(2\theta_1 - \pi)] + \pi \sin(2\theta_1)\} \tag{23d}$$

$$D_1^* = \frac{1}{4} q\&c1.\{\alpha[2(\&c1. - 1) + \sin(2\theta_1)(2\theta_1 - \pi)] + \sin(2\theta_1)\pi\} \tag{23e}$$

$$A_2^* = A_1^* \qquad B_2^* = B_1^* \qquad C_2^* = C_1^* \qquad D_2^* = D_1^* \tag{23f}$$

If $\theta_1 = -\theta_2 = 90^\circ$ there are

$$\overline{B}_1 = \overline{B}_2 = \overline{D}_1 = \overline{D}_2 = -\frac{q}{4(\alpha - 2\beta)} \tag{24a}$$

$$\overline{A}_1 = \overline{A}_2 = \overline{C}_1 = \overline{C}_2 = 0 \tag{24b}$$

and

$$\sigma_{x0} = \tau_{xy0} = 0 \tag{25a}$$

$$\sigma_{y0} = -\frac{q}{\alpha - 2\beta} \tag{25b}$$

If in Eq.(23) Z = 0, but $A_j^*, B_j^*, C_j^*, D_j^* \neq 0$, there is $\sigma_{ij0} \to \infty$. The corresponding conditions for $\sigma_{ij0} \to \infty$ are

$$\alpha = \beta(1 - \frac{1}{\cos(2\theta_1)}) \tag{26a}$$

for $\cos(2\theta_1) \neq 0$ and

$$\beta = 0 \tag{26b}$$

for $\cos(2\theta_1) = 0$.

In Fig.2 to Fig.5 σ^* is plotted in the α, β diagram for different combinations of θ_1 and θ_2, where σ^* is defined as

$$\sigma^* = -\frac{\sigma_0}{(E_1' + E_2')\Delta\alpha\Delta T} \tag{27}$$

All possible values of the Dundurs parameters α, β were considered.

4.2 $\theta_1 = \pi$ and θ_2 is arbitrary

In this case the coefficients are

$$Z = \{\alpha^2[\sin(2\theta_2)(\theta_2+\pi)+\cos(2\theta_2)-1]$$

$$-2\alpha[(\cos(2\theta_2)-1)+\sin(2\theta_2)\theta_2]+[\sin(2\theta_2)(\theta_2-\pi)+(\cos(2\theta_2)-1)]\} \qquad (28a)$$

$$B_1^* = \frac{1}{2}q\{\alpha[(\cos(2\theta_2)-1)+\sin(2\theta_2)(\theta_2+\pi)]$$

$$+[\sin(2\theta_2)(\pi-\theta_2)-(\cos(2\theta_2)-1)]\} \qquad (28b)$$

$$D_1^* = -B_1^* \qquad (28c)$$

$$A_1^* = C_1^* = A_2^* = B_2^* = C_2^* = D_2^* = 0 \qquad (28d)$$

and

$$\bar{B}_1 = \frac{q}{2(\alpha-1)} \qquad (28e)$$

where $\theta_2 \neq -\pi$. The regular stress terms are

$$\sigma_{y0} = \tau_{xy0} = 0 \qquad (29a)$$

$$\sigma_{1x0} = \frac{2q}{\alpha-1} \qquad (29b)$$

$$\sigma_{2x0} = 0 \qquad (29c)$$

For $\alpha = 1$ there is $\sigma_{ij0} \to \infty$.

If $\theta_1 = -\theta_2 = \pi$ the coeffients are

$$Z = 16\pi^2 \qquad (30a)$$

$$B_1^* = -D_1^* = -4q\pi^2 \qquad (30b)$$

$$B_2^* = -D_2^* = 4q\pi^2 \qquad (30c)$$

$$A_1^* = C_1^* = A_2^* = C_2^* = 0 \qquad (30d)$$

The regular stress terms follow

$$\sigma_{y0} = \tau_{xy0} = 0 \qquad (31a)$$

$$\sigma_{1x0} = -q \qquad (31b)$$

$$\sigma_{2x0} = q \qquad (31c)$$

In this case, the regular stress terms are independent of β. In Fig.6 σ^* is plotted versus α for $|\theta_2| \neq \pi$ and $|\theta_2| = \pi$. As in this case $B_1 + D_1 = 0$, i.e. $\sigma_{y0} = 0$, σ^* is defined as

$$\sigma^* = -\frac{\sigma_{1x0}}{(E_1' + E_2')\Delta\alpha\Delta T} \tag{32}$$

In Fig.7 the stress exponents ω_k ($\omega_k = \{Re(\lambda_n) \mid -0.5 \le Re(\lambda_n) \le 0.5\}$) are plotted versus $|\theta_2|$ for $\theta_1 = 180°$ and the material data are

$$E_1 = 280(GPa) \qquad \nu_1 = 0.26$$

$$E_2 = 72(GPa) \qquad \nu_2 = 0.30$$

It can be seen that if $|\theta_2| \neq 180°$ but close to 180°, there are three non-zero stress exponents ω_k in the range $-0.5 \le \omega_k \le 0.5$. Thus, even if $|\theta_2|$ changes from $|\theta_2| \neq 180°$ to $|\theta_2| = 180°$ there is a jump of the regular stress $\sigma_{ij0}(\theta)$ at $|\theta_2| = 180°$, the stresses are continuous.

5. Stress distribution in the near field of the singularity point

As an example, a joint with $\theta_1 = 115°$, $\theta_2 = -45°$ is considered which is shown in Fig.8. The material data are

$$E_1 = 280(GPa) \qquad \nu_1 = 0.26 \qquad \alpha_1 = 2.5 \times 10^{-6}/K$$

$$E_2 = 14000(GPa) \qquad \nu_2 = 0.30 \qquad \alpha_2 = 18.95 \times 10^{-6}/K$$

and $\Delta T = -100K$. For the given material data and plane strain the stress exponents are

$$\omega_1 = 0.0879$$

$$\omega_2 = -0.059259$$

The distribution of the regular stress term is shown in Fig.9 and the constant stress σ_0 is

$$\sigma_0 = 30611 \text{ MPa}$$

The angular functions f_{ijk} for each stress exponent are given in Fig.10.

The FEM was used to determine the stress intensity factors K_k (k = 1, 2). By application of the method given in [3], the values of K_k were obtained. For this example the stress intensity factors K_k are

$$K_1 = -10433 \text{ MPa}$$

$$K_2 = -20960 \text{ MPa}$$

From Eq.(1) the stress distributions were calculated along different lines, where N=2. The stress term σ_{ij1}, σ_{ij2}, σ_{ij0} was drawn as σ_1, σ_2, and σ_0 respectively in the following figures. In Fig.11 the stress distributions σ_θ and $\tau_{r\theta}$ for $\theta = 0$ are indicated. In Fig.12 and in Fig.13 the stress distribution σ_r for $\theta = 115^\circ$ and $\theta = -45^\circ$ is shown.

It can be seen that the regular stress term $\sigma_{ij0}(\theta)$ is very important to the stress distribution in the near field of the singularity point under conditions of thermal loading($\Delta T \neq 0$).

6. Conclusions

The stresses in the near field of the singularity point can be described analytically by

$$\sigma_{ij}(r, \theta) = \sum_{k=1}^{N} \frac{K_k}{(r/L)^{\omega_k}} f_{ijk}(\theta) + \sigma_{ij0}(\theta)$$

Under thermal loading conditions the regular stress term is very important to the stress distribution in the near field of the singularity point.

For an arbitrary geometry with angles θ_1 and θ_2 the regular stress term can be calculated analytically by the method given in this paper.

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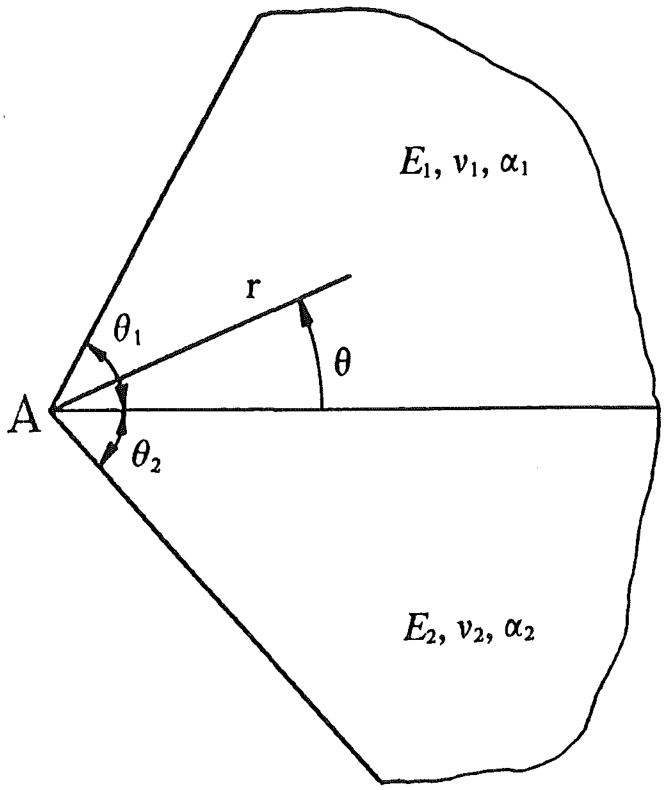

Fig.1 The general configuration of a joint of dissimilar materials.

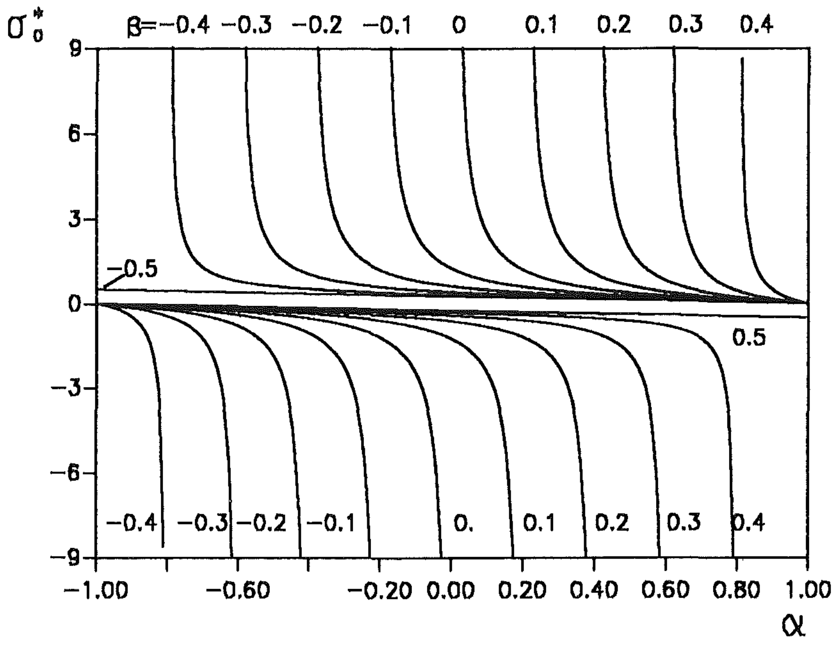

Fig.2 The σ_0^* distribution for the joint with $\theta_1 = 90°$, $\theta_2 = -90°$.

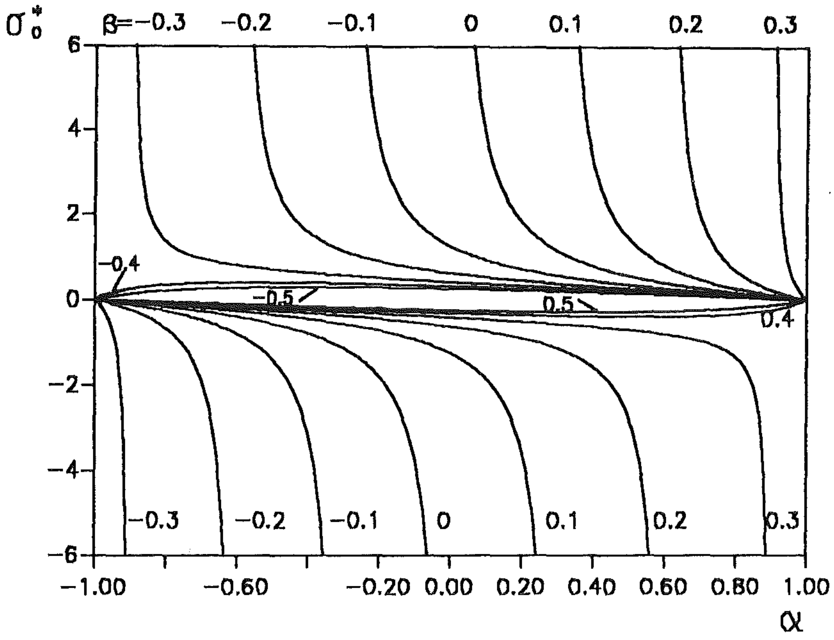

Fig.3 The σ_0^* distribution for the joint with $\theta_1 = 120^\circ$, $\theta_2 = -60^\circ$.

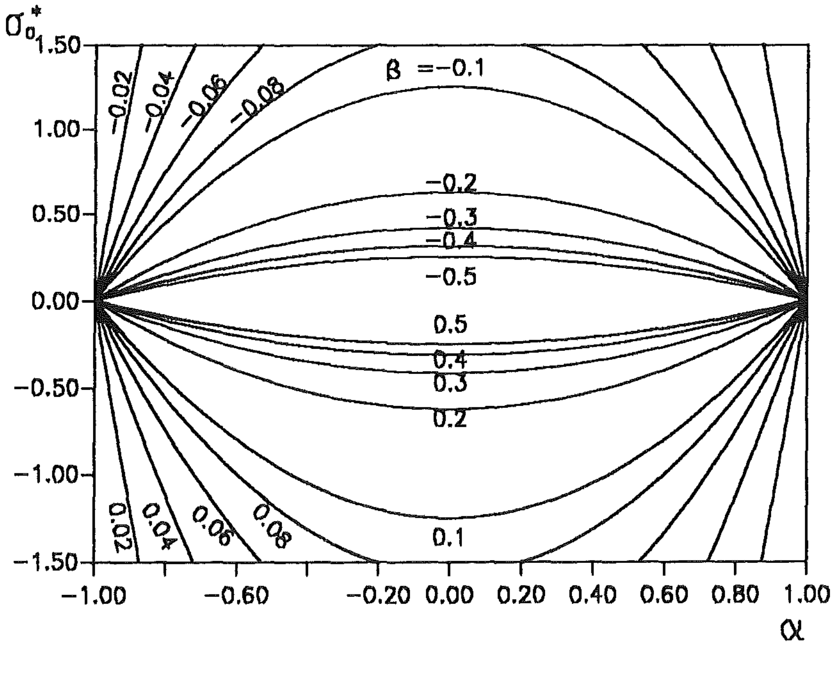

Fig.4 The σ_0^* distribution for the joint with $\theta_1 = 135^\circ$, $\theta_2 = -45^\circ$.

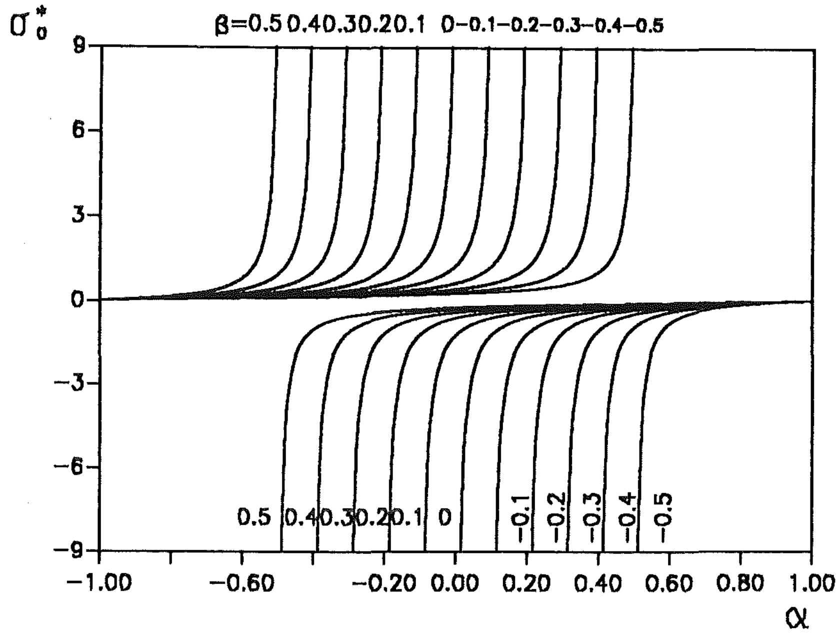

Fig.5 The σ_0^* distribution for the joint with $\theta_1 = 150^\circ$, $\theta_2 = -30^\circ$.

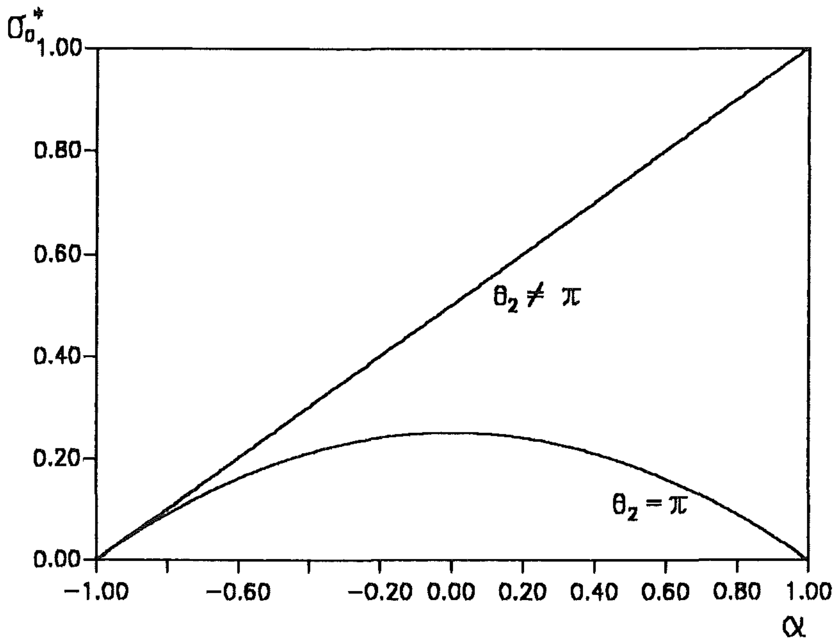

Fig.6 The σ_0^* distribution for the joint with $\theta_1 = \pi$.

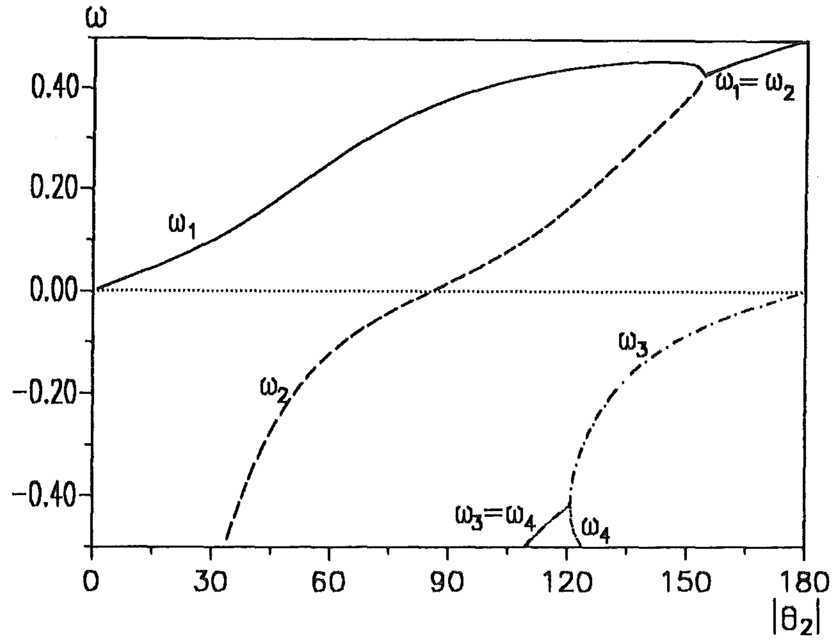

Fig.7 **The ω_k distribution for the joint with $\theta_1 = \pi$.**

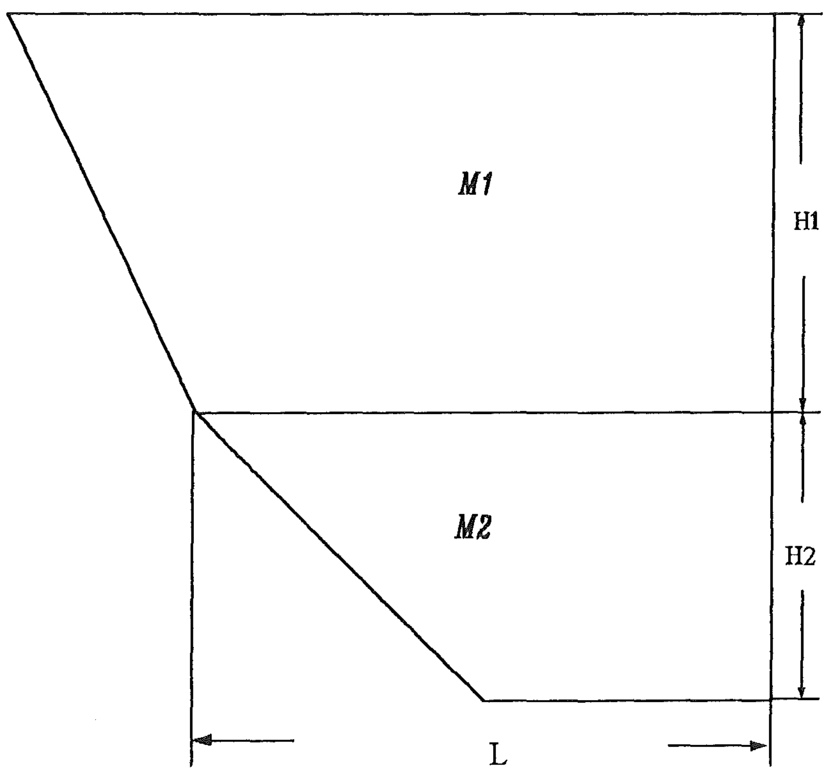

Fig.8 **The geometry of the joint with $\theta_1 = 115^\circ$, $\theta_2 = -45^\circ$.**

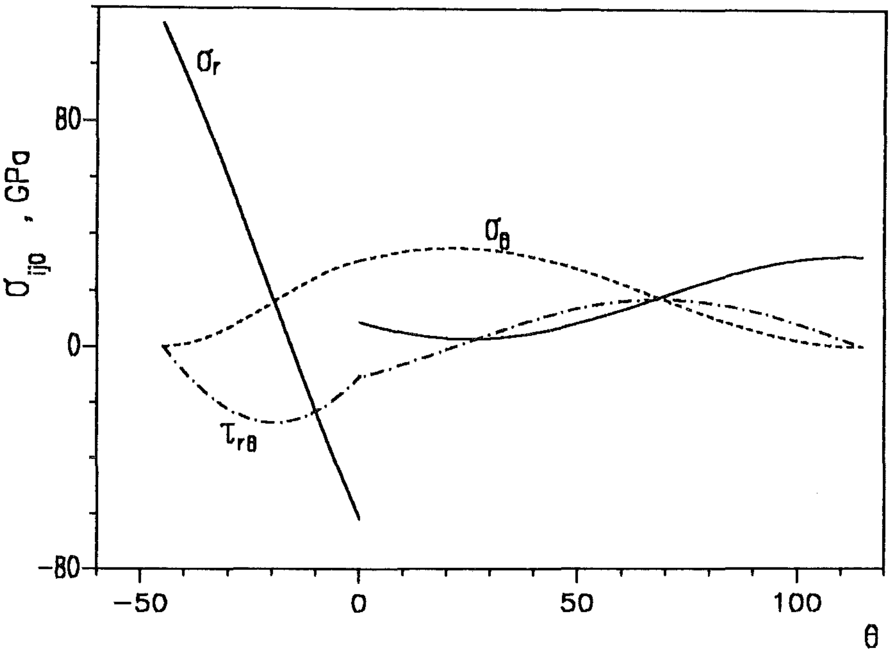

Fig.9 The regular stress term for the example.

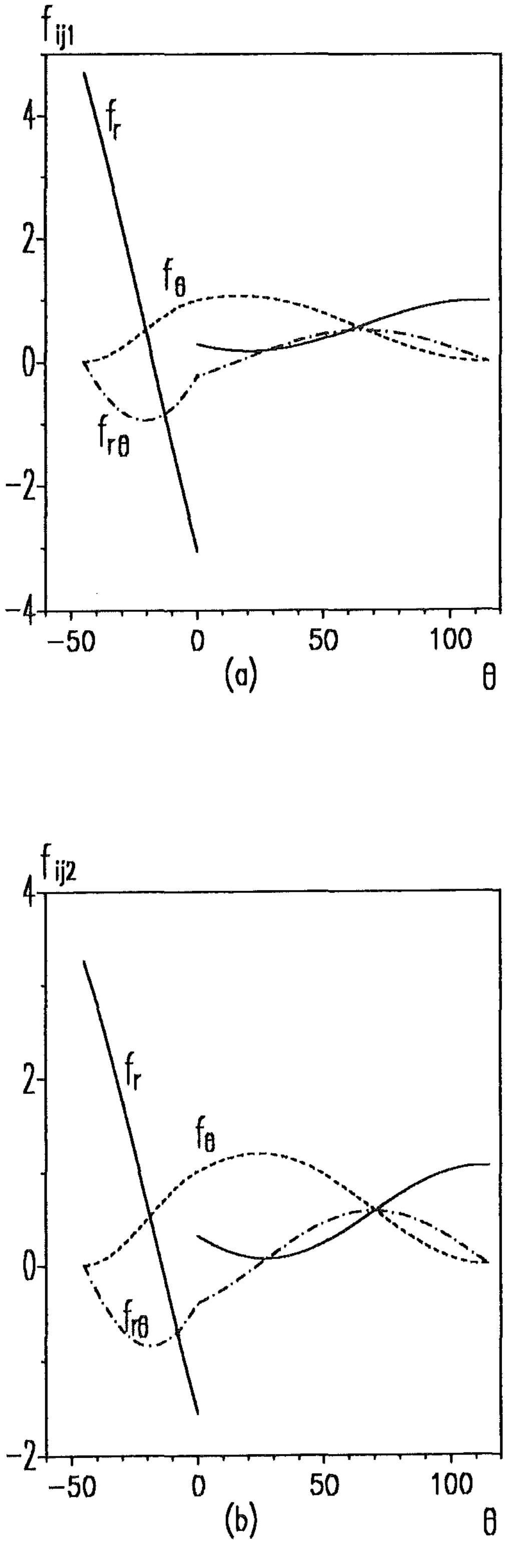

Fig.10 The angular functions for ω_1 and ω_2.

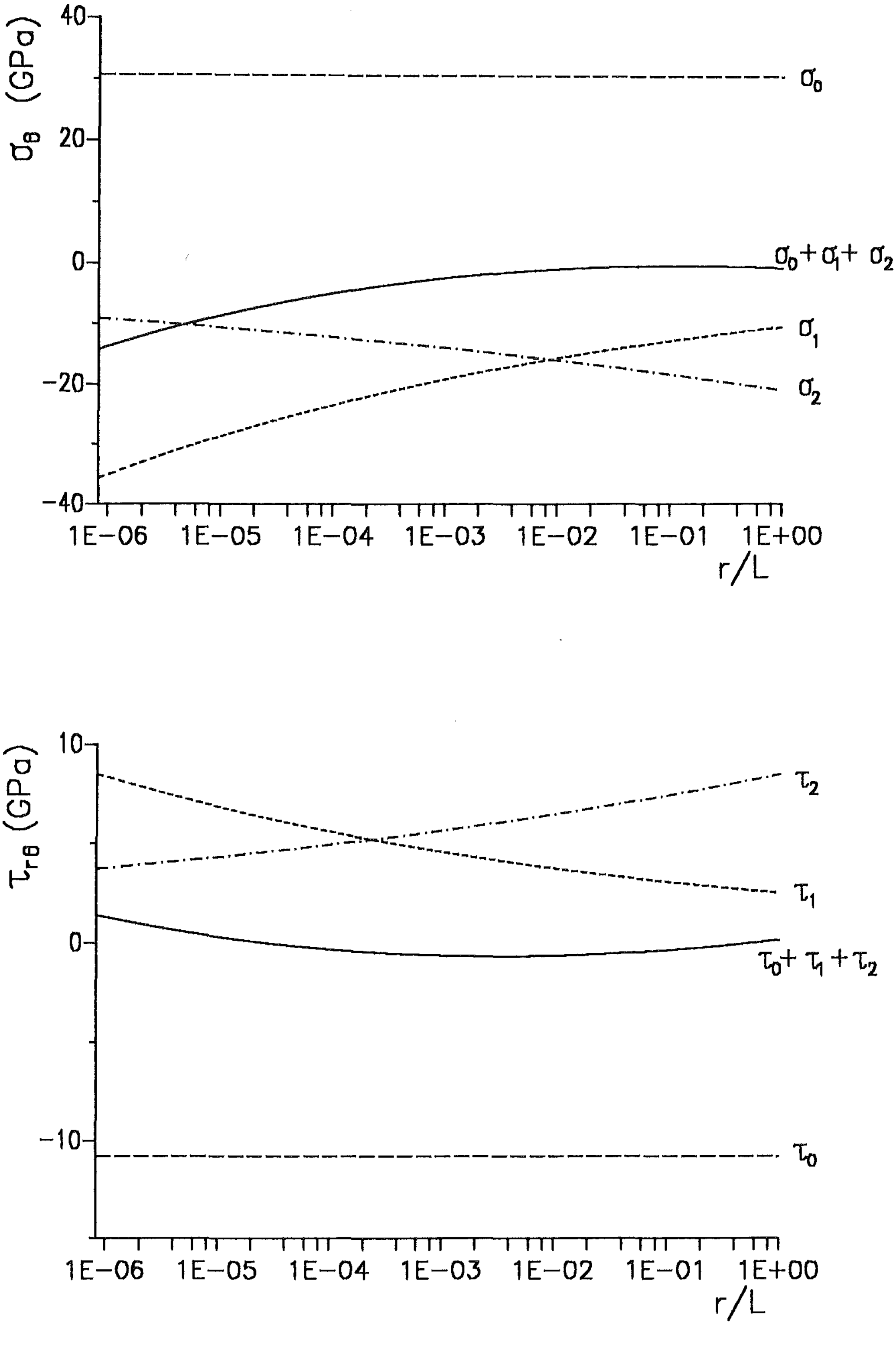

Fig.11 The stress distribution σ_θ and $\tau_{r\theta}$ for $\theta = 0$.

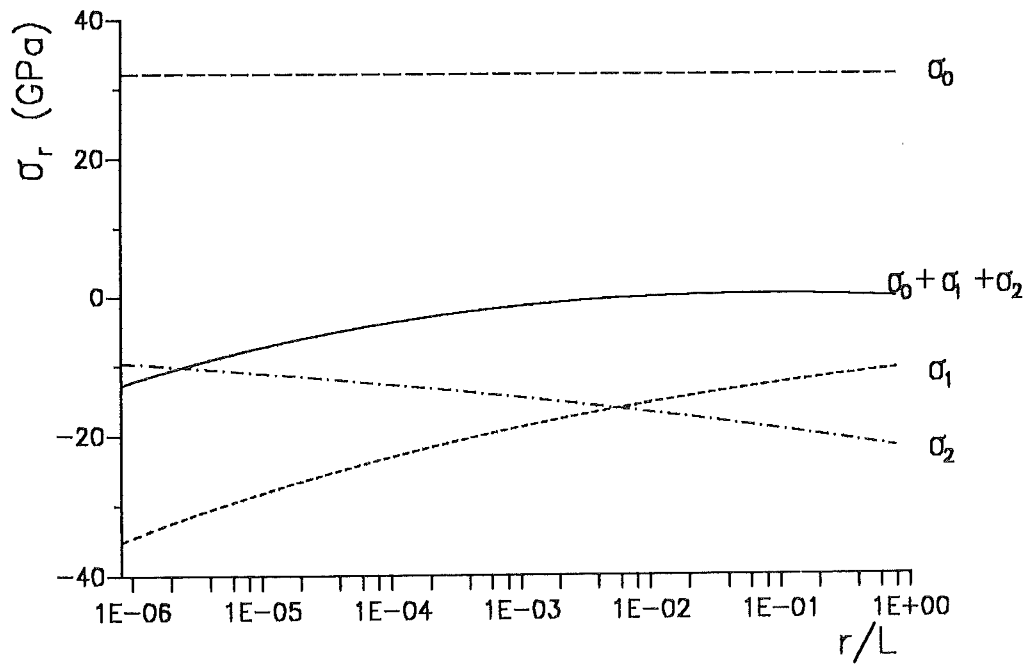

Fig.12 The stress distribution σ_r for $\theta = 115°$.

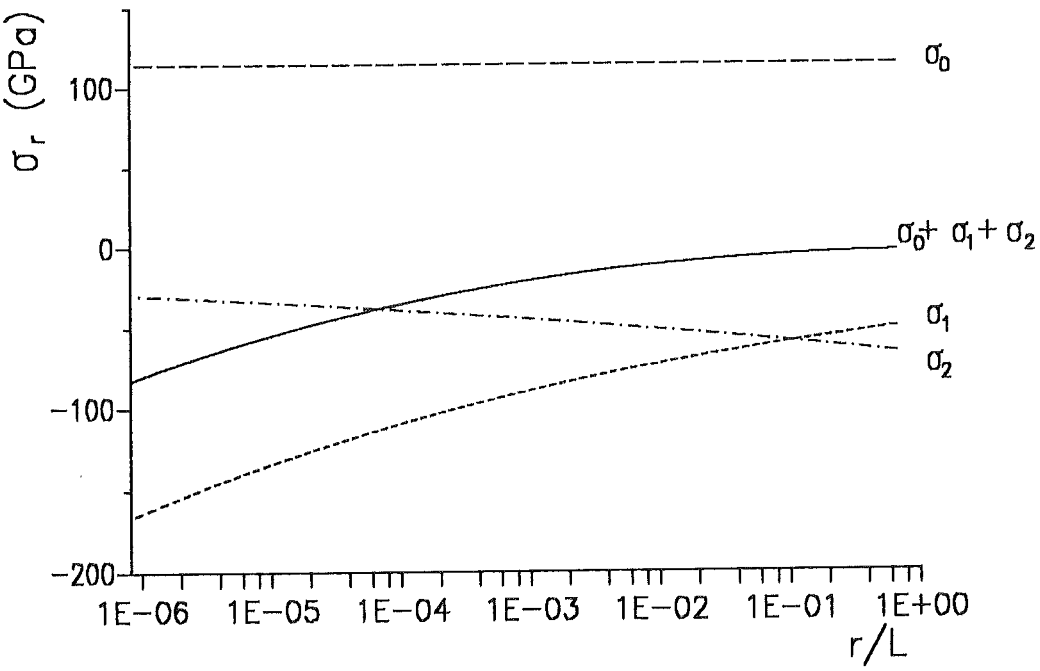

Fig.13 The stress distribution σ_r for $\theta = -45°$.