KfK 5193
September 1993

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ISSN 0303-4003

Thermodynamische Konsistenz viskoplastischer Materialmodelle, die Änderungsgeschwindigkeiten der externen Variablen in den Evolutionsgleichungen der internen Variablen enthalten

Zusammenfassung

Ziel der Studie ist es, thermodynamische Restriktionen für eine besondere Klasse Interner Variablen Modelle abzuleiten und zu untersuchen. Deren Evolutionsgleichungen bestehen aus zwei Anteilen: dem üblichen irreversiblen Anteil, der nur vom gegenwärtigen Zustand bestimmt wird, und einem reversiblen aber wegabhängigen Anteil, der linear von den Raten der externen Variablen abhängt (Evolutionsgleichungen vom "gemischten Typ"). Die thermodynamische Analyse beruht zunächst auf der klassischen Clausius-Duhem Entropieungleichung und der Coleman-Noll Schlußweise. Diese Analyse ist auf kleine Verzerrungen und Rotationen beschränkt.

Die Resultate werden spezialisiert und auf eine generelle Klasse von elastisch-viskoplastischen Materialmodellen übertragen. Anschließend werden sie angewandt auf mehrere viskoplastische Modelle vom "gemischten Typ", wie sie in der Literatur vorgeschlagen oder diskutiert wurden (Robinson et al., Krempl et al., Freed et al.) und es wird gezeigt, daß einige dieser Modelle thermodynamisch inkonsistent sind.

Es wird demonstriert, daß eine nachträgliche Analyse der thermodynamischen Konsistenz von Materialmodellen, die nicht in einem thermodynamischen Rahmen eingebettet sind, eine schwierige Aufgabe darstellen und mißliche Überraschungen zu Tage fördern kann. Deshalb sollten thermodynamische Konzepte bei der Entwicklung von Modellen von vornherein berücksichtigt werden. Dies gilt insbesondere dann, wenn die Modellentwicklung außerhalb üblicher Wege erfolgt, wie es bei Modellen mit Evolutionsgleichungen vom "gemischten Typ" der Fall ist.

Die Studie wird mit der Auswertung der erweiterten Clausius-Duhem Entropieungleichung (Müllers Konzept), bei der dem Entropiefluß eine eigene konstitutive Gleichung zugeordnet wird, abgeschlossen; auch werden die einschränkenden Bilanzgleichungen mit Hilfe der Methode der Lagrangeschen Multiplikatoren (Liu´s Vorgehensweise) explizit berücksichtigt. Diese Analyse wird für ein viskoplastisches Materialmodell vom "gemischten Typ" durchgeführt. Es wird gezeigt, daß diese Auswertung sehr viel komplizierter ist als die der klassischen Clausius-Duhem Entropieungleichung mit der Coleman-Noll Argumentation. Ihre Vorteile, nachgewiesen für andere, hier nicht diskutierte konstitutive Klassen,

rechtfertigen diese Schwierigkeit. Im vorliegenden Falle bestätigt sie die thermodynamischen Restriktionen, wie sie mit dem klassischen Vorgehen abgeleitet wurden.

Thermodynamic Consistency of Viscoplastic Material Models Involving External Variable Rates in the Evolution Equations for the Internal Variables

Summary

The objective of this study is to derive and investigate thermodynamic restrictions for a particular class of internal variable models. Their evolution equations consist of two contributions: the usual irreversible part, depending only on the present state, and a reversible but path dependent part, linear in the rates of the external variables (evolution equations of "mixed type"). In the first instance the thermodynamic analysis is based on the classical Clausius-Duhem entropy inequality and the Coleman-Noll argument. The analysis is restricted to infinitesimal strains and rotations.

The results are specialized and transferred to a general class of elastic-viscoplastic material models. Subsequently, they are applied to several viscoplastic models of "mixed type", proposed or discussed in the literature (Robinson et al., Krempl et al., Freed et al.), and it is shown that some of these models are thermodynamically inconsistent.

It is demonstrated that a subsequent analysis of the thermodynamic consistency of material models not embedded in a thermodynamic frame may be a cumbersome exercise and may produce awkward surprises. Therefore, the models should be developed observing thermodynamical concepts from the beginning. This is especially so if the development of the model is outside the usual route which is the case for evolution equations of the "mixed type".

The study is closed with the evaluation of the extended Clausius-Duhem entropy inequality (concept of Müller) where the entropy flux is governed by an assumed constitutive equation in its own right; also the constraining balance equations are explicitly accounted for by the method of Lagrange multipliers (Liu's approach). This analysis is done for a viscoplastic material model with evolution equations of the "mixed type". It is shown that this approach is much more involved than the evaluation of the classical Clausius-Duhem entropy inequality with the Coleman-Noll argument. Its merits, proven for other constitutive classes than discussed here, justifies this inconvenience. In the present case it confirms the thermodynamic restrictions derived with the classical procedure.

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1. Introduction

It is well known that the response of many materials depends on their past history. One possibility to account for the material memory is to simply state that the response, for example the stress, internal energy and heat flux at time t, are *functionals* of the entire past history, up to and including the present time t of the relevant deformation measures, temperature and temperature gradient. This representation puts aside the classical state space representation; instead it considers function spaces.

Another approach to account for the material memory is to use the concept of internal state variables. Internal variables cannot be easily determined and direct control is not possible since these internal (hidden or conceiled) variables are not directly connected to any external force variable. Thus, the form of the balance equations for linear momentum, moment of momentum and energy are the same whether or not internal variables are accounted for. Examples for interpretations of internal variables are various [1], e.g.,

- quantities related to density, motion, and arrangement of dislocations which give rise to inelastic behavior of metals
- or internal stresses or displacements in rheological models.

The internal variables supplement the external (or observable) variables, e.g. a deformation measure, the temperature and temperature gradient. Internal and external variables are supposed to completely define the present non-equilibrium state at a material point of a body. Thus, the responses like stress, internal energy or heat flux are *functions* of the external and internal variables. The history dependence of the response functions comes into play by requiring that the time evolution of the internal variables is governed by ordinary differential equations such that the instant rate of an internal variable is uniquely determined by instant values of external and internal variables. These relations are the constitutive equations for the internal variables.

For example, let α_v, $v = 1, \ldots, N$ be scalar, vectorial or tensorial internal variables, then the evolution equations are given by

$$\overset{\nabla}{\alpha}_\varsigma = f_\varsigma(F_{mn}, T, g_k, \alpha_\nu), \quad \varsigma = 1, \ldots, N \qquad (1.1)$$

where T is the temperature and F_{mn} and g_k are the deformation and temperature gradient and the symbol $(\overset{\nabla}{.})$ characterizes an objective time derivative.

The relative merits of the representation of memory effects by functionals and by hidden variables have been described by Nemat-Nasser [2]; here internal variables offer certain definite advantages. In fact, the actual development in solid mechanics - plasticity, viscoplasticity and continuum damage mechanics* - has given clear preference to this approach.

On account of these applications the intimate relationship of the notion of internal variables with the description of *dissipative* phenomena should be noted. Therefore, it is necessary to embed this phenomenological approach into a thermodynamic frame, since continuum thermodynamics imposes restrictions on the constitutive equations characterizing the material behavior; otherwise the model may be quite empty.

These thermodynamic restrictions derive from an evaluation of an entropy principle. However, the suitable formulation of the entropy principle or the entropy production inequality has raised many controversies which persist at the present time [5 - 12].** Very often it is taken to be the Clausius-Duhem entropy inequality. In its time and space integral form it reads

$$\left[\int_V \rho\eta\, dV\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\left\{\int_V \frac{\rho r}{T}\, dV - \oint_A \frac{q_k n_k}{T}\, dA\right\} dt \geq 0 \qquad (1.2)$$

where η is the specific entropy, q_k are components of the heat flux, r is the external heat supply, T the (absolute) temperature, ρ the density and V, A, n_k are the volume, the surface area of the body and the external unit vector on the surface. The usual summation convention applies to repeated subscripts.

For sufficiently smooth fields the divergence theorem and localization in space and time yield the local instant Clausius-Duhem entropy inequality

$$\sigma := \rho\dot{\eta} + \left(\frac{q_k}{T}\right)_{,k} - \frac{\rho r}{T} \geq 0 ; \qquad (1.3)$$

the comma $(\,.\,)_{,k}$ denotes partial differentiation with respect to the cartesian coordinate x_k and the dot marks material time differentiation.

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* Maugin and Drouot [4] point out various other phenomena and applications - relaxation in dielectrics and magnets, phase transition, heat conduction, superfluids etc. - were the usefulnes of the internal variable formalism has been established.

** This list of references contain only publications which have the character of review articles.

Coleman and Noll [13] introduced an argument which allowed to draw conclusions from this inequality. The Coleman-Noll argument considers a large variety of admissible thermodynamic processes, i.e., all smooth solutions of the balance equations with proper account for the assumed constitutive relations. Then a "dissipation postulate" is required to hold: "For all admissible thermodynamic processes and for every time interval and every part of the body the entropy production must be non-negative". The decisive words are the quantifiers "all" and "every". The quantifier "every" requires that the *local instant* form (1.2) should be satisfied, and the quantifier "all" requires an identical satisfaction of (1.2) which makes the postulate a restrictive condition on the constitutive assumptions. From a physical standpoint an admissible process is controlled by prescribing the body force and heat supply as well as the initial and boundary conditions. In the approach of Coleman and Noll the body force and heat supply are required to be assigneable in any way and this allows to satisfy the balance equation identically; but initial and boundary conditions are not to be considered explicitly.

The concept of Coleman and Noll, advanced by Truesdell [14] and others and applied numerous times, has been criticized for different reasons:

(a) Entropy and absolute temperature are well defined, derived quantities only in *thermostatics* (equilibrium states). Their transfer to non-equilibrium processes involves hypothetical generalizations [5 - 12, 15 - 17]. However, it should be noted that recent work on the foundations of *thermodynamics* has focussed on justifying the use of entropy and absolute temperature (and even internal energy) for non-classical materials. This task requires the formulations of versions of the Second (and First) Laws that do not mention entropy (and energy) and that are sufficiently general to be applicable to non-classical materials (entropy free thermodynamics, e.g. [6, 16, 18 - 20]).

(b) Even if the existence of entropy and absolute temperature is accepted, the form of the entropy transfer, i.e., the entropy flux q_k/T, has been questioned. This is so since the Coleman-Noll approach with the classical C-D inequality imposes severe restrictions on the dependence of constitutive functions on spatial or temporal derivatives of the primary state variables, e.g., strain and temperature [22 - 24]. Such derivatives are quantities which typically do not vanish in non-equilibrium situations. However, it is expected that the response functions "stress", "internal energy" etc. could also depend on such non-equilibrium variables beyond the equilibrium state variables and internal variables.

(c) In the Coleman-Noll approach the body force and heat supply are required to be manipulatable arbitrarily but initial and boundary conditions need not to be considered explicitly. This has been critized by Woods [25 - 27]: A fictitious body force and heat supply are required unconnected with any genuine physical sources. These distributions could be supplied only by some imponderable medium permeating the body ("phlogiston axiom") and are beyond control at interior points of the body. Green [28] did respond to this criticism but without much success [26, 27].

Some of the criticism, especially the aspect (b) and (c), has been overcome by a new concept developed by Müller (e.g. [29 - 31]):

- Entropy is still a primitive quantity assumed to exist in non-equilibrium processes. The temperature is understood as a measure how hot a particle is (empirical temperature [31, p. 1]). The instant entropy inequality has the following form:

$$\frac{d}{dt}\int_V \rho\eta \, dV - \int_V \rho s \, dV + \oint_A \phi_k n_k \, dA \geq 0 \tag{1.4}$$

where the entropy flux ϕ_k is assumed to be governed by a constitutive equation in its own right. Thus, the classical relation $\phi_k = q_k/T$ is not assumed ab initio. Note that

$\phi_k n_k < 0$: Flow of entropy into the volume V

$\phi_k n_k > 0$: Flow of entropy out of the volume V .

Further, s is the entropy supply due to prescribed heat sources r (notably $r = 0$ [31]) such that $s = r/T$.

The introduction of the entropy flux ϕ_k as an independent quantity can be motivated by different arguments: Hutter [6] shows that theories of mixture and electro-mechanical interactions as well as arguments coming from the kinetic theory of gases require an entropy flux which is not collinear with the heat flux. It is also obvious that an entropy flux as an independent quantity introduces more flexibility. However, at the same time this may generate the problem of insufficient information for a unique determination of all constitutive functions. To get around this problem Müller considers the "ideal contact" of the material model at hand and another material whose (equilibrium) properties are known.

- The evaluation of the entropy inequality requires a multitude of admissible thermodynamic processes. In Müller's approach the body force and heat supply are fixed quantities; notably, they are assumed to vanish. Consequently, the multitude of processes is produced by an arbitrary choice of initial conditions. It has been shown by the author [32] that, depending on the structure of the assumed constitutive equations, this may not be sufficient but appropriate *boundary conditions* are required too, which are arbitrarily assigneable. Now, the balance equations are not satisfied identically anymore. They are constraining conditions for a thermodynamic process and are explicitly accounted for in the evaluation of the entropy inequality. This can be done in a direkt way or in a formal way by the method of Lagrange multipliers (Liu's approach [33]).

It has been amply shown that this approach is less restrictive than a Rational Thermodynamics based on the *classical* Clausius-Duhem entropy inequality.

It should be noted that Müller's modification of the entropy flux is related to an alternative approach by Dunn and Serrin [34] as well as Dunn [35]. They preserve the structure of the classical Clausius-Duhem entropy inequality and of the balance equation of linear momentum along with the notion that the local contact force between the subparts of the body is delivered in the usual way by the Cauchy stress tensor. Thus, they modified only the energy balance equation by postulating an "interstitial work flux w_k" such that the integral energy balance is given by

$$\begin{aligned} \frac{d}{dt}\int_V \rho\left\{\epsilon + \frac{1}{2}\dot{u}_k\dot{u}_k\right\}dV = \int_V \rho\left\{b_k\dot{u}_k + r\right\}dV - \\ - \oint_A \left\{\dot{u}_k\sigma_{kl}n_l + w_k n_k - q_k n_k\right\}dA, \end{aligned} \tag{1.5}$$

where ϵ is the specific internal energy, $\dot{u}_k$ and b_k are the components of the velocity and the body force, respectively, and σ_{kl} are the components of the Cauchy stress tensor.

The flux w_k is thought of as mechanical in nature and is attributed to long range spatial interactions; it represents a rate of supply of energy across every material surface beyond the usual mechanical supply due to the power of the surface tractions. In conventional thermodynamics the surface energy flux h_k, beyond that due to the power of the surface tractions, is just the heat flux q_k and is thus inseperably linked to the surface entropy flux $\phi_k = q_k/T$. The introduction of the interstitial work flux w_k changes the surface energy flux h_k to $h_k = \phi_k - w_k$ while ϕ_k is still given by q_k/T. Therefore, mathematically this theory is similar or equivalent to a theory in which h_k is kept equal to and interpreted as the heat flux q_k but the entropy flux ϕ_k is taken to be given by $\phi_k = q_k/T + k_k$ where k_k is due to long range spatial interactions.

Coming back to internal variable models, we note that Coleman and Gurtin [36] have used the classical C-D inequality and the Coleman-Noll argument to derive thermodynamic restrictions for the constitutive equations of nonlinear materials with internal variables whose temporal evolution is governed by ordinary differential equations of the form (1.1); there the a_p's were scalar variables, such that the objective time derivative reduces to the ordinary material time derivative. An alternative derivation was given in [32] using an extended form of Müller's approach but with the classical C-D inequality (i.e. $\phi_k = q_k/T$). Nevertheless, the results were the same as those derived by Coleman and Gurtin.

The thermodynamics with internal state variables formulated by Coleman and Gurtin has been used by Perzyna et al. [37 - 40] and Kratochvil and Dillon [41 - 42] to analyse the thermodynamic restrictions imposed on some elastic-viscoplastic or elastic-plastic material models. Recently, the author [1, 43 - 44] studied the thermodynamic consistence of a thermodynamic extension of an early version of an elastic-viscoplastic material by Robinson et al. [45 - 47] on the basis of the classical C-D inequality and the Coleman-Noll argument. A peculiar aspect of this model is the discontinuity of the evolution equations which required due consideration in the analysis. Necessary and sufficient conditions on the material parameters were derived to assure compatibility with the classical C-D inequality for all processes and all states. The study closed by pointing out future theoretical work to enhance understanding of the phenomenological Robinson model. We recall here the first two items [1]:

- So far only a simplified version of the Robinson model has been analysed. The thermodynamical consistency analysis must be extended to include isotropic hardening as well as the thermomechanical path dependence of the drag stress. This last aspect is modeled such that the evolution equation for the drag stress involves the rate of the observable variable "temperature" in a non-integrable form. This requires special attention when the Clausius-Duhem entropy inequality is evaluated.

- In this analysis the Clausius-Duhem inequality has been used in its traditional form in connection with the Coleman-Noll argument. Both concepts are subject to criticism. A more advanced and less restrictive entropy principle and a physically more realistic method for exploiting the entropy principle should be used.

In fact in recent years the evolution equations for the *internal variables of viscoplastic models have been extended to include the rates of the external (observable) variables* temperature [47, 48, 49, 51, 54, 58, 59, 61] and total strain [50, 52, 53, 56] or stress [55 - 59] in a linear form to account for the thermal history dependence [47, 54] or to improve the predictions under ratchetting conditions [52, 53]. Also a nonlinear dependence on external variable rates has been proposed [60, 61]. Such a dependence requires appropriate considerations when deriving constitutive restrictions from the C-D inequality. This aspect has been studied by Lubliner [62, 63] to some extend for a general internal variable model with evolution equations of the form

$$\dot{\alpha}_{\rho} = f_{\rho}(\Lambda) + \underset{\rho}{F}_{kl}(\Lambda)\,\dot{C}_{kl} + \underset{\rho}{H}(\Lambda)\,\dot{T}\,, \quad \rho = 1, \ldots, N \qquad (1.6)$$

where Λ stands for the ordered set $\Lambda = \{ C_{mn}, T, \alpha_v \}$; C_{mn} are the components of the right Cauchy-Green tensor.

Lubliner [63] made the following observations:

- If external variable rates are not included in the evolution equations, then, for a wide class of functions f_{ρ}, the solutions $\alpha_{\rho}(t)$ are functionals of only the past histories and not the present values of the external variables.

- Without external variable rates the evolution equations are not form invariant under a transformation to a new set of internal variables $\alpha'_{\rho} = \tilde{\alpha}'_{\rho}(\Lambda)$, which may be of physical significance. The simplest form of the evolution equations, which is invariant under such a transformation is one which contains first-order rates linearly.

- The linear dependence of the evolution equations on the first-order external variable rates represents a "reversible behavior".

The latter aspect is made evident as follows. Assume that $f_\rho \equiv 0$. Then (1.6) may be written in the incremental form

$$d\alpha_\rho = \underset{\rho}{F}_{kl}(\Lambda)\, dC_{kl} + \underset{\rho}{H}(\Lambda)\, dT \tag{1.7}$$

which shows that the time scale is unimportant for the change in α_ρ. Starting off from an initial state Λ_0 and prescribing the history of the external variables $C_{kl}(t)$, $T(t)$ in some time-interval $t_0 \leq t \leq t_1$, equ. (1.7) may be integrated which yields the internal variables α_ρ as a function of t in the interval $t_0 \leq t \leq t_1$. The final value $\alpha_{\rho 1} = \alpha_\rho(t_1)$ depends on the topology of the "path" $[\, C_{kl}(t), T(t)\,]$ in the $\{\, C_{kl}, T\, \}$-space, where t is the curve or path parameter, and not on the velocity this path is run through. Starting off from the state $\Lambda_1 = \{\, C_{kl1}, T_1, \alpha_{\rho 1}\, \}$ and following the same path of the external variables in the $\{\, C_{kl}, T\, \}$-space in the reverse direction, which is always possible, one reaches the *original initial state* Λ_0. In this sense the response is reversible but path dependent. This behavior may be visualized also in the Λ-space, the complete state space: Starting off from an initial state Λ_0 and assuming that the "path" of the external variables is given in the $\{\, C_{kl}, T\, \}$-subspace and passes through the "projection" $\{\, C_{kl0}, T_0\, \}$ of Λ_0 and ends at $\{\, C_{kl1}, T_1\, \}$, then (1.7) and the path determine a trajectory $\Lambda_{(s)}$ in the Λ-space passing through Λ_0 and Λ_1; s is the curve parameter. The path of the external variables is the projection $\Pi_{(s)}$ of the trajectory $\Lambda_{(s)}$. Reversibility means that starting at Λ_1 and passing through the same projection $\Pi_{(s)}$ in the reverse direction, then the state Λ follows the same trajectory $\Lambda_{(s)}$ in the reverse direction.

Path-independence is only assured if the functions $\underset{\rho}{F}_{kl}(\Lambda)$ and $\underset{\rho}{H}(\Lambda)$ satisfy certain conditions which are derived as follows. If path independence exists, then the α_ρ's can be represented as functions of the external variables, i.e.

$$\alpha_\rho = \tilde{\alpha}_\rho\,(C_{mn}, T). \tag{1.8}$$

Thus

$$\underset{\rho}{F}_{kl}(\Lambda) = \underset{\rho}{\tilde{F}}_{kl}(C_{mn}, T), \quad \underset{\rho}{H}(\Lambda) = \underset{\rho}{\tilde{H}}(C_{mn}, T) \tag{1.9}$$

and (1.7) takes the form

$$d\alpha_{\varrho} = \underset{\varrho}{\tilde{F}}_{kl}(C_{mn},T)\,dC_{kl} + \underset{\varrho}{\tilde{H}}(C_{mn},T)\,dT \tag{1.10}$$

and the right side is required to represent a total differential. The necessary and sufficient conditions are

$$\frac{\partial \underset{\varrho}{\tilde{F}}_{kl}}{\partial C_{mn}} = \frac{\partial \underset{\varrho}{\tilde{F}}_{mn}}{\partial C_{kl}} \quad , \qquad \frac{\partial \underset{\varrho}{\tilde{F}}_{kl}}{\partial T} = \frac{\partial \underset{\varrho}{\tilde{H}}}{\partial C_{kl}} \ ; \tag{1.11}$$

in terms of the implicit functions $\underset{\varrho}{F}_{kl}$, $\underset{\varrho}{H}$ we have

$$\begin{aligned}
\frac{\partial \underset{\varrho}{\tilde{F}}_{kl}}{\partial C_{mn}} &= \frac{\partial \underset{\varrho}{F}_{kl}}{\partial C_{mn}} + \frac{\partial \underset{\varrho}{F}_{kl}}{\partial \tilde{\alpha}_{\nu}} \frac{\partial \tilde{\alpha}_{\nu}}{\partial C_{mn}} \\
\frac{\partial \underset{\varrho}{\tilde{F}}_{kl}}{\partial T} &= \frac{\partial \underset{\varrho}{F}_{kl}}{\partial T} + \frac{\partial \underset{\varrho}{F}_{kl}}{\partial \tilde{\alpha}_{\nu}} \frac{\partial \tilde{\alpha}_{\nu}}{\partial T} \\
\frac{\partial \underset{\varrho}{\tilde{H}}}{\partial C_{mn}} &= \frac{\partial \underset{\varrho}{H}}{\partial C_{mn}} + \frac{\partial \underset{\varrho}{H}}{\partial \tilde{\alpha}_{\nu}} \frac{\partial \tilde{\alpha}_{\nu}}{\partial C_{mn}}
\end{aligned} \tag{1.12}$$

From (1.8) and (1.10) we obtain

$$\begin{aligned}
d\alpha_{\varrho} &= \frac{\partial \tilde{\alpha}_{\varrho}}{\partial C_{mn}}\,dC_{mn} + \frac{\partial \tilde{\alpha}_{\varrho}}{\partial T}\,dT \\
&= \underset{\varrho}{\tilde{F}}_{mn}\,dC_{mn} + \underset{\varrho}{\tilde{H}}\,dT
\end{aligned}$$

so that

$$\frac{\partial \tilde{\alpha}_{\varrho}}{\partial C_{mn}} = \underset{\varrho}{\tilde{F}}_{mn} \quad , \qquad \frac{\partial \tilde{\alpha}_{\varrho}}{\partial T} = \underset{\varrho}{\tilde{H}} \ . \tag{1.13}$$

Finally, with (1.12), (1.13) and (1.9) the integrability conditions (1.11) read in terms of the functions $\underset{\varrho}{F}_{kl}$ and $\underset{\varrho}{H}$

$$\frac{\partial \underset{\varrho}{F}_{kl}}{\partial C_{mn}} + \frac{\partial \underset{\varrho}{F}_{kl}}{\partial \alpha_{\nu}} \underset{\nu}{F}_{mn} = \frac{\partial \underset{\varrho}{F}_{mn}}{\partial C_{kl}} + \frac{\partial \underset{\varrho}{F}_{mn}}{\partial \alpha_{\nu}} \underset{\nu}{F}_{kl}$$

$$\frac{\partial \underset{\varrho}{F}_{kl}}{\partial T} + \frac{\partial \underset{\varrho}{F}_{kl}}{\partial \alpha_{\nu}} \underset{\nu}{H} = \frac{\partial \underset{\varrho}{H}}{\partial C_{kl}} + \frac{\partial \underset{\varrho}{H}}{\partial \alpha_{\nu}} \underset{\nu}{F}_{kl}$$

or

$$\frac{\partial F_{\rho kl}}{\partial C_{mn}} - \frac{\partial F_{\rho mn}}{\partial C_{kl}} - \left[F_{\nu kl} \frac{\partial F_{\rho mn}}{\partial \alpha_\nu} - F_{\nu mn} \frac{\partial F_{\rho kl}}{\partial \alpha_\nu} \right] = 0$$

$$\frac{\partial F_{\rho kl}}{\partial T} - \frac{\partial H_\rho}{\partial C_{kl}} - \left[F_{\nu kl} \frac{\partial H_\rho}{\partial \alpha_\nu} - H_\nu \frac{\partial F_{\rho kl}}{\partial \alpha_\nu} \right] = 0. \tag{1.14}$$

This is the desired set of conditions.

Consequently, at every instant the internal variables are determined by the instantaneous external variables. In this case the α_ρ's lose their character as internal variables.

The first term f_ρ in (1.6) represents irreversible behavior. Thus dropping the rate terms in (1.6) we get

$$\overset{\circ}{\alpha}_\rho = f_\rho(\Lambda). \tag{1.15}$$

At any state Λ this relation uniquely defines the rates $\dot{\alpha}_\rho$. The segment of a trajectory is given by

$$dC_{kl} = \overset{\circ}{C}_{kl}(t)\,dt\,, \quad dT = \overset{\circ}{T}(t)\,dt\,, \quad d\alpha_\rho = f_\rho(\Lambda(t))\,dt\,.$$

Thus, reversing the history of the external variables, i.e., reversing the sign of the rate $\overset{\circ}{C}_{kl}$ and $\overset{\circ}{T}$, changes the sign of the increments dC_k and dT but the $d\alpha_\rho$'s are not affected. Therefore, a different segment corresponding to a different trajectory results; this represents an irreversible response.

These observations justify to call the evolution equations (1.6) to be of "mixed type" (Lehmann [64]).

The purpose of the present study is to derive and discuss thermodynamic restrictions for a class of general internal variable models with evolution equations which contain the rates of external variables total strain or stress, temperature and possibly temperature gradient linearly. It should be noted that the analysis is restricted to infinitesimal strains and rotations. The study is presented in three major sections 2, 3, 4.

In section 2 the thermodynamic approach is based on the classical Clausius-Duhem entropy inequality and the Coleman-Noll argument. For a general internal variable model of "mixed type", where the strain is one of the independent external variables, thermodynamic restrictions are derived. It is shown* that the classical potential relations for the stress and energy, involving the Helmholtz function or free energy, do not hold any more. Following Lubliner [63], explicit conditions are developed which assure the reduceability of the evolution equations of mixed type to the purely irreversible type; this implies the transformation of the internal variables. As a consequence the classical potential relations hold again.

Alternatively, evolution equations are studied which involve the stress as an independent external variable instead of the strain. This reflects the common formulation in plasticity and viscoplasticity. Then the satisfaction of the Clausius-Duhem entropy inequality for arbitrary thermomechanical processes may pose a problem since the arbitrary choice of a stress field may not be associated with a compatible strain field. Appropriate conditions are derived which assure that this variable choice is acceptable. Also the conditions for the reduceability of the evolution equations are given. For some special evolution equations of mixed type explicit transformation functions for the internal variables are determined.

Also, the conditions for the thermodynamic consistency of assumed strain-stress relations, as proposed in purely mechanical theories, are studied.

Finally, the internal variable model is extended to represent a class of elastic-viscoplastic materials. The total strain is additively composed of the thermoelastic and viscoplastic strain. The evolution equation for the viscoplastic strain is solely determined by an irreversible contribution, whereas the evolution equations for the internal variables, which control, e.g., the hardening, are of the "mixed type". The irreversible part in the evolution equations may change according to switch conditions, which define, e.g., an elastic range. However, these conditions should depend only on the independent state variables; an influence of rates of external variables is excluded. The thermodynamic results obtained above are translated to this case.

In section 3 the results obtained are applied to different constitutive models, the model of Robinson et al. [47] and Krempl et al. [52] and models discussed by Freed, Chaboche and Walker [57, 82]. The first two models were proposed without a thermodynamic embedding. Their evolution equations of the in-

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* This result has been derived preveously, e.g. [62, 63]

ternal variables are of the mixed type: In the Robinson model the rate of temperature is involved and the model of Krempl et al. involves the rate of total strain or, equivalently, the stress rate. The two models discussed by Freed et al. in a thermodynamic frame involve the stress rate in the evolution equations. These assumptions require an analysis of the thermodynamic consistency of these models. With respect to the Robinson model this analysis supplements the previous work [1].

Finally, in section 4 the extended Clausius-Duhem entropy inequality, using the concept of Müller and the method of Lagrange multipliers (Liu's approach) to take into account explicitly the constraining balance equations, is used to derive the thermodynamic restrictions of an elastic-viscoplastic material model with evolution equations of mixed type. This analysis is done to obtain further support for the general results derived in section 2.3.

2. Thermodynamics on the Basis of the Classical Clausius-Duhem Entropy Inequality and the Coleman-Noll Argument

2.1 Strain as an Independent External Variable

2.1.1 Derivation of Thermodynamic Restrictions for a General Internal Variable Model

In the following a general internal variable model is considered which involves the rates of the external variables, i.e., $\dot{\varepsilon}_{kl}$, $\dot{T}$, $\dot{g}_k$ such that

$$\left.\begin{aligned}
\sigma_{kl} &= \hat{\sigma}_{kl}\,(\varepsilon_{mn},\, T,\, g_k\,,\, \dot{\varepsilon}_{kl}\,,\, \dot{T},\, \dot{g}_k\,, \alpha_\gamma) && \text{stress}\\
q_k &= \hat{q}_k\;(\text{---------- '' ----------}) && \text{heat flux}\\
\epsilon &= \hat{\epsilon}\;\;(\text{---------- '' ----------}) && \text{internal energy}\\
\eta &= \hat{\eta}\;\;(\text{---------- '' ----------}) && \text{entropy}
\end{aligned}\right\} \quad (2.1)$$

and the evolution equations for the internal variables are given by

$$\dot{\alpha}_\gamma = A_\gamma\,(\varepsilon_{mn},\, T,\, g_k,\, \dot{\varepsilon}_{kl}\,,\, \dot{T},\, \dot{g}_k\,,\, \alpha_\gamma)\,,\; \gamma = 1,\ldots,N. \quad (2.2)$$

If the internal variables α_γ are not present in the model such that (2.2) is skipped and (2.1) does not involve the α_γ's, the model belongs to the "materials of differential type" [66].

In the later course of the development a special case is primarily considered where (2.1) does not depend on the rates and (2.2) involves the rates only linearly. Thus

$$\left.\begin{aligned}
\sigma_{kl} &= \hat{\sigma}_{kl}\,(\varepsilon_{mn},\, T,\, g_k\,,\, \alpha_\gamma)\\
q_k &= \hat{q}_k\;(\text{------ '' ------})\\
\epsilon &= \hat{\epsilon}\;\;(\text{------ '' ------})\\
\eta &= \hat{\eta}\;\;(\text{------ '' ------})
\end{aligned}\right\} \quad (2.3)$$

and

$$\begin{aligned}
\dot{\alpha}_\gamma &= \hat{f}_\gamma\,(\varepsilon_{mn},\, T,\, g_k,\, \alpha_\nu)\\
&\quad + \hat{E}_{\gamma\, mn}\,(\text{------ '' ------})\,\dot{\varepsilon}_{mn} \;\; +
\end{aligned}$$

$$\left.\begin{aligned} &\vdots \\ &+ \underset{\gamma}{\hat{H}}(\varepsilon_{mn}, T, g_k, \alpha_\nu)\,\dot{T} \\ &+ \underset{\gamma}{\hat{G}}_l(\text{———} \,''\, \text{———})\,\dot{g}_l . \end{aligned}\right\} \quad (2.4)$$

Several of the proposed viscoplastic material models are special cases of this restricted class, and their thermodynamic consistency will be a primary subject of this report. However, the thermodynamic consistency analysis will initially be done assuming (2.1) combined with the evolution equations (2.4); this somewhat more general set of relations allows to point out some important conditions.

For sufficiently smooth processes the local dissipation inequality reads (e.g. [1])

$$\gamma := \sigma_{kl}\,\dot{\varepsilon}_{kl} - \rho\,\dot{\psi} - \rho\eta\,\dot{T} - \frac{q_k g_k}{T} \geq 0 . \tag{2.5}$$

Here the free energy is given by

$$\psi = \epsilon - T\eta \tag{2.6}$$

and

$$\gamma = T\sigma \tag{2.7}$$

denotes the dissipation where σ is the entropy production rate. With (2.1) the free energy is a function of the variables

$$\Pi = \{\varepsilon_{mn}, T, g_k, \alpha_\nu, \dot{\varepsilon}_{mn}, \dot{T}, \dot{g}_k\}, \tag{2.8}$$

i.e.,

$$\psi = \hat{\psi}(\Pi). \tag{2.9}$$

Its material time derivative is given by

$$\begin{aligned} \dot{\psi} = {} & \frac{\partial\hat{\psi}}{\partial\varepsilon_{mn}}\dot{\varepsilon}_{mn} + \frac{\partial\hat{\psi}}{\partial T}\dot{T} + \frac{\partial\hat{\psi}}{\partial g_k}\dot{g}_k \\ & + \frac{\partial\hat{\psi}}{\partial\dot{\varepsilon}_{mn}}\ddot{\varepsilon}_{mn} + \frac{\partial\hat{\psi}}{\partial\dot{T}}\ddot{T} + \frac{\partial\hat{\psi}}{\partial\dot{g}_k}\ddot{g}_k + \frac{\partial\hat{\psi}}{\partial\alpha_\nu}\dot{\alpha}_\nu . \end{aligned} \tag{2.10}$$

With (2.10) and (2.4) inequality (2.5) may be put into the following form

$$\begin{aligned}
\gamma = & \Big[\sigma_{mn} - \rho_0 \Big(\frac{\partial\hat\psi}{\partial\varepsilon_{mn}} + \frac{\partial\hat\psi}{\partial\alpha_v}\hat E_{vmn}\Big)\Big]_{sym}\dot\varepsilon_{mn} \\
& -\rho\Big[\frac{\partial\hat\psi}{\partial T} + \frac{\partial\hat\psi}{\partial\alpha_v}\hat H_v + \hat\eta\Big]\dot T \\
& -\rho\Big[\frac{\partial\hat\psi}{\partial g_k} + \frac{\partial\hat\psi}{\partial\alpha_v}\hat G_{vk}\Big]\dot g_k \qquad (2.11) \\
& -\rho\Big\{\Big(\frac{\partial\hat\psi}{\partial\dot\varepsilon_{mn}}\Big)_{sym}\ddot\varepsilon_{mn} + \frac{\partial\hat\psi}{\partial\dot T}\ddot T + \frac{\partial\hat\psi}{\partial\dot g_k}\ddot g_k\Big\} \\
& -\rho\,\frac{\partial\hat\psi}{\partial\alpha_v}\hat f_v - \frac{1}{T}\hat q_k g_k \qquad \geq 0 .
\end{aligned}$$

In this expression all hatched quantities represent constitutive functions depending on some or all of the variables (2.8). Therefore, the dissipation γ is a function of the quantities

$$\Pi^* = \{\varepsilon_{kl}, T, g_k, \alpha_v, \dot\varepsilon_{kl}, \dot T, \dot g_k, \ddot\varepsilon_{kl}, \ddot T, \ddot g_k\}, \qquad (2.12)$$

i.e.

$$\gamma = \hat\gamma(\Pi^*) \qquad (2.13)$$

if the constitutive functions could be arbitrarily chosen. It should be noted that $\hat\gamma$ depends *linearly* on the accelerations $\ddot\varepsilon_{kl}$, $\ddot T$, $\ddot g_k$ but generally *nonlinearly* on the varibles Π , equ. (2.8).

Within the frame of Rational Thermodynamics it is required that the dissipation inequality (2.11) is identically satisfied for *all* smooth admissible thermodynamic processes; any smooth solution of the balance equations (momentum, energy*) together with the assumed constitutive relations (2.1) and (2.4) represents such a process. The Coleman-Noll argument [13, 32] implies that the externally prescribed body forces and heat supply may be arbitrarily adjusted such that the balance equations can always be satisfied. As a consequence the quantities Π*

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* The mass balance equation need not to be considered since the analysis is restricted to infinitesimal strains and rotations.

are allowed to take arbitrary values locally, that is at a material point.

Consider processes where at some instant the values of the variables Π are the same but which differ in the acceleration $\ddot{\varepsilon}_{kl}$, $\ddot{T}$, $\ddot{g}_k$ instantly. Thus, in the expression for γ all quantities are fixed except the accelerations. Since these may take arbitrary values and are only *linearly* involved, their appropriate coefficients are required to vanish identically; otherwise the dissipation inequality could be violated in some processes. Thus,

$$\left(\frac{\partial \hat{\psi}}{\partial \dot{\varepsilon}_{mn}}\right)_{sym} = 0 \; , \quad \frac{\partial \hat{\psi}}{\partial \dot{T}} = 0 \; , \quad \frac{\partial \hat{\psi}}{\partial \dot{g}_k} = 0 \tag{2.14}$$

that is the *free energy is independent of the rates*

$$R = \{ \dot{\varepsilon}_{kl} , \dot{T} , \dot{g}_k \} \tag{2.15}$$

i.e.,

$$\psi = \hat{\psi}(\varepsilon_{kl}, T, g_k, \alpha_\nu) . \tag{2.16}$$

Obviously, the *free energy is still allowed to depend on the temperature gradient* g_k. The dissipation inequality reduces to

$$\begin{aligned} \gamma = {} & \left[\hat{\sigma}_{mn} - \rho\left(\frac{\partial \hat{\psi}}{\partial \varepsilon_{mn}} + \frac{\partial \hat{\psi}}{\partial \alpha_\nu} \hat{E}_{\nu mn}\right)_{sym}\right] \dot{\varepsilon}_{mn} \\ & - \rho \left[\frac{\partial \hat{\psi}}{\partial T} + \frac{\partial \hat{\psi}}{\partial \alpha_\nu} \hat{H}_\nu + \hat{\eta}\right] \dot{T} \\ & - \rho \left[\frac{\partial \hat{\psi}}{\partial g_k} + \frac{\partial \hat{\psi}}{\partial \alpha_\nu} \hat{G}_{\nu k}\right] \dot{g}_k \\ & - \rho \frac{\partial \hat{\psi}}{\partial \alpha_\nu} \hat{f}_\nu - \frac{1}{T} \hat{q}_k g_k \qquad \geq 0 \end{aligned} \tag{2.17}$$

where

$$\gamma = \tilde{\gamma}(\Pi) \tag{2.18}$$

is generally a nonlinear function of the arguments Π. Of course, the dissipation inequality (2.17) is required to be satisfied for all sets Π but the derivations of further constitutive restrictions is rather difficult, if not impossible, if further assumptions on the mathematical structure of the constitutive functions are not made.

However, it is still possible to derive some consequences for the constitutive functions at an *equilibrium state* presumably some (fairly) general conditions are met. Following [1, 36], an equilibrium state is defined by

$$\left.\begin{aligned} &\dot{\varepsilon}_{kl} = 0 , \qquad \dot{T} = 0 \quad , \qquad \dot{g}_k = 0 \\ &\text{and} \\ &\qquad\qquad g_k = 0 \end{aligned}\right\} \tag{2.19}$$

as well as

$$\hat{f}_\gamma (\varepsilon_{mn} , T , 0 , \alpha_\nu) = 0 \tag{2.20}$$

such that $\dot{\alpha}_\gamma$ vanishes. Conditions (2.20) are satisfied by equilibrium values

$$\varepsilon_{mn}^E \; , \; T^E \; , \; \alpha_\nu^E \; .$$

With (2.13, 2.20) the dissipation

$$\gamma = \tilde{\gamma} (\Pi^E)$$

where

$$\Pi^E = \{ \varepsilon_{kl}^E , T^E , 0 , \alpha_\nu^E , 0 , 0 , 0 \} \tag{2.21}$$

vanishes at an equilibrium state.

We assume that the material model under consideration is such that the dissipation is positive except at an equilibrium state. Assuming further *continuity* and *differentiability* of the constitutive functions such that their properties apply also to $\tilde{\gamma}$ (Π), the conditions

$$\tilde{\gamma} (\Pi^E) = 0 \quad , \qquad \tilde{\gamma} (\Pi \neq \Pi^E) > 0 \tag{2.22}$$

imply that the first variation of $\tilde{\gamma}$ vanishes at an equilibrium state and the second variation is positive definite, i.e.,

$$(\delta \tilde{\gamma})_E = 0 \quad , \quad (\delta^2 \tilde{\gamma})_E > 0 \, . \tag{2.23}$$

The left hand side of $(2.23)_1$ is given by

$$[\delta\tilde{\gamma}]_E = \Big[\frac{\partial\tilde{\gamma}}{\partial\varepsilon_{mn}}\delta\varepsilon_{mn} + \frac{\partial\tilde{\gamma}}{\partial T}\delta T + \frac{\partial\tilde{\gamma}}{\partial g_k}\delta g_k + \frac{\partial\tilde{\gamma}}{\partial\alpha_\nu}\delta\alpha_\nu$$

$$+ \frac{\partial\hat{\gamma}}{\partial\dot{\varepsilon}_{mn}}\delta\dot{\varepsilon}_{mn} + \frac{\partial\tilde{\gamma}}{\partial\dot{T}}\delta\dot{T} + \frac{\partial\tilde{\gamma}}{\partial\dot{g}_k}\delta\dot{g}_k \Big]_E$$

$$= -\varrho\Big[\frac{\partial\hat{\psi}}{\partial\alpha_\nu}\;\frac{\partial\hat{f}_\nu}{\partial\varepsilon_{mn}}\Big]_{E\atop sym}\delta\varepsilon_{mn} - \varrho\Big[\frac{\partial\hat{\psi}}{\partial\alpha_\nu}\;\frac{\partial\hat{f}_\nu}{\partial T}\Big]_E\delta T$$

$$-\varrho\Big[\frac{\partial\hat{\psi}}{\partial\alpha_\nu}\;\frac{\partial\hat{f}_\nu}{\partial g_k} + \frac{1}{\varrho T}\hat{q}_k\Big]_E\delta g_k$$

$$-\varrho\Big[\frac{\partial\hat{\psi}}{\partial\alpha_\nu}\;\frac{\partial\hat{f}_\nu}{\partial\alpha_\gamma}\Big]_E\delta\alpha_\gamma$$

$$+\Big[\hat{\sigma}_{mn} - \varrho\Big(\frac{\partial\hat{\psi}}{\partial\varepsilon_{mn}} + \frac{\partial\hat{\psi}}{\partial\alpha_\nu}\hat{E}_{\nu mn}\Big)_{sym}\Big]_E\delta\dot{\varepsilon}_{mn}$$

$$-\varrho\Big[\frac{\partial\hat{\psi}}{\partial T} + \frac{\partial\hat{\psi}}{\partial\alpha_\nu}\hat{H}_\nu + \hat{\eta}\Big]_E\delta\dot{T}$$

$$-\varrho\Big[\frac{\partial\hat{\psi}}{\partial g_k} + \frac{\partial\hat{\psi}}{\partial\alpha_\nu}\hat{G}_{\nu k}\Big]\delta\dot{g}_k$$

Thus, the condition

$$[\delta\tilde{\gamma}]_E = 0 \qquad \forall\; \delta\varepsilon_{mn}, \delta T, \delta g_k, \delta\alpha_\gamma, \delta\dot{\varepsilon}_{mn}, \delta\dot{T}, \delta\dot{g}_k$$

yields

$$\left.\begin{aligned}
&\Big[\frac{\partial\hat{\psi}}{\partial\alpha_\nu}\,\frac{\partial\hat{f}_\nu}{\partial\varepsilon_{mn}}\Big]_{\substack{E\\ sym}} &&= 0\\
&\Big[\frac{\partial\hat{\psi}}{\partial\alpha_\nu}\,\frac{\partial\hat{f}_\nu}{\partial T}\Big]_{E} &&= 0\\
&\Big[\frac{\partial\hat{\psi}}{\partial\alpha_\nu}\,\frac{\partial\hat{f}_\nu}{\partial g_k} + \frac{1}{\rho T}\,\hat{q}_k\Big]_{E} &&= 0\\
&\Big[\frac{\partial\hat{\psi}}{\partial\alpha_\nu}\,\frac{\partial\hat{f}_\nu}{\partial\alpha_\gamma}\Big]_{E} &&= 0\\
&\Big[\hat{\sigma}_{mn} - \rho\Big(\frac{\partial\hat{\psi}}{\partial\varepsilon_{mn}} + \frac{\partial\hat{\psi}}{\partial\alpha_\nu}\,E_{\nu\,mn}\Big)_{sym}\Big]_{E} &&= 0\\
&\Big[\frac{\partial\hat{\psi}}{\partial T} + \frac{\partial\hat{\psi}}{\partial\alpha_\nu}\,\hat{H}_\nu + \hat{\eta}\Big]_{E} &&= 0\\
&\Big[\frac{\partial\hat{\psi}}{\partial g_k} + \frac{\partial\hat{\psi}}{\partial\alpha_\nu}\,\hat{G}_{\nu k}\Big]_{E} &&= 0\,.
\end{aligned}\right\}\quad(2.24)$$

Comparing these conditions with the corresponding conditions of the simpler constitutive class discussed in [1] we observe that conditions $(2.24)_1$ to $(2.24)_4$ are formally the same as (2.21) in ref. [1]. But conditions $(2.24)_5$ to $(2.24)_7$ are new. Some more specific conclusions can be obtained from (2.24) if further conditions on the constitutive response are imposed. We will not start a discussion analogous to that in ref. [1] but simply assume that the considered class of material models is locally asymptotically stable under constant strain and temperature. Thus, whenever

$$\varepsilon_{kl} = const.\,, \qquad T = const.\,, \qquad g_k = 0$$

the solution $a_y(t)$ of

$$\dot{\alpha}_\gamma = \hat{f}_\gamma(\varepsilon_{mn}^E, T^E, 0, \alpha_\rho)$$

$$\alpha_\rho = \alpha_\rho^0 \qquad \text{at } t=0$$

tends to the equilibrium value α_ρ^E with increasing time

$$\alpha_\rho(t) \longrightarrow \alpha_\rho^E \quad , \qquad t \to \infty$$

whatever the choice of the initial values α_ρ^0 in the neighborhood of the equilibrium values. For such an isothermal relaxation process the dissipation (2.17) reduces to

$$\gamma = -\rho \frac{\partial \hat{\psi}}{\partial \alpha_\nu} \hat{f}^\nu = -\rho \frac{\partial \hat{\psi}}{\partial \alpha_\nu} \dot{\alpha}_\nu = -\rho \frac{d\psi}{dt} > 0 \; ; \tag{2.25}$$

therefore, the free energy is a decreasing function of time. This result was already obtained for the less general constitutive class analysed in [1]. It is now not necessary to duplicate the results of [1]. We simply state that the above observation and the assumed continuity and differentiability of the free energy implies that the free energy has a relative minimum with respect to all variations of the internal variables, i.e.,

$$\left(\delta_\alpha \hat{\psi}\right)_E = 0 \; , \qquad \left(\delta^2_\alpha \hat{\psi}\right)_E > 0 \tag{2.26}$$

and thus

$$\left(\frac{\partial \hat{\psi}}{\partial \alpha_\gamma}\right)_E = \left(\frac{\partial \hat{\psi}}{\partial \alpha_\gamma}(\varepsilon_{mn}, T, g_k, \alpha_\rho)\right)_{\varepsilon_{mn}^E, T^E, g_k^E = 0, \alpha_\rho^E} = 0 . \tag{2.27}$$

Enforcing this requirement, condition $(2.24)_{1,2,4}$ is satisfied and $(2.24)_3$ yields

$$\left(\hat{q}_k\right)_E = 0 .$$

From $(2.24)_{5,6,7}$ one obtains

$$\left.\begin{aligned} \left(\hat{\sigma}_{mn} - \rho \frac{\partial \hat{\psi}}{\partial \varepsilon_{mn}}\right)^E_{sym} &= 0 \\ \left(\frac{\partial \hat{\psi}}{\partial T} + \hat{\eta}\right)_E &= 0 \\ &\;\vdots \end{aligned}\right\} \tag{2.28}$$

$$\left.\left(\frac{\partial\hat{\psi}}{\partial g_k}\right)_E = 0. \quad\right\rbrace \qquad (2.28)$$

It appears that these relations are the same as those obtained in the classical purely thermoelastic case without internal variables. However, it should be noted that equs. (2.28) are valid only at an equilibrium state; further the hatched functions still depend on the equilibrium values of the internal variables. But it is necessary to recall that the equilibrium values $\varepsilon_{kl}{}^E$, T^E, and $\alpha_\rho{}^E$ are related by the equilibrium condition (2.20)

$$\hat{f}_\gamma(\varepsilon_{mn}{}^E, T^E, 0, \alpha_\varsigma{}^E) = 0.$$

If this admits a unique solution for $\alpha_\rho{}^E$ * such that

$$\alpha_\varsigma{}^E = h_\varsigma(\varepsilon_{mn}{}^E, T^E), \qquad (2.29)$$

then we have

$$\left.\begin{aligned}
(\hat{\psi})_E &= \hat{\psi}(\varepsilon^E_{mn}, T^E, 0, \alpha_\varsigma{}^E) = \overset{E}{\psi}(\varepsilon^E_{mn}, T^E)\\
(\hat{\eta})_E &= \hat{\eta}(\varepsilon^E_{mn}, T^E, 0, \alpha^E_\varsigma, 0,0,0) = \overset{E}{\eta}(\varepsilon^E_{mn}, T^E)\\
(\hat{\sigma}_{mn})_E &= \hat{\sigma}_{mn}(\varepsilon^E_{mn}, T^E, 0, \alpha^E_\varsigma, 0,0,0) = \overset{E}{\sigma}_{mn}(\varepsilon^E_{mn}, T^E).
\end{aligned}\right\rbrace \qquad (2.30)$$

Consequently

$$\frac{\partial\overset{E}{\psi}}{\partial\varepsilon^E_{kl}} = \frac{\partial\hat{\psi}}{\partial\varepsilon^E_{mn}} + \frac{\partial\hat{\psi}}{\partial\alpha^E_\varsigma}\,\frac{\partial h_\varsigma}{\partial\varepsilon^E_{mn}}\,. \qquad (2.31)$$

With (2.27) one has

$$\frac{\partial\hat{\psi}}{\partial\alpha_\varsigma{}^E} = \left(\frac{\partial\hat{\psi}}{\partial\alpha_\varsigma{}^E}\right)_E = 0$$

and thus

$$\left(\frac{\partial\hat{\psi}}{\partial\varepsilon_{mn}}\right)_E = \frac{\partial\hat{\psi}}{\partial\varepsilon^E_{mn}} = \frac{\partial\overset{E}{\psi}}{\partial\varepsilon^E_{mn}}$$

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* A sufficient condition is

and $(2.28)_{1,2}$ reduces to

$$\left.\begin{aligned} \overset{E}{\sigma}_{mn} - \varrho\,\frac{1}{2}\left(\frac{\partial \overset{E}{\psi}}{\partial \varepsilon^{E}_{mn}} + \frac{\partial \overset{E}{\psi}}{\partial \varepsilon^{E}_{nm}}\right) &= 0 \\ \frac{\partial \overset{E}{\psi}}{\partial T^{E}} + \overset{E}{\eta} &= 0 \end{aligned}\right\} \tag{2.32}$$

where ψ^E is independent of the temperature gradient. Thus, at an equilibrium state the constitutive restrictions are equivalent to those of the thermoelastic case.

The conditions on the second variation of the dissipation and of the free energy remains to be analysed but this will not be done at this level of generality.

We now turn the attention to a constitutive class characterized by (2.3) and (2.4) that is, $\hat{\sigma}_{kl}$, $\hat{q}_k$, $\hat{\varepsilon}$ *and* $\hat{\eta}$ *do not depend on the rates* $\mathbb{R}$ of the external variables by assumption. Then the free energy $\hat{\psi}$ does not depend on these variables, too. Note that this was proved to be the case for the more general case (2.3) and (2.4). The dissipation inequality (2.17) still applies, however, with the important difference that none of the hatched constitutive functions depend on the rates $\mathbb{R}$. Therefore, the dissipation γ *depends only linearly on the rates* $\mathbb{R}$ and this allows to derive further important conditions. Since γ is required to be non-negative for all rates $\mathbb{R}$ the following necessary and sufficient conditions are obtained:

$$\left.\begin{aligned} \hat{\sigma}_{mn} - \varrho\left(\frac{\partial \hat{\psi}}{\partial \varepsilon_{mn}} + \frac{\partial \hat{\psi}}{\partial \alpha_{\nu}}\,\hat{E}_{\nu\, mn}\right)_{sym} &= 0 \\ \hat{\eta} + \frac{\partial \hat{\psi}}{\partial T} + \frac{\partial \hat{\psi}}{\partial \alpha_{\nu}}\,\hat{H}_{\nu} &= 0 \\ \frac{\partial \hat{\psi}}{\partial g_k} + \frac{\partial \hat{\psi}}{\partial \alpha_{\nu}}\,G_{\nu k} &= 0 \end{aligned}\right\} \tag{2.33}$$

and the residual dissipation inequality

$$-\varrho\,\frac{\partial \hat{\psi}}{\partial \alpha_{\nu}}\,\hat{f}_{\nu} - \frac{1}{T}\,\hat{q}_k\, g_k \geq 0\,. \tag{2.34}$$

If the evolution equations for the internal variables do not involve the rate $\dot{g}_k$, i.e., then (2.33)$_3$ yields

$$\left.\begin{aligned} \underset{\nu}{\hat{G}}_k &\equiv 0 \\ \frac{\partial\hat{\psi}}{\partial g_k} &= 0 . \end{aligned}\right\} \quad (2.35)$$

Further, if the evolution function $\hat{f}_\nu$ does not depend on the temperature gradient g_k,

$$f_\nu = \hat{f}_\nu(\varepsilon_{mn}, T, \alpha_\nu) , \qquad (2.36)$$

then, observing (2.34), the residual inequality splits up into two separate inequalities, as discussed in [1], i.e.,

$$\left.\begin{aligned} &\text{mechanical dissipation} \quad \gamma_M := -\varrho \frac{\partial\hat{\psi}}{\partial\alpha_\nu}\hat{f}_\nu \geq 0 \\ &\text{thermal dissipation} \quad \gamma_T := -\frac{1}{T}\hat{q}_k g_k \geq 0 . \end{aligned}\right\} \quad (2.37)$$

The relations (2.37)$_1$ and (2.37)$_2$ remain the same as (2.8) in ref. [1], and, stating explicitly the dependence on the parameters, equations (2.33)$_{1+2}$ read

$$\left.\begin{aligned} &\hat{\sigma}_{mn}(\varepsilon_{kl}, T, g_k, \alpha_s) - \varrho\left(\frac{\partial\hat{\psi}(\varepsilon_{kl}, T, \alpha_s)}{\partial\varepsilon_{mn}} + \right. \\ &\qquad\left. + \frac{\partial\hat{\psi}}{\partial\alpha_\nu}(\varepsilon_{kl}, T, \alpha_s)\,\underset{\nu}{\hat{E}}_{mn}(\varepsilon_{kl}, T, g_k, \alpha_s)\right)_{sym} = 0 \\ &\hat{\eta}(\varepsilon_{kl}, T, g_k, \alpha_s) + \frac{\partial\hat{\psi}(\varepsilon_{kl}, T, \alpha_s)}{\partial T} + \frac{\partial\hat{\psi}}{\partial\alpha_\nu}\underset{\nu}{\hat{H}}(\varepsilon_{kl}, T, g_k, \alpha_s) = 0 \end{aligned}\right\} \quad (2.38)$$

Note that here still the temperature gradient is involved; (2.38) has to be satisfied for all state variables ε_{kl}, T, g_k, α_ρ. If the hatched functions could be choosen arbitrarily then (2.38) would represent 6+1 relations between the 10+n variables ε_{kl}, T, g_k, α_ν and this is in contradiction to their assumed independence. Therefore, the hatched functions cannot be chosen at will.

In the following the constitutive functions $\underset{\nu}{\hat{E}}_{mm}$ and $\underset{\nu}{\hat{H}}$ are assumed to be independent of the temperature gradient. Thus, the evolution equations (2.4) are neither affected by the gradient g_k nor by its rate $\dot{g}_k$:

$$\left.\begin{aligned} &f_\gamma = \hat{f}_\gamma(\varepsilon_{kl}, T, \alpha_s) , \quad \underset{\nu}{E}_{mn} = \underset{\nu}{\hat{E}}_{mn}(\varepsilon_{kl}, T, \alpha_s) \\ &\underset{\nu}{H} = \underset{\nu}{\hat{H}}(\varepsilon_{kl}, T, \alpha_s) , \quad \underset{\nu}{\hat{G}}_k \equiv 0 . \end{aligned}\right\} \quad (2.39)$$

Then condition (2.38) shows that stress and entropy are not allowed to depend on the temperature gradient and (2.38) reduces to

$$\left.\begin{aligned} &\overset{A}{\sigma}_{mn}(\varepsilon_{kl},T,\alpha_s) - \rho\Big(\frac{\partial\overset{A}{\psi}(\varepsilon_{kl},T,\alpha_s)}{\partial\varepsilon_{mn}} + \\ &\qquad + \frac{\partial\overset{A}{\psi}(\varepsilon_{kl},T,\alpha_s)}{\partial\alpha_\nu}\overset{A}{\underset{\nu}{E}}_{mn}(\varepsilon_{kl},T,\alpha_s)\Big)_{sym} = 0 \\ &\overset{A}{\eta}(\varepsilon_{kl},T,\alpha_s) + \frac{\partial\overset{A}{\psi}(\varepsilon_{kl},T,\alpha_s)}{\partial T} + \frac{\partial\overset{A}{\psi}(\varepsilon_{kl},T,\alpha_s)}{\partial\alpha_\nu}\overset{A}{\underset{\nu}{H}}(\varepsilon_{kl},T,\alpha_s) = 0 \end{aligned}\right\} \quad (2.40)$$

Finally, it should be noted that the "equilibrium part" (Lehmann [64]) which characterizes the reversible behavior (section 1) of the evolution equations (2.4) plays its part only in the restriction (2.37), (2.38) or (2.40) but not in the residual dissipation inequality (2.34). *Therefore, it does not contribute to the dissipation but changes the structure of the classical potential relations for the stress and entropy.*

This change of the structure has an important consequence: The Legendre transformation, so important for the classical relations, is not valid any more. We illustrate this in the following.

The classical potential relations read

$$\left.\begin{aligned} &\overset{A}{\sigma}_{mn}(\varepsilon_{kl},T,\alpha_s) - \rho\Big(\frac{\partial\overset{A}{\psi}(\varepsilon_{kl},T,\alpha_s)}{\partial\varepsilon_{mn}}\Big)_{sym} = 0 \\ &\overset{A}{\eta}(\varepsilon_{kl},T,\alpha_s) + \frac{\partial\overset{A}{\psi}(\varepsilon_{kl},T,\alpha_s)}{\partial T} = 0 \end{aligned}\right\} \quad (2.41)$$

We assume that (2.41) is invertible such that both relations can be solved uniquely for ε_{kl} and T, i.e.,

$$\left.\begin{aligned} \varepsilon_{kl} &= \check{\varepsilon}_{kl}(\sigma_{mn},\eta,\alpha_s) \\ T &= \check{T}(\sigma_{mn},\eta,\alpha_s). \end{aligned}\right\} \quad (2.42)$$

This allows to introduce the function ϕ

$$\rho\,\phi(\sigma_{kl},\eta,\alpha_s) = \sigma_{mn}\check{\varepsilon}_{mn} - \rho\,\eta\,\check{T} - \rho\,\overset{A}{\psi}(\check{\varepsilon}_{kl},\check{T},\alpha_s). \quad (2.43)$$

The first variation of ϕ with respect to the independent variables* σ_{mn}, η, α_p is

$$\rho\,.\,\delta\phi = \rho\frac{\partial\check{\phi}}{\partial\sigma_{mn}}\delta\sigma_{mn} + \rho\frac{\partial\check{\phi}}{\partial\eta}\delta\eta + \rho\frac{\partial\check{\phi}}{\partial\alpha_\rho}\delta\alpha_\rho \tag{2.44}$$

and the r.h.s. of (2.43) gives

$$\begin{aligned}\rho\,\delta\phi = {} & \Big[\check{\varepsilon}_{mn} + \Big(\sigma_{kl} - \rho\frac{\partial\hat{\psi}}{\partial\varepsilon_{kl}}\Big)\frac{\partial\check{\varepsilon}_{kl}}{\partial\sigma_{mn}} - \rho\Big(\eta + \frac{\partial\hat{\psi}}{\partial T}\Big)\frac{\partial\check{T}}{\partial\sigma_{mn}}\Big]\delta\sigma_{mn} + \\ & + \Big[-\rho\check{T} + \Big(\sigma_{kl} - \rho\frac{\partial\hat{\psi}}{\partial\varepsilon_{kl}}\Big)\frac{\partial\check{\varepsilon}_{kl}}{\partial\eta} - \rho\Big(\eta + \frac{\partial\hat{\psi}}{\partial T}\Big)\frac{\partial\check{T}}{\partial\eta}\Big]\delta\eta + \\ & + \Big[-\rho\frac{\partial\hat{\psi}}{\partial\alpha_\rho} + \Big(\sigma_{kl} - \rho\frac{\partial\hat{\psi}}{\partial\varepsilon_{kl}}\Big)\frac{\partial\check{\varepsilon}_{kl}}{\partial\alpha_\rho} - \rho\Big(\eta + \frac{\partial\hat{\psi}}{\partial T}\Big)\frac{\partial\check{T}}{\partial\alpha_\rho}\Big]\delta\alpha_\rho .\end{aligned} \tag{2.45}$$

Observing (2.41), this yields

$$\rho\,\delta\phi = \check{\varepsilon}_{mn}\,\delta\sigma_{mn} - \rho\,\check{T}\,\delta\eta - \rho\frac{\partial\hat{\psi}}{\partial\alpha_\rho}\delta\alpha_\rho . \tag{2.45'}$$

Equating (2.44) and (2.45)' and requiring validity for all variations $\delta\sigma_{mn}$, $\delta\eta$, $\delta\alpha_p$ one gets

$$\left.\begin{aligned} \check{\varepsilon}_{mn} - \rho\Big(\frac{\partial\check{\phi}}{\partial\sigma_{mn}}\Big)_{sym} &= 0 \\ \check{T} + \frac{\partial\check{\phi}}{\partial\eta} &= 0 \\ \Big(\frac{\partial\hat{\psi}}{\partial\alpha_\rho}\Big)_{\substack{\varepsilon=\check{\varepsilon}\\ T=\check{T}}} + \frac{\partial\check{\phi}}{\partial\alpha_\rho} &= 0 . \end{aligned}\right\} \tag{2.46}$$

The structure of the equations $(2.46)_1$ and $(2.46)_2$ are analogous to (2.41) and this is an essential property of a Legendre transformation.

On the other hand, if the evolution equations involve rates of external variables, then the thermodynamic restrictions apply. Using (2.43) as a definition for ϕ again, we obtain (2.44) and (2.45). Accounting for (2.40), equ. (2.44) gives

$$\rho\,\delta\phi = \Big[\check{\varepsilon}_{mn} + \rho\frac{\partial\hat{\psi}}{\partial\alpha_\rho}\hat{E}_{\rho kl}\frac{\partial\check{\varepsilon}_{kl}}{\partial\sigma_{mn}} + \rho\frac{\partial\hat{\psi}}{\partial\alpha_\rho}\hat{H}_\rho\frac{\partial\check{T}}{\partial\sigma_{mn}}\Big]\delta\sigma_{mn} +$$

---

* It should be observed that the set of independent "state"-variables should be truely *minimal*, i.e., should not involve *superfluous independent variables*.

$$\left.\begin{aligned}
&\ldots + \left[-\varrho \check{T} + \varrho \frac{\partial \hat{\psi}}{\partial \alpha_\varrho} \underset{\varrho}{E}_{kl} \frac{\partial \check{\varepsilon}_{kl}}{\partial \eta} + \varrho \frac{\partial \hat{\psi}}{\partial \alpha_\varrho} \underset{\varrho}{\hat{H}} \frac{\partial \check{T}}{\partial \eta} \right] \delta \eta + \\
&+ \left[-\varrho \frac{\partial \hat{\psi}}{\partial \alpha_\nu} + \varrho \frac{\partial \hat{\psi}}{\partial \alpha_\varrho} \underset{\varrho}{E}_{kl} \frac{\partial \check{\varepsilon}_{kl}}{\partial \alpha_\nu} + \varrho \frac{\partial \hat{\psi}}{\partial \alpha_\varrho} \underset{\varrho}{\hat{H}} \frac{\partial \check{T}}{\partial \alpha_\nu} \right] \delta \alpha_\nu .
\end{aligned}\right\} \quad (2.47)$$

Equating (2.44) and (2.47) finally yields

$$\left.\begin{aligned}
&\check{\varepsilon}_{mn} - \varrho \left(\frac{\partial \check{\phi}}{\partial \sigma_{mn}} \right)_{sym} + \varrho \frac{\partial \hat{\psi}}{\partial \alpha_\varrho} \left(\underset{\varrho}{\hat{E}}_{kl} \frac{\partial \check{\varepsilon}_{kl}}{\partial \sigma_{mn}} + \underset{\varrho}{\hat{H}} \frac{\partial \check{T}}{\partial \sigma_{mn}} \right)_{sym} = 0 \\
&\check{T} + \varrho \left(\frac{\partial \check{\phi}}{\partial \eta} \right) - \varrho \frac{\partial \hat{\psi}}{\partial \alpha_\varrho} \left(\underset{\varrho}{\hat{E}}_{kl} \frac{\partial \check{\varepsilon}_{kl}}{\partial \eta} + \underset{\varrho}{\hat{H}} \frac{\partial \check{T}}{\partial \eta} \right) = 0 \\
&\frac{\partial \hat{\psi}}{\partial \alpha_\nu} + \frac{\partial \check{\phi}}{\partial \alpha_\nu} - \frac{\partial \hat{\psi}}{\partial \alpha_\varrho} \left(\underset{\varrho}{\hat{E}}_{kl} \frac{\partial \check{\varepsilon}_{kl}}{\partial \alpha_\nu} + \underset{\varrho}{\hat{H}} \frac{\partial \check{T}}{\partial \alpha_\nu} \right) = 0 .
\end{aligned}\right\} \quad (2.48)$$

Here it is understood that the arguments ε_{kl} and T in the functions $\hat{\psi}$, $\underset{\varrho}{\hat{E}}_{mm}$ and $\underset{\varrho}{\hat{H}}$ are expressed in terms of σ_{mn}, η, α_ν via the inverse relation (2.42). Comparing (2.48) and (2.40), it is seen that an analogous structure does not exist.

At this point it seems to be rather obvious to inquire into those *conditions which enforce a vanishing of the extra terms*

$$\frac{\partial \hat{\psi}}{\partial \alpha_\nu} \underset{\nu}{\hat{E}}_{mn} = 0 , \qquad \frac{\partial \hat{\psi}}{\partial \alpha_\nu} \underset{\nu}{\hat{H}} = 0 . \tag{2.49}$$

These conditions mean that the function $\underset{\varrho}{\hat{E}}_{mm}$ and $\underset{\varrho}{\hat{H}}_\nu$ are *orthogonal* to $\partial\hat{\psi}/\partial\alpha_\nu$, so an "intimate" relation between those functions and $\partial\hat{\psi}/\partial\alpha_\nu$ is imposed. The conditions would allow the classical Legendre transformation

$$\varrho \, \check{\phi} (\sigma_{mn}, \eta, \alpha_\nu) = \sigma_{mn} \, \varepsilon_{mn} - \eta T + \varrho \hat{\psi} (\varepsilon_{mn}, T, \alpha_\nu)$$

yielding

$$\varepsilon_{mn} = \varrho \left(\frac{\partial \check{\phi}}{\partial \sigma_{mn}} \right)_{sym}$$

$$T = - \varrho \frac{\partial \check{\phi}}{\partial \eta} .$$

This property is mathematically rather valuable, however, the physical implications of (2.49) are not clear. On the other hand, if only *one internal variable* $\nu = 1$ is present, then (2.49) implies that

$$\hat{E}_{mn} = \hat{E}_{1\,mn} = 0, \qquad \hat{H} = \hat{H}_1 = 0, \qquad \text{if } \frac{\partial\hat{\psi}}{\partial\alpha} \neq 0;$$

thus the influence of the rates of external variables in the evolution equations drops out. Thus, the conditions (2.49), imposing restrictions on $\hat{E}_{mn}$ and $\hat{H}$, is too far going.

The origin of the additional terms $\partial\hat{\psi}/\partial\alpha_\nu\, \hat{E}_{mn}$ etc. is solely due to the assumed influence of rates of external variables in the evolution equations (2.4). Assume that it is possible to find a new set of internal variables α_γ' by an invertible transformation

$$\alpha_\gamma' = h_\gamma(\varepsilon_{mn}, T, g_k, \alpha_\nu) \tag{2.50}$$

which allows to reduce the evolution equation (2.4) to the standard form

$$\dot{\alpha}'_\gamma = \hat{f}'_\gamma(\varepsilon_{mn}, T, g_k, \alpha_\nu'). \tag{2.51}$$

Then all constitutive functions will depend on the external variables and on the transformed internal variables and, most important, the constitutive restrictions (2.38) reduce to the classical potential form.

Specifically, provided (2.39) holds, the evaluation of the dissipation inequality yields

$$\hat{\sigma}_{mn}(\varepsilon_{kl}, T, \alpha_\nu') - \varrho \frac{\partial\hat{\psi}'(\varepsilon_{kl}, T, \alpha_\nu')}{\partial\varepsilon_{mn}} = 0$$

$$\hat{\eta}(\text{—}\,{''}\,\text{—}) + \frac{\partial\hat{\psi}'}{\partial T}(\text{—}\,{''}\,\text{—}) = 0$$

$$\frac{\partial\hat{\psi}'}{\partial g_k} = 0$$

and the two seperate residual inequalities (2.37)

$$\gamma_M' = -\varrho \frac{\partial \hat{\psi}'}{\partial \alpha_\nu'} \hat{f}_\nu' \geq 0$$

$$\gamma_T' = -\frac{1}{T} \hat{q}_k' g_k \geq 0 .$$

Therefore, in the following section the reduceability of the evolution equations is analysed.

2.1.2 Reduceability of the Evolution Equations

For the following we still consider the more general evolution equations (2.4)

$$\begin{aligned} \dot{\alpha}_\gamma = {} & \hat{f}_\gamma (\varepsilon_{kl}, T, g_k, \alpha_\varsigma) \\ & + \hat{E}_{\gamma\, mn} (\varepsilon_{kl}, T, g_k, \alpha_\varsigma)\, \dot{\varepsilon}_{mn} + \\ & + \hat{H}_\gamma (\varepsilon_{kl}, T, g_k, \alpha_\varsigma)\, \dot{T} + \\ & + \hat{G}_\gamma (\varepsilon_{kl}, T, g_k, \alpha_\varsigma)\, \dot{g}_k . \end{aligned} \tag{2.52}$$

We will now derive the conditions which assure that the evolution equations (2.52) can be reduced to the standard form

$$\dot{\alpha}_\gamma' = \hat{f}_\gamma' (\varepsilon_{kl}, T, g_k, \alpha_\varsigma')$$

where the α_γ' are suitably transformed internal variables

$$\alpha_\gamma' = h_\gamma (\varepsilon_{kl}, T, g_k, \alpha_\varsigma) . \tag{2.53}$$

Here we assume that (2.53) is invertible

$$\alpha_\gamma = h_\gamma^{-1} (\varepsilon_{kl}, T, \alpha_\varsigma'); \tag{2.54}$$

this is assured if

$$\det \left(\frac{\partial h_\gamma}{\partial \alpha_\mu} \right) \neq 0 . \tag{2.55}$$

Observing (2.52), the time derivative of (2.53) yields

$$\begin{aligned}\overset{\circ}{\alpha}{}'_\gamma &= \frac{\partial h_\gamma}{\partial \varepsilon_{mn}}\overset{\circ}{\varepsilon}_{mn} + \frac{\partial h_\gamma}{\partial T}\overset{\circ}{T} + \frac{\partial h_\gamma}{\partial g_m}\overset{\circ}{g}_m + \frac{\partial h_\gamma}{\partial \alpha_\varrho}\overset{\circ}{\alpha}_\varrho \\ &= \frac{\partial h_\gamma}{\partial \alpha_\varrho}\hat{f}_\varrho + \left(\frac{\partial h_\gamma}{\partial \varepsilon_{mn}} + \frac{\partial h_\gamma}{\partial \alpha_\varrho}\underset{\varrho}{\hat{E}}_{mn}\right)\overset{\circ}{\varepsilon}_{mn} + \\ &\quad + \left(\frac{\partial h_\gamma}{\partial T} + \frac{\partial h_\gamma}{\partial \alpha_\varrho}\underset{\varrho}{\hat{H}}\right)\overset{\circ}{T} + \left(\frac{\partial h_\gamma}{\partial g_m} + \frac{\partial h_\gamma}{\partial \alpha_\varrho}\underset{\varrho}{\hat{G}}_m\right)\overset{\circ}{g}_m .\end{aligned} \tag{2.56}$$

Thus, if a transformation h_γ exists which statisfies

$$\left.\begin{aligned}\frac{\partial h_\gamma}{\partial \varepsilon_{mn}} + \frac{\partial h_\gamma}{\partial \alpha_\varrho}\underset{\varrho}{\hat{E}}_{mn} = 0, \quad \frac{\partial h_\gamma}{\partial T} + \frac{\partial h_\gamma}{\partial \alpha_\varrho}\underset{\varrho}{\hat{H}} = 0 \\ \frac{\partial h_\gamma}{\partial g_m} + \frac{\partial h_\gamma}{\partial \alpha_\varrho}\underset{\varrho}{\hat{G}}_m = 0,\end{aligned}\right\} \tag{2.57}$$

then the reduced form (2.52) exists where

$$\hat{f}'_\gamma(\varepsilon_{mn}, T, g_m, \alpha'_\varrho) = \left[\frac{\partial h_\gamma}{\partial \alpha_\varrho}(\varepsilon_{mn}, T, g_m, \alpha_\nu)\,\hat{f}_\varrho(\varepsilon_{mn}, T, g_m, \alpha_\nu)\right]_{\alpha_\nu = h_\nu^{-1}} . \tag{2.58}$$

Here it is understood that α_ν on the r.h.s. is expressed in terms of ε_{mm}, T, g_m,α_p' via (2.54). The set of equations (2.57) represents an overdetermined system of partial differential equations for each of the transformation functions h_γ, γ = 1, ..., N. Consequently, the functions $\underset{\varrho}{\hat{E}}_{mm}$, $\underset{\varrho}{\hat{H}}$ and $\underset{\varrho}{\hat{G}}_m$ have to satisfy certain *integrability conditions* if a transformation should exist. Of course, the satisfaction of the integrability conditions will not assure a unique set of functions h_γ.

To make the analysis more concise a compact notation is used. We introduce the arrays v_i and $F_{\rho i}$

$$\left.\begin{aligned} v_i &= \{\varepsilon_{mn} = \varepsilon_{nm}, T, g_m\} \\ \underset{\varrho}{F}_i &= \{\underset{\varrho}{\hat{E}}_{mn} = \underset{\varrho}{\hat{E}}_{nm}, \underset{\varrho}{\hat{H}}, \underset{\varrho}{\hat{G}}_m\}, \quad i = 1, \ldots, 10 \\ &= \underset{\varrho}{F}_i(v_j, \alpha_\mu)\end{aligned}\right\} \tag{2.59}$$

and

$$h_\gamma = \check{h}_\gamma(v_i, \alpha_\varrho) \tag{2.60}$$

Then the system (2.57) takes the form

$$\frac{\partial \breve{h}_\gamma}{\partial v_i} + \frac{\partial \breve{h}_\gamma}{\partial \alpha_\rho} \underset{\rho}{F}{}_i = 0 \; ; \quad \begin{matrix} \gamma, \rho = 1, \dots, N \\ i = 1, \dots, 10 . \end{matrix} \tag{2.61}$$

For a *fixed* γ we put

$$\Gamma \equiv \breve{h}_\gamma \tag{2.62}$$

which should satisfy

$$\frac{\partial \Gamma}{\partial v_i} + \frac{\partial \Gamma}{\partial \alpha_\rho} \underset{\rho}{F}{}_i = 0 \; , \quad i = 1, \dots, 10 . \tag{2.63}$$

If

$$\Gamma = \breve{\Gamma}(v_i , \alpha_\rho) \tag{2.64}$$

is a common differentiable solution of the system (2.63), then the second order partial derivatives are interchangeable. Observing (2.63), they are given by

$$\left.\begin{aligned} \frac{\partial^2 \breve{\Gamma}}{\partial v_i \, \partial v_j} &= -\frac{\partial}{\partial v_j}\left(\frac{\partial \breve{\Gamma}}{\partial \alpha_\rho} \underset{\rho}{F}{}_i\right) = -\frac{\partial \breve{\Gamma}}{\partial \alpha_\rho} \frac{\partial \underset{\rho}{F}{}_i}{\partial v_j} - \frac{\partial^2 \breve{\Gamma}}{\partial \alpha_\rho \, \partial v_j} \underset{\rho}{F}{}_i \\ \frac{\partial^2 \breve{\Gamma}}{\partial v_j \, \partial v_i} &= -\frac{\partial}{\partial v_i}\left(\frac{\partial \breve{\Gamma}}{\partial \alpha_\rho} \underset{\rho}{F}{}_j\right) = -\frac{\partial \breve{\Gamma}}{\partial \alpha_\rho} \frac{\partial \underset{\rho}{F}{}_j}{\partial v_i} - \frac{\partial^2 \breve{\Gamma}}{\partial \alpha_\rho \, \partial v_i} \underset{\rho}{F}{}_j \end{aligned}\right\} \tag{2.65}$$

and

$$\frac{\partial^2 \breve{\Gamma}}{\partial v_i \, \partial \alpha_\nu} = -\frac{\partial}{\partial \alpha_\nu}\left(\frac{\partial \breve{\Gamma}}{\partial \alpha_\rho} \underset{\rho}{F}{}_i\right) = -\frac{\partial^2 \breve{\Gamma}}{\partial \alpha_\rho \, \partial \alpha_\nu} \underset{\rho}{F}{}_i - \frac{\partial \breve{\Gamma}}{\partial \alpha_\rho} \frac{\partial \underset{\rho}{F}{}_i}{\partial \alpha_\nu} . \tag{2.66}$$

Changing the indices ν → ρ, ρ → ν and i → j and observing the interchangeability of partial derivatives

$$\frac{\partial^2 \breve{\Gamma}}{\partial v_i \, \partial \alpha_\nu} = \frac{\partial^2 \breve{\Gamma}}{\partial \alpha_\nu \, \partial v_i} ,$$

equ. (2.66) yields

$$\frac{\partial^2 \breve{\Gamma}}{\partial \alpha_\rho \, \partial v_i} = -\left\{\frac{\partial^2 \breve{\Gamma}}{\partial \alpha_\nu \, \partial \alpha_\rho} \underset{\nu}{F}{}_j + \frac{\partial \breve{\Gamma}}{\partial \alpha_\nu} \frac{\partial \underset{\nu}{F}{}_j}{\partial \alpha_\rho}\right\} . \tag{2.67}$$

The interchangeability of the left hand sides of (2.65) and the substraction of $(2.65)_1$ and $(2.65)_2$ gives

$$-\frac{\partial\check{\Gamma}}{\partial\alpha_\rho}\left[\frac{\partial \underset{\rho}{F}_i}{\partial v_j}-\frac{\partial \underset{\rho}{F}_j}{\partial v_i}\right]-\frac{\partial^2\check{\Gamma}}{\partial\alpha_\rho\partial v_j}\underset{\rho}{F}_i-\frac{\partial^2\check{\Gamma}}{\partial\alpha_\rho\partial v_i}\underset{\rho}{F}_j=0. \qquad (2.68)$$

The mixed derivatives

$$\frac{\partial^2\check{\Gamma}}{\partial\alpha_\rho\partial v_j}\ ,\quad \frac{\partial^2\check{\Gamma}}{\partial\alpha_\rho\partial v_i}$$

are given by (2.67); inserting them into (2.68) several terms chancel and (2.68) reduces to

$$+\frac{\partial\check{h}_\gamma}{\partial\alpha_\rho}\left\{\left[\frac{\partial \underset{\rho}{F}_i}{\partial v_j}-\frac{\partial \underset{\rho}{F}_j}{\partial v_i}\right]-\left[\underset{\nu}{F}_i\frac{\partial \underset{\rho}{F}_j}{\partial\alpha_\nu}-\underset{\nu}{F}_j\frac{\partial \underset{\rho}{F}_i}{\partial\alpha_\nu}\right]\right\}=0 \qquad (2.69)$$

where (2.62) has been observed. For fixed i and j the equations (2.69) represents a homogeneous system of linear equations for the terms in { ... }-brackets with the coefficient matrix [$\partial\check{h}_\gamma/\partial\alpha_\rho$]. With the condition of non-singularity (2.55) we obtain for any set of subscripts (i, j) the trivial solution

$$\left[\frac{\partial \underset{\rho}{F}_i}{\partial v_j}-\frac{\partial \underset{\rho}{F}_j}{\partial v_i}\right]-\left[\underset{\nu}{F}_i\frac{\partial \underset{\rho}{F}_j}{\partial\alpha_\nu}-\underset{\nu}{F}_j\frac{\partial \underset{\rho}{F}_i}{\partial\alpha_\nu}\right]=0\ ,\ \nu=1,2,..N \qquad (2.70)$$

This condition involves only the functions $\underset{\rho}{F}_i$ or alternatively $\underset{\rho}{\hat{E}}_{mm}$, $\underset{\rho}{\hat{H}}$, and $\underset{\rho}{\hat{G}}_m$, respectively, and represents *necessary* restrictions on these functions to allow a reduction of the evolution equations to the standard form. That these conditions are also sufficient conditions for the existence of a solution $\check{\Gamma}$ could possibly be shown similar to the approach described in [67].

The integrability conditions (2.70) are now explicitly formulated in terms of the functions $\underset{\rho}{\hat{E}}_{mn}$ etc. using the definitions (2.59). Then we have

$$\left.\begin{array}{llll}
v_1 \ \dots\ v_6 & \widehat{=} & \varepsilon_{mn}=\varepsilon_{nm} & \\
\quad\ \ v_7 & \widehat{=} & T & \\
v_8 \ \dots\ v_{10} & \widehat{=} & g_m & \\
\underset{\rho}{F}_1 \ \dots\ \underset{\rho}{F}_6 & \widehat{=} & \underset{\rho}{\hat{E}}_{mn}=\underset{\rho}{\hat{E}}_{nm} & m,n=1,2,3 \\
\quad\ \ \underset{\rho}{F}_7 & \widehat{=} & \underset{\rho}{\hat{H}} & \\
\underset{\rho}{F}_8 \ \dots\ \underset{\rho}{F}_{10} & \widehat{=} & \underset{\rho}{\hat{G}}_m &
\end{array}\right\} \qquad (2.71)$$

and (2.70) yields the following essential integrability conditions

$$\left.\begin{aligned}
&\frac{\partial \underset{\varsigma}{\hat{E}}_{mn}}{\partial \varepsilon_{kl}} - \frac{\partial \underset{\varsigma}{\hat{E}}_{kl}}{\partial \varepsilon_{mn}} - \left(\underset{\nu}{\hat{E}}_{mn} \frac{\partial \underset{\varsigma}{\hat{E}}_{kl}}{\partial \alpha_\nu} - \underset{\nu}{\hat{E}}_{kl} \frac{\partial \underset{\varsigma}{\hat{E}}_{mn}}{\partial \alpha_\nu} \right) = 0 \\
&\frac{\partial \underset{\varsigma}{\hat{E}}_{mn}}{\partial T} - \frac{\partial \underset{\varsigma}{\hat{H}}}{\partial \varepsilon_{mn}} - \left(\underset{\nu}{\hat{E}}_{mn} \frac{\partial \underset{\varsigma}{\hat{H}}}{\partial \alpha_\nu} - \underset{\nu}{\hat{H}} \frac{\partial \underset{\varsigma}{\hat{E}}_{mn}}{\partial \alpha_\nu} \right) = 0 \\
&\frac{\partial \underset{\varsigma}{\hat{E}}_{mn}}{\partial g_k} - \frac{\partial \underset{\varsigma}{\hat{G}}_k}{\partial \varepsilon_{mn}} - \left(\underset{\nu}{\hat{E}}_{mn} \frac{\partial \underset{\varsigma}{\hat{G}}_k}{\partial \alpha_\nu} - \underset{\varsigma}{\hat{G}}_k \frac{\partial \underset{\varsigma}{\hat{E}}_{mn}}{\partial \alpha_\nu} \right) = 0 \\
&\frac{\partial \underset{\varsigma}{\hat{H}}}{\partial g_k} - \frac{\partial \underset{\varsigma}{\hat{G}}_k}{\partial T} - \left(\underset{\nu}{\hat{H}} \frac{\partial \underset{\varsigma}{\hat{G}}_k}{\partial \alpha_\nu} - \underset{\nu}{\hat{G}}_k \frac{\partial \underset{\varsigma}{\hat{H}}}{\partial \alpha_\nu} \right) = 0
\end{aligned}\right\} \quad (2.72)$$

Inspection shows that the conditions $(2.72)_{1+2}$ resemble the conditions of path-independence , equ. (1.14), of the "reversible" contribution to the internal variable rate, discussed in section 1. Indeed, the two different aspects, i.e., path-independence and reduceability of the evolution equations, are closely related and it is worth to discuss this in the following *supplement*.

The time derivative of the inverse of the transformation function (2.54)

$$\alpha_\gamma = \overset{-1}{h_\gamma}(\varepsilon_{kl}, T, g_k, \alpha'_\varsigma) = \tilde{k}_\gamma(\varepsilon_{kl}, T, g_k, \alpha'_\varsigma) \qquad (S2.1)$$

is given by

$$\dot{\alpha}_\gamma = \frac{\partial \tilde{k}_\gamma}{\partial \varepsilon_{mn}} \dot{\varepsilon}_{mn} + \frac{\partial \tilde{k}_\gamma}{\partial T} \dot{T} + \frac{\partial \tilde{k}_\gamma}{\partial g_m} \dot{g}_m + \frac{\partial \tilde{k}_\gamma}{\partial \alpha'_\varsigma} \dot{\alpha}'_\varsigma . \qquad (S.2.2)$$

In the right side of the evolution equations (2.52) the internal variables may be expressed in terms of the set {εkl, T, gk, α'v} with the inverse (S2.1).

Thus (2.52) takes the form

$$\begin{aligned}
\dot{\alpha}_\gamma = {} & \tilde{f}_\gamma(\varepsilon_{kl}, T, g_k, \alpha'_\nu) + \\
& + \underset{\gamma}{\tilde{E}}_{mn}(\text{———} \; '' \; \text{———}) \, \dot{\varepsilon}_{mn} + \\
& + \underset{\gamma}{\tilde{H}} \, (\text{———} \; '' \; \text{———}) \, \dot{T} + \\
& + \underset{\gamma}{\tilde{G}}_k \, (\text{———} \; '' \; \text{———}) \, \dot{g}_k
\end{aligned} \qquad (S2.3)$$

where

$$\left.\begin{aligned} \tilde{f}_\gamma(\varepsilon_{kl},T,g_k,\alpha_\nu') &= \hat{f}_\gamma(\varepsilon_{kl},T,g_k,\tilde{k}_\varsigma(\varepsilon_{kl},\dots)) \\ \underset{\gamma}{\tilde{E}}_{mn}(\text{——}\;''\;\text{——}) &= \underset{\gamma}{\hat{E}}_{mn}(\text{——}\;''\;\text{——}) \\ \underset{\gamma}{\tilde{H}}(\text{——}\;''\;\text{——}) &= \underset{\gamma}{\hat{H}}(\text{——}\;''\;\text{——}) \\ \underset{\gamma}{\tilde{G}}_k(\text{——}\;''\;\text{——}) &= \underset{\gamma}{\hat{G}}_k(\text{——}\;''\;\text{——}) \end{aligned}\right\}\text{(S2.4)}$$

Combining (S2.2) and (S2.3) one gets

$$\dot{\alpha}_\gamma = \frac{\partial\tilde{k}_\gamma}{\partial\alpha_\varsigma'}\dot{\alpha}_\varsigma' + \underbrace{\frac{\partial\tilde{k}_\gamma}{\partial\varepsilon_{mn}}\dot{\varepsilon}_{mn} + \frac{\partial\tilde{k}_\gamma}{\partial T}\dot{T} + \frac{\partial\tilde{k}_\gamma}{\partial g_k}\dot{g}_k}_{=[d\tilde{k}_\gamma/dt]_{\alpha_\varsigma'=\text{const.}}} = \tag{S2.5}$$

$$= \tilde{f}_\gamma + \underset{\gamma}{\tilde{E}}_{mn}\dot{\varepsilon}_{mn} + \underset{\gamma}{\tilde{H}}\dot{T} + \underset{\gamma}{\tilde{G}}_k\dot{g}_k .$$

If reduceability is possible, then functions $\tilde{k}_\gamma$ exist such that terms involving rates of external variables drop out in equ. (S2.5) for arbitrary rates $\dot{\varepsilon}_{mn}$, $\dot{T}$, $\dot{g}_k$, i.e.

$$\left[\frac{d\tilde{k}_\gamma}{dt}\right]_{\alpha_\varsigma'=\text{const.}} = \frac{\partial\tilde{k}_\gamma}{\partial\varepsilon_{mn}}\dot{\varepsilon}_{mn} + \frac{\partial\tilde{k}_\gamma}{\partial T}\dot{T} + \frac{\partial\tilde{k}_\gamma}{\partial g_k}\dot{g}_k$$

$$= \underset{\gamma}{\tilde{E}}_{mn}\dot{\varepsilon}_{mn} + \underset{\gamma}{\tilde{H}}\dot{T} + \underset{\gamma}{\tilde{G}}_k\dot{g}_k$$

or

$$\left[d\tilde{k}_\gamma \right]_{\alpha'_\rho = \mathrm{const.}} = \underset{\gamma}{\tilde{E}}_{mn}\, d\varepsilon_{mn} + \underset{\gamma}{\tilde{H}}\, dT + \underset{\gamma k}{\tilde{G}}\, dg_k . \tag{S2.6}$$

Thus the r.h.s.'s are required to be *total differentials* in the space of the external variables ε_{mn}, T, g_k with α'_ρ = const. The associated potentials are $\tilde{k}_\gamma$ (ε_{mn}, T, g_m, α'_ρ) with constant parameter α'_ρ. This, of course, implies then that the integrals of the right hand sides are *path-independent*

$$\tilde{k}_\gamma(\varepsilon_{mn}, T, g_k, \alpha'_\rho) - \tilde{k}_\gamma(\varepsilon_{mn_0}, T_0, g_{k_0}, \alpha'_\rho) =$$

$$= \int_{\varepsilon_{mn_0}, T_0, g_{k_0}}^{\varepsilon_{mn}, \ldots} \left\{ \underset{\gamma}{\tilde{E}}_{mn}\, d\varepsilon_{mn} + \underset{\gamma}{\tilde{H}}\, dT + \underset{\gamma k}{\tilde{G}}\, dg_k \right\}_{\alpha'_\rho = \mathrm{const.}} . \tag{S2.7}$$

The existence of the "potentials" $\tilde{k}_\gamma$ means that the internal variables α_γ = $\tilde{k}_\gamma$ (ε_{mn}, T, g_m, α'_ν), *with the parameters* α'_ν *assumed to be constant*, are just functions of the present values of the external variables. Consequently

$$\frac{\partial \tilde{k}_\gamma}{\partial \varepsilon_{mn}} = \underset{\gamma}{\tilde{E}}_{mn} , \quad \frac{\partial \tilde{k}_\gamma}{\partial T} = \underset{\gamma}{\tilde{H}} , \quad \frac{\partial \tilde{k}_\gamma}{\partial g_k} = \underset{\gamma k}{\tilde{G}} . \tag{S2.8}$$

Necessary and sufficient conditions for the existence of the potentials $\tilde{k}_\gamma$ are given by the integrability conditions:

$$\left.\begin{aligned} \frac{\partial \underset{\gamma}{\tilde{E}}_{mn}}{\partial \varepsilon_{kl}} &= \frac{\partial \underset{\gamma}{\tilde{E}}_{kl}}{\partial \varepsilon_{mn}} , & \frac{\partial \underset{\gamma}{\tilde{E}}_{mn}}{\partial T} &= \frac{\partial \underset{\gamma}{\tilde{H}}}{\partial \varepsilon_{mn}} \\ \frac{\partial \underset{\gamma}{\tilde{E}}_{mn}}{\partial g_k} &= \frac{\partial \underset{\gamma k}{\tilde{G}}}{\partial \varepsilon_{mn}} , & \frac{\partial \underset{\gamma}{\tilde{H}}}{\partial g_k} &= \frac{\partial \underset{\gamma k}{\tilde{G}}}{\partial T} ; \end{aligned}\right\} \tag{S2.9}$$

here the transformed internal variables α'_ρ are helt constant. These conditions can be written in terms of the original (implicit) functions $\underset{s}{\hat{E}}_{mn}$ etc., i.e., the r.h.s. of (S2.4). Thus, we obtain for example

$$\frac{\partial \underset{\gamma}{\tilde{E}}_{mn}}{\partial \varepsilon_{kl}} = \frac{\partial \underset{\gamma}{\hat{E}}_{mn}}{\partial \varepsilon_{kl}} + \frac{\partial \underset{\gamma}{\hat{E}}_{mn}}{\partial \tilde{h}_\nu}\,\frac{\partial \tilde{k}_\nu}{\partial \varepsilon_{kl}}$$

and with (S2.8) and (S2.4) this takes the form

$$\frac{\partial \underset{\gamma}{\tilde{E}}_{mn}}{\partial \varepsilon_{kl}} = \frac{\partial \underset{\gamma}{\hat{E}}_{mn}}{\partial \varepsilon_{kl}} + \frac{\partial \underset{\gamma}{\hat{E}}_{mn}}{\partial \alpha_\nu}\,\underset{\nu}{\hat{E}}_{kl}\,. \tag{S2.10}$$

Similarly all other expressions in (S2.9) may be transformed. It is then shown that (S2.9) is equivalent to (2.72). Thus, reduceability and path-independence as described above are equivalent.

At this place it should also be noted that the transformation (2.53) does not affect the mechanical and the thermal dissipation. We show this as follows. After transformation the mechanical and the thermal dissipations are given by

$$\gamma_M' = -\rho\,\frac{\partial \hat{\psi}'}{\partial \alpha_\gamma'}\,\hat{j}_\gamma' \ , \quad \gamma_T' = -\frac{1}{T}\,\hat{q}_k'\,g_k \tag{S2.11}$$

where generally with (2.58)

$$\left.\begin{aligned} \psi = \hat{\psi}'(\varepsilon_{mn}, T, g_k, \alpha_\nu') &= \hat{\psi}(\varepsilon_{mn}, T, g_k, \alpha_\nu) \\ \hat{j}_\gamma'(\text{—}\ ''\ \text{—}) &= \frac{\partial h_\gamma}{\partial \alpha_\rho}\,\hat{j}_\rho(\text{—}\ ''\ \text{—}) \\ \hat{q}_k'(\text{—}\ ''\ \text{—}) &= \hat{q}_k(\text{—}\ ''\ \text{—})\,. \end{aligned}\right\} \tag{S2.12}$$

With (2.53) and (S2.1), i.e.,

$$\alpha_\nu' = h_\nu(\varepsilon_{mn}, T, g_k, \alpha_\nu)\ , \quad \alpha_\nu = \tilde{k}_\nu(\varepsilon_{mn}, T, g_k, \alpha_\nu')$$

and

$$\frac{\partial \alpha_\mu}{\partial \alpha_\nu} = \frac{\partial h_\mu}{\partial \alpha_\varrho'}\,\frac{\partial \alpha_\varrho'}{\partial \alpha_\nu} = \frac{\partial h_\mu}{\partial \alpha_\varrho'}\;\;\frac{\partial h_\varrho}{\partial \alpha_\nu} = \delta_{\mu\nu} \tag{S2.13}$$

we obtain

$$\frac{\partial \hat{\psi}'}{\partial \alpha'_\varrho} = \frac{\partial \hat{\psi}}{\partial \alpha_\gamma}\;\frac{\partial \alpha_\gamma}{\partial \alpha_\varrho'} = \frac{\partial \hat{\psi}}{\partial \alpha_\gamma}\;\frac{\partial h_\gamma}{\partial \alpha_\varrho'}$$

such that

$$\begin{aligned}\gamma_M' &= -\varrho\,\frac{\partial \hat{\psi}'}{\partial \alpha_\varrho'}\,\hat{f}_\varrho' = -\varrho\,\frac{\partial \hat{\psi}}{\partial \alpha_\gamma}\,\overbrace{\frac{\partial h_\gamma}{\partial \alpha_\varrho'}\;\frac{\partial h_\varrho}{\partial \alpha_\mu}}^{\delta_{\gamma\mu}}\,\hat{f}_\mu = \\ &= -\varrho\,\frac{\partial \hat{\psi}}{\partial \alpha_\gamma}\,\hat{f}_\gamma = \gamma_M\end{aligned} \tag{S2.14}$$

which proves the equality of the mechanical dissipations. This is clearly obvious for the thermal dissipation.

We now return to the discussion of the integrability conditions (2.72). Inspection of (2.72) shows that these conditions are identically satisfied whenever the quantities $\underset{\varrho}{\hat{E}}_{mn}$, $\underset{\varrho}{\hat{H}}$ and $\underset{\varrho}{\hat{G}}_k$ are strictly constant. In the following three cases are treated with decreasing complexity.

Case I: $\underset{\varrho}{\hat{G}}_k = 0,\ \underset{\varrho}{\hat{H}} \neq 0,\ \underset{\varrho}{\hat{E}}_{mn} \neq 0$

In this case an influence of the rate of the temperature gradient on the rates of the internal variables is excluded. Condition $(2.72)_3$ and $(2.72)_4$ reduces to

$$\frac{\partial \underset{\varrho}{\hat{E}}_{mn}}{\partial g_k} = 0\,, \qquad \frac{\partial \underset{\varrho}{\hat{H}}}{\partial g_k} = 0 \tag{2.73}$$

that is, the functions $\underset{\varrho}{\hat{E}}_{mm}$ and $\underset{\varrho}{\hat{H}}$ are required to be independent of the temperature gradient. The conditions $(2.72)_1$ and $(2.72)_2$ remain. Equ.'s $(2.72)_1$ are requirements solely for the functions $\underset{\varrho}{\hat{E}}_{mm}$ but $(2.72)_2$ involves both $\underset{\varrho}{\hat{E}}_{mm}$ and $\underset{\varrho}{\hat{H}}$.

Case II: $\underset{\varrho}{\hat{G}}_k = 0,\ \underset{\varrho}{\hat{H}} = 0,\ \underset{\varrho}{\hat{E}}_{mn} \neq 0$

Here only the strain rate affects the evolution of the internal variables. Of course, again the conditions (2.73) have to be satisfied as well as

$$\frac{\partial \underset{\gamma}{\hat{E}}_{mn}}{\partial T} = 0 \qquad (2.74)$$

which follows from $(2.72)_2$. Equ. (2.74) requires temperature independence of $\hat{E}_{mm}$; thus

$$\underset{\gamma}{E}_{mn} = \underset{\gamma}{\hat{E}}_{mn}(\varepsilon_{kl}, \alpha_v). \qquad (2.75)$$

Further, the complex relation $(2.72)_1$ remains.

Case III: $\underset{\gamma}{\hat{G}}_k = 0,\ \underset{\gamma}{\hat{H}} \neq 0,\ \underset{\gamma}{\hat{E}}_{mm} = 0$

The conditions $(2.72)_1$ and $(2.72)_3$ are identically satisfied and $(2.72)_2$ and $(2.72)_4$ reduce to

$$\frac{\partial \underset{\gamma}{\hat{H}}}{\partial \varepsilon_{mn}} = 0\,, \qquad \frac{\partial \underset{\gamma}{\hat{H}}}{\partial g_k} = 0 \qquad (2.76)$$

which implies

$$\underset{\gamma}{H} = \underset{\gamma}{\hat{H}}(T, \alpha_v). \qquad (2.77)$$

From this it is immediately obvious that any evolution law which involves only the temperature allows a reduction to the standard form if the constitutive function $\hat{H}$ depends only on the temperature and the internal variables.

2.1.3 Thermodynamic Consistency of Stress-Strain Relations

We assume that an ad hoc-model has been proposed by prescribing the functions

$$\hat{\sigma}_{kl}(\varepsilon_{kl}, T, \alpha_v)\,, \quad \hat{\epsilon}(\text{---}''\text{---})\,, \quad \hat{\eta}(\text{---}''\text{---})$$

$$\underset{\gamma}{\hat{f}}(\text{---}''\text{---})\,, \quad \underset{\gamma}{\hat{E}}_{mn}(\text{---}''\text{---})\,, \quad \underset{\gamma}{\hat{H}}(\text{---}''\text{---})$$

where the functions $\underset{\gamma}{\hat{E}}_{mn}$ and $\underset{\gamma}{\hat{H}}$ are such that the evolutions equations (2.52) are not reduceable. Clearly, the choice is not necessarily compatible with the two restrictions (2.40): Thermodynamics requires that there exists a Helmholtz function ψ (free energy) which should satisfy the two conditions (2.40). With the above functions given, the conditions (2.40) may be regarded as an overdetermined system of partial differential equations for the single function ψ.

Therefore, certain integrability conditions have to be satisfied by $\hat{\sigma}_{kl}$ and $\hat{E}_{kl}$ as well as $\hat{\eta}$ and $\hat{H}$. To prevent a duplication of work we leave this question here and come back to it again in section (2.2.4), where the stress is used as an independent variable.

2.2 Stress as an Independent External Variable

2.2.1 The Problem of the Derivation of General Restrictions

Frequently the stress-strain relation $(2.3)_1$ is invertible:

$$\varepsilon_{mn} = \check{\varepsilon}_{mn}(\sigma_{kl}, T, g_k, \alpha_\nu). \tag{2.78}$$

This allows to use in all the other constitutive equations, including the evolution equations, the set $\{\sigma_{kl}, T, g_k, \alpha_\nu\}$ as independent variables. Such an approach would automatically take proper account of the constitutive restrictions if the exchange of variables is done *after* the evaluation of the entropy inequality.

On the other hand, one may consider a constitutive model which uses the stress as an independent external variable and the strain as a dependent one from the beginning. Then, the evolution equations which involves rates of external variables will depend, among others, also on the stress rate. For such a model the evaluation of the Clausius-Duhem inequality has to be repeated. It will be shown that under such conditions certain peculiar conceptual problems may occur which usually have been overlooked in the literature [41].

We will restrict the discussion to a *homogeneous* body with the following type of constitutive relations:

$$\begin{aligned}
\varepsilon_{mn} &= \check{\varepsilon}_{mn}(\sigma_{kl}, T, g_k, \alpha_\nu) \\
\epsilon &= \check{\epsilon}\,(\text{------ '' ------}) \\
\eta &= \check{\eta}\,(\text{------ '' ------}) \\
q_k &= \check{q}_k\,(\text{------ '' ------})
\end{aligned} \tag{2.79}$$

and the evolution equations

$$
\begin{aligned}
\dot{\alpha}_{\gamma} = {} & \check{f}_{\gamma}(\sigma_{kl}, T, g_k, \alpha_\nu) \\
& + \check{E}_{\gamma mn}(\text{——''——})\,\dot{\sigma}_{mn} + \\
& + \check{H}_{\gamma}(\text{——''——})\,\dot{T} + \\
& + \check{G}_{\gamma k}(\text{——''——})\,\dot{g}_k \, .
\end{aligned}
\tag{2.80}
$$

The lokal, instant entropy inequality is given by

$$\sigma := \rho\dot{\eta} + \left(\frac{q_k}{T}\right)_{,k} - \frac{\rho r}{T} \geq 0$$

and the lokal, instant energy balance reads

$$\rho\dot{\epsilon} = \sigma_{kl}\dot{\varepsilon}_{kl} - q_{k,k} + \rho r$$

and this allows to eliminate the heat supply from the entropy inequality

$$\sigma = \rho\dot{\eta} - \frac{1}{T}\rho\dot{\epsilon} + \frac{1}{T}\sigma_{kl}\dot{\varepsilon}_{kl} - \frac{q_k g_k}{T^2} \geq 0.$$

Thus, the dissipation γ is given by

$$\gamma := T\sigma = \rho T\dot{\eta} - \rho\dot{\epsilon} + \sigma_{kl}\dot{\varepsilon}_{kl} - \frac{q_k g_k}{T} \geq 0 . \tag{2.81}$$

We now introduce the Gibbs function (or free enthalpy [68]) using the notation of Gibbs [69]

$$\zeta = \check{\zeta}(\sigma_{kl}, T, g_k, \alpha_\nu) = \check{\epsilon} - T\check{\eta} - \frac{1}{\rho}\sigma_{kl}\check{\varepsilon}_{kl} = \check{\psi} - \frac{1}{\rho}\sigma_{kl}\check{\varepsilon}_{kl} \tag{2.82}$$

The material time derivative is given by

$$\frac{d}{dt}\zeta = \dot{\zeta} = \dot{\epsilon} - \dot{T}\eta - T\dot{\eta} - \frac{1}{\rho}\dot{\sigma}_{kl}\varepsilon_{kl} - \frac{1}{\rho}\sigma_{kl}\dot{\varepsilon}_{kl} \tag{2.83}$$

and this allows to write the dissipation inequality (2.81) in the following form

$$\gamma = -\rho\left(\dot{\zeta} + \eta\dot{T}\right) - \dot{\sigma}_{kl}\varepsilon_{kl} - \frac{q_k g_k}{T} \geq 0 . \tag{2.84}$$

In terms of the Gibbs function $\check{\zeta}$ we have

$$\dot{\zeta} = \frac{\partial\check{\zeta}}{\partial\sigma_{kl}}\dot{\sigma}_{kl} + \frac{\partial\check{\zeta}}{\partial T}\dot{T} + \frac{\partial\check{\zeta}}{\partial g_k}\dot{g}_k + \frac{\partial\check{\zeta}}{\partial\alpha_\nu}\dot{\alpha}_\nu . \qquad (2.85)$$

Inserting this into (2.84) and collecting terms with the appropriate rates, yields

$$\begin{aligned}\gamma = &-\left(\rho\frac{\partial\check{\zeta}}{\partial\sigma_{kl}} + \check{\varepsilon}_{kl}\right)_{sym}\dot{\sigma}_{kl} - \rho\left(\frac{\partial\check{\zeta}}{\partial T} + \check{\eta}\right)\dot{T}\\ &-\rho\frac{\partial\check{\zeta}}{\partial g_k}\dot{g}_k - \rho\frac{\partial\check{\zeta}}{\partial\alpha_\nu}\dot{\alpha}_\nu - \frac{\check{q}_k g_k}{T} \geq 0.\end{aligned} \qquad (2.86)$$

The rates of internal variables are determined by (2.80), i.e., they are completely defined by the instant values of all the independent variables and the rates of the external independent variables. Accounting for this fact, the dissipation inequality reads

$$\begin{aligned}\gamma = &-\left[\rho\frac{\partial\check{\zeta}}{\partial\sigma_{kl}} + \rho\frac{\partial\check{\zeta}}{\partial\alpha_\nu}\underset{\vee}{\check{E}}_{kl} + \check{\varepsilon}_{kl}\right]_{sym}\dot{\sigma}_{kl}\\ &-\rho\left[\frac{\partial\check{\zeta}}{\partial T} + \frac{\partial\check{\zeta}}{\partial\alpha_\nu}\underset{\vee}{\check{H}} + \check{\eta}\right]\dot{T}\\ &-\rho\left[\frac{\partial\check{\zeta}}{\partial g_k} + \frac{\partial\check{\zeta}}{\partial\alpha_\nu}\underset{\vee}{\check{G}}_k\right]\dot{g}_k\\ &-\rho\frac{\partial\check{\zeta}}{\partial\alpha_\nu}\check{f}_\nu \qquad\qquad - \frac{\check{q}_k g_k}{T} \geq 0.\end{aligned} \qquad (2.87)$$

Here we have to recall that the dissipation inequality has to be satisfied identically for all (smooth) admissible thermomechanic processes, i.e., solutions of the balance equations together with the assumed constitutive relations as well as appropriate initial and boundary conditions. These solutions actually represent fields in space and time. Physically an arbitrary process is controlled and determined by a choice of body forces and heat supplies and initial and boundary conditions. On the other hand, any smooth displacement and temperature field u_k (x_m, t) and T (x_k, t) and initial values $\underset{\bullet}{g}_{\gamma}$ (x_k) of the internal variables completely defines an admissible process [32, 36]. Especially the strain and strain rate fields are given by

$$\varepsilon_{kl} = \frac{1}{2}\left(u_{k,l} + u_{l,k} \right)$$

$$\dot{\varepsilon}_{kl} = \frac{1}{2}\left(\dot{u}_{k,l} + \dot{u}_{l,k} \right)$$

and satisfy the compatibility conditions automatically. Since u_k (x_m, t) is an *arbitrary function*, the strain and strain rates are *locally* and *instantly arbitrary* and *independent.*

On the other side, not every smooth stress and temperature field σ_{kl} (x_m, t) and T (x_m, t) and set of initital values $\underset{o}{\alpha}_Y$ (x_m) corresponds to an admissible process. Of course, the evolution law (2.80) can be integrated in principle to yield the field of internal variables α_v (x_m, t); however, the strain field ε_{kl} (x_m, t) obtained from $(2.79)_1$ generally does not satisfy the compatibility conditions: The strain field is not deriveable from a displacement field. Therefore, a smooth and unique velocity and acceleration field generally does not exist if the above choice is made.

Consequently Cauchy's first law of motion (balance of linear momentum) cannot be satisfied. In short, the above choice is non-realizable.

However, there exists a class of peculiar processes which legitimates the use of the stress as one of the quantities characterizing a thermomechanical process. Assume that the stress and temperature field as well as initial values $\underset{o}{\alpha}_Y$ are *homogeneous,* that is independent of the spatial coordinates. Thus, the temperature gradient g_k vanishes everywhere

$$g_k = 0$$

and the strain field calculated from $(2.79)_1$ is uniform. However, for a uniform strain field a compatible displacement field can always be constructed. Thus, to some choice of homogeneous fields σ_{kl}, T and $\underset{o}{\alpha}_Y$ there alway exists a process defined by u_k, T, and $\underset{o}{\alpha}_Y$. For an arbitrary stress rate and temperature rate it then follows from (2.87)

$$\left.\begin{aligned} &\left[\varrho \frac{\partial \check{\xi}}{\partial \sigma_{kl}} + \varrho \frac{\partial \check{\xi}}{\partial \alpha_\nu} \underset{\nu}{\check{E}}_{kl} + \check{\varepsilon}_{kl}\right]_{\substack{sym \\ g_k=0}} &= 0 \\ &\left[\frac{\partial \check{\xi}}{\partial T} + \frac{\partial \check{\xi}}{\partial \alpha_\nu} \underset{\nu}{\check{H}} + \check{\eta}\right]_{g_k=0} &= 0 \end{aligned}\right\} \quad (2.88)$$

and

$$\left[-\varrho \frac{\partial \check{\xi}}{\partial \alpha_\nu} \check{f}_\nu\right]_{g_k=0} \geq 0 \qquad (2.89)$$

but a statement about the term

$$\left[\frac{\partial \check{\xi}}{\partial g_k} + \frac{\partial \check{\xi}}{\partial \alpha_\nu} \underset{\nu}{\check{G}}_k\right]_{g_k=0} \qquad (2.90)$$

cannot be made. This would require an arbitrary choice of g_k and especially $\dot{g}_k$ and this would involve generally an incompatible strain field even if the stress field is homogeneous. Obviously, equ. (2.88) to (2.90) are limited results.

However, there is an important exception. If $(2.79)_1$ to $(2.79)_3$ and (2.80) are assumed *independent of the temperature gradient and its rate ab initio,* then the choice of homogeneous fields σ_{kl}, T and $\underset{\circ}{\alpha}_\gamma$ yields essential constitutive restrictions of the form (2.88) and (2.89) valid for *any* value of g_k since then these expression are independent of g_k.

In general the above results show that the characterization of an admissible thermomechanic process by the displacement and temperature field as well as the initial values $\underset{\circ}{\alpha}_\gamma$ is much more preferable. This possibly explains why Coleman and Gurtin [36] did not use the stress as an independent variable.

Taking this conclusion as a starting point we will now show under what conditions a direct evaluation* of (2.87) is still justified. The material derivative of $(2.79)_1$ yields

$$\dot{\varepsilon}_{kl} = \frac{\partial \check{\varepsilon}_{kl}}{\partial \sigma_{mn}} \dot{\sigma}_{mn} + \frac{\partial \check{\varepsilon}_{kl}}{\partial T} \dot{T} + \frac{\partial \dot{\varepsilon}_{kl}}{\partial g_k} \dot{g}_k + \frac{\partial \check{\varepsilon}_{kl}}{\partial \alpha_\nu} \dot{\alpha}_\nu = \ldots$$

---

* The direct evaluation is based on the independence and arbitrariness of $\dot{\sigma}_{kl}$, $\dot{T}$ and $\dot{g}_k$

$$\begin{aligned} \dots = & \left(\frac{\partial \check{\varepsilon}_{kl}}{\partial \sigma_{mn}} + \frac{\partial \check{\varepsilon}_{kl}}{\partial \alpha_s} \underset{s}{\check{E}}_{mn} \right)_{sym} \overset{\circ}{\sigma}_{mn} \\ & + \left(\frac{\partial \check{\varepsilon}_{kl}}{\partial T} + \frac{\partial \check{\varepsilon}_{kl}}{\partial \alpha_s} \underset{s}{\check{H}} \right) \dot{T} \\ & + \left(\frac{\partial \check{\varepsilon}_{kl}}{\partial g_m} + \frac{\partial \check{\varepsilon}_{kl}}{\partial \alpha_s} \underset{s}{\check{G}}_m \right) \dot{g}_m \\ & + \frac{\partial \check{\varepsilon}_{kl}}{\partial \alpha_s} \check{f}_s \end{aligned} \qquad (2.91)$$

where the evolution equation (2.80) is observed.

We put

$$C_{klmn} = \frac{\partial \check{\varepsilon}_{kl}}{\partial \sigma_{mn}} + \frac{\partial \check{\varepsilon}_{kl}}{\partial \alpha_s} \underset{s}{\check{E}}_{mn} \qquad (2.92)$$

which is a fourth order tensor symmetric in (kl) and (mn). Its "inverse" $C^{-1}{}_{pqkl}$ is defined such that its multiplication with (2.91) and contraction yields

$$\overset{-1}{C}_{pqkl} \overset{\circ}{\varepsilon}_{kl} = \overset{\circ}{\sigma}_{pq} + \overset{-1}{C}_{pqkl} \left\{ \left(\frac{\partial \check{\varepsilon}_{kl}}{\partial T} + \frac{\partial \check{\varepsilon}_{kl}}{\partial \alpha_s} \underset{s}{\check{H}} \right) \dot{T} + \dots \right\}$$

which implies

$$\overset{-1}{C}_{pqkl} C_{klmn} \overset{\circ}{\sigma}_{mn} = \overset{\circ}{\sigma}_{pq}$$

or

$$\overset{-1}{C}_{pqkl} C_{klmn} \overset{\circ}{\sigma}_{mn} - \frac{1}{2} \left(\overset{\circ}{\sigma}_{pq} + \overset{\circ}{\sigma}_{qp} \right) = 0$$

and thus

$$\left[\overset{-1}{C}_{pqkl} C_{klmn} - \frac{1}{2} \left(\delta_{pm} \delta_{qn} + \delta_{qm} \delta_{pn} \right) \right] \overset{\circ}{\sigma}_{mn} = 0.$$

Therefore

$$\overset{-1}{C}_{pqkl} C_{klmn} = \frac{1}{2} \left(\delta_{pm} \delta_{qn} + \delta_{qm} \delta_{pn} \right) = \tilde{\delta}_{pqmn} \qquad (2.93)$$

is a "unit tensor" which reflects the symmetry properties of the subscripts.

The necessary and sufficient conditions for the existence of an "inverse" can be obtained as follows. The relation (2.91) represents a system of linear equations for the stress rates $\dot{\sigma}_{mn} = \dot{\sigma}_{nm}$. Collecting the stress and strain rates in column matrixes

$$\dot{\boldsymbol{\sigma}}^T = \{\dot{\sigma}_{11}, \dot{\sigma}_{22}, \dot{\sigma}_{33}, \dot{\sigma}_{12}, \dot{\sigma}_{23}, \dot{\sigma}_{31}\}$$

$$\dot{\boldsymbol{\varepsilon}}^T = \{\dot{\varepsilon}_{11}, \dots \quad , \dot{\varepsilon}_{12}, \dots, \dot{\varepsilon}_{31}\}$$

and introducing

$$\mathbb{K}^T = \{K_{11}, K_{22}, K_{33}, K_{12}, K_{23}, K_{31}\}$$

where

$$K_{kl} = \left(\frac{\partial \check{\varepsilon}_{kl}}{\partial T} + \frac{\partial \check{\varepsilon}_{kl}}{\partial \alpha_s} \underset{s}{\check{H}}\right)\dot{T} + \\ + \left(\frac{\partial \check{\varepsilon}_{kl}}{\partial g_{mn}} + \frac{\partial \check{\varepsilon}_{kl}}{\partial \alpha_s} \underset{s}{\check{G}}_{mn}\right)\dot{g}_{mn} + \\ + \frac{\partial \check{\varepsilon}_{kl}}{\partial \alpha_s} \check{f}_s$$

the relation (2.91) reads

$$\dot{\boldsymbol{\varepsilon}}^T = \mathbb{C}\,\dot{\boldsymbol{\sigma}} + \mathbb{K}\,; \tag{2.94}$$

here the matrix $\mathbb{C}$ contains the elements of $C_{kl\,mn}$

$$\mathbb{C} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & 2C_{1112} & 2C_{1123} & 2C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & 2C_{2212} & 2C_{2223} & 2C_{2231} \\ \cdots\cdots & & & \cdots\cdots & & \\ C_{1211} & C_{1222} & C_{1233} & 2C_{1212} & 2C_{1223} & 2C_{1231} \\ \cdots\cdots & & & \cdots\cdots & & \\ C_{3111} & C_{3111} & C_{3133} & 2C_{3112} & 2C_{3123} & 2C_{3131} \end{pmatrix} \tag{2.95}$$

Consequently, a unique solution for $\dot{\sigma}$ is assured if the 6x6 matrix is non-singular

$$\det(\mathbb{C}) \neq 0. \tag{2.96}$$

Provided this non-singularity is satisfied, equ. (2.91) or (2.94) can be solved for the stress rates. This allows to eliminate the stress rate from the dissipation inequality (2.87); we obtain

$$\begin{aligned}
\gamma = & -\Big[\Big(\varrho\frac{\partial\check{\zeta}}{\partial\sigma_{mn}} + \varrho\frac{\partial\check{\zeta}}{\partial\alpha_s}\underset{s}{\check{E}}_{mn} + \check{\varepsilon}_{mn}\Big)_{sym}{C'}^{-1}_{mnkl}\Big]\dot{\varepsilon}_{kl}\\
& -\Big[\Big(\varrho\frac{\partial\check{\zeta}}{\partial T} + \varrho\frac{\partial\check{\zeta}}{\partial\alpha_s}\underset{s}{\check{H}} + \varrho\check{\eta}\Big)\\
& \quad -\Big(\varrho\frac{\partial\check{\zeta}}{\partial\sigma_{mn}} + \varrho\frac{\partial\check{\zeta}}{\partial\alpha_s}\underset{s}{\check{E}}_{mn} + \check{\varepsilon}_{mn}\Big)_{sym}{C'}^{-1}_{mnkl}\Big(\frac{\partial\check{\varepsilon}_{kl}}{\partial T} + \frac{\partial\check{\varepsilon}_{kl}}{\partial\alpha_s}\underset{s}{\check{H}}\Big)\Big]\dot{T}\\
& -\Big[\Big(\varrho\frac{\partial\check{\zeta}}{\partial g_k} + \varrho\frac{\partial\check{\zeta}}{\partial\alpha_s}\underset{s}{\check{G}}_k\Big)\\
& \quad -\Big(\varrho\frac{\partial\check{\zeta}}{\partial\sigma_{mn}} + \varrho\frac{\partial\check{\zeta}}{\partial\alpha_s}\underset{s}{\check{E}}_{mn} + \check{\varepsilon}_{mn}\Big)_{sym}{C'}^{-1}_{mnkl}\Big(\frac{\partial\check{\varepsilon}_{kl}}{\partial g_p} + \frac{\partial\check{\varepsilon}_{kl}}{\partial\alpha_s}\underset{s}{\check{G}}_p\Big)\Big]\dot{g}_p\\
& -\Big[\Big(\varrho\frac{\partial\check{\zeta}}{\partial\alpha_s}\check{f}_s + \frac{\check{q}_k g_k}{T}\Big) -\\
& \quad -\Big(\varrho\frac{\partial\check{\zeta}}{\partial\sigma_{mn}} + \varrho\frac{\partial\check{\zeta}}{\partial\alpha_s}\underset{s}{\check{E}}_{mn} + \check{\varepsilon}_{mn}\Big)_{sym}{C'}^{-1}_{mnkl}\Big(\frac{\partial\check{\varepsilon}_{kl}}{\partial\alpha_s}\check{f}_s\Big)\Big]\\
\geq & \; 0.
\end{aligned} \tag{2.97}$$

However, this inequality still contains the stress as an independent variable since the functions $\check{\zeta}$, $\check{\varepsilon}_{kl}$, $\check{\eta}$, $\check{f}_p$, $\underset{s}{\check{E}}_{mm}$, $\underset{s}{\check{H}}$, $\underset{s}{\check{G}}_m$ etc. as well as $\check{C}'_{mmkl}$ depend explicitly on the stress. We recall the requirement that the thermomechanical process must be compatible which is assured if u_k, T and $\underset{o}{g}_p$ are taken as the primary fields. Therefore, the strain-stress relation $(2.79)_1$ must be invertible to define a unique stress field in terms of ε_{kl}, T, g_k and α_p

$$\sigma_{kl} = \check{\sigma}_{kl}\left(\varepsilon_{mn},\ T,\ g_m,\ \alpha_s\right). \tag{2.98}$$

Thus, the Jacobian of $(2.79)_1$ is required to be non-singular*

$$\det\left(\frac{\partial \check{\varepsilon}_{kl}}{\partial \sigma_{mn}}\right) \neq 0. \tag{2.99}$$

In inequality (2.97) it is understood that the stress is represented by the function (2.98). Then for arbitrary and locally independent rates

$$\overset{\circ}{\varepsilon}_{kl},\ \overset{\circ}{T},\ \overset{\circ}{g}_k$$

the dissipation inequality yields the following restrictions

$$\left.\begin{aligned}
&\left(\varrho \frac{\partial \check{\xi}}{\partial \sigma_{mn}} + \varrho \frac{\partial \check{\xi}}{\partial \alpha_s} \underset{s}{\check{E}}_{mn} + \check{\varepsilon}_{mn}\right)_{sym} C^{-1}_{mnkl} = 0 \\
&\left(\varrho \frac{\partial \check{\xi}}{\partial T} + \varrho \frac{\partial \check{\xi}}{\partial \alpha_s} \underset{s}{\check{H}} + \varrho \check{\eta}\right) - \\
&\quad - \left(\varrho \frac{\partial \check{\xi}}{\partial \sigma_{mn}} + \dots\right)_{sym} C^{-1}_{mnkl} \left(\frac{\partial \check{\varepsilon}_{kl}}{\partial T} + \frac{\partial \check{\varepsilon}_{kl}}{\partial \alpha_s} \underset{s}{\check{H}}\right) = 0 \\
&\left(\varrho \frac{\partial \check{\xi}}{\partial g_k} + \varrho \frac{\partial \check{\xi}}{\partial \alpha_s} \underset{s}{\check{G}}_k\right) - \\
&\quad - \left(\varrho \frac{\partial \check{\xi}}{\partial \sigma_{mn}} + \dots\right)_{sym} C^{-1}_{mnkl} \left(\frac{\partial \check{\varepsilon}_{kl}}{\partial g_p} + \frac{\partial \check{\varepsilon}_{kl}}{\partial \alpha_s} \underset{s}{\check{G}}_p\right) = 0.
\end{aligned}\right\} \tag{2.100}$$

Multiplication of $(2.100)_1$ with C_{klpq} and observing (2.93) gives

---

* It is understood that $\partial \varepsilon_{kl} / \partial \sigma_{mn}$ is represented by a two-dimensional array.

$$\left(\varrho \frac{\partial \check{\zeta}}{\partial \sigma_{mn}} + \varrho \frac{\partial \check{\zeta}}{\partial \alpha_\varrho} \underset{\varrho}{\check{E}}_{mn} + \check{\varepsilon}_{mn}\right) \underset{sym}{\tilde{\delta}_{mnpq}} = \tag{2.101}$$

$$= \left(\varrho \frac{\partial \check{\zeta}}{\partial \sigma_{pq}} + \varrho \frac{\partial \check{\zeta}}{\partial \alpha_\varrho} \underset{\varrho}{E}_{pq} + \check{\varepsilon}_{pq}\right)_{sym} = 0.$$

Here the symmetry of $\partial\check{\zeta} / \partial\sigma_{mn}$, $\underset{\varrho}{\check{E}}_{mn}$, $\check{\varepsilon}_{mn}$ is accounted for. Consequently, the conditions $(2.100)_{2/3}$ reduce to

$$\begin{aligned} \varrho \frac{\partial \check{\zeta}}{\partial T} + \varrho \frac{\partial \check{\zeta}}{\partial \alpha_\varrho} \underset{\varrho}{\check{H}} + \varrho \check{\eta} &= 0 \\ \varrho \frac{\partial \check{\zeta}}{\partial g_k} + \varrho \frac{\partial \check{\zeta}}{\partial \alpha_\varrho} \underset{\varrho}{\check{G}}_k &= 0 \end{aligned} \tag{2.102}$$

and the dissipation inequality (2.97) simplifies to the residual dissipation inequality

$$-\varrho \frac{\partial \check{\zeta}}{\partial \alpha_\varrho} \check{f}_\varrho - \frac{\check{q}_k g_k}{T} \geq 0. \tag{2.103}$$

Obviously, the conditions (2.101) to (2.103) would have been obtained if inequality (2.87) were evaluated directly for arbitrary $\dot{\sigma}_{kl}$, $\dot{T}$, and $\dot{g}_k$.

The above results can be summarized in the following *conclusion:*

> The use of the *stress as an independent* variable in the constitutive equations and a *direct evaluation of the dissipation inequality* with an arbitrary variation of the stress rate is formally justified provided this approach can be interpreted in terms of a variation of a kinematically compatible thermomechanical process. For homogeneous bodies this is assured if the strain-stress relation is invertible, i.e., which is assured if
>
> $$det\left(\frac{\partial \check{\varepsilon}_{kl}}{\partial \sigma_{mn}}\right) \neq 0 \tag{2.104}$$
>
> and if the incremental strain rate - stress rate relation, with due account of the evolution equations for the internal variables, is invertible, and that is the case if

$$\det(\mathbb{C}) \mathrel{\hat{=}} \det\left(\frac{\partial \check{\varepsilon}_{kl}}{\partial \sigma_{mn}} + \frac{\partial \check{\varepsilon}_{kl}}{\partial \alpha_\varsigma} \underset{\varsigma}{\check{E}}_{mn}\right) \neq 0. \tag{2.105}$$

For the more special case that the evolution equations for the internal variables do not depend on the stress rate, i.e.,

$$\underset{\varsigma}{\check{E}}_{mn} \equiv 0$$

then the condition (2.105) reduces to (2.104).

2.2.2 Reduceability of the Evolution Equations

We will now indicate the conditions which assure that the evolution equations (2.80) can be reduced to the standard form

$$\dot{\alpha}_\gamma'' = \check{f}_\gamma''(\sigma_{kl}, T, g_k, \alpha_\varsigma'') \tag{2.106}$$

where the α_γ'' are transformed internal variables

$$\alpha_\gamma'' = \check{h}_\gamma(\sigma_{kl}, T, g_k, \alpha_\nu). \tag{2.107}$$

It is immediately obvious that the derivation of these conditions is entirely analogous to the approach described in section (2.12). Thus, with the appropriate notation we get, analogous to (2.72), the following *integrability conditions*

$$\left.\begin{aligned}
&\frac{\partial \underset{\varsigma}{\check{E}}_{mn}}{\partial \sigma_{kl}} - \frac{\partial \underset{\varsigma}{\check{E}}_{kl}}{\partial \sigma_{mn}} - \left(\underset{\nu}{\check{E}}_{mn} \frac{\partial \underset{\varsigma}{\check{E}}_{kl}}{\partial \alpha_\nu} - \underset{\nu}{\check{E}}_{kl} \frac{\partial \underset{\varsigma}{\check{E}}_{mn}}{\partial \alpha_\nu}\right) = 0 \\
&\frac{\partial \underset{\varsigma}{\check{E}}_{mn}}{\partial T} - \frac{\partial \underset{\varsigma}{\check{H}}}{\partial \sigma_{mn}} - \left(\underset{\nu}{\check{E}}_{mn} \frac{\partial \underset{\varsigma}{\check{H}}}{\partial \alpha_\nu} - \underset{\nu}{\check{H}} \frac{\partial \underset{\varsigma}{\check{E}}_{mn}}{\partial \alpha_\nu}\right) = 0 \\
&\frac{\partial \underset{\varsigma}{\check{E}}_{mn}}{\partial g_k} - \frac{\partial \underset{\varsigma}{\check{G}}_k}{\partial \sigma_{mn}} - \left(\underset{\nu}{\check{E}}_{mn} \frac{\partial \underset{\varsigma}{\check{G}}_k}{\partial \alpha_\nu} - \underset{\nu}{\check{G}}_k \frac{\partial \underset{\varsigma}{\check{E}}_{mn}}{\partial \alpha_\nu}\right) = 0 \\
&\frac{\partial \underset{\varsigma}{\check{H}}}{\partial g_k} - \frac{\partial \underset{\varsigma}{\check{G}}_k}{\partial T} - \left(\underset{\nu}{\check{H}} \frac{\partial \underset{\varsigma}{\check{G}}_k}{\partial \alpha_\nu} - \underset{\varsigma}{\check{G}}_k \frac{\partial \underset{\varsigma}{\check{H}}}{\partial \alpha_\nu}\right) = 0,
\end{aligned}\right\} \tag{2.108}$$

Again the three special cases are considered with decreasing complexity and which are presented in an abridged form.

Case I'' $\quad \underset{s}{\check{G}}_k = 0, \quad \underset{s}{\check{H}} \neq 0, \quad \underset{s}{\check{E}}_{mm} \neq 0$

(2.108)$_1$ and (2.108)$_2$ remain

(2.108)$_3$ reduces to $\dfrac{\partial \underset{s}{\check{E}}_{mn}}{\partial g_k} = 0$

(2.108)$_4$ reduces to $\dfrac{\partial \underset{s}{\check{H}}}{\partial g_k} = 0$

(2.109)

Case II'' $\quad \underset{s}{\check{G}}_k = 0, \quad \underset{s}{\check{H}} = 0, \quad \underset{s}{\check{E}}_{mm} \neq 0$

(2.108)$_1$ remains

(2.108)$_2$ reduces to $\dfrac{\partial \underset{s}{\check{E}}_{mn}}{\partial T} = 0$

(2.108)$_3$ reduces to $\dfrac{\partial \underset{s}{\check{E}}_{mn}}{\partial g_k} = 0$

(2.108)$_4$ identically satisfied.

(2.110)

Here it should be stated explicitly that, provided only the stress rate is involved in the evolutions equation, than the function $\underset{s}{\check{E}}_{mn}$ should not only be independent of the temperature gradient but also independent of the temperature; further, the complicated integrability condition (2.108)$_1$ remains to be satisfied.

It appears that the required temperature independence of the function $\underset{s}{\check{E}}_{mn}$ is not acceptable on physical grounds: It is very unlikely, that phenomenological functions like $\underset{s}{\check{E}}_{mn}$ should not depend on the absolute temperature; this is a property which characterizes very basic material constants usually beyond the phenomenological level.

It is worthwhile to indicate explicitly some simple functions $\underset{s}{\check{E}}_{mn}$ which satisfy all the conditions (2.110):

(α) Assume $\underset{s}{\check{E}}_{mn}$ independent of all variables, i.e.,

$$\underset{s}{\check{E}}_{mn} = const. ,$$

then its derivatives vanish and (2.108), is identically satisfied.

(β) Assume $\underset{s}{\check{E}}_{mn}$ independent of α_μ, T, g_k and linear in σ_{pq}, i.e.,

$$\underset{s}{\check{E}}_{mn} = \underset{s}{C}_{mnpq}\,\sigma_{pq} \quad , \quad \underset{s}{C}_{mnpq} \,\hat{=}\, const. .$$

Then $(2.108)_1$ yields the condition

$$\underset{s}{C}_{mnkl} - \underset{s}{C}_{klmn} = 0 ;$$

for example

$$\underset{s}{C}_{klmn} = \underset{s}{A}\,\delta_{kl}\,\delta_{mn}$$

such that (2.80) reads

$$\overset{\circ}{\alpha}_\gamma = f_\gamma + \underset{\gamma}{A}\,\sigma_{kk}\,\dot{(\sigma_{mn})} .$$

(γ) Assume E_{mn} independent of σ_{kl}, T, g_k and linear in α_μ, i.e.,

$$\underset{s}{\check{E}}_{mn} = \underset{s}{S}_{mn\mu}\,\alpha_\mu \quad , \quad \underset{s}{S}_{mn\mu} \,\hat{=}\, const.$$

From $(2.108)_1$ it follows

$$\left(\underset{\nu}{S}_{mn\mu}\,\underset{s}{S}_{kl\nu} - \underset{\nu}{S}_{kl\mu}\,\underset{s}{S}_{mn\nu} \right) = 0$$

which is still a complex requirement. If

$$\underset{\nu}{S}_{mn\mu} = T_{mn}\,\underset{\nu}{A}_\mu \quad , \qquad \underset{\nu}{A}_\mu \,\hat{=}\, const. , \; T_{mn} \,\hat{=}\, const.$$

then the above requirement takes the form

$$\left(T_{mn}\,\underset{\nu}{A}_\mu\,T_{kl}\,\underset{s}{A}_\nu - T_{kl}\,\underset{\nu}{A}_\mu\,T_{mn}\,\underset{s}{A}_\nu \right) = 0$$

which is identically satisfied. Then

$$\overset{\circ}{\alpha}_\gamma = f_\gamma + \underset{\gamma}{A}_\mu\,\alpha_\mu\,T_{mn}\,\dot{\sigma}_{mn} .$$

(δ) Assume $\overset{\vee}{\underset{s}{E}}_{mn}$ independent of T, g_k and linear in α_μ and σ_{kl}, such that

$$\overset{\vee}{\underset{s}{E}}_{mn} = C_{mnpq}\,\sigma_{pq}\,\underset{s}{A}_\mu\,\alpha_\mu\;;$$

this implies that the influence of the stress is the same for all functions $\overset{\vee}{\underset{s}{E}}_{mn}$ $\rho = 1 \dots N$. Provided $C_{mn\,pq}$ is symmetric

$$C_{mnpq} = C_{pqmn}$$

$(2.108)_2$ is satisfied, since each of the two terms of $(2.108)_1$ vanishes identically.

Case III'' $\quad \overset{\vee}{\underset{s}{G}}_k = 0, \quad \overset{\vee}{\underset{s}{H}} \neq 0, \quad \overset{\vee}{\underset{s}{E}}_{mm} = 0$

$$\left.\begin{array}{ll} (2.108)_1 \text{ identically satisfied} & \\ (2.108)_2 \text{ reduces to} & \dfrac{\partial \overset{\vee}{\underset{s}{H}}}{\partial \sigma_{mn}} = 0 \\ (2.108)_3 \text{ identically satisfied} & \\ (2.108)_4 \text{ reduces to} & \dfrac{\partial \overset{\vee}{\underset{s}{H}}}{\partial g_k} = 0\,. \end{array}\right\} \quad (2.111)$$

Consequently, a reduction to the standard form of the evolution equation is only possible if the function $\overset{\vee}{\underset{s}{H}}$ is independent of the stress and temperature gradient and is atmost a function of the temperature and internal variables

$$\underset{s}{H} = \overset{\vee}{\underset{s}{H}}(T, \alpha_\nu)\,. \qquad (2.112)$$

2.2.3 Determination of Transformation Functions

In this section we will give some general information how suitable transformation function can be determined if the integrability conditions are satisfied. However, the discussion will be restricted to the cases III'' and II'' only, i.e.,

$$\underset{s}{\check{H}} \neq 0 \qquad \text{or} \qquad \underset{s}{\check{E}}_{mn} \neq 0 .$$

2.2.3.1 *Case III''*: $\underset{s}{\check{G}}_k = 0, \quad \underset{s}{\check{H}} \neq 0, \quad \underset{s}{\check{E}}_{mm} = 0$

The assumed evolution equation is given by

$$\dot{\alpha}_\gamma = \check{f}_\gamma(\sigma_{mn}, T, g_k, \alpha_\nu) + \underset{\gamma}{\check{H}}(T, \alpha_\nu)\,\dot{T}. \tag{2.113}$$

Integrability is assured since the condition (2.112) is satisfied. The transformation relation is taken to be

$$\alpha''_\gamma = \check{h}_\gamma(\sigma_{kl}, T, g_k, \alpha_\nu) \tag{2.114}$$

and its time derivative is

$$\begin{aligned}
\dot{\alpha}''_\gamma &= \frac{\partial \check{h}_\gamma}{\partial \sigma_{kl}}\dot{\sigma}_{kl} + \frac{\partial \check{h}_\gamma}{\partial T}\dot{T} + \frac{\partial \check{h}_\gamma}{\partial g_k}\dot{g}_k + \frac{\partial \check{h}_\gamma}{\partial \alpha_\nu}\dot{\alpha}_\nu \\
&= \frac{\partial \check{h}_\gamma}{\partial \alpha_\nu}\check{f}_\nu + \left(\underset{\nu}{\check{H}}\frac{\partial \check{h}_\gamma}{\partial \alpha_\nu} + \frac{\partial \check{h}_\gamma}{\partial T}\right)\dot{T} + \\
&\quad + \frac{\partial \check{h}_\gamma}{\partial \sigma_{kl}}\dot{\sigma}_{kl} + \frac{\partial \check{h}_\gamma}{\partial g_k}\dot{g}_k .
\end{aligned} \tag{2.115}$$

Reduceability to the standard form implies

$$\frac{\partial \check{h}_\gamma}{\partial \sigma_{kl}} = 0 \quad , \quad \frac{\partial \check{h}_\gamma}{\partial g_k} = 0 \tag{2.116}$$

and

$$\frac{\partial \check{h}_\gamma}{\partial T} + \underset{s}{\check{H}}\frac{\partial \check{h}_\gamma}{\partial \alpha_s} = 0 . \tag{2.117}$$

The conditions (2.116) require that $\check{h}_\gamma$ is of the form

$$\check{h}_\gamma = \check{h}_\gamma\,(T,\alpha_\nu). \tag{2.118}$$

For a *fixed index* γ we put

$$\Gamma = \check{h}_\gamma\,(T,\alpha_\nu) = \check{\Gamma}(T,\alpha_\nu) \tag{2.119}$$

and (2.117) reads

$$\frac{\partial\check{\Gamma}}{\partial T} + \check{H}_\rho\,\frac{\partial\check{\Gamma}}{\partial\alpha_\rho} = 0 \tag{2.120}$$

which is a homogeneous linear partial differential equation for $\check{\Gamma}$. Any transformation function $\check{h}_\gamma$ must reflect the mathematical structure of the representative solution $\check{\Gamma}$.

From the theory of partial differential equations [70] it is known that the theory of any linear partial differential equation is equivalent to the theory of systems of ordinary differential equations and this also allows the construction of a solution. This system is given by

$$\frac{dT(s)}{ds} = 1\,, \qquad \frac{d\alpha_\rho(s)}{ds} = \check{H}_\rho\,(T,\alpha_1,\dots,\alpha_N) \tag{2.121}$$

and

$$\frac{d\check{\Gamma}(T(s),\alpha_\nu(s))}{ds} = 0\,. \tag{2.122}$$

The functions $T(s), \alpha_\rho(s)$ represent the characteristic base curves in the $(T, \alpha_1, \dots, \alpha_N)$-space and s is the curve parameter. Equ. (2.122) states that $\check{\Gamma}$ is constant along a characteristic base curve. The initial conditions for the system (2.121) are

$$s = 0\,, \qquad T = \underset{0}{T}\,, \qquad \alpha_\rho = \underset{0}{\alpha}_\rho\;; \tag{2.123}$$

and these values define a single point in the $(N+1)$-dimensional $(T, \alpha_1, \dots, \alpha_N)$-space. Because of the uniqueness of the solution of (2.121) only a single characteristic base curve passes through the point (2.123). We consider an infinite set of initial points on an $[(N+1)-1 = N]$-dimensional surface (N-dimensional manifold) in the $(T, \alpha_1, \dots, \alpha_N)$-space. This surface is given by the parameter representation

$$T = \underset{0}{T}(r_1,\dots,r_N)\,, \qquad \alpha_\rho = \underset{0}{\alpha}_\rho(r_1,\dots,r_N)\,, \quad \rho = 1,\dots,N$$

where $r_1, \dots r_N$ are the surface parameters. Further, if to every point of this surface a value

$$\Gamma = \underset{0}{\Gamma}\,(r_1,\dots,r_N) \tag{2.124}$$

is attached, then each characteristic base curve emanating from the surface, "transports" its value Γ_0 to other points in the $(N+1)$-dimensional space. Of course, here it is assumed that the surface of initial values does not contain any of the characteristic base curves (is "non-characteristic").

The solution $\Gamma = \check{\Gamma}(T, \alpha_1, \ldots, \alpha_N)$ is obtained follows. The general solution of the ordinary system of differential equations (2.121) under due account of the initial values is given by

$$\left.\begin{aligned} T &= s + T_0(r_1, \ldots, r_N) \\ \alpha_\varrho &= \tilde{\alpha}_\varrho\left(s, \alpha_{01}(r_1, \ldots, r_N), \ldots, \alpha_{0N}(r_1, \ldots, r_N)\right) \\ &\qquad \varrho = 1, \ldots, N. \end{aligned}\right\} \quad (2.125)$$

Thus

$$s = T - T_0(r_1, \ldots, r_N).$$

Inserting this into $(2.125)_2$ and solving for $r_1, \ldots r_N$ we get*

$$r_\varrho = \tilde{r}_\varrho(T, \alpha_1, \ldots, \alpha_N), \quad \varrho = 1, \ldots, N$$

and from (2.124) we finally obtain

$$\Gamma = \check{\Gamma}(T, \alpha_1, \ldots, \alpha_N) = \Gamma_0\left(\tilde{r}_1(T, \alpha_1, \ldots, \alpha_N), \ldots, \tilde{r}_N(T, \alpha_1, \ldots, \alpha_N)\right). \quad (2.126)$$

With $(2.125)_1$ the differential equations $(2.121)_2$ read

$$\frac{d\alpha_\varrho}{ds} = \check{H}_\varrho\left(s + T_0(r_\nu), \alpha_1, \ldots, \alpha_N\right), \quad \varrho = 1, \ldots, N. \quad (2.127)$$

For a restricted class of $\check{H}_p$-functions a general closed form solution can be obtained. Assume that H_p *is separable in the variables* T *and* α_p; then

$$\check{H}_\varrho = H(T)\, L_\varrho(\alpha_1, \ldots, \alpha_N) \quad (2.128)$$

and a variable transformation

$$s^* = \int_0^s H(T)\, ds = \int_0^s H(s + T_0)\, ds = \int_{T_0}^{T} H(T)\, dT \quad (2.129)$$

yields

---

* This solution is possible in principle since the surface of initial values is non-characteristic.

$$\frac{d\alpha_\rho}{ds^*} = L_\rho(\alpha_1, \ldots, \alpha_N). \tag{2.130}$$

In addition it is assumed that L_ρ *is linear in* α_ν

$$L_\rho = L_{\rho\nu}\,\alpha_\nu \tag{2.131}$$

where $L_{\rho\nu}$ is a constant matrix. The general solution of (2.130) with (2.131) is

$$\alpha_\rho = C_{\rho 1}\,e^{\lambda_1 s^*} + \ldots + C_{\rho N}\,e^{\lambda_N s^*} \tag{2.132}$$

where the λ_i are the eigenvalues of the matrix $L_{\rho\mu}$

$$[\,L_{\rho\mu} - \lambda_i\,\delta_{\rho\mu}\,]\,C_{\mu i} = 0. \tag{2.133}$$

Here it is assumed that (N) distinct eigenvalues are obtained. The column matrices $C_{\rho 1}$, $C_{\rho 2}$, ... are the eigenvectors corresponding to the eigenvalues α_1, α_2, etc. We introduce normalized eigenvectors $C_{\rho i}'$

$$\sum_\rho (C'_{\rho i})^2 = 1 \tag{2.134}$$

such that

$$C_{\rho i} = c_i\,C'_{\rho i} \qquad \text{(no summation)}\;. \tag{2.135}$$

Then the general solution (2.142) reads

$$\alpha_\rho = \sum_i C_{\rho i}\,e^{\lambda_i s^*} = \sum_i c_i\,C'_{\rho i}\,\exp\left(\lambda_i \int_{T_0}^{T} H(\tau)\,d\tau\right) \tag{2.136}$$

and with the initial condition we obtain

$$\alpha_\rho(s=0) = \alpha_\rho(s^*=0) = \sum_{i=1}^{N} c_i\,C'_{\rho i} \overset{!}{=} \overset{\circ}{\alpha}_\rho(r_1, \ldots, r_N). \tag{2.137}$$

The eigenvectors are linearly independent and thus the inverse of $C_{\rho i}'$ exists; therefore

$$c_i = \sum_{\rho=1}^{N} \overset{-1}{C'_{i\nu}}\,\overset{\circ}{\alpha}_\nu. \tag{2.138}$$

Consequently,

$$\alpha_\rho = \tilde{\alpha}_\rho(s, r_1, \ldots, r_N) = \sum_{i=1}^{N}\sum_{\nu=1}^{N} \overset{\circ}{\alpha}_\nu(r_1, \ldots, r_N)\,\overset{-1}{C}_{i\nu}\,C'_{\rho i}\,\exp\left(\lambda_i \int_0^s H(s+T_0)\,ds\right) \tag{2.139}$$

and this gives the solution $(2.125)_2$ explicitly in terms of the parameters s and r_1, ... r_N.

To obtain the representative solution $\check{\Gamma}$ (T, α_ρ) further assumptions, concerning the initial values $\underset{o}{T}$ (r_ν) and $\underset{o}{\alpha}_\rho$ (r_ν) as well as $\underset{o}{\Gamma}$ (r_ν), have to be made. Since we intend to find a simple transformation function which allows a reduction to the standard form, the following assumptions are made

$$\begin{aligned} T_o(r_1, \dots r_N) &= \underset{o}{T} = const. \\ \underset{o}{\alpha}_\varrho(r_1, \dots, r_N) &= r_\varrho \,. \end{aligned} \tag{2.140}$$

Thus

$$\begin{aligned} \alpha_\varrho &= \sum_{i=1}^{N} \sum_{\nu=1}^{N} r_\nu \, \overset{-1}{C'_{i\nu}} \, C'_{\varrho i} \exp\Big(d_i \int_{\underset{o}{T}}^{T} H(T)\, dT\Big) \\ &= \sum_{\nu=1}^{N} r_\nu \, D_{\nu\varrho}(T, \underset{o}{T}) \end{aligned} \tag{2.141}$$

with

$$D_{\nu\varrho}(T, \underset{o}{T}) = \sum_{i=1}^{N} \overset{-1}{C'_{\nu i}} \, C'_{\varrho i} \exp\Big(d_i \int_{\underset{o}{T}}^{T} H(T)\, dT\Big). \tag{2.142}$$

Assuming $D_{\nu\rho}$ to be non-singular, one gets from (2.141)

$$r_\nu = \sum_{\varrho=1}^{N} \overset{-1}{D}_{\nu\varrho}(T, \underset{o}{T}) \, \alpha_\varrho \,. \tag{2.143}$$

Further, assuming still an *arbitrary* distribution $\Gamma = \Gamma_o(r_1, \dots, r_N)$ on the initial value surface, one obtains the following solution

$$\begin{aligned} \Gamma = \check{\Gamma}(T, \alpha_\varrho) &= \\ = \Gamma_o\Big(\sum_\varrho \overset{-1}{D}_{1\varrho}(T, \underset{o}{T})\alpha_\varrho, \sum_\varrho \overset{-1}{D}_{2\varrho}(T, \underset{o}{T})\alpha_\varrho, \dots, \dots \\ \dots, \sum_\varrho \overset{-1}{D}_{N\varrho}(T, \underset{o}{T})\alpha_\varrho \Big) . \end{aligned} \tag{2.144}$$

Consequently, any set of continuous differentiable functions $\Gamma_{o\gamma}(r_1, \dots r_N)$.

$$\Gamma_{o\gamma}(r_\nu) = \Gamma_{o\gamma}\Big(\sum_\rho \overset{-1}{D}_{1\rho}(T,T_0)\alpha_\rho \,,\, \sum_\rho \overset{-1}{D}_{2\rho}(T,T_0)\alpha_\rho \,,\, \dots,\, \sum_\rho \overset{-1}{D}_{N\rho}(T,T_0)\alpha_\rho\Big)$$

represents a suitable set of transformation relations for the internal variables

$$\alpha_\gamma'' = \overset{\vee}{h}_\gamma(T, \alpha_1, \dots, \alpha_N) = \Gamma_{o\gamma}\Big(\sum_\rho \overset{-1}{D}_{1\rho}(T,T_0)\alpha_\rho, \dots\Big). \quad (2.145)$$

The simplest case is given by putting

$$\Gamma_{o\gamma}(r_1, \dots, r_N) = b\, r_\gamma = b \sum_\rho \overset{-1}{D}_{\gamma\rho}(T,T_0)\alpha_\rho \quad (2.146)$$

where (b) is a constant. Thus,

$$\alpha_\gamma'' = b \sum_\rho \overset{-1}{D}_{\gamma\rho}(T,T_0)\alpha_\rho \quad (2.147)$$

which is a *linear* relation between the primary internal variables and the transformed ones. However, the matrix $D_{\gamma\rho}$ is generally a rather complicated function of the temperature.

In the following a rather simple case is considered, i.e.,

$$L_{\rho\nu} = \delta_{\rho\nu} \quad \Longrightarrow \quad \overset{\vee}{H}_\rho(\alpha_\nu, T) = H(T)\alpha_\rho . \quad (2.148)$$

Thus, the evolution equation is of the form

$$\dot{\alpha}_\gamma = \overset{\vee}{f}_\gamma(\sigma_{mn}, T, g_k, \alpha_\nu) + H(T)\,\alpha_\gamma\,\dot{T}. \quad (2.149)$$

The solution of the characteristic equation

$$\frac{d\alpha_\rho}{H(T)\,ds} = \frac{d\alpha_\rho}{ds^*} = \alpha_\rho \,, \qquad \rho = 1, \dots, N$$

reads

$$\alpha_\rho = c_\rho\, e^{s^*} = c_\rho \exp\Big(\int_0^s H(s+T_0)\,ds\Big). \quad (2.150)$$

Observing the initial conditions $(2.140)_2$, we get

$$\alpha_\rho = r_\rho \exp\Big(\int_0^s H(s+T_0)\,ds\Big)$$

and with $(2.125)_1$ we find

$$r_\rho = \alpha_\rho \exp\Big(-\int_{T_0}^{T} H(T)\,dT\Big). \quad (2.151)$$

With the assumption (2.146) we get finally the following transformation relation

$$\alpha_\gamma'' = \check{h}_\gamma(T, \alpha_\nu) = b\,\alpha_\gamma \exp\Big(-\int_{T_0}^{T} H(T)\,dT\Big) \tag{2.152}$$

where b ≠ 0 is arbitrary. This is an outstanding simple relation.

For this simple case the reduced evolution equation takes the following form

$$\begin{aligned} \mathring{\alpha}_\gamma'' &= \frac{\partial \check{h}_\gamma}{\partial \alpha_\varsigma}\check{f}_\varsigma = \check{f}_\gamma''(\sigma_{mn}, T, g_k, \alpha_\nu'') \\ &= b \exp\Big[-\int_{T_0}^{T} H(T)\,dT\Big]\check{f}_\varsigma\Big(\sigma_{mn}, T, g_k, \tfrac{1}{b}\alpha_\nu'' \exp\Big[-\int_{T_0}^{T} H(T)\,dT\Big]\Big). \end{aligned} \tag{2.153}$$

In section 3.1 we will discuss an example from the literature which falls in this simple category.

Finally, we analyse the simple case where the functions $\check{H}_\gamma$ do not depend on the internal variables and have a common temperature dependence; thus

$$\check{H}_\gamma = H(T)\,C_\varsigma \quad , \qquad C_\varsigma \mathrel{\hat{=}} \text{const.} \tag{2.154}$$

With the substitution

$$\chi = \int_{T_0}^{T} H(T)\,dT \;, \qquad d\chi = H\,dT \tag{2.155}$$

the condition (2.117) takes the form

$$\frac{\partial h_\gamma}{\partial \chi} + \frac{\partial h_\gamma}{\partial \alpha_\varsigma} C_\varsigma = 0 . \tag{2.156}$$

With Γ denoting any of the functions h_γ we have

$$\frac{\partial \Gamma}{\partial \chi} + \frac{\partial \Gamma}{\partial \alpha_\varsigma} C_\varsigma = 0 .$$

The associated characteristic equations are

$$\frac{d\chi}{ds} = 1 \;, \qquad \frac{d\alpha_\varsigma}{ds} = C_\varsigma \mathrel{\hat{=}} \text{const}, \quad \frac{d\Gamma}{ds} = 0 \tag{2.157}$$

their solutions are

$$\left.\begin{aligned} \chi &= s + \underset{0}{\chi}(r_1, \ldots, r_N) \\ \alpha_\varrho &= C_\varrho s + \underset{0}{\alpha}_\varrho(r_1, \ldots, r_N) \\ \Gamma &= \Gamma_0(r_1, \ldots, r_N). \end{aligned}\right\} \quad (2.158)$$

In the (1 + N)-dimensional (χ, α_1, ..., α_N)-space the solutions $\chi(s)$, $\alpha_\rho(s)$ represent a characteristic base curve with curve parameter s "starting off" (s = 0) from a point (χ_0, α_0, ..., α_{0N}) which is located on a N-dimensional surface with surface coordinates r_1, ..., r_N.

Elimination of s between $(2.158)_{1,2}$ yields

$$\alpha_\varrho = C_\varrho \cdot (\chi - \underset{0}{\chi}) + \underset{0}{\alpha}_\varrho . \quad (2.159)$$

We make the especially simple assumptions

$$\left.\begin{aligned} \underset{0}{\chi}(r_1, \ldots, r_N) &= \chi_0 = \text{const.} = 0 \\ \underset{0}{\alpha}_\varrho(r_1, \ldots, r_N) &= r_\varrho \end{aligned}\right\} \quad (2.160)$$

With (2.158) this yields

$$r_\varrho = \alpha_\varrho - C_\varrho \cdot (\chi - \chi_0) = \alpha_\varrho - C_\varrho \chi . \quad (2.161)$$

Since Γ stands for any of the transformations h_γ

$$\Gamma \triangleq h_\gamma ,$$

we have

$$\Gamma = \Gamma_0(r_1, \ldots, r_N) \Longrightarrow h_\gamma = \underset{0}{h}_\gamma(r_1, \ldots, r_N); \quad (2.162)$$

a very simple choice is

$$\underset{0}{h}_\gamma(r_1, \ldots, r_N) = C\, r_\gamma \quad , \; C = \text{const.} . \quad (2.163)$$

Combining $(2.162)_2$, (2.163) and (2.161), the transformation (2.114) with (2.118) reads

$$\begin{aligned} \alpha_\gamma'' &= h_\gamma(T, \alpha_\nu) = \underset{0}{h}_\gamma(r_1, \ldots, r_N) = C\, r_\gamma = \\ &= c\,[\alpha_\gamma - C_\varrho X]. \end{aligned} \tag{2.164}$$

Thus, the transformation of the internal variables α_γ consists simply in a temperature dependent shift $C_\rho X$ of the variables α_γ and a change of scale with scale factor C if $C \neq 1$. Note that this transformation relation is not the only one which satisfies (2.156) but it is very simple.

2.2.3.2 *Case II'': $\underset{\varrho}{\check{G}}_m \equiv 0$, $\underset{\varrho}{\check{H}} \equiv 0$, $\underset{\varrho}{\check{E}}_{mn} \neq 0$*

The assumed evolution equation is given by

$$\dot{\alpha}_\gamma = \check{f}_\gamma(\sigma_{mn}, T, g_k, \alpha_\nu) + \underset{\gamma}{\check{E}}_{pq}(\sigma_{mn}, \alpha_\varrho)\,\dot{\sigma}_{pq};$$

provided $\underset{\gamma}{\check{E}}_{pq}$ satisfies equ. $(2.110)_1$, integrability is assured. The transformation is given by

$$\alpha_\gamma'' = \check{h}_\gamma(\sigma_{kl}, T, g_l, \alpha_\mu)$$

and its time derivative reads

$$\begin{aligned} \dot{\alpha}_\gamma'' &= \frac{\partial \check{h}_\gamma}{\partial \sigma_{kl}}\dot{\sigma}_{kl} + \frac{\partial \check{h}_\gamma}{\partial T}\dot{T} + \frac{\partial \check{h}_\gamma}{\partial g_l}\dot{g}_l + \frac{\partial \check{h}_\gamma}{\partial \alpha_\mu}\dot{\alpha}_\mu = \\ &= \frac{\partial \check{h}_\gamma}{\alpha_\mu}\check{f}_\mu + \left(\frac{\partial \check{h}_\gamma}{\partial \sigma_{kl}} + \underset{\nu}{\check{E}}_{kl}\frac{\partial \check{h}_\gamma}{\partial \alpha_\nu}\right)\dot{\sigma}_{kl} + \\ &\quad + \frac{\partial \check{h}_\gamma}{\partial T}\dot{T} + \frac{\partial \check{h}_\gamma}{\partial g_l}\dot{g}_l . \end{aligned}$$

Reduceability requires

$$\frac{\partial \check{h}_\gamma}{\partial T} = 0 \quad , \quad \frac{\partial \check{h}_\gamma}{\partial g_l} = 0 \tag{2.165}$$

$$\frac{\partial \check{h}_\gamma}{\partial \sigma_{kl}} + \underset{\nu}{\check{E}}_{kl}\frac{\partial \check{h}_\gamma}{\partial \alpha_\nu} = 0 . \tag{2.166}$$

Thus

$$\alpha_\gamma'' = \check{h}_\gamma(\sigma_{kl}, \alpha_\mu). \tag{2.167}$$

To find a solution $\check{h}_\gamma(\sigma_{kl}, \alpha_\mu)$ satisfying (2.166) is generally a formidable problem. We present here only a transformation function $\check{h}_\gamma$ when E_{mn} is a set of constant values

$$\underset{\nu}{\check{E}}_{mn} \sim const. \tag{2.168}$$

independent of the temperature. Assume that $\check{h}_\gamma$ is a linear function in α_μ and σ_{kl} such that

$$\check{h}_\gamma = \underset{\gamma}{A}_{pq}\,\sigma_{pq} + \underset{\gamma}{B}_\mu\,\alpha_\mu \tag{2.169}$$

where A_{pq} and B_μ are sets of constant values. Inserting (2.169) in (2.166) gives

$$\underset{\gamma}{A}_{mn} + \underset{\nu}{\check{E}}_{mn}\,\underset{\gamma}{B}_\nu = 0. \tag{2.170}$$

We choose

$$\underset{\gamma}{A}_{mn} = -\underset{\nu}{\check{E}}_{mn}\,\underset{\gamma}{B}_\nu$$

and (2.166) is satisfied. Consequently

$$\check{h}_\gamma = \underset{\gamma}{B}_\mu \cdot \left(\alpha_\mu - \underset{\mu}{\check{E}}_{pq}\,\sigma_{pq}\right) \tag{2.171}$$

is an admissible transformation function where the B_μ are arbitrary constant factors. However, it should be noted that any set of functions G_ρ of $\check{h}_\gamma$

$$\check{h}_\rho^* = G_\rho(\check{h}_\gamma)$$

yields another set of admissible transformation functions $\check{h}_\rho^*$.

2.2.4 Thermodynamic Consistency of Assumed Strain-Stress Relations

In section 2.2.1 the assumed constitutive relations are (2.79) and (2.80) with the strain-stress relation given by $(2.79)_1$. In the following a less general class will be assumed, i.e.,

$$\varepsilon_{mn} = \check{\varepsilon}_{mn}(\sigma_{kl}, T, \alpha_\nu)$$

$$\left.\begin{aligned} \epsilon &= \check{\epsilon}(\sigma_{kl}, T, \alpha_{\nu}) \\ \eta &= \check{\eta}(\text{—''—}) \\ q_k &= \check{q}_k(\text{—''—}) \end{aligned}\right\} \quad (2.172)$$

and the evolution equations

$$\begin{aligned} \dot{\alpha}_{\gamma} = {} & \check{j}_{\gamma}(\sigma_{kl}, T, \alpha_{\nu}) + \\ & + \underset{\gamma}{\check{E}}_{mn}(\text{—''—})\,\dot{\sigma}_{mn} + \\ & + \underset{\gamma}{\check{H}}(\text{—''—})\,\dot{T}\,. \end{aligned} \quad (2.173)$$

With these assumptions a *direct evaluation* of the Clausius-Duhem entropy inequality is possible using homogeneous fields for σ_{kl}, T, and $\underset{\circ}{\alpha}_{\gamma}$; this yields the following necessary and sufficient conditions

$$\left.\begin{aligned} \rho\frac{\partial\check{\zeta}}{\partial\sigma_{kl}} + \rho\frac{\partial\check{\zeta}}{\partial\alpha_{\nu}}\underset{\nu}{\check{E}}_{kl} + \check{\varepsilon}_{kl} &= 0 \\ \frac{\partial\check{\zeta}}{\partial T} + \frac{\partial\check{\zeta}}{\partial\alpha_{\nu}}\underset{\nu}{\check{H}} + \check{\eta} &= 0 \end{aligned}\right\} \quad (2.174)$$

and

$$-\rho\frac{\partial\check{\zeta}}{\partial\alpha_{\nu}}\check{j}_{\nu} \geq 0 \,, \qquad -\frac{\check{q}_k\, g_k}{T} \geq 0\,. \quad (2.175)$$

Note that the functions ζ, $\check{\eta}$, $\check{\varepsilon}_{kl}$, $\underset{\gamma}{\check{H}}$ depend only on the variables σ_{kl}, T, and α_{ν}.

In the following we assume that an *ad hoc-model* has been proposed by prescribing the functions

$$\check{\varepsilon}_{kl}(\sigma_{mn}, T, \alpha_{\nu}) \,, \quad \check{j}_{\gamma}(\text{—''—}) \,, \quad \underset{\gamma}{\check{E}}_{mn}(\text{—''—}) \,, \quad \underset{\gamma}{\check{H}}(\text{—''—})$$

and the functions $\underset{\gamma}{\check{E}}_{mn}$ and $\underset{\gamma}{\check{H}}$ are such that the *evolution equations (2.173) are not reduceable*. Such a choice of functions is done when purely mechanical theories are developed without a thermodynamic embedding; then, of course, the internal energy $\check{\varepsilon}$ and entropy $\check{\eta}$ are not a matter of concern. Nevertheless, one should require that the mechanical theory is thermodynamically consistent in the sense that it can be considered as a special case of a more embracing thermodynamic theory. Thus, one should require that there exists a Gibbs function satisfying the restriction $(2.174)_1$. We will extend this question by assuming that also $\check{\eta}$ is given by an ad hoc choice. Thus, the ad hoc model consists of a choice of

$$\check{\varepsilon}_{kl}\ ,\ \check{f}_{\gamma}\ ,\ \underset{\gamma}{\check{E}}_{mn}\ ,\ \underset{\gamma}{\check{H}}\ ,\ \check{\eta} \tag{2.176}$$

all of them functions of $(\sigma_{mn}, T, \alpha_\nu)$. It is evident that such a choice is not necessarily compatible with the two restrictions (2.174): Thermodynamics requires that there exists a Gibbs function ζ satisfying both equations (2.174). Since the functions (2.176) are given, the equations (2.174) represent an *overdetermined* set of 6 + 1 partial differential equations for single function ζ. Therefore, not every choice of the functions (2.176) allows the existence of a common solution ζ.

With respect to the (6 + 1 + N) derivatives

$$\frac{\partial\check{\zeta}}{\partial\sigma_{kl}} = \frac{\partial\check{\zeta}}{\partial\sigma_{lk}}\ ,\ \frac{\partial\check{\zeta}}{\partial T}\ ,\ \frac{\partial\check{\zeta}}{\partial\alpha_\nu}\ ,\quad \nu = 1,\dots,N$$

the equations (2.174) are an *underdetermined* system of (6 + 1) linear equations. Thus, equ. (2.174) allows to solve for $\partial\check{\zeta}/\partial\sigma_{kl}$ and $\partial\check{\zeta}/\partial T$ in terms of all other quantities including $\partial\check{\zeta}/\partial\alpha_\nu$; but $\partial\check{\zeta}/\partial\alpha_\nu$ cannot be obtained from (2.74). Therefore, the theory of *overdetermined* systems of linear partial differential equations of first order [67], which yields necessary and sufficient conditions for the existence of a common solution, is not simply applicable.

We consider α_ν as a set of parameters and $\partial\check{\zeta}/\partial\alpha_\nu$ as functions, which we do not explicitely prescribe*, except that they should not necessarily vanish. Then (2.174) is a system of (6 + 1) differential equations for the single function $\check{\zeta}$ with (6 + 1) independent variables σ_{kl} and T.

---

* In irreversible thermodynamics the derivatives $-\rho\ \partial\check{\zeta}/\partial\alpha_\nu$ are identified as the affinities A_ν or thermodynamic forces. If had hoc assumptions $A_\nu = \check{A}_\nu(\alpha_{mn}, T, \alpha_\nu)$ are made, then an additional set of equations is available.

We restrict the further discussion to the equ. $(2.174)_1$ which will, in fact, show the essentials. The inclusion of $(2.174)_2$ is done by a proper naming of the independent variables and the given functions since both equations (2.174) have the same structure.

Equ.'s $(2.174)_1$ represent six partial differential equations for the Gibbs function. If a solution $\check{\zeta}$ is required to exist, the functions $\underset{\nu}{\check{E}}_{kl}$ and $\check{\varepsilon}_{kl}$ are not arbitrary but have to satisfy certain conditions. Provided

$$\underset{\nu}{\check{E}}_{kl} \equiv 0 \tag{2.177}$$

the existence of a differentiable common solution $\check{\zeta}$ of $(2.174)_1$ implies that the second order partial differentials are interchangeable, i.e.,

$$\frac{\partial^2 \check{\zeta}}{\partial \sigma_{kl}\, \partial \sigma_{mn}} = \frac{\partial^2 \check{\zeta}}{\partial \sigma_{mn}\, \partial \sigma_{kl}}$$

and this yields the necessary integrability condition

$$\frac{\partial \check{\varepsilon}_{kl}}{\partial \sigma_{mn}} = \frac{\partial \check{\varepsilon}_{mn}}{\partial \sigma_{kl}} \tag{2.178}$$

which is also a sufficient condition (see [67]). For this simple situation this relation represents the thermodynamic consistency condition for the strain-stress function. If $\underset{\nu}{\check{E}}_{kl}$ is nonzero,we obtain with (2.174)

$$\begin{aligned} \frac{\partial^2 \zeta}{\partial \sigma_{pq}\, \partial \sigma_{rs}} &= \frac{\partial}{\partial \sigma_{rs}} \left[- \frac{\partial \check{\zeta}}{\partial \alpha_\nu} \underset{\nu}{\check{E}}_{pq} - \frac{1}{\rho} \check{\varepsilon}_{pq} \right] = \\ &= - \frac{\partial^2 \check{\zeta}}{\partial \alpha_\nu\, \partial \sigma_{rs}} \underset{\nu}{\check{E}}_{pq} - \frac{\partial \check{\zeta}}{\partial \alpha_\nu} \frac{\partial \underset{\nu}{\check{E}}_{pq}}{\partial \sigma_{rs}} - \frac{1}{\rho} \frac{\partial \check{\varepsilon}_{pq}}{\partial \sigma_{rs}} = \\ &= - \frac{\partial^2 \zeta}{\partial \alpha_\nu\, \partial \alpha_\gamma} \underset{\nu}{\check{E}}_{pq} - \frac{\partial \check{\zeta}}{\partial \alpha_\nu} \frac{\partial \underset{\nu}{\check{E}}_{pq}}{\partial \alpha_\gamma} - \frac{1}{\rho} \frac{\partial \check{\varepsilon}_{pq}}{\partial \alpha_\gamma} . \end{aligned} \tag{2.179}$$

With the change of indices from (p,q) to (r,s), (γ) to (v), and (v) to (ρ) we get from $(2.179)_2$

$$\frac{\partial^2 \check{\xi}}{\partial \alpha_v \, \partial \sigma_{rs}} = - \frac{\partial^2 \check{\xi}}{\partial \alpha_\varrho \, \partial \alpha_v} \underset{\varrho}{\check{E}}_{rs} - \frac{\partial \check{\xi}}{\partial \alpha_\varrho} \frac{\partial \underset{\varrho}{\check{E}}_{rs}}{\partial \alpha_v} - \frac{1}{\varrho} \frac{\partial \check{\varepsilon}_{rs}}{\partial \alpha_v} \, . \tag{2.180}$$

Exchanging (p,q) and (r,s) in $(2.179)_1$ and substracting this from $(2.179)_1$ we obtain

$$\frac{\partial^2 \check{\xi}}{\partial \sigma_{pq} \, \partial \sigma_{rs}} - \frac{\partial^2 \check{\xi}}{\partial \sigma_{rs} \, \partial \sigma_{pq}} = - \frac{\partial \check{\xi}}{\partial \alpha_v} \left(\frac{\partial \underset{v}{\check{E}}_{pq}}{\partial \sigma_{rs}} - \frac{\partial \underset{v}{\check{E}}_{rs}}{\partial \sigma_{pq}} \right) -$$

$$- \frac{\partial^2 \check{\xi}}{\partial \alpha_v \, \partial \sigma_{rs}} \underset{v}{\check{E}}_{pq} + \frac{\partial^2 \check{\xi}}{\partial \alpha_v \, \partial \sigma_{pq}} \underset{v}{\check{E}}_{rs} - \tag{2.181}$$

$$- \frac{1}{\varrho} \left(\frac{\partial \check{\varepsilon}_{pq}}{\partial \sigma_{rs}} + \frac{\partial \check{\varepsilon}_{rs}}{\partial \sigma_{pq}} \right) = 0 \, .$$

Inserting (2.180) in (2.181) with a proper choice of the indices, several terms drop out and we derive the following relation

$$\left\{ \frac{\partial \underset{v}{\check{E}}_{pq}}{\partial \sigma_{rs}} - \frac{\partial \underset{v}{\check{E}}_{rs}}{\partial \sigma_{pq}} - \left(\frac{\partial \underset{v}{\check{E}}_{rs}}{\partial \alpha_\varrho} \underset{\varrho}{\check{E}}_{pq} - \frac{\partial \underset{v}{\check{E}}_{pq}}{\partial \alpha_\varrho} \underset{\varrho}{\check{E}}_{rs} \right) \right\} \frac{\partial \check{\xi}}{\partial \alpha_v} +$$

$$+ \frac{1}{\varrho} \left[\frac{\partial \check{\varepsilon}_{pq}}{\partial \sigma_{rs}} - \frac{\partial \check{\varepsilon}_{rs}}{\partial \sigma_{pq}} - \left(\frac{\partial \check{\varepsilon}_{rs}}{\partial \alpha_\varrho} \underset{\varrho}{\check{E}}_{pq} - \frac{\partial \check{\varepsilon}_{pq}}{\partial \alpha_\varrho} \underset{\varrho}{\check{E}}_{rs} \right) \right] =$$

$$= 0 \, . \tag{2.182$_1$}$$

The conditions for the complete set of equations (2.174) are obtained by supplementing $(2.182)_1$ with the conditions

$$\left\{ \frac{\partial \underset{v}{\check{E}}_{pq}}{\partial T} - \frac{\partial \underset{v}{\check{H}}}{\partial \sigma_{pq}} - \left(\frac{\partial \underset{v}{\check{H}}}{\partial \alpha_\varrho} \underset{\varrho}{\check{E}}_{pq} - \frac{\partial \underset{v}{\check{E}}_{pq}}{\partial \alpha_\varrho} \underset{\varrho}{\check{H}} \right) \right\} \frac{\partial \check{\xi}}{\partial \alpha_v} +$$

$$+ \frac{1}{\varrho} \left[\frac{\partial \check{\varepsilon}_{pq}}{\partial T} - \varrho \frac{\partial \check{\eta}}{\partial \sigma_{pq}} - \left(\varrho \frac{\partial \check{\eta}}{\partial \alpha_\varrho} \underset{\varrho}{\check{E}}_{pq} - \frac{\partial \check{\varepsilon}_{pq}}{\partial \alpha_\varrho} \underset{\varrho}{\check{H}} \right) \right] =$$

$$= 0 \, . \tag{2.182$_2$}$$

This was obtained from $(2.182)_1$ by a proper renaming of the functions in $(2.182)_1$ according to the similarity of $(2.179)_1$ and $(2.179)_2$. We write (2.182) in a more compact form using matrix notation. The derivatives $\partial\zeta/\partial\alpha_\nu$, $\nu = 1, \ldots N$ are collected in the N-dimensional column matrix

$$\mathbb{z} := \left\{ \frac{\partial \breve{\zeta}}{\partial \alpha_\nu} , \quad \nu = 1, \ldots, N \right\}. \tag{2.183}$$

The associated coefficients, i.e. the { ... }-brackets in (2.182) are collected in a (M, N)-matrix $\mathbb{K}$. Since σ_{rs} is symmetric and $\underset{s}{\breve{E}}_{pq}$ is assumed to be symmetric, the { ... }-brackets in $(2.182)_1$ represent $(6\cdot6 - 6)/2 = 15$ terms for each greek subscript ν. In $(2.182)_2$ the { ... }-brackets represent six terms for each ν. Therefore, the number of rows of the matrix $\mathbb{K}$ is $M = 15 + 6 = 21$ and the number of columns is N. In general N, the number of internal variables, is much less than 21

$$N < M = 21 . \tag{2.184}$$

The [...]-brackets in (2.182) are collected in a column vector $\mathbb{c}$ with dimension 21,

$$\mathbb{c} \mathrel{\hat{=}} \left\{ \begin{array}{l} -\frac{1}{\varrho}\Big[\frac{\partial \breve{\varepsilon}_{pq}}{\partial \sigma_{rs}} - \frac{\partial \breve{\varepsilon}_{rs}}{\partial \sigma_{pq}} - \Big(\frac{\partial \breve{\varepsilon}_{rs}}{\partial \alpha_s} \underset{s}{\breve{E}}_{pq} - \frac{\partial \breve{\varepsilon}_{pq}}{\partial \alpha_s} \underset{s}{\breve{E}}_{rs}\Big) \\ \\ -\frac{1}{\varrho}\Big[\frac{\partial \breve{\varepsilon}_{pq}}{\partial T} - \varrho \frac{\partial \breve{\eta}}{\partial \sigma_{pq}} - \Big(\varrho \frac{\partial \breve{\eta}}{\partial \alpha_s} \underset{s}{\breve{E}}_{pq} - \frac{\partial \breve{\varepsilon}_{pq}}{\partial \alpha_s} \underset{s}{\breve{H}}\Big)\Big] \end{array} \right\} \tag{2.185}$$

such that (2.182) takes the form

$$\mathbb{K}\, \mathbb{z} = \mathbb{c} . \tag{2.186}$$

With (2.184) this represents a system of *overdetermined* linear equations for the unknown vector $\mathbb{z}$.

The conditions (2.182) or (2.186) are based on the assumed existence of a common solution ζ of the system (2.174) and on the interchangeability of the partial derivatives of ζ. They are not integrability conditions in the usual sense since they still involve the derivatives $\partial\zeta/\partial\alpha_\nu$. It is evident that not every choice of functions $\underset{\nu}{\breve{E}}_{pq}$, $\underset{\nu}{\breve{H}}$, $\breve{\varepsilon}_{pq}$, $\breve{\eta}$ or matrix $\mathbb{K}$ and $\mathbb{c}$ admits a solution $\mathbb{z}$; but, of course, the existence of a solution is required, otherwise a function ζ satisfying (2.174) does not exist. In fact the existence of a solution $\mathbb{z}$ represents a necessary condition.

From the theory of linear equations, where the elements of $\mathbb{K}$ and $\mathbb{c}$ are usually fixed numbers, we know that a solution $\mathbb{z}$ exists under the following conditions [71]:

- In general the system (2.186) with M>N allows a solution only for special vectors c; a solution need not exist. If there is a solution, it is not necessarily unique; it contains D = N - R arbitrary constants where D is the defect und R is the rank of the matrix $\mathbb{K}$.

  A solution exists for the inhomogeneous system then and only then when the equations representing the system are compatible with each other. This is the case when the rank of the extended matrix ($\mathbb{K}, \mathbb{c}$) is not larger than that of $\mathbb{K}$.

According to [71], the practical numerical determination of this last property is rather simple when the matrices are given by specific numerical values. However, if the elements of the matrices are themselves functions of various variables (here σ_{mn}, T, a_v), a prove of the existence of a solution $\mathbb{z}$ must be obtained for *all* admissible choices of the variables (σ_{mn}, T, a_v). This can be done numerically only for a few selected states and an analytical prove appears to be feasible only for simple cases.

We will now consider several special cases.

Case I: $\mathbb{K} \equiv \mathbb{0}$

We observe that the first { ... }-brackets in (2.182) or alternatively the matrix $\mathbb{K}$ involve only the $\overset{\vee}{\underset{\vee}{E}}_{pq}$- and $\overset{\vee}{\underset{\vee}{H}}$-functions and they control the reduceability of the evolution equations (see equ. $(2.108)_{1+2}$); in fact the identical vanishing of the elements of $\mathbb{K}$

$$\mathbb{K} = \mathbb{0} \tag{2.187}$$

are here the only *reduceability conditions* since $\overset{\vee}{\underset{\vee}{G}}_k \equiv 0$ and $\overset{\vee}{\underset{\vee}{E}}_{pq}$ as well as $\overset{\vee}{\underset{\vee}{H}}$ are independent of the temperature gradient.

If reduceability of the evolution equations for the internal variables is possible, then $\mathbb{K} \equiv \mathbb{0}$ and (2.186) yields

$$\mathbb{C} \equiv \mathbb{0} \tag{2.188}$$

or explicitely

$$\left.\begin{aligned} &\frac{\partial \check{\varepsilon}_{pq}}{\partial \sigma_{rs}} - \frac{\partial \check{\varepsilon}_{rs}}{\partial \sigma_{pq}} - \left(\frac{\partial \check{\varepsilon}_{rs}}{\partial \alpha_{s}} \underset{s}{\check{E}}_{pq} - \frac{\partial \check{\varepsilon}_{pq}}{\partial \alpha_{s}} \underset{s}{\check{E}}_{rs} \right) = 0 \\ &\frac{\partial \check{\varepsilon}_{pq}}{\partial T} - \rho \frac{\partial \check{\eta}}{\partial \sigma_{pq}} - \left(\rho \frac{\partial \check{\eta}}{\partial \alpha_{s}} \underset{s}{\check{E}}_{pq} - \frac{\partial \check{\varepsilon}_{pq}}{\partial \alpha_{s}} \underset{s}{\check{H}} \right) = 0. \end{aligned}\right\} (2.188)'$$

Thus, if reduceability is possible, the condition (2.186) yields the integrability conditions (2.188) which assure the existence of a Gibbs function ζ depending on internal variables ($\mathbf{z} \neq \mathbf{0}$). In this case the equs. (2.188) represent the thermodynamic consistency condition for the strain and entropy functions $\check{\varepsilon}_{pq}$ and $\check{\eta}$.

Reduceability implies that there exists appropriately transformed internal variables α_γ'

$$\alpha_\gamma' = h_\gamma (\sigma_{pq}, T, \alpha_s) \quad , \quad \gamma = 1, \dots, N$$

such that the evolution equations are of the form

$$\dot{\alpha}_\gamma' = \frac{\partial h_\gamma}{\partial \alpha_s} \check{f}_s = \check{f}_\gamma'(\sigma_{pq}, T, \alpha').$$

Then all constitutive functions can be represented as functions of σ_{pq}, T, α_v' and the classical potential relations for the strains $\check{\varepsilon}'_{mn}$ (σ_{pq}, T, α_v') and entropy $\check{\eta}'$ (σ_{pq}, T, α_v') apply. The corresponding integrability conditions are

$$\frac{\partial \check{\varepsilon}'_{mn}}{\partial \sigma_{pq}} - \frac{\partial \check{\varepsilon}'_{pq}}{\partial \sigma_{mn}} = 0 \quad , \quad \frac{\partial \check{\varepsilon}'_{pq}}{\partial T} - \rho \frac{\partial \check{\eta}'}{\partial \sigma_{pq}} = 0. \qquad (2.189)$$

Note that the conditions (2.189) are just alternative formulations of (2.188)' in terms of the new internal variables.

Case II: $\mathbf{c} \equiv \mathbf{0}$

The general inhomogeneous problem (2.186) will not be treated in the following; here the remarks on page 67 may suffice. Instead, we consider the special case that the given strain and entropy functions $(2.176)_1$ and $(2.176)_5$ are independent of the internal variables and are deriveable from a potential function $\zeta^e(\sigma_{mn}, T)$ such that

$$\check{\varepsilon}_{kl} = -\varrho \frac{\partial \xi^e(\sigma_{mn},T)}{\partial \sigma_{kl}} \quad , \quad \check{\eta} = -\frac{\partial \xi^e(\sigma_{mn},T)}{\partial T} \quad . \tag{2.190}$$

Although this appears to be a peculiar case some mechanical models proposed in the literature are special cases of $(2.190)_1$, for example Hooke's law. Naturally, their thermodynamic consistence needs to be analyzed; an example is studied in section 3.3. This motivates the following general analysis. If $\check{\varepsilon}_{kl}$ and $\check{\eta}$ are chosen such that (2.190) applies, then

$$\mathbb{C} = \mathbb{0} \tag{2.191}$$

and (2.186) simplifies to the homogeneous system

$$\mathbb{K}\mathbb{z} = \mathbb{0} \tag{2.192}$$

with the (M, N)-matrix $\mathbb{K}$ (M > N).

The theory of overdetermined homogeneous linear systems of equations yields the following theorem [71]:

> A homogeneous system with N unknowns admits always the trivial solution $\mathbb{z} = \mathbb{0}$. Non-trivial solutions $\mathbb{z} \neq \mathbb{0}$ exist then and only then when the column matrices $\mathbb{b}_\nu$ of the matrix $\mathbb{K}$.
>
> $$\mathbb{K} = (\mathbb{b}_1, \ldots, \mathbb{b}_N) \tag{2.193}$$
>
> are *linear dependent*, i.e., the rank is less than N or the defect D = N - R is positive. The general non-trivial solution $\mathbb{z}$ is not unique; it consists of D > 0 linearly independent solutions $\mathbb{z}_k$, such that
>
> $$\mathbb{z} = a_1 \mathbb{z}_1 + a_2 \mathbb{z}_2 + \cdots a_D \mathbb{z}_D$$
> $$\mathbb{K}\mathbb{z}_k = \mathbb{0} \quad , \qquad k = 1, \ldots, D \tag{2.194}$$
>
> with free parameters a_k.
>
> If the column matrices $\mathbb{b}_\nu$ are linearly independent, i.e., R = N < M, then only a trivial solution $\mathbb{z} = \mathbb{0}$ exists.

From this statement the following conclusions can be drawn with respect to the satisfaction of the condition (2.192).

Case II.1: The trivial case $\mathbb{z} \equiv 0$

Obviously, if

$$\mathbb{z} = \{\partial \check{\zeta}/\partial \alpha_\nu\} \equiv \mathbb{0}, \tag{2.195}$$

the condition (2.192) is trivially satisfied formally whatever the structure of the matrix $\mathbb{K}$. But this implies that the Gibbs function *does not depend* on any of the internal variables α_ν. The existence of a Gibbs function ζ satisfying (2.174) is a basic thermodynamic requirement. However, whether or not the Gibbs function should depend on the internal variables α_ν is a matter of a physical assumption. If the independence of ζ on α_ν, $\nu = 1 \ldots N$ is acceptable on physical grounds, then the condition (2.192) yields no restrictions on the elements of the matrix $\mathbb{K}$, that is on the constitutive functions $\underset{\varrho}{\check{E}}_{pq}$ and $\underset{\varrho}{\check{H}}$, and the Gibbs function is simply given by

$$\check{\zeta} = \zeta^e(\sigma_{mn}, T) \ . \tag{2.196}$$

Of course, the internal energy ϵ is given by (see (2.82))

$$\check{\epsilon} = \check{\zeta} + T\check{\eta} + \frac{1}{\varrho}\sigma_{kl}\check{\varepsilon}_{kl} \tag{2.197}$$

and is not allowed to depend on the internal variables, too. Then the assumed constitutive equations (2.172) and (2.173) are thermodynamically consistent provided the residual dissipation inequalities are satisfied.

Case II.2: The non-trivial case $\mathbb{z} = \{\partial\zeta/\partial\alpha_\nu\} \not\equiv \mathbb{0}$

If the Gibbs function is required to depend on the internal variables because of physical reasons, and this is usually the case, then the trivial case $\mathbb{z} \equiv \mathbb{0}$ is not acceptable. That is, we require that (2.192) admits a non-trivial solution $\mathbb{z} \not\equiv 0$. However, this imposes restrictions on the properties of the (M, N)-matrix $\mathbb{K}$.

Case II.2.1: $\mathbb{K} \equiv \mathbb{0}$ $\Rightarrow$ Reduceability of evolution equations

The conditions (2.192) are trivially satisfied if all elements of $\mathbb{K}$ vanish identically. But these are exactly the conditions of reduceability. From the above theorem on homogeneous systems of linear equations it is clear that $\mathbb{K} \equiv 0$ is only a sufficient condition. Thus, reduceability is not a necessary consequence of (2.192) if $\mathbb{z} \not\equiv 0$ is required. Reduceability imposes restrictions on the functions $\underset{\varrho}{\check{E}}_{pq}$ and $\underset{\varrho}{\check{H}}$. But the assumption $\mathbb{z} \not\equiv 0$ implies that the internal energy ϵ, equ. (2.197), depends on internal variables. Thus, we have a somewhat exceptional situation that $\check{\zeta}$ and $\check{\epsilon}$

depends on internal variables but $\check{\varepsilon}_{kl}$ and $\check{\eta}$ are independent by assumption. Formal thermodynamic consistence is achieved if $\underset{s}{\check{E}}_{pq}$ and $\underset{s}{\check{H}}$ satisfy the reduceability conditions and if also the dissipation inequalities are satisfied.

Case II.2.2: $\mathbb{K} \not\equiv \mathbb{0}$

We assume that the functions $\underset{s}{\check{E}}_{pq}$ and $\underset{s}{\check{H}}$ do not satisfy the reduceability conditions and therefore $\mathbb{K} \not\equiv \mathbb{0}$. Since $\mathbb{z} \not\equiv 0$ is required, then, according to the above theorem, the elements of $\mathbb{K}$ must be related to each other in a special way: The column matrices $\mathbb{b}_v$, $v = 1, \dots, N$ of $\mathbb{K}$ are required to be linearly dependent, i.e., there should be N quantities y_k, $k = 1, \dots, N$ such that

$$y_1 \mathbb{b}_1 + y_2 \mathbb{b}_2 + \dots + y_N \mathbb{b}_N = \mathbb{0} \quad , \quad \sum_{k=1}^{N} (y_k)^2 \neq 0 .$$

Since the $\mathbb{b}_v$ depend on $\underset{s}{\check{E}}_{pq}$ and $\underset{s}{\check{H}}$ which are functions of the independent state variables (σ_{mn}, T, α_p), the linear dependence is required for all state variables. Therefore, the quantities y_k may be state functions themselves.

In general, the linear dependence appears to be a rather unlikely situation. Nevertheless, it cannot be excluded. However, an analytical approach to test the linear dependence for all sets of state variables (σ_{mn}, T, α_v) is not feasible, except for simple cases. But a numerical analysis [71] for one or a few sets of state variables may suffice.

If it is found that the rank of the (M, N)-matrix $\mathbb{K}$ is equal to N,

$$R := \mathrm{Rank}(\mathbb{K}) = N ,$$

then the $\mathbb{b}_v$ are linearly independent. Thus, condition (2.192) cannot be satisfied under the assumption $\mathbb{z} \neq \mathbb{0}$. *This means that the functions $\underset{s}{\check{E}}_{pq}$, $\underset{s}{\check{H}}$ which determine the elements of $\mathbb{K}$ are thermodynamically inconsistent or the assumptions (2.190) are inadmissible or both.*

If, on the other hand, it is found that the rank is less than N for some finite domain $\mathcal{D}$ in the (σ_{mn}, T, α_p)-state space,

$$R < N \quad , \qquad \forall\, (\sigma_{mn}, T, \alpha_p) \in \mathcal{D}$$

then the $\mathbb{b}_v$ are linearly dependent. Therefore, there exists a non-trivial solution $\mathbb{z}$.

The solution $\mathbb{z}$ depends on the elements of $\mathbb{K}$ and D = N - R free parameters a_k. Both D and a_k, $k = 1, \dots, D$ may depend on the state variables. Thus, if z_v is a component of $\mathbb{z}$, then

$$z_\nu = \sum_{k=1}^{D} a_k \, \mathfrak{z}_{k\nu} \tag{2.198}$$

where $\mathfrak{z}_{k\nu}$ are the components of the D linearly independent column matrices $\mathbf{z}_k$, equ. (2.194). This satisfies the condition (2.192) with $\mathbb{K} \neq \mathbf{0}$ and $\mathbf{z} \neq \mathbf{0}$ in a formal way.

However, this is not sufficient to assure the existence of a Gibbs function ζ depending on the internal variables since (2.192) is only necessary. For example, it is also required that the z_ν's are derivatives of ζ, i.e.,

$$z_\nu = \frac{\partial \zeta}{\partial \alpha_\nu} \, , \quad \nu = 1, \dots, N \tag{2.199}$$

which is only the case if

$$\frac{\partial z_\nu}{\partial \alpha_\mu} = \frac{\partial z_\mu}{\partial \alpha_\nu} \, . \tag{2.200}$$

These additional requirements are made more explicit in the following.

We will take a different starting point for the study of Case II.2. Since we require $\mathbf{z} \neq \mathbf{0}$, the Gibbs function ζ can always be represented in the following form

$$\check{\zeta}(\sigma_{mn}, T, \alpha_s) = \zeta^e(\sigma_{mn}, T) + \zeta^\alpha(\sigma_{mn}, T, \alpha_s) . \tag{2.201}$$

Introducing this into (2.174), we get with (2.190)

$$\left.\begin{aligned} \frac{\partial \zeta^\alpha}{\partial \sigma_{kl}} + \frac{\partial \zeta^\alpha}{\partial \alpha_\nu} \underset{\nu}{\check{E}}_{kl} &= 0 \\ \frac{\partial \zeta^\alpha}{\partial T} + \frac{\partial \zeta^\alpha}{\partial \alpha_\nu} \underset{\nu}{\check{H}} &= 0 \, . \end{aligned}\right\} \tag{2.202}$$

Formally, these equations are analogous to the determining differential equations (2.57) for a single transformation function $\Gamma = h_\gamma$. Thus the conditions (2.69) derived previously are applicable here and take the following explicit form

$$\left.\begin{aligned}
&\frac{\partial \check{\xi}^{\alpha}}{\partial \alpha_{\nu}}\left\{\frac{\partial \check{\underset{\nu}{E}}_{pq}}{\partial \sigma_{rs}}-\frac{\partial \check{\underset{\nu}{E}}_{rs}}{\partial \sigma_{pq}}-\left(\frac{\partial \check{\underset{\nu}{E}}_{rs}}{\partial \alpha_{\varrho}}\check{\underset{\varrho}{E}}_{pq}-\frac{\partial \check{\underset{\nu}{E}}_{pq}}{\partial \alpha_{\varrho}}\check{\underset{\varrho}{E}}_{rs}\right)\right\}=0\\
&\frac{\partial \check{\xi}^{\alpha}}{\partial \alpha_{\nu}}\left\{\frac{\partial \check{\underset{\nu}{E}}_{pq}}{\partial T}-\frac{\partial \check{\underset{\nu}{H}}}{\partial \sigma_{pq}}-\left(\frac{\partial \check{\underset{\nu}{H}}}{\partial \alpha_{\varrho}}\check{\underset{\varrho}{E}}_{pq}-\frac{\partial \check{\underset{\nu}{E}}_{pq}}{\partial \alpha_{\varrho}}\check{\underset{\varrho}{H}}\right)\right\}=0
\end{aligned}\right\}\quad(2.203)$$

or equivalently

$$\mathbb{K}\,\mathbb{Z}^{\alpha}=\mathbb{0} \tag{2.204}$$

with

$$\mathbb{Z}^{\alpha}=\left\{\partial\check{\xi}/\partial\alpha_{\nu}\right\}. \tag{2.205}$$

Of course, equ. (2.204) follows also immediately from (2.192) since with (2.201)

$$\mathbb{Z}^{\alpha}=\left\{\partial\check{\xi}/\partial\alpha_{\nu}\right\}=\mathbb{0}+\mathbb{Z}^{\alpha}=\mathbb{Z}^{\alpha}. \tag{2.206}$$

The case studies Case II.2.1 and 2.2 are now supplemented by the following statements.

Supplement Case II.2.1: $\mathbb{K}\equiv\mathbb{0}$

The identical vanishing of the matrix $\mathbb{K}$ implies reduceability of the evolution equations of the internal variables and the existence of a set of transformation functions $\check{h}_{\gamma}$,

$$\alpha_{\gamma}''=\check{h}_{\gamma}\left(\sigma_{mn},T,\alpha_{\nu}\right) \tag{2.207}$$

is assumed. Each single transformation function $\check{h}_{\gamma}$ satisfies, analogous to $(2.57)_{1+2}$, the system

$$\left.\begin{aligned} \frac{\partial \check{h}_\gamma}{\partial \sigma_{kl}} + \frac{\partial \check{h}_\gamma}{\partial \alpha_\nu} \check{E}_{\nu kl} &= 0 \\ \frac{\partial \check{h}_\gamma}{\partial T} + \frac{\partial \check{h}_\gamma}{\partial \alpha_\nu} \check{H}_\nu &= 0 . \end{aligned}\right\} \quad (2.208)$$

Because of the similarity of (2.202) and (2.208), the partial potential $\zeta^\alpha(\sigma_{mn}, T, \alpha_\nu)$ does not only exist but is required to be some function of any or all of the admissible transformation functions $\check{h}_\gamma$, i.e.,

$$\zeta^\alpha = \check{\zeta}^\alpha(\check{h}_\gamma(\sigma_{mn}, T, \alpha_\nu)) . \quad (2.209)$$

Therefore, introducing the new internal variables α_γ'', equ.(2.207), not only the evolution equations for the internal variables are reduced to a form free of rates of the external variables, but also the additional potential ζ^α is simplified to a function of only the new internal variables α_γ''. Thus the Gibbs function takes the form

$$\zeta = \zeta^e(\sigma_{mn}, T) + \zeta^\alpha(\alpha_\gamma'') \quad (2.210)$$

where the form of the function ζ^α is arbitrary.

Supplement Case II.2.2: $\mathbb{K} \neq 0$

We assume that the rank of $\mathbb{K}$ is less than N in some finite domain $\mathcal{D}$ of the state space,

$$R := \mathrm{Rank}(\mathbb{K}) < N \quad , \quad \forall (\sigma_{mn}, T, \alpha_\nu) \in \mathcal{D} . \quad (2.211)$$

Then equ. (2.204) admits a non-trivial solution of the form

$$\mathbf{Z}^\alpha = a_1 \mathbf{Z}_1^\alpha + \ldots + a_D \mathbf{Z}_D^\alpha$$

or componentwise (see (2.198))

$$Z^\alpha_\nu = \sum_{k=1}^{D} a_k Z_{k\nu} . \quad (2.212)$$

If an additional function $\zeta^\alpha(\sigma_{mn}, T, \alpha_\nu)$ depending on α_ν should exist with the property

$$\frac{\partial \zeta^\alpha}{\partial \alpha_\nu} = Z_\nu^\alpha = \sum_{k=1}^{D} a_k Z_{k\nu} , \quad (2.213)$$

then z_ν^α has to satisfy

$$\frac{\partial z_\nu^\alpha}{\partial \alpha_\mu} = \frac{\partial z_\mu^\alpha}{\partial \alpha_\nu} . \tag{2.214}$$

With (2.202) we also get

$$\left.\begin{aligned} \frac{\partial \zeta^\alpha}{\partial \sigma_{kl}} &= - \frac{\partial \zeta^\alpha}{\partial \alpha_\nu} \underset{\nu}{\check{E}}_{kl} = - z_\nu^\alpha \underset{\nu}{\check{E}}_{kl} \\ \frac{\partial \zeta^\alpha}{\partial T} &= - \frac{\partial \zeta^\alpha}{\partial \alpha_\nu} \underset{\nu}{\check{H}} = - z_\nu^\alpha \underset{\nu}{\check{H}} . \end{aligned}\right\} \tag{2.215}$$

Compatibility of the non-trivial solution z_ν^α with the requirement (2.213) can be checked further by taking the partial derivatives of (2.215), i.e.,

$$\left.\begin{aligned} \frac{\partial^2 \zeta^\alpha}{\partial \alpha_\varrho \partial \sigma_{kl}} &= - \frac{\partial}{\partial \alpha_\varrho} \left(z_\nu^\alpha \underset{\nu}{\check{E}}_{kl} \right) = \frac{\partial}{\partial \sigma_{kl}} z_\varrho^\alpha \\ \frac{\partial^2 \zeta^\alpha}{\partial \alpha_\varrho \partial T} &= - \frac{\partial}{\partial \alpha_\varrho} \left(z_\nu^\alpha \underset{\nu}{\check{H}} \right) = \frac{\partial}{\partial T} z_\varrho^\alpha . \end{aligned}\right\} \tag{2.216}$$

Thus, any non-trivial solution z^α_ν , equ. (2.212), must satisfy also (2.214) and (2.216) if it should represent the α_ν-derivative of the additional Gibbs function ζ^α in the domain $\mathcal{D}$. The conditions (2.214) and (2.216) are certainly necessary conditions. A formal prove that (2.214) and (2.216), provided (2.212) holds, are also sufficient, has not be obtained.

If these conditions are violated, the non-trivial solution z_ν^α is irrelevant since an additional potential function ζ^α, to which z_ν^α can be related, does not exist. Then, either one accepts that the Gibbs function ζ is independent of α_ν, i.e.

$$\zeta = \zeta^e(\sigma_{mn}, T) , \qquad \zeta^\alpha \equiv 0 ,$$

so that

$$Z^\alpha \equiv 0 \quad , \qquad Z_\nu^\alpha \equiv 0$$

or, with $\zeta^\alpha \not\equiv 0$, the assumed constitutive relations are thermodynamically inconsistent.

Note that (2.212) involves the undetermined functions a_k. This complicates the evaluation of (2.214) and (2.216) and may pose a problem. Therefore, the evaluation of (2.214) and (2.216) will provide only then definite conclusions if they are independent of specific choices for the undetermined functions a_k.

2.3 Extension to a Class of Elastic-Viscoplastic Material Models with Internal Variables

The assumed constitutive relations (2.172) and (2.173) are still fairly general. It should be noted that the internal variables α_ν have not yet been identified. In fact, they may represent scalar or tensorial variables, e.g., second order tensors.

In viscoplasticity, assuming infinitesimal deformations, usually the total strain tensor ε_{kl} is partitioned into its thermoelastic and viscoplastic part

$$\varepsilon_{kl} = \varepsilon^e_{kl} + \varepsilon^p_{kl} \tag{2.217}$$

where ε^e_{kl} includes the thermal strain ε^{th}_{kl}, e.g.

$$\varepsilon^{th}_{kl} = A^{th}(T - T_0)\,\delta_{kl} \tag{2.218}$$

with the thermal expansion coefficient A^{th}. Then

$$\varepsilon^e_{kl} = \varepsilon^{el}_{kl} + \varepsilon^{th}_{kl}\,; \tag{2.219}$$

ε^{el}_{kl} denotes the purely elastic strain controlled by the stresses. For the thermoelastic strain ε^e_{kl} a constitutive equation of the general form

$$\varepsilon^e_{kl} = \check{\varepsilon}^e_{kl}\,(\sigma_{mn}, T, \beta_\nu, \varepsilon^p_{mn}) \tag{2.220}$$

is assumed where the β_ν are scalar or tensorial internal variables. Here an explicit dependence on the viscoplastic strains ε^p_{mn} is still included; usually this is excluded on other grounds. The viscoplastic strain ε^p_{kl} is controlled by an evolution law

$$\dot{\varepsilon}^p_{kl} = \check{h}_{kl}\,(\sigma_{mn}, T, \beta_\nu, \varepsilon^p_{mn}). \tag{2.221}$$

The internal variables β_γ are governed by a set of evolution equations similar to (2.173)

$$\begin{aligned}\dot{\beta}_\gamma = {} & \check{f}_\gamma(\sigma_{mn}, T, \beta_\nu, \varepsilon^p_{mn}) + \\ & + \check{E}_{\gamma mn}(\text{———}\,''\,\text{———})\,\dot{\sigma}_{mn} + \\ & + \check{H}_\gamma(\text{——}\,''\,\text{——})\,\dot{T}\end{aligned} \tag{2.222}$$

they may also involve the viscoplastic strains explicitly. Here the irreversible part, i.e. the functions $\check{h}_{kl}$ and $\check{f}_\gamma$ may change in form according to certain switch conditions, which are required to depend only on the independent state variables and not on the rates of the external variables. The definition of an elastic range by a yield function $F(\sigma_{kl}, T, \beta_\nu, \varepsilon_{mn}{}^p)$ such that $\check{h}_{kl} \equiv 0$ if $F < 0$ is an example. To be definite we also assume that the functions $\check{h}_{kl}$ and $\check{f}_\gamma$ are continuous at these transitions.

It has been shown in ref. [1] and [41] that such a mathematical structure fits into the general frame of (2.172) and (2.173) if the viscoplastic strains are interpreted as internal variables such that the α_ν in (2.172) and (2.173) represent the following set

$$\alpha_\nu = \{\beta_\varsigma, \varepsilon^p_{kl}\}. \tag{2.223}$$

Mathematically this is perfectly admissible if only the form of the constitutive functions in (2.172) and (2.173) is properly constructed. However, in material science circles the viscoplastic strain is given an extra status since it is not considered to be an independent state variable on empirical grounds, i.e., it does not affect the internal energy and entropy if the stress is used as an independent state variable. In other words only the difference between total and viscoplastic strain is an independent state variable.

To be explicit the assumed constitutive relations take the following form:

$$\left.\begin{aligned}\varepsilon^e_{kl} &= \check{\varepsilon}^e_{kl}(\sigma_{mn}, T, \beta_\nu) \\ \epsilon &= \check{\epsilon}(\text{——}\,''\,\text{——}) \\ \eta &= \check{\eta}(\text{——}\,''\,\text{——}) \\ q_k &= \check{q}_k(\sigma_{mn}, T, g_k, \beta_\nu)\end{aligned}\right\} \tag{2.224}$$

with the evolution for the viscoplastic strain

$$\dot{\varepsilon}^p_{kl} = \check{h}_{kl}(\sigma_{mn}, T, \beta_\nu) \tag{2.225}$$

and for the internal variables β_ν

$$\begin{aligned} \dot{\beta}_\gamma = {} & \check{f}_\gamma(\sigma_{kl}, T, \beta_\nu) + \\ & + \check{E}_{\gamma\,mn}(\sigma_{kl}, T, \beta_\nu)\dot{\sigma}_{mn} + \\ & + \check{H}_\gamma(\sigma_{kl}, T, \beta_\nu)\dot{T} \quad , \quad \nu = 1, 2, \dots, N . \end{aligned} \tag{2.226}$$

The dissipation inequality (2.81) is now with (2.217)

$$\gamma := \varrho T \dot{\eta} - \varrho \dot{\epsilon} + \sigma_{kl}\dot{\varepsilon}^e_{kl} + \sigma_{kl}\dot{\varepsilon}^p_{kl} - \frac{q_k g_k}{T} \geq 0 . \tag{2.227}$$

We introduce the Gibbs function as follows (compare (2.82))

$$\xi = \check{\xi}(\sigma_{kl}, T, \beta_\nu) = \check{\epsilon} - T\check{\eta} - \frac{1}{\varrho}\sigma_{kl}\check{\varepsilon}^e_{kl} \tag{2.228}$$

such that

$$\frac{d}{dt}\xi = \dot{\xi} = \dot{\epsilon} - \dot{T}\eta - T\dot{\eta} - \frac{1}{\varrho}\dot{\sigma}_{kl}\varepsilon^e_{kl} - \frac{1}{\varrho}\sigma_{kl}\dot{\varepsilon}^e_{kl}$$

and

$$\dot{\xi} = \frac{\partial\check{\xi}}{\partial\sigma_{kl}}\dot{\sigma}_{kl} + \frac{\partial\check{\xi}}{\partial T}\dot{T} + \frac{\partial\check{\xi}}{\partial\beta_\nu}\dot{\beta}_\nu$$

so that (2.227) reads

$$\gamma = -\varrho\dot{\xi} - \varrho\dot{T}\eta - \dot{\sigma}_{kl}\varepsilon^e_{kl} + \sigma_{kl}\dot{\varepsilon}^p_{kl} - \frac{q_k g_k}{T} =$$

$$= \dots$$

$$
\begin{aligned}
&= -\left(\rho \frac{\partial \check{\xi}}{\partial \sigma_{kl}} + \rho \frac{\partial \check{\xi}}{\partial \beta_\varrho} \underset{\varrho}{\check{E}}_{kl} + \check{\varepsilon}^e_{kl}\right)\dot{\sigma}_{kl} - \\
&- \rho\left(\frac{\partial \check{\xi}}{\partial T} + \frac{\partial \check{\xi}}{\partial \beta_\varrho} \underset{\varrho}{\check{H}} + \check{\eta}\right)\dot{T} + \\
&+ \sigma_{kl}\,\dot{\varepsilon}^p_{kl} - \rho \frac{\partial \check{\xi}}{\partial \beta_\varrho} \check{f}_\varrho - \frac{\check{q}_k g_k}{T} \geq 0
\end{aligned}
\tag{2.229}
$$

At this place we recall the discussion in section 2.2.1. Thus, an arbitrary variation of the *total strain rate field* $\dot{\varepsilon}_{kl}$ (x_m, t) associated to a compatible displacement or velocity field is in general preferable compared to a variation in the stress rate field $\dot{\sigma}_{kl}$. However, with (2.217), $(2.224)_1$, (2.225) and (2.226)

$$
\begin{aligned}
\dot{\varepsilon}_{kl} &= \frac{\partial \check{\varepsilon}^e_{kl}}{\partial \sigma_{mn}} \dot{\sigma}_{mn} + \frac{\partial \check{\varepsilon}^e_{kl}}{\partial T}\dot{T} + \frac{\partial \check{\varepsilon}^e}{\partial \beta_v}\dot{\beta}_v + \dot{\varepsilon}^p_{kl} = \\
&= \left(\frac{\partial \check{\varepsilon}^e_{kl}}{\partial \sigma_{mn}} + \frac{\partial \check{\varepsilon}^e_{kl}}{\partial \beta_v} \underset{v}{\check{E}}_{mn}\right)\dot{\sigma}_{mn} + \\
&+ \left(\frac{\partial \check{\varepsilon}^e_{kl}}{\partial T} + \frac{\partial \check{\varepsilon}^e_{kl}}{\partial \beta_v} \underset{v}{\check{H}}\right)\dot{T} + \\
&+ \frac{\partial \check{\varepsilon}^e_{kl}}{\partial \beta_v} \check{f}_v + \check{h}_{kl}(\sigma_{mn}, T, \beta_v).
\end{aligned}
\tag{2.230}
$$

Assume that locally at some instant the values σ_{mn}, T, β_v and $\varepsilon^p{}_{mn}$ and the temperature rate $\dot{T}$ are somehow prescribed. Then (2.230) represents a system of linear equations for the stress rate if the total strain rate $\dot{\varepsilon}_{kl}$ is given. Thus, a locally arbitrary modulation of $\dot{\varepsilon}_{kl}$ induces a stress rate $\dot{\sigma}_{kl}$ in a *unique* way if

$$
\left(\frac{\partial \varepsilon^e_{kl}}{\partial \sigma_{mn}} + \frac{\partial \varepsilon^e_{kl}}{\partial \beta_v} \underset{v}{E}_{mn}\right) \hat{=} \quad \text{invertible} , \tag{2.231}
$$

This can be put also in a suitable matrix formulation analogous to section 2.2.1 but this is not done here. Thus, provided (2.231) is possible, an arbitrary local modulation of $\dot{\varepsilon}_{kl}$ or $\dot{\sigma}_{kl}$ is mathematically equivalent. Therefore, with the usual argumentation we obtain from (2.229) as necessary and sufficient conditions

$$\begin{aligned} \rho \frac{\partial \check{\xi}}{\partial \sigma_{kl}} + \rho \frac{\partial \check{\xi}}{\partial \beta_\nu} \underset{\nu}{\check{E}}_{kl} + \check{\varepsilon}^e_{kl} &= 0 \\ \rho \frac{\partial \check{\xi}}{\partial T} + \rho \frac{\partial \check{\xi}}{\partial \beta_\nu} \underset{\nu}{\check{H}} + \rho \check{\eta} &= 0 \end{aligned} \qquad (2.232)$$

and

$$\begin{aligned} \sigma_{kl} \dot{\varepsilon}_{kl} - \rho \frac{\partial \check{\xi}}{\partial \beta_\rho} \check{f}_\rho &\geq 0 \\ - \frac{\check{q}_k g_k}{T} &\geq 0 \end{aligned} \qquad (2.233)$$

where the possiblility of separating the residual dissipation inequality is observed.

With respect to the evolution equations (2.226) for the internal variables β_ν all results of the sections 2.2.2 and 2.2.3 apply if we rename the variables, i.e., $\alpha_\nu \rightarrow \beta_\nu$, and if we drop the dependence on the temperature gradient in these sections.

Similarly, for the question of thermodynamic consistency of an assumed elastic strain-stress relation $(2.224)_1$ and an assumed entropy relation $(2.224)_3$ all results of section 2.2.4 can be used if the following renaming is done

$$\alpha_\nu \Longrightarrow \beta_\nu \,, \qquad \check{\varepsilon}_{kl} \Longrightarrow \check{\varepsilon}^e_{kl} \,, \qquad \check{\xi} \Longrightarrow \check{\xi} \,. \qquad (2.234)$$

We note especially the results described in the last part of section 2.2.4, here translated to the present situation.

Assume that the proposed elastic strain-stress relation and the entropy temperature relation are independent of the internal variables β_ν and are derivable from a partial potential $\xi^e(\sigma_{mn}, t)$, i.e.,

$$\check{\varepsilon}^e_{kl} = - \rho \frac{\partial \xi^e}{\partial \sigma_{kl}} \,, \qquad \check{\eta} = - \frac{\partial \xi^e}{\partial T} \,. \qquad (2.235)$$

Since we require the existence of a Gibbs function ξ depending on the internal variables, ξ can always be put in the form

$$\xi(\sigma_{mn}, T, \beta_s) = \xi^e(\sigma_{mn}, T) + \xi^\beta(\sigma_{mn}, T, \beta_s). \tag{2.236}$$

Introducing this into (2.232), one obtains

$$\left.\begin{aligned} \frac{\partial \xi^\beta}{\partial \sigma_{kl}} + \frac{\partial \xi^\beta}{\partial \beta_v} \underset{v}{\check{E}}_{kl} &= 0 \\ \frac{\partial \xi^\beta}{\partial T} + \frac{\partial \xi^\beta}{\partial \beta_v} \underset{v}{\check{H}} &= 0. \end{aligned}\right\} \tag{2.237}$$

If a common solution ξ^β of (2.237) exists, then the interchangeability of the partial derivatives yields (see (2.203)) the necessary condition

$$\left.\begin{aligned} \frac{\partial \xi^\beta}{\partial \beta_v}\left\{\left(\frac{\partial \underset{v}{\check{E}}_{pq}}{\partial \sigma_{rs}} - \frac{\partial \underset{v}{\check{E}}_{rs}}{\partial \sigma_{pq}}\right) - \left(\frac{\partial \underset{v}{\check{E}}_{rs}}{\partial \beta_s} \underset{s}{\check{E}}_{pq} - \frac{\partial \underset{v}{\check{E}}_{pq}}{\partial \beta_s} \underset{s}{\check{E}}_{rs}\right)\right\} &= 0 \\ \frac{\partial \xi^\beta}{\partial \beta_v}\left\{\left(\frac{\partial \underset{v}{\check{E}}_{pq}}{\partial T} - \frac{\partial \underset{v}{\check{H}}}{\partial \sigma_{pq}}\right) - \left(\frac{\partial \underset{v}{\check{H}}}{\partial \beta_s} \underset{s}{\check{E}}_{pq} - \frac{\partial \underset{v}{\check{E}}_{pq}}{\partial \beta_s} \underset{s}{\check{H}}\right)\right\} &= 0 \\ \text{or} \qquad \mathbb{K}\,\mathbf{z}^\beta = \mathbf{0} \quad , \quad \mathbf{z}^\beta = \left\{\partial \xi^\beta / \partial \beta_v\right\}. \end{aligned}\right\} \tag{2.238}$$

Since the discussion in section 2.2.4 applies, if the "translation" (2.234) is observed, the following statements can be made.

Case II.2.1: $\mathbf{z}^\beta \neq 0$, $\mathbb{K} \equiv 0$

Provided the evolution equations (2.226) are reduceable, then the { ... }-brackets in (2.238) vanish identically ($\mathbb{K} \equiv 0$) and (2.238) is satisfied. Then (2.237) has a common solution ξ^β and the partial potential $\xi^\beta(\sigma_{mn}, T, \beta_\rho)$ is required to be some function of any or all of the transformation functions $\check{h}_\gamma(\sigma_{mn}, T, \beta_\rho)$, $\gamma = 1, \ldots, N$, i.e.,

$$\xi^\beta = \check{\xi}^\beta\left(\check{h}_\gamma(\sigma_{mn}, T, \beta_\varrho)\right) ; \qquad 2.239)$$

the existence of the transformation

$$\beta'_\gamma = \check{h}_\gamma(\sigma_{mn}, T, \beta_\varrho) \qquad (2.240)$$

assures the reduceability of the evolution equations. Consequently, the additional potential ξ^β is solely a function of the new transformed internal variable β'_ρ, i.e.,

$$\xi^\beta = \xi^\beta(\beta'_\varrho) . \qquad (2.241)$$

We summarize this result in the following *theorem*:

> If the thermoelastic strain $\check{\varepsilon}^e_{kl}$ and the entropy $\check{\eta}$ are assumed to be independent of internal variables β_ρ and deriveable from a potential $\xi^e(\sigma_{mn}, T)$, equ. (2.235)
>
> $$\check{\varepsilon}^e_{kl} = -\varrho \frac{\partial \xi^e}{\partial \sigma_{kl}} , \quad \check{\eta} = -\frac{\partial \xi^e}{\partial T} ,$$
>
> and if the evolution equations for the internal variables are reduceable to the standard form
>
> $$\overset{\circ}{\beta}{}'_\varrho = f'_\varrho(\sigma_{mn}, T, \beta'_\varrho)$$
>
> where
>
> $$\beta'_\varrho = \check{h}_\varrho(\sigma_{mn}, T, \beta_\nu)$$
>
> are suitable transformed internal variables, then a Gibbs function $\xi(\sigma_{mn}, \beta_\rho, T)$ - depending on internal variables β_ρ - exists and is required to have the form
>
> $$\xi = \xi^e(\sigma_{mn}, T) + \xi^\beta(\beta'_\varrho) .$$

Case II.2.2: $\mathbb{z}^\beta \neq \mathbb{0}$, $\mathbb{K} \neq \mathbb{0}$

Provided the rank of the MN-matrix $\mathbb{K}$ is equal to N, $R = N < M$, then (2.238) cannot be satisfied for $\mathbb{z}^\beta = \{\partial\xi^\beta/\partial\alpha_\nu\} \neq \mathbb{0}$. Thus, (2.237) has not a common solution ξ^β. Consequently, the functions $\underset{\vee}{\check{E}}_{pq}$ and $\underset{\vee}{\check{H}}$ are thermodynamically inconsistent or the assumptions (2.235) are inadmissible.

Provided the rank R of the matrix $\mathbb{K}$ is less than N, $R < N < M$, then (2.238) can be satisfied under the assumption $\mathbb{z}^\beta \neq \mathbb{0}$. The non-trivial solution $\mathbb{z}^\beta$ takes the form

$$\mathbb{z}^\beta = a_1 \mathbb{z}_1^\beta + a_2 \mathbb{z}_2^\beta + \dots + a_D \mathbb{z}_D^\beta , \quad D = N - R \qquad (2.242)$$

where the $\mathbb{z}^\beta_j$ are D linearly independent solutions of

$$\mathbb{K}\,\mathbb{z}_j^\beta = 0 , \quad j = 1, \dots, D \qquad (2.243)$$

and a_k are free parameters. Thus, the β_ν-derivatives of the partial potential ξ^β require the representation

$$\frac{\partial \xi^\beta}{\partial \beta_\nu} = \{\mathbb{z}^\beta\}_\nu = \sum_{j=1}^{D} a_j Z_{j\nu} \qquad (2.244)$$

where $Z_{j\nu}$ are the $\nu = 1, \dots, N$ components of each of the $j = 1, \dots, D$ linearly independent solutions $\mathbb{z}^\beta_j$; they depend on the { ... }-brackets in (2.238) in a complicated nonlinear way.

A positive defect $D = N - R > 0$ is necessary but not sufficient to assure the existence of a common solution ξ^β of the system (2.237). However, the compatibility of (2.224) and the system (2.237) can be checked by requiring the interchangeability of the partial derivatives, e.g.,

$$\frac{\partial^2 \xi^\beta}{\partial\beta_s\,\partial\beta_\nu} = \frac{\partial^2 \xi^\beta}{\partial\beta_s\,\partial\beta_\nu} , \quad \frac{\partial^2 \xi^\beta}{\partial\beta_\nu\,\partial\sigma_{kl}} = \frac{\partial^2 \xi^\beta}{\partial\sigma_{kl}\,\partial\beta_\nu} , \quad \frac{\partial^2 \xi^\beta}{\partial T\,\partial\beta_\nu} = \frac{\partial^2 \xi^\beta}{\partial\beta_\nu\,\partial T} \qquad (2.245)$$

(see equ. (2.214) ÷ (2.216) and the following remarks) which is necessary for a common solution.

If (2.244) is violated, then a common solution ξ^α of (2.237) does not exist. Then, either one accepts

$$\xi^{\beta} \equiv 0 \quad \Longrightarrow \quad \xi = \xi^{e}(\sigma_{mn}, T), \tag{2.246}$$

i.e., the Gibbs function is independent of β_V, or the assumed constitutive relations are thermodynamically inconsistent.

3. Applications to Several Constitutive Models

3.1 The Robinson Model

3.1.1 Formal Description

The Robinson model presupposes infinitesimal deformations and initially isotropic response. Here the version given in [47] is summarized.

The total strain tensor ε_{kl} is assumed to be separable into its elastic, viscoplastic, and thermal parts, i.e.,

$$\varepsilon_{kl} = \varepsilon^{el}_{kl} + \varepsilon^{p}_{kl} + \varepsilon^{th}_{kl}. \tag{3.1}$$

The thermal part is given by

$$\varepsilon^{th}_{kl} = A^{th}(T - T_0)\,\delta_{kl} \tag{3.2}$$

implying isotropic thermal expansion. The elastic strain ε^{el}_{kl} is related to the stress by Hooke's law

$$\sigma_{kl} = \lambda\,\varepsilon^{el}_{mm}\,\delta_{kl} + \mu\,(\varepsilon^{el}_{kl} + \varepsilon^{el}_{lk}). \tag{3.3}$$

Lame's constants λ and μ are given in terms of Young's modulus E and Poisson's number ν by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}\,, \qquad \mu = \frac{E}{2(1+\nu)}\,. \tag{3.4}$$

The essential part of the model characterizes the viscoplastic strain ε^{p}_{kl}, the back stress α_{kl} and the drag stress κ:

$$\dot{\varepsilon}^{p}_{kl} = \left\{ \begin{array}{lll} A\,F^{n}\,\dfrac{\Sigma_{kl}}{\sqrt{J_2}} \,; & F > 0 & \text{and} \quad S_{kl}\,\Sigma_{kl} > 0 \\ 0 \,; & F \leq 0 & \text{or} \quad S_{kl}\,\Sigma_{kl} \leq 0 \\ \vdots & & \end{array} \right. \tag{3.5}$$

$\vdots$

$$\dot{\alpha}_{kl} = \begin{cases} \dfrac{H}{G^{\beta}}\,\dot{\varepsilon}^{p}_{kl} - R\,G^{m-\beta}\dfrac{a_{kl}}{\sqrt{I_2}}\ ; & G>G_0 \text{ and } S_{kl}a_{kl}>0 \\ \dfrac{H}{G_0^{\beta}}\,\dot{\varepsilon}^{p}_{kl} - R\,G^{m-\beta}\dfrac{a_{kl}}{\sqrt{I_2}}\ ; & G\leq G_0 \text{ or } S_{kl}a_{kl}\leq 0 \end{cases}$$

$$\dot{\mathcal{K}} = \hat{\Gamma}(W^p,T)\,\dot{W}^p + \hat{\Theta}(W^p,T)\,\dot{T} \qquad (3.6)$$

with

$$\mathcal{K} = 3\varkappa^2 \qquad (3.7)$$

such that

$$\dot{\varkappa} = \frac{1}{6\varkappa}\left[\hat{\Gamma}(W^p,T)\,\dot{W}^p + \hat{\Theta}(W^p,T)\,\dot{T}\right] \qquad (3.8)$$

where

$S_{kl} = \sigma_{kl} - \frac{1}{3}\sigma_{mm}\delta_{kl}$	deviatoric stress
$a_{kl} = \alpha_{kl} - \frac{1}{3}\alpha_{mm}\delta_{kl}$	deviatoric back stress
$\Sigma_{kl} = S_{kl} - a_{kl}$	effective stress
$J_2 = \frac{1}{2}\Sigma_{kl}\Sigma_{lk}$	2nd invariant of effective stress
$I_2 = \frac{1}{2}a_{kl}\,a_{lk}$	2nd invariant of back stress
$\varkappa$	drag stress
$\varkappa_0$	drag stress at reference temperature T_0 and undeformed state
$G = I_2/\varkappa_0^2$	dimensionless 2nd invariant of back stress
T	absolute temperature

and F is the "yield" function

$$F = \frac{J_2}{\varkappa^2} - 1\ . \qquad (3.9)$$

The quantity W^p is the plastic work

$$W^p = \int s_{kl} \, \dot{\varepsilon}^p_{kl} \, dt \tag{3.10}$$

where

$$\dot{W}^p = s_{kl} \, \dot{\varepsilon}^p_{kl} \tag{3.11}$$

is the plastic stress power.

The quantities A, H, G_0, R, κ_0, m, β with (m - β) > 0 are positive temperature dependent parameters.

The equations $(3.5)_2$ describe the kinematic hardening whereas (3.6) or (3.8) model isotropic hardening. It is noteworthy that only the evolution law for the drag stress involves the rate of an external observable variables, that is the temperature rate $\dot{T}$. Clearly, this is a non-isothermal theory which requires non-isothermal experiments. Robinson proposed this scalar evolution law to account qualitatively for the thermomechanical path dependence observed in, for example, the cyclic hardening behavior of some alloys of interest [47].

Classically, the drag stress κ or K affecting the yield function (3.9) has been assumed to be an explicit function of W^p and T, i.e.,*

$$K = \hat{K}(W^p, T) \, .$$

This is equivalent to adopting the evolutionary equation for K as

$$\dot{K} = \frac{\partial \hat{K}}{\partial W^p} \dot{W}^p + \frac{\partial \hat{K}}{\partial T} \dot{T} \, .$$

This is a perfect differential and integrable independently of the thermomechanical history $\{W^p(t), T(t)\}$. The evolution law (3.6), however, is not required to satisfy the integrability condition**

$$\frac{\partial \hat{\Gamma}}{\partial T} = \frac{\partial \hat{\Theta}}{\partial W^p} \, .$$

---

* Alternatively the accumulated effective plastic strain has been used instead of W^p.

** Here it should be noted that a differential form with two independent variables such as

$$dK = \hat{\Gamma}(W^p, T) \, dW^p + \hat{\Theta}(W^p, T) \, dT$$

always admits a multitude of integrating factors $N_{(W^p, T)}$ such that

$$d\tilde{K} = N \hat{\Gamma} \, dW^p + N \hat{\Theta} \, dT$$

represents a total differential.

Without further comments we state the smooth functions $\hat{\Gamma}$ and $\hat{\theta}$ proposed by Robinson:

$$\left.\begin{aligned}
\hat{\Gamma}(W^p,T) &= \frac{K_s(T)-K_i(T)}{W_0(T)} \exp\left[-\frac{W^p}{W_0(T)}\right] > 0 \\
\hat{\theta}(W^p,T) &= -\frac{Q(W^p)}{T^2} K_1 \exp\left[-Q(W^p)\left(\frac{1}{T_0}-\frac{1}{T}\right)\right] < 0 \\
K_i(T) &= K_0 - K_1\left(1-\exp\left[-Q_0\left(\frac{1}{T_0}-\frac{1}{T}\right)\right]\right) \\
K_s(T) &= A + B(T-T_p) + C(T-T_p)^2 \\
W_0(T) &= \bar{A} + \bar{B}(T-T_p)^2
\end{aligned}\right\} \tag{3.12}$$

Here, K_0, K_1 and Q_0 are temperature independent material constants, whereas K_i, K_s and W_0 are temperature dependent. Note that K_i (T) is the temperature dependent hardening parameter in the virgin state (i.e. $W^p = 0$) of the material.K_s (T) is the saturated value of K in a deformation process at constant temperature T and where the plastic work has reached a very large value.

According to Robinson the function $Q(W^p)$ is not known presently but appears not to be a strong function of W^p; for the exploratory calculations in [47] it was taken to be a constant

$$Q(W^p) = Q(0) = Q_0 = \text{const.} \tag{3.13}$$

3.1.2 Extension of the Robinson Model within a Thermodynamic Frame

3.1.2.1 General Structure of Free Energy and Thermodynamic Restrictions

For the Robinson model without isotropic hardening a choice for the free energy had been made (see ref. [1]). This formulation contains an error since a distinction between the purely elastic strain

$$\varepsilon^{el}_{kl} := \varepsilon_{kl} - \varepsilon^{p}_{kl} - \varepsilon^{th}_{kl}$$

and the thermoelastic strain

$$\varepsilon^{e}_{kl} := \varepsilon_{kl} - \varepsilon^{p}_{kl} , \tag{3.14}$$

which contains the thermal strain ε^{th}_{kl}, had not been made. This error affects only part of the formulation in section 4.1 in ref. [1] but not the rest of the study. Also this previous work did account only for the kinematic hardening but not for the isotropic hardening.

In the following we present a corrected version which included the effects of both hardening mechanisms. With equ. (4.1), ref. [1], the free energy is assumed to be given by

$$\left.\begin{aligned} &\psi = \tilde{\psi}(\Pi^{*}) \\ \text{where} \quad &\\ &\Pi^{*} := \{\varepsilon^{e}_{kl}, T, a_{kl}, \varkappa\}. \end{aligned}\right\} \tag{3.15}$$

Note that we choose the drag stress κ as an internal variable but not its square $K = 3\kappa^2$.

Equ.´s (3.15) imply that the free energy does not depend explicitly on the total and the plastic strain but only on its difference, the thermoelastic strain ε^{e}_{kl}; also as usual the temperature gradient is excluded from the list of independent variables. The same assumption is made for the stress and entropy

$$\sigma_{kl} = \tilde{\sigma}_{kl}(\Pi^{*}) , \quad \eta = \tilde{\eta}(\Pi^{*}). \tag{3.16}$$

The material rate of the free energy is then given by

$$\dot{\psi} = \frac{\partial \tilde{\psi}}{\partial \varepsilon^{e}_{kl}} \dot{\varepsilon}^{e}_{kl} + \frac{\partial \tilde{\psi}}{\partial T} \dot{T} + \frac{\partial \tilde{\psi}}{\partial a_{kl}} \dot{a}_{kl} + \frac{\partial \tilde{\psi}}{\partial \varkappa} \dot{\varkappa} .$$

Then the dissipation inequality (2.5) takes the following form

$$\gamma = \left(\tilde{\sigma}_{kl} - \rho \frac{\partial \tilde{\psi}}{\partial \varepsilon^{e}_{kl}}\right)_{sym} \dot{\varepsilon}_{kl} + \left(\rho \frac{\partial \tilde{\psi}}{\partial \varepsilon^{e}_{kl}}\right)_{sym} \dot{\varepsilon}^{p}_{kl} - \left(\rho \frac{\partial \tilde{\psi}}{\partial T} + \rho \tilde{\eta}\right) \dot{T} - \\ - \left(\rho \frac{\partial \tilde{\psi}}{\partial a_{kl}}\right)_{sym} \dot{a}_{kl} - \rho \frac{\partial \tilde{\psi}}{\partial \kappa} \dot{\kappa} - \frac{q_k g_k}{T} \geq 0 . \tag{3.17}$$

The evolution equations for the plastic strain and the back stress do not contain rates of the external variables; only the rate of the drag stress depends on the temperature rate (see equ. (3.8)). Accounting for this, equ. (3.17) reads

$$\gamma = \left(\tilde{\sigma}_{kl} - \rho \frac{\partial \tilde{\psi}}{\partial \varepsilon^{e}_{kl}}\right)_{sym} \dot{\varepsilon}_{kl} - \left(\rho \frac{\partial \tilde{\psi}}{\partial T} + \rho \frac{\partial \tilde{\psi}}{\partial \kappa} \frac{\hat{\Theta}}{\sigma \kappa} + \rho \tilde{\eta}\right) \dot{T} + \\ + \left(\rho \frac{\partial \tilde{\psi}}{\partial \varepsilon^{e}_{kl}}\right)_{sym} \dot{\varepsilon}^{p}_{kl} - \left(\rho \frac{\partial \tilde{\psi}}{\partial a_{kl}}\right)_{sym} \dot{a}_{kl} - \rho \frac{\partial \tilde{\psi}}{\partial \kappa} \frac{\hat{\Gamma} \dot{W}^{p}}{\sigma \kappa} - \frac{q_k g_k}{T} \\ \geq 0 . \tag{3.18}$$

Here it should be noted that the evolution equation for the plastic strain and back stress, equ. (3.5), are *discontinuous at certain states* such that the rates $\dot{\varepsilon}p_{kl}$, $\dot{a}_{kl}$ and thus also the plastic stress power $\dot{W}p$ in (3.8) suffer a jump. At these states inequality (3.18) is not applicable since the rates $\dot{\varepsilon}p_{kl}$ etc. are not defined. A more general derivation of thermodynamic restrictions is required starting off from a time-integral dissipation inequality. This has been done in ref. [1] in general terms except that there the evolution equations were of the purely "irreversible type", i.e., did not contain rates of the external variables. However, it can be shown that this does not affect the general conclusion obtained in [1]. Thus, a time-integral form does not impose additional constitutive restrictions at states where a discontinuity in the rates of internal variables occur. The restrictions obtained for smooth processes from the classical *instant* form of the dissipation inequality are applicable to all states, and no other constitutive restrictions are to be observed.

Except for the states where the evolution functions are discontinuous, a smooth kinematically compatible thermomechanical process is defined by a smooth displacement and temperature history $u_k(x_m, t)$ and $T(x_m, t)$ as well as appropriate initial conditions for all evolution equations. Thus, an arbitrary choice of the functions u_k and T implies a locally arbitrary strain rate $\dot{\varepsilon}_{kl}$ and temperature rate $\dot{T}$. Then the necessary and sufficient conditions for the identical satisfaction of the dissipation inequality are*

$$\left.\begin{aligned} \tilde{\sigma}_{kl} &= \frac{1}{2}\varrho\left(\frac{\partial\tilde{\psi}}{\partial\varepsilon^e_{kl}} + \frac{\partial\tilde{\psi}}{\partial\varepsilon^e_{lk}}\right) \\ \tilde{\eta} &= -\frac{\partial\tilde{\psi}}{\partial T} - \frac{\partial\tilde{\psi}}{\partial\varkappa}\,\frac{\hat{\theta}}{\sigma\varkappa} \end{aligned}\right\} \quad (3.19)$$

and with (3.19)$_1$

$$\left.\begin{aligned} \gamma_M &:= \tilde{\sigma}_{kl}\,\dot{\varepsilon}^p_{kl} - \left(\varrho\frac{\partial\tilde{\psi}}{\partial a_{kl}}\right)_{sym}\dot{a}_{kl} - \varrho\frac{\partial\tilde{\psi}}{\partial\varkappa}\,\frac{\hat{\Gamma}\dot{W}^p}{\sigma\varkappa} \geq 0 \\ \gamma_T &:= -\frac{q_k g_k}{T} \geq 0 \end{aligned}\right\} \quad (3.20)$$

The separation into two inequalities is due to the fact that the mechanical dissipation γ_M does not depend on the temperature gradient.

Obviously, conditions (3.19)$_1$ represents the classical potential relation for the stress. However, (3.19)$_2$ does not correspond to the traditional relation between entropy and free energy.

3.1.2.2 Choice of Free Energy

In terms of the thermoelastic strain ε^e_{kl} and the thermal strain ε^{th}_{kl} the purely elastic strain ε^{el}_{kl} is

$$\varepsilon^{el}_{kl} = \varepsilon^e_{kl} - \varepsilon^{th}_{kl}\,. \quad (3.21)$$

Hooke's law (3.3) then takes the following form

$$\sigma_{kl} = \lambda\,\varepsilon^e_{mm}\,\delta_{kl} + \mu\left(\varepsilon^e_{kl} + \varepsilon^e_{lk}\right) - (3\lambda + 2\mu)\,\varepsilon^{th}_{kl}\,. \quad (3.22)$$

With the restriction (3.19)$_1$ we get

$$\frac{1}{2}\varrho\left(\frac{\partial\tilde{\psi}}{\partial\varepsilon^e_{kl}} + \frac{\partial\tilde{\psi}}{\partial\varepsilon^e_{lk}}\right) = \lambda\,\varepsilon^e_{mm}\,\delta_{kl} + 2\mu\,\varepsilon^e_{kl} - (3\lambda+2\mu)\,\alpha^{th}(T-T_0)\,\delta_{kl}\,.$$

---

* Note that the functions $\tilde{\psi}$, $\hat{\theta}$ and $\hat{\Gamma}$ are smooth

Thus, except for integration constants, the free energy $\tilde{\psi}$ may be obtained by partial integration in the domain of the thermoelastic strain. One finds

$$\left.\begin{aligned} \tilde{\psi}(\mathbb{T}^*) &= \tilde{\psi}^e(\varepsilon^e_{kl},T) + \tilde{\psi}^{IT}(a_{mn},\varkappa,T) \\ \tilde{\psi}^e(\varepsilon^e_{kl},T) &:= \frac{1}{\varrho}\frac{1}{2}\left(2\mu\,\varepsilon^e_{kl}\varepsilon^e_{lk} + \lambda\,\varepsilon^e_{mm}\varepsilon^e_{nn}\right) \\ &\quad - \frac{1}{\varrho}(3\lambda+2\mu)A^{th}(T-T_0)\,\varepsilon^e_{mm}. \end{aligned}\right\} \quad (3.23)$$

According to our basic assumption (3.3), the "integration constant" $\tilde{\psi}^{IT}$ depends only on the internal stresses a_{mn}, κ and the temperature T. It is required to be invariant under coordinate transformation. Therefore, implying isotropic material behavior only the invariants of the back stress, i.e.,

$$\begin{aligned} i_1 &= tr\,\underline{a} = a_{mm} = 0 \\ i_2 &= \frac{1}{2}(tr\,\underline{a})^2 - tr\,\underline{a}\,\underline{a} = -\frac{1}{2}a_{kl}a_{lk} = -I_2 \\ i_3 &= det\,\underline{a} \end{aligned}$$

are involved. But in the following the simplest assumption is made, that is, the back stress and the drag stress are only *quadratically* involved with *separate* contributions. Thus,

$$\tilde{\psi}^{IT} = \tilde{\psi}^T(T) + \frac{1}{\varrho}M(T)\underbrace{\frac{1}{2}a_{mn}a_{nm}}_{I_2} + \frac{1}{\varrho}N(T)\frac{1}{2}\varkappa^2 \quad (3.24)$$

Therefore, the contribution due to the internal stresses involves only two new temperature dependent material parameters $M_{(T)}$ and $N_{(T)}$. For the following the temperature dependent contribution $\tilde{\psi}^T$ need not to be specified in detail.

3.1.2.3 Reduceability of the Evolution Equations

The evolution equations (3.5) and (3.8) together with (3.11) represent a rather simple special case of the general scheme (2.80): The temperature rate is involved in only one evolution equation and, as usual, the temperature gradient does not play any role. This corresponds to the case III'' in section 2.2.2. Thus, the reduction of the evolution equation for the drag stress to the standard form is assured since the temperature rate term does not explicity depend on the stress and temperature gradient. In this context we may consider the plastic work as general internal variable determined by the evolution equation (3.11).

Consequently, a proper transformation of the internal variable "drag stress"

$$\varkappa' = h^{\varkappa}(\varkappa, W^p, T)$$

exists which allows the reduction of (3.8) to the standard form.

Instead of using the drag stress κ we use the variables K and its evolution equation (3.6). The transformation equation reads

$$\mathcal{K}' = h(\mathcal{K}, W^p, T). \tag{3.25}$$

Differentiation of (3.25) with respect to time yields

$$\dot{\mathcal{K}}' = \frac{\partial h}{\partial \mathcal{K}} \dot{\mathcal{K}} + \frac{\partial h}{\partial W^p} \dot{W}^p + \frac{\partial h}{\partial T} \dot{T} = \\ = \left(\frac{\partial h}{\partial W^p} + \frac{\partial h}{\partial \mathcal{K}} \hat{\Gamma} \right) \dot{W}^p + \left(\frac{\partial h}{\partial T} + \frac{\partial h}{\partial \mathcal{K}} \hat{\Theta} \right) \dot{T}. \tag{3.26}$$

Reduceability of (3.26) implies that the condition

$$\frac{\partial h}{\partial T} + \frac{\partial h}{\partial \mathcal{K}} \hat{\Theta}(W^p, T) = 0 \tag{3.27}$$

admits a solution. This is a linear first order partial differential equation for the transformation function h. The variable W_p plays here only the role of a parameter. Since the function $\hat{\theta}$ does not depend on the drag stress by assumption, a new temperature variable can be introduced

$$\vartheta = \int_{\mathring{T}}^{T} \hat{\Theta}(W^p,T)\,dT = -\int_{\mathring{T}}^{T} \frac{Q}{T^2}\mathcal{K}_1 \exp\left[-\left(\frac{Q}{T_0}-\frac{Q}{T}\right) dT\right.$$

$$= \mathcal{K}_1 \exp\left[-Q\left(\frac{1}{T_0}-\frac{1}{T}\right)\right]\Big|_{\mathring{T}}^{T} \tag{3.28}$$

where $\mathring{T}$ is some arbitrary initial value. Putting $\mathring{T} = T_0$ we get

$$\vartheta = \mathcal{K}_1 \left[\exp\left[-Q\left(\frac{1}{T_0}-\frac{1}{T}\right)\right] - 1\right]. \tag{3.29}$$

With

$$d\vartheta = \hat{\Theta}(W^p,T)\,dT$$

equ. (3.27) reads

$$\frac{\partial h}{\partial \vartheta} + \frac{\partial h}{\partial \mathcal{K}} = 0 . \tag{3.30}$$

The characteristic equations are rather simple:

$$\frac{d\vartheta}{ds} = 1 \quad , \quad \frac{d\mathcal{K}}{ds} = 1 \tag{3.31}$$

where s is the curve parameter along the characteristic base curve. Along a characteristic base curve the function h is constant

$$\frac{dh}{ds} = \frac{\partial h}{\partial \vartheta}\frac{d\vartheta}{ds} + \frac{\partial h}{\partial \mathcal{K}}\frac{d\mathcal{K}}{ds} = 0. \tag{3.32}$$

With the initial values

$$s = 0 \quad , \quad \vartheta = \mathring{\vartheta} \quad , \quad \mathcal{K} = \mathring{\mathcal{K}} \tag{3.33}$$

the solutions of (3.31) are

$$\vartheta = s + \mathring{\vartheta} \quad , \quad \mathcal{K} = s + \mathring{\mathcal{K}} \tag{3.34}$$

which gives

$$\mathcal{K} = \vartheta - \overset{\circ}{\vartheta} + \overset{\circ}{\mathcal{K}} ; \tag{3.35}$$

for fixed $\overset{\circ}{\vartheta}$ and arbitrary $\overset{\circ}{\mathrm{K}} = \tau$ equ. (3.35) represents a bunch of parallel straight lines in the (ϑ, K)-plane. If h is prescribed along the initial value curve $s = 0$, τ arbitrary, such that

$$\tilde{h}(s=0, \tau) = \overset{\circ}{h}(\tau) \tag{3.36}$$

then the general solution of (3.32) is

$$h(\vartheta, \mathcal{K}) = \overset{\circ}{h}(\tau) = \overset{\circ}{h}(\mathcal{K} - (\vartheta - \overset{\circ}{\vartheta})).$$

With (3.29) and the choice

$$\overset{\circ}{\vartheta} = 0 \tag{3.37}$$

we obtain

$$h = \hat{h}(\mathcal{K}, W^p, T) = \overset{\circ}{h}\left(\mathcal{K} + \mathcal{K}_1\left(1 - \exp\left[-Q(W^p)\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]\right)\right) \tag{3.38}$$

where $\overset{\circ}{h}(\,.\,)$ is an arbitrary differentiable function. The most simple transformation function is given by choosing

$$\overset{\circ}{h}(\tau) = \tau \tag{3.39}$$

which yields finally the following transformation law

$$\mathcal{K}' = \hat{h}(\mathcal{K}, W^p, T) = \mathcal{K} + \mathcal{K}_1\left(1 - \exp\left[-Q(W^p)\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]\right) \tag{3.40}$$

and thus

$$\mathcal{K} = \mathcal{K}' - \mathcal{K}_1\left(1 - \exp\left[-Q(W^p)\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]\right). \tag{3.41}$$

It is noteworthy that the transformation (3.40) consists simply in a temperature and work dependent shift of the origin (zero point).

With

$$\frac{\partial h}{\partial \mathcal{K}} = 1, \quad \frac{\partial h}{\partial W^p} = +\mathcal{K}_1 \frac{dQ}{dW^p}\left(\frac{1}{T_0} - \frac{1}{T}\right)\exp\left[-Q\left(\frac{1}{T_0} - \frac{1}{T}\right)\right] \tag{3.42}$$

and reduced evolution equation (3.26) reads

$$\dot{\mathcal{K}}' = \left\{ \frac{\partial h}{\partial \mathcal{K}} \hat{\Gamma} + \frac{\partial h}{\partial W^p} \right\} \dot{W}^p =$$

$$= \left\{ \hat{\Gamma}(W^p, T) + \mathcal{K}_1 \frac{d Q(W^p)}{d W^p} \left(\frac{1}{T_0} - \frac{1}{T} \right) \exp\left[-Q(W^p) \left(\frac{1}{T_0} - \frac{1}{T} \right) \right] \right\} \dot{W}^p . \tag{3.43}$$

Again the rate of the transformed quantity $\mathcal{K}'$ depends only on W^p and T but not on $\mathcal{K}'$ itself.

If Q is taken as a constant Q_0, as used by Robinson in ref. [47], then the reduced evolution equation (3.43) is simply

$$\dot{\mathcal{K}}' = \hat{\Gamma}(W^p, T) \dot{W}^p \tag{3.44}$$

with

$$\mathcal{K}' = \mathcal{K} + \mathcal{K}_1 \left(1 - \exp\left[-Q_0 \left(\frac{1}{T_0} - \frac{1}{T} \right) \right] \right)$$

or

$$\left. \mathcal{K} = \mathcal{K}' - \mathcal{K}_1 \left(1 - \exp\left[-Q_0 \left(\frac{1}{T_0} - \frac{1}{T} \right) \right] \right) \right\} \tag{3.45}$$

It is remarkable that the reduced evolution equation (3.44) corresponds to the classical work hardening model for the isotropic hardening. To complete the transformation, the quantity $\mathcal{K}$ or $3\kappa^2$ appearing in the yield function (3.9) and in the free energy (3.24), has to be expressed by the transformed variable $\mathcal{K}'$ according to (3.41) or $(3.45)_2$. Thus

$$\begin{aligned} \psi = {} & \tilde{\psi}^e + \frac{1}{9} M(T) \frac{1}{2} a_{mn} a_{nm} + \\ & + \frac{1}{9} N(T) \frac{1}{6} \left\{ \mathcal{K}' - \mathcal{K}_1 \left(1 - \exp\left[-Q \left(\frac{1}{T_0} - \frac{1}{T} \right) \right] \right) \right\} + \\ & + \tilde{\psi}^T(T) . \end{aligned} \tag{3.46}$$

If Q is taken to depend on the plastic work W^p, the free energy with this set of variables will depend on W^p. This result will be disturbing for the material scientist since it appears not to have a "physically sound" bases.

This objection cannot be made if Q is constant: Q = Q_0. Then also the following *equivalence statement* can be made:

> The extension of the classical work hardening law for isotropic hardening by a temperature rate term, i.e.,
>
> $$\dot{\mathcal{K}} = \hat{\Gamma}(W^p, T)\,\dot{W}^p + \hat{\Theta}_{(T)}\,\dot{T} \tag{3.47}$$
>
> work hardening temperature rate term
>
> is equivalent to the following alternative approach:
>
> - isotropic work hardening law as usual, i.e.,
>
> $$\dot{\mathcal{K}} = \hat{\Gamma}(W^p, T)\,\dot{W}^p$$
>
> - but an additive supplement of the quantity K by a temperature term in the yield function and free energy, i.e.,
>
> $$\mathcal{K} \longrightarrow \mathcal{K} - \vartheta = \mathcal{K} - \int_{T_0}^{T} \hat{\Theta}_{(T)}\,dT.$$
>
> This additive temperature term depends only on the instant temperature T and a reference temperature T_0.

3.1.2.4 Evaluation of the Intrinsic Dissipation Inequality

The mechanical dissipation inequality is given by (3.20)

$$\gamma_M = \sigma_{kl}\,\dot{\varepsilon}^p_{kl} - \rho\,\frac{\partial\tilde{\psi}}{\partial a_{kl}}\,\dot{a}_{kl} - \rho\,\frac{\partial\tilde{\psi}}{\partial\varkappa}\,\frac{\hat{\Gamma}\dot{W}^p}{\sigma\varkappa} \geq 0. \tag{3.48}$$

With

$$\rho\,\frac{\partial\tilde{\psi}}{\partial a_{kl}} = M_{(T)}\,a_{kl}\,, \qquad \rho\,\frac{\partial\tilde{\psi}}{\partial\varkappa} = N_{(T)}\,\varkappa \tag{3.49}$$

and with the evolution equations (3.5) and (3.6) we get

$$\gamma_M = \gamma_P + \gamma_H + \gamma_R + \gamma_J \tag{3.50}$$

where

$$\gamma_P = \sigma_{kl} \dot{\varepsilon}^p_{kl} = S_{kl} \dot{\varepsilon}^p_{kl} \tag{3.51}$$

is the plastic stress power,

$$\gamma_H = -\Omega M a_{mn} \dot{\varepsilon}^p_{mn} \tag{3.52}$$

with

$$\Omega = \begin{cases} \Omega^+ = \dfrac{H}{G^\beta} > 0 \; ; \text{ if } G > G_0 \\ \qquad\qquad\qquad \text{and } S_{kl} a_{kl} > 0 \\ \\ \Omega^- = \dfrac{H}{G_0^\beta} > 0 \; ; \text{ if } G \leq G_0 \\ \qquad\qquad\qquad \text{or } S_{kl} a_{kl} \leq 0 \end{cases} \tag{3.53}$$

is the hardening contribution of the back stress, and

$$\gamma_R = + R M \frac{G^{m-\beta}}{\sqrt{I_2}} 2 I_2 \tag{3.54}$$

is a contribution due to static recovery; further

$$\gamma_T = -\rho \frac{\partial \tilde{\psi}}{\partial \kappa} \frac{\hat{\Gamma} \dot{W}^p}{\sigma \kappa} = - N_{(T)} \frac{\hat{\Gamma} \dot{W}^p}{\sigma} \tag{3.55}$$

is due to the work hardening part of the drag stress. Several observations can be made rather easily. The evolution equation for the plastic strain rate is such that the plastic stress power is non-negative for all processes [1], i.e.,

$$\gamma_P = \dot{W}^p \geq 0 . \tag{3.56}$$

Further, in the elastic regime $\dot{\varepsilon}^p_{kl} = 0$, and thus

$$\gamma_H = \gamma_R \geq 0 \qquad (\text{if } \dot{\varepsilon}^p_{kl} = 0)$$

consequently

$$M_{(T)}, R_{(T)} \geq 0 \quad , \quad \forall T;$$

to allow for recovery processes one must have R > 0 and therefore we derive

$$M_{(T)} \geq 0 . \tag{3.57}$$

The hardening term γ_H takes the form [1]

$$\gamma_H = -2 \frac{\Omega M A F^n}{\sqrt{J_2}} K_2 \tag{3.58}$$

where

$$K_2 := \frac{1}{2} a_{mn} \Sigma_{mn} \tag{3.59}$$

is one of the simultaneous invariants of the two tensors a_{mn} and Σ_{mn}.

The product term $\Omega MAF^n/\sqrt{J_2}$ is non-negative in the plastic region and therefore the invariant K_2 controls the sign of γ_H

$$\gamma_H = \begin{cases} < 0 , & K_2 > 0 \\ \geq 0 , & K_2 \leq 0 . \end{cases} \tag{3.60}$$

The drag stress contribution γ_J is negative since $\hat{\Gamma}$ and $\dot{W}^p$ are non-negative and since the parameter *N is assumed positive* analoguous to M; this appears physically sound (ψ increases with the increase of κ) although we do not have a physical model for the constitutive relations. Thus,

$$\gamma_J \leq 0 . \tag{3.61}$$

The mechanical dissipation then reads

$$\gamma_M = \underset{\geq 0}{\gamma_P} + \underset{\gtrless 0}{\gamma_H} + \underset{\geq 0}{\gamma_R} + \underset{\leq 0}{\gamma_J} . \tag{3.62}$$

With (3.51) to (3.55) we obtain

$$\gamma_M = \left[S_{kl} - M\Omega a_{kl} - \frac{N\hat{\Gamma}}{6} S_{kl} \right] \dot{\varepsilon}^{p}_{kl} +$$

$$+ RM G^{m-\beta} 2 \sqrt{I_2} = \tag{3.63}$$

$$= \frac{AF^{n}}{\sqrt{J_2}} \left[S_{kl} \left(1 - \frac{N}{6}\hat{\Gamma} \right) - M\Omega a_{kl} \right] \Sigma_{kl} + 2MR\, G^{m-\beta} \sqrt{I_2}.$$

With [1; p. 96]

$$S_{kl}\, \Sigma_{kl} = 2 \left(J_2 + K_2 \right) \tag{3.64}$$

equ. (3.63) yields

$$\gamma_M = \frac{AF^{n}}{\sqrt{J_2}} \left[2(J_2 + K_2)\left(1 - \frac{N}{6}\hat{\Gamma}\right) - 2M\Omega K_2 \right]$$

$$+ 2MR\, G^{m-\beta} \sqrt{I_2} =$$

$$= 2 \underbrace{\frac{AF^{n}}{\sqrt{J_2}}}_{\geq 0} \left[\underbrace{J_2}_{\geq 0} \cdot \left(1 - \frac{N}{6}\hat{\Gamma}\right) + \left(1 - \frac{N}{6}\hat{\Gamma} - M\Omega\right) \underbrace{K_2}_{>0} \right] \tag{3.65}$$

$$+ 2MR \left(\frac{I_2}{\varkappa_0^{2}} \right)^{m-\beta} \sqrt{I_2}.$$

In equ. (3.65) the signs of the various terms are as indicated; from (3.60) and (3.62) it is clear that the intrinsic dissipation is to be analyzed only for positive values of K_2.

If the dependence of the various functions in (3.65) is observed, one finds that the intrinsic dissipation is a function of a set of variables

$$\gamma_M = \hat{\gamma}_M(I_2\,, J_2\,, K_2\,, \mathcal{K}, W^p, T)$$

where

$$\left.\begin{aligned} I_2 &\geq 0 \\ 3J_2/\mathcal{K} &\geq 1 \\ K_2 &> 0 \\ \mathcal{K} &\geq 0 \\ W^p &\geq 0 \\ T &> 0\,. \end{aligned}\right\} \tag{3.66}$$

According to [1], the simultaneous invariant K_2 is related to I_2 and J_2 via the Schwarz inequality which, for positive K_2, reads

$$\left.\begin{aligned} 0 < K_2 &= \xi \sqrt{I_2}\sqrt{J_2} \\ 0 < \xi &\leq 1\,. \end{aligned}\right\} \tag{3.67}$$

Thus, the dissipation inequality requires γ_M to be non-negative,

$$\gamma_M = \tilde{\gamma}_M(I_2\,, J_2\,, \xi\,, \mathcal{K}, W^p, T) \geq 0 \tag{3.68}$$

for this set of arguments which may take independently arbitrary values within the assigned ranges.

Putting $I_2 = 0$, then (3.67) yields $K_2 = 0$. Then inequality (3.65) yields a *necessary condition* for the material parameters

$$1 - \frac{N(T)}{6}\,\hat{\Gamma}(W^p,T) \geq 0\,, \qquad \forall\, W^p, T. \tag{3.69}$$

For a given temperature T the function $\hat{\Gamma}$ has the upper bound at $W^p = 0$, i.e.,

$$\hat{\Gamma}(W^p,T) \leq \hat{\Gamma}_{max}(T) = \frac{\mathcal{K}_s(T) - \mathcal{K}_t(T)}{W_0(T)} > 0. \tag{3.70}$$

Therefore, a *necessary upper bound* for $N_{(T)}$ is obtained from (3.69)

$$N(T) \leq \frac{6}{\hat{\Gamma}_{max}(T)} = \frac{6\, W_o(T)}{\mathcal{H}_s(T) - \mathcal{H}_i(T)} \; . \tag{3.71}$$

Further, equ. (3.65) yields a *sufficient condition* for the material parameters if the factor of K_2 is required to be non-negative:

$$\begin{aligned} &\left(1 - \frac{N(T)}{6} \hat{\Gamma}(W^p, T) - M(T)\, \Omega(I_2, T)\right) \geq 0, \\ &\forall\, W^p, T, I_2 . \end{aligned} \tag{3.72}$$

According to (3.53), the discontinuous function Ω has the upper bound

$$\Omega(I_2, T) \leq \Omega_{max}(T) = \frac{H}{G_o^{\beta}} \; . \tag{3.73}$$

Therefore, taking the smallest value of the left hand side of (3.72), we deduce a *sufficient condition* which involves only the temperature as a state variable

$$1 - N(T) \frac{\hat{\Gamma}_{max}(T)}{6} - M(T)\, \Omega_{max}(T) \geq 0 \; , \quad \forall T. \tag{3.74}$$

This is a linear relation for the two parameters $M_{(T)}$ and $N_{(T)}$ which defines a triangular region in the (N, M)-plane (Fig. 1), the "region of sufficiency": Every pair of free energy parameters (N, M) within this region and on its boundary assures that the intrinsic dissipation γ_M is non-negative for all types of processes.

This results proves that the Robinson model, including the isotropic hardening, and its non-isothermal component is formally thermodynamically consistent in the following sence:

> *The model can be supplemented by thermodynamic functions such that the extended model is compatible with the requirement of non-negative entropy production. But obviously the model may still fail when confronted with caloric measurements.*

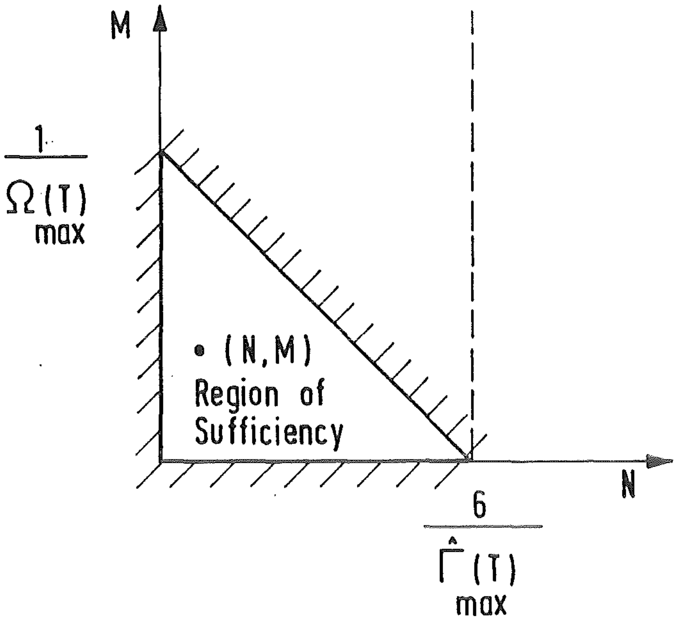

Fig. 1: Graphical Representation of the Sufficient Condition (3.74) for Non-Negative Mechanical Dissipation

The derivation of more stringent conditions for the material parameters to yield a non-negative dissipation deserves some comments. The previous discussion shows that there exists a lower bound for the intrinsic dissipation if we put $\xi = 1$ and observe the necessary condition (3.69):

$$\begin{aligned}\gamma_{LB} = {} & 2A\,\overset{m}{F}(J_2,\mathcal{K})\Big[\sqrt{J_2}\Big(1-\frac{N(T)}{6}\hat{\Gamma}(W^p,T)\Big)+\\ & +\sqrt{I_2}\Big(1-\frac{N(T)}{6}\hat{\Gamma}(W^p,T)-M(T)\,\Omega(I_2,T)\Big)\Big]+\\ & +2\,M(T)\,R(T)\Big(\frac{1}{\kappa_0^2}\Big)^{m-\beta}(I_2)^{m-\beta+1/2}\\ & \geq 0 \end{aligned} \qquad (3.75)$$

Thus, the lower bound intrinsic dissipation γ_{LB} may become negative solely due the term $M\Omega$ in relation to the other terms.

We will now introduce dimensionless quantities

$$\left.\begin{array}{l} \tilde{I}_2 = I_2/\varkappa_0^2 = G, \quad \tilde{J}_2 = J_2/\varkappa_0^2, \quad \tilde{K}_2 = K_2/\varkappa_0^2 \\ \text{and} \\ \tilde{\varkappa}^2 = \varkappa^2/\varkappa_0^2, \quad \tilde{\mathcal{K}} = \mathcal{K}/\varkappa_0^2 \end{array}\right\} \tag{3.76}$$

such that

$$\tilde{\mathcal{K}} = 3\tilde{\varkappa}^2$$

and

$$x = \sqrt{\tilde{I}_2}, \quad y = \sqrt{\tilde{J}_2}, \quad x_0 = \sqrt{\tilde{I}_{02}} = \sqrt{G_0}. \tag{3.77}$$

Then equ. (3.75) reads

$$\begin{aligned} \gamma_{LB} = {} & 2A\, F^n(y,\tilde{\mathcal{K}})\, \varkappa_0 \Big[y \Big(1 - \frac{N(T)}{6} \hat{\Gamma}(W^p,T)\Big) \\ & + x \Big(1 - \frac{N(T)}{6} \hat{\Gamma}(W^p,T) - M(T)\, \Omega(x,T)\Big)\Big] + \\ & + 2\, H(T)\, R(T)\, \varkappa_0\, (x^2)^{m-\beta+1/2} \end{aligned} \tag{3.78}$$

where

$$F = \frac{\tilde{J}_2}{\tilde{\varkappa}^2} - 1 = \frac{y^2}{\tilde{\mathcal{K}}/3} - 1 \tag{3.79}$$

$$\Omega = \left\{ \begin{array}{ll} \dfrac{H}{\tilde{I}_\ell^{\,\beta}} = \dfrac{H}{x^{2\beta}}\,, & \text{if } x > x_0 \text{ and } \tilde{K}_\ell + \tilde{I}_\ell > 0 \\[2ex] \dfrac{H}{\tilde{I}_{0\ell}^{\,\beta}} = \dfrac{H}{x_0^{2\beta}}\,, & \text{if } x \le x_0 \text{ or } \tilde{K}_\ell + \tilde{I}_\ell \le 0. \end{array} \right\} \tag{3.79}$$

Here the relation

$$S_{h\ell}\, Q_{h\ell} = 2\left(K_\ell + I_\ell\right)$$

is used in the switch conditions. Taking (K̃, Wp, T) as parameters, γ_{LB} is a function of x and y. The lower bound dissipation represents a surface in the (x, y, γ_{LB})-space and a discussion in geometrical terms is possible analogous to [1].

If the [...]-bracket in (3.78) is allowed to be negative, a critical region exists in the first quadrant of the (x, y)-plane (Fig. 2); in this region γ_{LB} may become negative. Its boundary is defined by

$$\left. \begin{array}{lll} & & \text{Boundary:} \\ y = \sqrt{\tilde{K}_\ell/3} & , & \overline{AB'B''} \\[2ex] y = \left(\dfrac{HM}{x^{2\beta}}\, \dfrac{1}{1-\frac{N}{6}\hat{p}} - 1 \right) x & , & \overline{B''C} \\[2ex] y = \left(\dfrac{HM}{x_0^{2\beta}}\, \dfrac{1}{1-\frac{N}{6}\hat{p}} - 1 \right) x & , & \overline{CA} \end{array} \right\} \tag{3.80}$$

Assuming

$$\beta \ge 1/2 \tag{3.81}$$

implies that the shape of the critical region approximates a triangle (Fig. 2).

Equations $(3.80)_2$ and $(3.80)_3$ define the geometrical locus in the (x, y)-plane where the [...]-bracket in equ. (3.78) vanishes, i.e., changes its sign. Outside the critical region the lower bound dissipation is definitely positive.

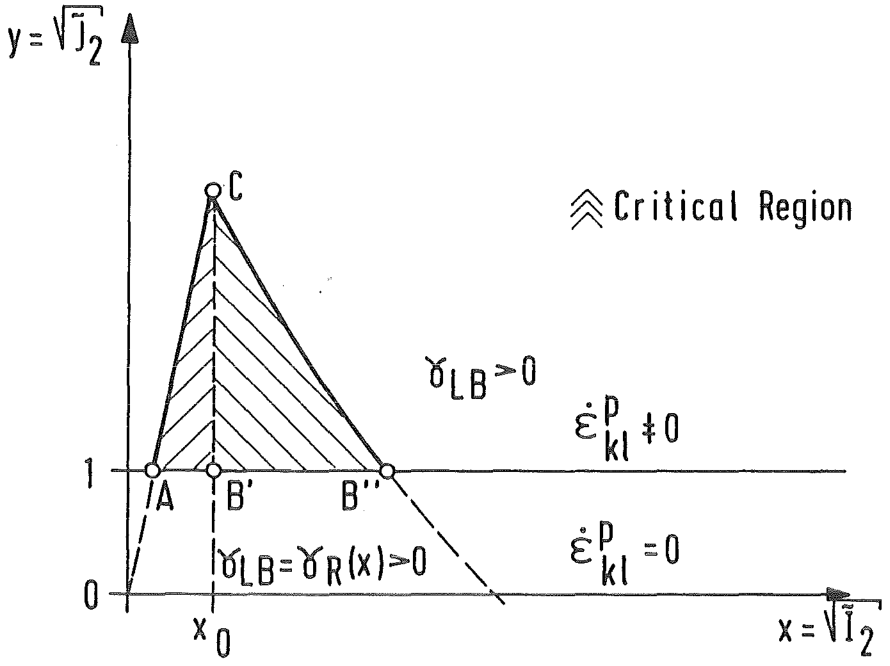

Fig. 2: Critical Region in the (x, y)-Plane

It is obvious that a critical region does exist if and only if the point C is above point B' (Fig. 2), provided (3.81) is valid. Thus, the following criteria apply

$$\text{A kritical region} \left\{ \begin{matrix} \text{exists} \\ \\ \text{does not exist} \end{matrix} \right\} \text{ if } \left\{ \begin{matrix} \gamma_c := \left(\frac{HM}{x_0^{2/\beta}} \frac{1}{1-\frac{N}{6}\hat{r}} - 1 \right) x_0 > \sqrt{\tilde{K}/3} \\ \\ \gamma_c \leq \sqrt{\tilde{K}/3} \end{matrix} \right. \qquad (3.82)$$

Consequently, the condition $(3.82)_2$ is a *sufficient condition* for a non-negative intrinsic dissipation everywhere in the (x, y)-state space. Equ. $(3.82)_2$ is formulated as an upper bound condition for M

$$M \leq \frac{x_0^{2/\beta}}{H} \underbrace{\left(1 - \frac{N}{6}\hat{r}\right)}_{\geq 0} \left(1 + \frac{\sqrt{\tilde{K}/3}}{x_0}\right). \qquad (3.83)$$

Unfortunately, this condition still depends on the variable W^p, the plastic work, and on the drag stress variable $\tilde{K}$; therefore, it does not represent a proper restriction on temperature dependent material parameters.

The work hardening function $\hat{\Gamma}(W^p, T)$ takes values in the interval

$$\underset{W^p \to \infty}{0} < \hat{\Gamma} \leq \underset{W^p = 0}{\hat{\Gamma}_{max}} = \frac{\mathcal{K}_s(T) - \mathcal{K}_i(T)}{W_0(T)} . \tag{3.84}$$

However, generally such bounds do not exist for the drag stress variable $\tilde{K}$ except that $\tilde{K}$ cannot become negative. Therefore, and with the maximum value of $\hat{\Gamma}$ a *sufficient upper bound for M* is given by

$$M \leq \frac{\varkappa_0^{2/\beta}}{H} \left(1 - \frac{N}{6} \hat{\Gamma}_{max}\right) = \frac{1}{\Omega_{max}} \left(1 - \frac{N}{6} \hat{\Gamma}_{max}\right). \tag{3.85}$$

This result has already been noted above (Fig. 1). For *strictly isothermal processes* ($\dot{T} = 0$) this bound can be improved. The initial value of the drag stress variable $\tilde{K}$ for a virgin (undeformed) material is

$$\begin{aligned} \tilde{\mathcal{K}}_i &= \frac{\mathcal{K}_i}{\varkappa_0^2} = \frac{1}{\varkappa_0^2}\left[\mathcal{K}_0 - \mathcal{K}_1 \left(1 - \exp\left[-Q_0 \left(\frac{1}{T_0} - \frac{1}{T_A}\right)\right]\right)\right] = \\ &= 3 - \frac{\mathcal{K}_1}{\varkappa_0^2}\left[1 - \exp\left[-Q_0\left(\frac{1}{T_0} - \frac{1}{T_A}\right)\right]\right] \end{aligned} \tag{3.86}$$

where T_A is the actual temperature. The value $\tilde{K}_i$ may be below the reference value

$$\tilde{\mathcal{K}}_0 = \frac{\mathcal{K}_0}{\varkappa_0^2} = 3$$

which corresponds to the reference temperature T_0. The evolution equation (3.6) simplifies to

$$\dot{\mathcal{K}} = \hat{\Gamma}(W^p, T_A)\, \dot{W}^p \tag{3.87}$$

with the initial value

$$W^p = 0 \; , \quad \mathcal{K}(0, T_A) = \mathcal{K}_i .$$

The ingration of (3.87) yields simply

$$\mathcal{K} = \mathcal{K}_i + (\mathcal{K}_s - \mathcal{K}_i)\left(1 - \exp\left[-W^p/W_0\right]\right)$$

or

$$\left.\mathcal{K} = \mathcal{K}_s - (\mathcal{K}_s - \mathcal{K}_i)\exp[-W^p/W_0]. \right\} \qquad (3.88)$$

The Robinson model has the inherent property that the plastic work is a monotonously increasing non-negative quantity and for a virgin material we have $W^p = 0$. Therefore, the drag stress variable K is also a monotonously increasing non-negative quantity for strictly isothermal processes. Its value lies in the interval

$$\underset{(W^p=0)}{\mathcal{K}_i} \leq \mathcal{K} \leq \underset{W^p \to \infty}{\mathcal{K}_s}$$

For isothermal processes the right hand side of (3.88) is only a function of W^p. Inspection shows that it is in fact monotonously increasing with W^p; its absolute minimum is at $W^p = 0$. Thus,

$$M \leq \frac{X_0^{2/\beta}}{H}\left(1 - \frac{N}{6}\overbrace{\frac{\mathcal{K}_s - \mathcal{K}_i}{W_0}}^{\hat{\Gamma}_{max}}\right)\left(1 + \frac{1}{X_0}\sqrt{\frac{\mathcal{K}_i}{3\kappa_0^2}}\right) \qquad (3.89)$$

This inequality assures only that a critical region does not exist and therefore it is only a *sufficient criterion* for non-negative dissipation. But more important, it is only valid for isothermal processes, only then the drag stress variable K is a monotonously increasing quantity even if $\hat{\theta} \equiv 0$.

With these remarks we close the evaluation of the intrinsic dissipation inequality.

3.1.2.5 Conclusions

The Robinson model, as described in [47], was developed to account qualitatively for thermomechanical path depedence of the isotropic hardening. Therefore, the evolution equation for the drag stress involves a temperature rate term linearly and thus it belongs to the evolution equation of "mixed type" (see section1). Clearly the theory is non-isothermal. However, the theory, as originally described, was not embedded in a thermodynamic frame and therefore its thermodynamic consistence was unresolved. For an early version of the purely "irreversible type" [45] a thorough analysis ([1], see also [43, 44]) proved its compatibility with the classical Clausius-Duhem entropy inequality. The analysis of the thermodynamic consistence of the present version requires an extension of the model, i.e. an appropriate choice for the Helmholtz free energy. Here a very simple assumption is made: The backstress and the drag stresss are only quadratically involved which implies only two additional parameters, i.e., $M_{(T)}$ for the backstress and $N_{(T)}$ for the drag stress.

The present analysis shows that this "mixed type" formulation gives the classical potential for the stress but destroyes that for the entropy. However, it is demonstrated that the evolution equation for the drag stress is reduceable to the purely "irreversible type" by a simple transformation of the internal variable "drag stress". Thus the classical potential relation for the entropy is retained again. As a consequence the free energy becomes a function also of the plastic work, a property which may raise objections on empirical grounds in material science circles. From the residual dissipation inequality a necessary condition on the parameter N is derived, for the parameter M only sufficient conditions are deduced. No restrictions for the parameters of the original Robinson model [47] are obtained.

These results demonstrate that also the advanced version [47] is formally consistent with thermodynamics.

3.2 A Model of Krempl et al.

3.2.1 Formal Description of the Model

Krempl et al. [52] developed a visco-plastic model based on overstress. It belongs to the class of "unified" theories, i.e., "plasticity" and "creep" are modeled with a *single* set of equations, rather than taking the traditional approach of seperate sets of equations to predict "time-independent plastic" strains and "time-dependent creep" strains. Predicting both "plasticity" and "creep" within a single variable is the primary distinguishing feature of the unified constitutive equations approach [72]. The model of Krempl et al. does not use a yield surface and special loading and unloading conditions. Both elastic and inelastic strain rates are active at all times.

In ref. [52] the formulation is given for the uniaxial state of stress but in ref. [53] a multiaxial state is assumed and this is the basis for the following description.

The total strain ε_{kl} is additively composed of the thermoelastic strain ε^{e}_{kl}, which includes the thermal strain ε^{th}_{kl}, and the viscoplastic strain ε^{p}_{kl}

$$\varepsilon_{kl} = \varepsilon^{e}_{kl} + \varepsilon^{p}_{kl} \tag{3.90}$$

where

$$\left.\begin{aligned} \varepsilon^{e}_{kl} &= \frac{1+\nu}{E}\,\sigma_{kl} - \frac{\nu}{E}\,\sigma_{mm}\,\delta_{kl} + \varepsilon^{th}_{kl} \\ \varepsilon^{th}_{kl} &= A^{th}\,(T - T_0)\,\delta_{kl}\,. \end{aligned}\right\} \quad (3.91)$$

The evolution equation for the viscoplastic strain is given by

$$\dot{\varepsilon}^{p}_{kl} = \frac{3}{2}\,\frac{1}{E\,k[\Gamma]}\,(s_{kl} - g_{kl}) \qquad (3.92)$$

with the deviatoric stress s_{kl} and the deviatoric equilibrium stress g_{kl}

$$s_{kl} = \sigma_{kl} - \frac{1}{3}\,\sigma_{mm}\,\delta_{kl}\,, \qquad g_{kl} = \gamma_{kl} - \frac{1}{3}\,\gamma_{mm}\,\delta_{kl}\,. \qquad (3.93)$$

It involves only one type of internal variable: the symmetric deviatoric equilibrium or back stress g_{kl}. Note that $\dot{\varepsilon}^{p}_{kl}$ and, with appropriate initial conditions, also ε^{p}_{kl} are deviatoric. The equilibrium stress is governed by the evolution equation

$$\dot{g}_{kl} = \frac{2}{3}\,\psi[\Gamma]\,\dot{e}_{kl} - \frac{\dot{\phi}}{b[\Gamma]}\left(g_{kl} - \frac{2}{3}\,E_t\,e_{kl}\right) \qquad (3.94)$$

where

$$\left.\begin{aligned} \Gamma &= \left\{\frac{3}{2}\,(s_{kl} - g_{kl})(s_{kl} - g_{kl})\right\}^{1/2} \\ b[\Gamma] &= A/(\psi[\Gamma] - E_t) \\ \dot{\phi} &= \left\{\frac{2}{3}\,\dot{\varepsilon}^{p}_{kl}\,\dot{\varepsilon}^{p}_{kl}\right\}^{1/2} \end{aligned}\right\} \quad (3.95)$$

and e_{kl} is the total deviatoric strain

$$e_{kl} = \varepsilon_{kl} - \frac{1}{3}\,\varepsilon_{mm}\,\delta_{kl}\,. \qquad (3.96)$$

E and ν are the elastic constants; E_t is the tangent modulus at the maximum strain of interest, and A is a positive constant [52] or an additional internal variable controlled by a separate evolution law [53] if cyclic hardening is to be modeled. The shape function $\psi[\Gamma]$ has the properties

$$0 < \psi[\Gamma] , \quad \psi[0] = \bar{E} < E , \quad \frac{d\psi}{d\Gamma} < 0. \tag{3.97}$$

The viscosity function $k[\Gamma]$ is also positive, bounded, and decreasing, i.e.,

$$0 < k[\Gamma] , \quad \max k[\Gamma] \leq K , \quad \frac{dk}{d\Gamma} < 0. \tag{3.98}$$

It should be noted that the model of Krempl et al. is actually formulated for constant temperature and the temperature dependence of the various terms is not explicitly mentioned.

Comparing the evolution law (3.94) for the equilibrium stress with other unified models (e.g. [72]), an important difference is seen in the first term: Krempl et al. use the total deviatoric strain rate $\dot{e}_{ij}$ instead of the inelastic strain rate $\dot{\varepsilon}p_{ij}$ which is the standard approach: Usually it is argued that g_{ij} is a state variable which should grow only when inelastic deformations occur and therefore the initial term is disturbing to material scientists and appears not to be "physical". Krempl et al. realize this criticism [52]. The reason for their choice is a better qualitative agreement of model predictions, i.e., almost linear elastic behavior under unloading and reloading conditions in strain controlled experiments.

Here it should be noted that the presence of the total strain rate in the evolution equations for the equilibrium or back stress implies that the solutions $g_{kl}(t)$ are not only functionals of the past history but also of the present values of total strain. Thus, a jump in the external variable "Cauchy stress" or "total strain" induces a jump in the back stress. Following microstructural considerations [73], one may take the position that this is not physically meaningful. But with Lubliner's observation (section 1) of the non-invariance of the evolution equations under transformations of internal variables, this argument does actually not hit the point: It would be more appropriate to require that only those internal variables are physically meaningful which allow to write the evolution equations in a rate-free form. This means that the evolution equations should be reduceable to the standard form (purely irreversible type).

3.2.2 Thermodynamic Consistence: Reduceability and Existence of a Gibbs Function

In the following we will not discuss the question whether or not the model is acceptable under microstructural considerations. Instead the thermodynamic consistence is of interest and here we will use the results of section 2.

To come as close as possible to the nomenclature of section 2 we write the evolution equation (3.94) in a form which involves the rate of the deviatoric stress instead of the total strain rate. With the strain partition (3.90) and Hooke's law (3.91) we find

$$\left.\begin{aligned} &\dot g_{kl} = f_{kl}(S_{mn}, g_{mn}, \varepsilon^p_{mn}) + \frac{1+\nu}{E}\,\frac{2}{3}\,\psi[r]\,\dot S_{kl} \\ &\text{where} \\ &f_{kl}(S_{mn}, g_{mn}, \varepsilon^p_{mn}) = \frac{2}{3}\,\psi[r]\,\dot\varepsilon^p_{kl} - \frac{\dot\phi}{b[r]}\left(g_{kl} - \frac{2}{3}E_t\left(e^e_{kl} + \varepsilon^p_{kl}\right)\right) \end{aligned}\right\} \quad (3.99)$$

with the deviatoric elastic strain e^e_{kl} to be obtained from $(3.91)_1$ and ε^p_{kl} found by integration of (3.92). Equ. $(3.99)_1$ may be written as

$$\dot g_{kl} = f_{kl}(S_{mn}, g_{mn}, \varepsilon^p_{mn}) + \check{E}_{klmn}[r]\,\dot\sigma_{mn} \quad (3.100)$$

where

$$\check{E}_{klmn}[r] = \frac{1+\nu}{E}\,\frac{2}{3}\,\psi[r]\left\{\underbrace{\frac{1}{2}\left(\delta_{km}\delta_{ln} + \delta_{lm}\delta_{kn}\right)}_{\tilde\delta_{klmn}} - \frac{1}{3}\,\delta_{kl}\delta_{mn}\right\} \quad (3.101)$$

In a *formal* sense here the internal variables are g_{kl} and ε^p_{kl}. Clearly, only (3.100) and not (3.92) involve the rate of the external variable stress, and the temperature rate does not occur in the evolution equations. However, in the following the viscoplastic strain ε^p_{kl} will not be assumed to be a state variable which affects the Gibbs function or Helmholtz free energy explicitly; it is simply a "process parameter" governed by the evolution law (3.92). Therefore, a transformation of the internal variables g_{kl} to g'_{kl} should not depend on ε^p_{kl} explicitly.

Then equ. (3.100) is analogous to (2.226) if we consider the viscoplastic strains ε^p_{kl} in (3.100) as additional "parameters" and if we use the following interpretation

$$\beta_\nu \longrightarrow g_{kl}\,, \quad f_\nu \longrightarrow f_{kl}\,, \quad \check{\Xi}_{mn} \longrightarrow E_{klmn}\,, \quad \check{H} \equiv 0. \tag{3.102}$$

The first term in the { ... } bracket of (3.101) is the 4. order unit tensor, which reflects the symmetry properties of the subscripts and the second term assures that the contracted product $E_{klmn}\,\dot{\sigma}_{mn}$ yields the deviatoric stress rate. The tensor E_{klm} has the following properties

$$E_{kkmn} = 0\,, \quad E_{klmm} = 0.$$

If the theory of Krempl et al. is thermodynamically consistent, then a Gibbs function ξ should exist which allows to derive the thermoelastic strain-stress relation (3.91) from it and an entropy temperature relation such that

$$\left.\begin{aligned} \check{\varepsilon}^{e}_{kl} &= -\varrho \left\{ \frac{\partial \check{\xi}}{\partial \sigma_{kl}} + \frac{\partial \check{\xi}}{\partial g_{mn}} \acute{E}_{mnkl} \right\} \\ \check{\eta} &= - \frac{\partial \check{\xi}}{\partial T} \end{aligned}\right\} \tag{3.103}$$

and with a mechanical dissipation inequality

$$\sigma_{kl}\,\dot{\varepsilon}^{p}_{kl} - \varrho \frac{\partial \check{\xi}}{\partial g_{mn}} \check{f}_{mn} \geq 0 \tag{3.104}$$

to be satisfied by the constitutive equations for all states. These conditions simply follow from (2.232) and (2.233) when properly translated to present case and with $\check{H}_p \rightarrow \check{H}_{kl} \equiv 0$.

The left hand side of $(3.103)_1$ is given by $(3.91)_1$, the classical isotropic thermoelastic strain-stress relation, which does not depend on the internal variables "equilibrium stress" g_{kl}.

Of course, the entropy function $\check{\eta}$ is not identified in the mechanical model of Krempl et al. We will assume that these were given by the classical thermoelastic entropy-temperature relation *independent of internal variables.* Then $\check{\varepsilon}_{kl}$ and $\check{\eta}$ are derivable from the partial potential ξ^e

$$\begin{aligned} \xi^{e} = &- \frac{1}{\varrho}\,\frac{1}{2} \left(\frac{\nu}{E} \sigma_{mm} \sigma_{nn} - \frac{1+\nu}{E} \sigma_{mn} \sigma_{nm} \right) - \\ &- \frac{1}{\varrho} A^{kk} (T - T_0)\, \sigma_{mm} + \overset{T}{\xi}(T) \end{aligned} \tag{3.105}$$

such that

$$\check{\varepsilon}^e_{kl} = -\varrho \frac{\partial \xi^e}{\partial \sigma_{kl}}, \qquad \check{\eta} = -\frac{\partial \xi^e}{\partial T}. \tag{3.106}$$

The Gibbs function $\check{\xi}$ (σ_{mn}, T, g_{kl}), depending on the internal variables g_{kl}, may be written in the following form

$$\check{\xi}(\sigma_{mn}, T, g_{mn}) = \xi^e(\sigma_{mn}, T) + \xi^g(\sigma_{mn}, T, g_{mn}) \tag{3.107}$$

with the partial potential ξ^g, tentatively assumed to be a function of all independent state variables. Inserting this into the restrictions (3.103) yields the following conditions

$$\left.\begin{aligned} -\left(\frac{\partial \xi^g}{\partial \sigma_{kl}} + \frac{\partial \xi^g}{\partial g_{mn}} E_{mnkl}\right) &= 0 \\ -\frac{\partial \xi^g}{\partial T} &= 0. \end{aligned}\right\} \tag{3.108}$$

Thus, the partial potential ξ^g is required to be independent of the temperature T.

Equation $(3.108)_1$ is an overdetermined system of partial differential equations for the single function ξ^g. Since we require that a Gibbs function depending on internal variables should exist, the system $(3.108)_1$ must admit a common solution ξ^g (σ_{mn}, g_{pq}).

Using the results of section (2.3), conditions (2.238) must be satisfied. With the present variables they read

$$\left.\begin{aligned} &\frac{\partial \xi^g}{\partial g_{mn}}\left\{\left(\frac{\partial E_{mnpq}}{\partial \sigma_{rs}} - \frac{\partial E_{mnrs}}{\partial \sigma_{pq}}\right) - \left(\frac{\partial E_{mnrs}}{\partial g_{kl}} E_{klpq} - \frac{\partial E_{mnpq}}{\partial g_{kl}} E_{klrs}\right)\right\} = 0 \\ &\frac{\partial \xi^g}{\partial g_{mn}}\left\{\frac{\partial E_{mnpq}}{\partial T}\right\} = 0. \end{aligned}\right\} \tag{3.109}$$

For non-vanishing $\partial\xi^g/\partial g_{kl}$ this is assured when either one or the other of the following conditions is satisfied:

- The evolution equations (3.100) for the internal variables g_{pq} are reduceable
- The defect $D = N - R$ of the MN-matrix $\mathbb{K}$ is positive

Reduceability:

The evolution equations (3.100) are reduceable then and only then when { ... } - brackets in (3.109) vanish identically

$$\left.\begin{aligned}
&\left(\frac{\partial E_{mnpq}}{\partial \sigma_{rs}} - \frac{\partial E_{mnrs}}{\partial \sigma_{pq}}\right) - \left(\frac{\partial E_{mnrs}}{\partial g_{kl}} E_{klpq} - \frac{\partial E_{mnpq}}{\partial g_{kl}} E_{klrs}\right) = 0 \\
&\frac{\partial E_{mnpq}}{\partial T} = 0 \; .
\end{aligned}\right\} \quad (3.110)$$

We note that the fourth order tensor $E_{mn\,pq}$, equ. (3.101), is not a constant so the conditions (3.110) are not trivially satisfied. If the elastic moduli E, ν are independent of the temperature, then $(3.110)_2$ is obeyed. The analysis of condition $(3.110)_1$ requires some calculation. With

$$\begin{aligned}
\frac{\partial \Gamma}{\partial \sigma_{rs}} &= \frac{\partial \Gamma}{\partial \Sigma_{kl}} \frac{\partial \Sigma_{kl}}{\partial \sigma_{rs}} = \frac{1}{2} \frac{1}{\Gamma} 3 \Sigma_{kl} \frac{\partial S_{kl}}{\partial \sigma_{rs}} \\
&= \frac{3}{2} \frac{1}{\Gamma} \left(S_{kl} - g_{kl}\right)\left(\delta_{kr} \delta_{ls} - \frac{1}{3} \delta_{kl} \delta_{rs}\right) \\
&= \frac{3}{2} \frac{1}{\Gamma} \left(S_{rs} - g_{rs}\right)
\end{aligned}$$

$$\frac{\partial \Gamma}{\partial g_{rs}} = -\frac{3}{2} \frac{1}{\Gamma} \left(S_{rs} - g_{rs}\right)$$

one finds

$$\frac{\partial E_{mnpq}}{\partial \sigma_{rs}} = \frac{1+\nu}{E}\,\frac{2}{3}\,\frac{d\Psi}{d\Gamma}\,\frac{\partial \Gamma}{\partial S_{rs}}\left[\tilde{\delta}_{mnpq} - \frac{1}{3}\delta_{mn}\delta_{pq}\right] =$$

$$= \frac{1+\nu}{E}\,\frac{d\Psi}{d\Gamma}\,\frac{1}{\Gamma}\left(S_{rs} - g_{rs}\right)\left[\tilde{\delta}_{mnpq} - \frac{1}{3}\delta_{mn}\delta_{pq}\right]$$

$$\frac{\partial E_{mnrs}}{\partial g_{kl}} = -\frac{1+\nu}{E}\,\frac{d\Psi}{d\Gamma}\,\frac{1}{\Gamma}\left(S_{kl} - g_{kl}\right)\left[\tilde{\delta}_{mnrs} - \frac{1}{3}\delta_{mn}\delta_{rs}\right]$$

with

$$\tilde{\delta}_{mnpq} = \frac{1}{2}\left(\delta_{mp}\delta_{nq} + \delta_{np}\delta_{mq}\right)$$

such that

$$\frac{\partial E_{mnrs}}{\partial g_{kl}}\,E_{klpq} = -\frac{1+\nu}{E}\,\frac{d\Psi}{d\Gamma}\,\frac{1}{\Gamma}\,\frac{1+\nu}{E}\,\frac{2}{3}\,\Psi\left(S_{pq} - g_{pq}\right)\left[\tilde{\delta}_{mnrs} - \frac{1}{3}\delta_{mn}\delta_{rs}\right]$$

Consequently, condition $(3.110)_1$ takes the following form

$$\frac{1+\nu}{E}\,\frac{d\Psi}{d\Gamma}\,\frac{1}{\Gamma}\Big[\!\Big[\left(S_{rs} - g_{rs}\right)\left[\tilde{\delta}_{mnpq} - \frac{1}{3}\delta_{mn}\delta_{pq}\right] -$$

$$-\left(S_{pq} - g_{pq}\right)\left[\tilde{\delta}_{mnrs} - \frac{1}{3}\delta_{mn}\delta_{rs}\right]$$

$$+\frac{1+\nu}{E}\,\frac{2}{3}\,\Psi\left\{\left(S_{pq} - g_{pq}\right)\left[\tilde{\delta}_{mnrs} - \frac{1}{3}\delta_{mn}\delta_{rs}\right]\right.$$

$$\left.-\left(S_{rs} - g_{rs}\right)\left[\tilde{\delta}_{mnpq} - \frac{1}{3}\delta_{mn}\delta_{pq}\right]\right\}\Big]\!\Big] =$$

$$= \frac{1+\nu}{E}\,\frac{d\Psi}{d\Gamma}\,\frac{1}{\Gamma}\left(1 - \frac{1+\nu}{E}\,\frac{2}{3}\,\Psi\right)\Big[\!\Big[\left(S_{rs} - g_{rs}\right)\left[\tilde{\delta}_{mnpq} - \frac{1}{3}\delta_{mn}\delta_{pq}\right]$$

$$-\left(S_{pq} - g_{pq}\right)\left[\tilde{\delta}_{mnrs} - \frac{1}{3}\delta_{mn}\delta_{rs}\right]\Big]\!\Big] =$$

$$\overset{!}{=} 0 \qquad (3.111)$$

With

$$\varphi := \frac{1+\nu}{E}\frac{d\Psi}{d\Gamma}\frac{1}{\Gamma}\left(1-\frac{1+\nu}{E}\frac{2}{3}\Psi\right) \neq 0 \qquad (3.112)$$

we obtain from (3.111)

$$(S_{rs}-g_{rs})\left[\tilde{\delta}_{mnpq}-\tfrac{1}{3}\delta_{mn}\delta_{pq}\right]-(S_{pq}-g_{pq})\left[\tilde{\delta}_{mnrs}-\tfrac{1}{3}\delta_{mn}\delta_{rs}\right]=0 \qquad (3.113)$$

which should be satisfied for all choices of subscripts. Assume, for example,

$$m = p \neq n = q \quad , \quad r = s \; ,$$

then the l.h.s of (3.113) reduces to

$$\tfrac{1}{2}\left(S_{rr}-g_{rr}\right) \qquad \text{(no summation);} \qquad (3.114)$$

clearly, this does not vanish identically. Therefore, the evolution equations (3.100) are not reduceable.

Defect of the MN-matrix$\mathbb{K}$

If the { ... } -brackets in (3.109) are not identically zero, which is the case, then (3.109) can be regarded as an overdetermined system of linear equations for the six unknowns $\partial\xi^g/\partial g_{mn}$

$$\frac{\partial \xi^g}{\partial g_{mn}} = \frac{\partial \xi^g}{\partial g_{nm}} \; .$$

Since $\partial E_{mn\,pq}/\partial T = 0$, the system is given by $(3.109)_1$. The { ... } -bracket is given by the l.h.s. of (3.111) and so $(3.109)_1$ takes the form

$$\varphi\cdot(S_{rs}-g_{rs})X_{pq} - \varphi\cdot(S_{pq}-g_{pq})X_{rs} = 0 \qquad (3.115)$$

where

$$\begin{aligned} X_{pq} &= \frac{\partial \xi^g}{\partial g_{mn}}\left(\tilde{\delta}_{mnpq}-\tfrac{1}{3}\delta_{mn}\delta_{pq}\right)= \\ &= \frac{\partial \xi^g}{\partial g_{pq}} - \frac{1}{3}\frac{\partial \xi^g}{\partial g_{mn}}\delta_{mn}\delta_{pq} = X_{qp} \end{aligned} \qquad (3.116)$$

is deviatoric. With

$$\Sigma_{rs} := (S_{rs} - g_{rs}) \cdot \varphi$$

$$\left[\Sigma_{rs}\right] \;\hat{=}\; \begin{bmatrix} \Sigma_1 & \Sigma_4 & \Sigma_6 \\ \Sigma_4 & \Sigma_2 & \Sigma_5 \\ \Sigma_6 & \Sigma_5 & \Sigma_3 \end{bmatrix}, \quad \left[X_{pq}\right] \;\hat{=}\; \begin{bmatrix} X_1 & X_4 & X_6 \\ X_4 & X_2 & X_5 \\ X_6 & X_5 & X_3 \end{bmatrix} \tag{3.117}$$

equ. (3.115) may also be written as

$$\Sigma_K X_L - \Sigma_L X_K = 0, \quad K, L = 1, \dots, 6 \tag{3.118}$$

This has the following matrix form

$$\mathbb{K}\, \mathbb{z} = \mathbb{0} \tag{3.119}$$

$$\underbrace{\begin{bmatrix} \Sigma_2 & -\Sigma_1 & 0 & 0 & 0 & 0 \\ \Sigma_3 & 0 & -\Sigma_1 & 0 & 0 & 0 \\ \Sigma_4 & 0 & 0 & -\Sigma_1 & 0 & 0 \\ \Sigma_5 & 0 & 0 & 0 & -\Sigma_1 & 0 \\ \Sigma_6 & 0 & 0 & 0 & 0 & -\Sigma_1 \\ 0 & \Sigma_3 & -\Sigma_2 & 0 & 0 & 0 \\ 0 & \Sigma_4 & 0 & -\Sigma_2 & 0 & 0 \\ 0 & \Sigma_5 & 0 & 0 & -\Sigma_2 & 0 \\ 0 & \Sigma_6 & 0 & 0 & 0 & -\Sigma_2 \\ 0 & 0 & \Sigma_4 & -\Sigma_3 & 0 & 0 \\ 0 & 0 & \Sigma_5 & 0 & -\Sigma_3 & 0 \\ 0 & 0 & \Sigma_6 & 0 & 0 & -\Sigma_3 \\ 0 & 0 & 0 & \Sigma_5 & -\Sigma_4 & 0 \\ 0 & 0 & 0 & \Sigma_6 & 0 & -\Sigma_4 \end{bmatrix}}_{\mathbb{K}} \underbrace{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{bmatrix}}_{\mathbb{z}} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}}_{\mathbb{0}} \tag{3.120}$$

To simplify the further discussion we choose an orthogonal coordinate system where Σ_{rs} takes its diagonal form. This is defined by the principal axes of Σ_{rs}. Then

$$\left[\Sigma_{rs}\right] \Longrightarrow \left[\Sigma^{*}_{rs}\right] = \begin{bmatrix} \Sigma_1^{*} & 0 & 0 \\ 0 & \Sigma_2^{*} & 0 \\ 0 & 0 & \Sigma_3^{*} \end{bmatrix}. \tag{3.121}$$

Of course, the principal axes of X_{pq} are not necessarily identical to those of Σ_{rs}. Then the system (3.120) reduces to

$$\begin{bmatrix} \Sigma_2^{*} & -\Sigma_1^{*} & 0 & 0 & 0 & 0 \\ \Sigma_3^{*} & 0 & -\Sigma_1^{*} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Sigma_1^{*} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Sigma_1^{*} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\Sigma_1^{*} \\ 0 & \Sigma_3^{*} & -\Sigma_2^{*} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Sigma_2^{*} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Sigma_2^{*} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\Sigma_2^{*} \\ 0 & 0 & 0 & -\Sigma_3^{*} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Sigma_3^{*} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\Sigma_3^{*} \end{bmatrix} \begin{bmatrix} X_1^{*} \\ X_2^{*} \\ X_3^{*} \\ X_4^{*} \\ X_5^{*} \\ X_6^{*} \end{bmatrix} = \left[0, \ldots, 0\right]^{T} \tag{3.122}$$

or explicitly

$$\left.\begin{aligned} \Sigma_2^{*} X_1^{*} - \Sigma_1^{*} X_2^{*} &= 0 \\ \Sigma_3^{*} X_1^{*} - \Sigma_1^{*} X_3^{*} &= 0 \\ \Sigma_3^{*} X_2^{*} - \Sigma_2^{*} X_3^{*} &= 0 \end{aligned}\right\} \tag{3.123}$$

and

$$X_4^* = 0\,, \quad X_5^* = 0\,, \quad X_6^* = 0\,. \tag{3.124}$$

Formally, the equ. (3.123) represent three equations for three unknowns X_1^*, X_2^*, X_3^*. The determinant vanishes identically. The nontrivial solution is

$$X_1^* = \frac{\Sigma_1^*}{\Sigma_3^*} X_3^*\,, \qquad X_2^* = \frac{\Sigma_2^*}{\Sigma_3^*} X_3^*$$

or

$$\frac{X_1^*}{\Sigma_1^*} = \frac{X_2^*}{\Sigma_2^*} = \frac{X_3^*}{\Sigma_3^*} = C = Const.$$

and

$$\begin{bmatrix} X_1^* \\ X_2^* \\ X_3^* \end{bmatrix} = \begin{bmatrix} \Sigma_1^* \\ \Sigma_2^* \\ \Sigma_3^* \end{bmatrix} C\,. \tag{3.125}$$

The defect is D = 1. If we account for the fact that the Σ_K^* are deviatoric,

$$\Sigma_1^* + \Sigma_2^* + \Sigma_3^* = 0$$

then we also have

$$X_1^* + X_2^* + X_3^* = 0\,.$$

From this result we see that for a general coordinate system we have

$$\left.\begin{aligned} [X_1, X_2, \ldots X_6] &= C\,[\Sigma_1, \Sigma_2, \ldots, \Sigma_6] \\ X_K = C\,\Sigma_K\,, &\qquad K = 1, \ldots, 6\,. \end{aligned}\right\} \tag{3.126}$$

It should be pointed out that (3.126) determines only the deviatoric part of $\partial\xi^g/\partial g_{pq}$. Thus,

$$\left.\begin{aligned} \frac{\partial \xi^g}{\partial g_{pq}} &= c\varphi \cdot (s_{pq} - g_{pq}) + G\delta_{pq} \\ G :&= \frac{1}{3} \frac{\partial \xi^g}{\partial g_{mn}} \delta_{mn} \end{aligned}\right\} \quad (3.127)$$

where G remains undetermined. Further, the evolution equations for the viscoplastic strain and equilibrium stress involve only the deviatoric Cauchy stress and the deviatoric equilibrium stress; the hydrostatic part of the Cauchy stress affects only the elastic volume change. Therefore, the partial potential ξ^g, if it exists, should only depend on the deviatoric quantities s_{mn} and g_{mn}.

Then (3.127) yields for $p \neq q$

$$\frac{\partial \xi^g}{\partial g_{pq}} = c\,\varphi \cdot (s_{pq} - g_{pq}) \quad (3.128)$$

and from $(3.108)_1$ we get with (3.101) and $k \neq l$

$$\frac{\partial \xi^g}{\partial s_{kl}} = -\frac{\partial \xi^g}{\partial g_{kl}} \frac{1+\nu}{E} \frac{2}{3} \psi(\Gamma) = -c\varphi(s_{kl} - g_{kl}) \frac{1+\nu}{E} \frac{2}{3} \psi(\Gamma) \quad (3.129)$$

Interchangeability of the partial derivatives requires

$$\frac{\partial \xi^g}{\partial g_{pq}\, \partial \sigma_{kl}} - \frac{\partial \xi^g}{\partial s_{kl}\, \partial g_{pq}} = 0 \qquad \forall \text{ subcript combinations} \quad (3.130)$$

For $p = k \neq q = l$ one gets

$$c\varphi + \frac{\partial c\varphi}{\partial s_{kl}}(s_{kl} - g_{kl}) - \left(c\varphi \frac{1+\nu}{E} \frac{2}{3}\psi - \frac{\partial}{\partial g_{kl}}\left[c\varphi \frac{1+\nu}{E}\frac{2}{3}\psi\right](s_{kl} - g_{kl})\right) = 0 \quad (3.131)$$

(no summation)

If we choose $s_{kl} - g_{kl} = 0$ for a given pair of subscripts kl, $(k \neq l)$, then

$$c\varphi\left(1 - \frac{1+\nu}{E}\frac{2}{3}\psi(\Gamma)\right) = c\frac{1+\nu}{E}\frac{d\psi}{d\Gamma}\frac{1}{\Gamma}\left(1 - \frac{1+\nu}{E}\frac{2}{3}\psi(\Gamma)\right)^2 = 0 \quad (3.132)$$

which certainly cannot be satisfied since ψ is not a constant. Thus, a common nontrivial solution $\xi^g \equiv 0$ depending on g_{mn} does not exist. *This means that the model of Krempl et al. does not admit a Gibbs function depending on the equilibrium stress. However, if this is required, then the model is thermodynamically inconsistent.* Consistency would require either a change in the evolution equation

(3.94) or an extension of the thermoelastic strain-stress relation (3.91) which would involve an extra term containing the equilibrium stress.

Of course, it is possible to drop the requirement that the Gibbs function ξ is depending on the equilibrium stress g_{mn}. Then ξ exists and is given by

$$\xi = \xi^{e}(\sigma_{mn}, T).$$

Whether this assumption is consistent with the dissipation inequality is studied in the next section.

3.2.3 Thermodynamic Consistence: Dissipation Inequality

At first sight it appears to be possible to refuse the internal variable "equilibrium stress" g_{mn} to be a true thermodynamic state variable, i.e., a variable affecting the state functions. Then

$$\frac{\partial \xi}{\partial g_{kl}} \equiv 0 \qquad (3.133)$$

and a Gibbs function is found to exist. But then the mechanical dissipation inequality (3.104) simplifies to

$$\gamma_M = s_{kl}\,\dot{\varepsilon}^{p}_{kl} \geq 0 \; ; \qquad (3.134)$$

that is, the plastic stress power should be non-negative for all states.

In the following we prove that there exists states which do not satisfy the condition (3.134). Consider a uniaxial state of stress such that

$$[s_{kl}] = \begin{bmatrix} \frac{2}{3}\sigma_{11} & 0 & 0 \\ 0 & -\frac{1}{3}\sigma_{11} & 0 \\ 0 & 0 & -\frac{1}{3}\sigma_{11} \end{bmatrix}, \quad [g_{kl}] = \begin{bmatrix} \frac{2}{3}g_{11} & 0 & 0 \\ 0 & -\frac{1}{3}g_{11} & 0 \\ 0 & 0 & -\frac{1}{3}g_{11} \end{bmatrix} \qquad (3.135)$$

and

$$\Gamma = \left\{ \frac{3}{2} (s_{kl} - g_{kl})(s_{kl} - g_{kl}) \right\}^{1/2} = |\sigma_{11} - g_{11}| . \qquad (3.136)$$

Then

$$\left[\dot{\varepsilon}^{p}_{kl}\right] = \begin{bmatrix} \dot{\varepsilon}^{p}_{11} & & \\ & \dot{\varepsilon}^{p}_{22} & \\ & & \dot{\varepsilon}^{p}_{33} \end{bmatrix} = \frac{1}{Ek}\begin{bmatrix} \sigma_{11}-\gamma_{11} & & \\ & -\frac{1}{2}(\sigma_{11}-\gamma_{11}) & \\ & & -\frac{1}{2}(\sigma_{11}-\gamma_{11}) \end{bmatrix} =$$

$$= \begin{bmatrix} \dot{\varepsilon}^{p}_{11} & & \\ & -\frac{1}{2}\dot{\varepsilon}^{p}_{11} & \\ & & -\frac{1}{2}\dot{\varepsilon}^{p}_{11} \end{bmatrix} \tag{3.137}$$

$$\left[\overset{\circ}{\gamma}_{kl}\right] = \frac{2}{3}\begin{bmatrix} \overset{\circ}{\gamma}_{11} & & \\ & -\frac{1}{2}\overset{\circ}{\gamma}_{11} & \\ & & -\frac{1}{2}\overset{\circ}{\gamma}_{11} \end{bmatrix} =$$

$$= \left[f_{kl}\right] + \frac{1+\nu}{E}\,\frac{2}{3}\,\psi[r]\,\frac{2}{3}\begin{bmatrix} \overset{\circ}{\sigma}_{11} & & \\ & -\frac{1}{2}\overset{\circ}{\sigma}_{11} & \\ & & -\frac{1}{2}\overset{\circ}{\sigma}_{11} \end{bmatrix} \tag{3.138}$$

and thus

$$\dot{\varepsilon}^{p}_{11} = \frac{\sigma_{11}-\gamma_{11}}{E\,k(r)}\,, \quad \overset{\circ}{\gamma}_{11} = \frac{3}{2}f_{11} + \frac{1+\nu}{E}\,\frac{2}{3}\,\psi(r)\,\overset{\circ}{\sigma}_{11}\,. \tag{3.139}$$

It is clear that with (3.99) the term f_{11} is only a function of σ_{11}, γ_{11}, ε^{p}_{11}. The plastic stress power is

$$S_{kl}\,\dot{\varepsilon}^{p}_{kl} = \sigma_{11}\,\dot{\varepsilon}^{p}_{11} \tag{3.140}$$

and it is obviously negative, if for positive values of the stress, the viscoplastic strain rate $\dot{\varepsilon}^{p}_{11}$ is negative; this is the case when $\sigma_{11} - \gamma_{11} < 0$. Fig. 3 shows the first quadrant of the stress-equilibrium stress diagram. It is clear that above the graph $\sigma_{11} - \gamma_{11} = 0$ the plastic strain rate is positive and below negative with negative plastic stress power.

We now show that states in this domain can indeed be reached by a suitable process. Assume that the initial state is at the origin ($\sigma_{11} = 0$, $\gamma_{11} = 0$). We then perform a *fast* tensile test which raises the stress to the level $\overset{\prime}{\sigma}_{11}$ at $t = t_1$ and γ_{11} arrives at the value $\overset{\prime}{\gamma}_{11}$. This value can be found from $(3.139)_2$ by integration:

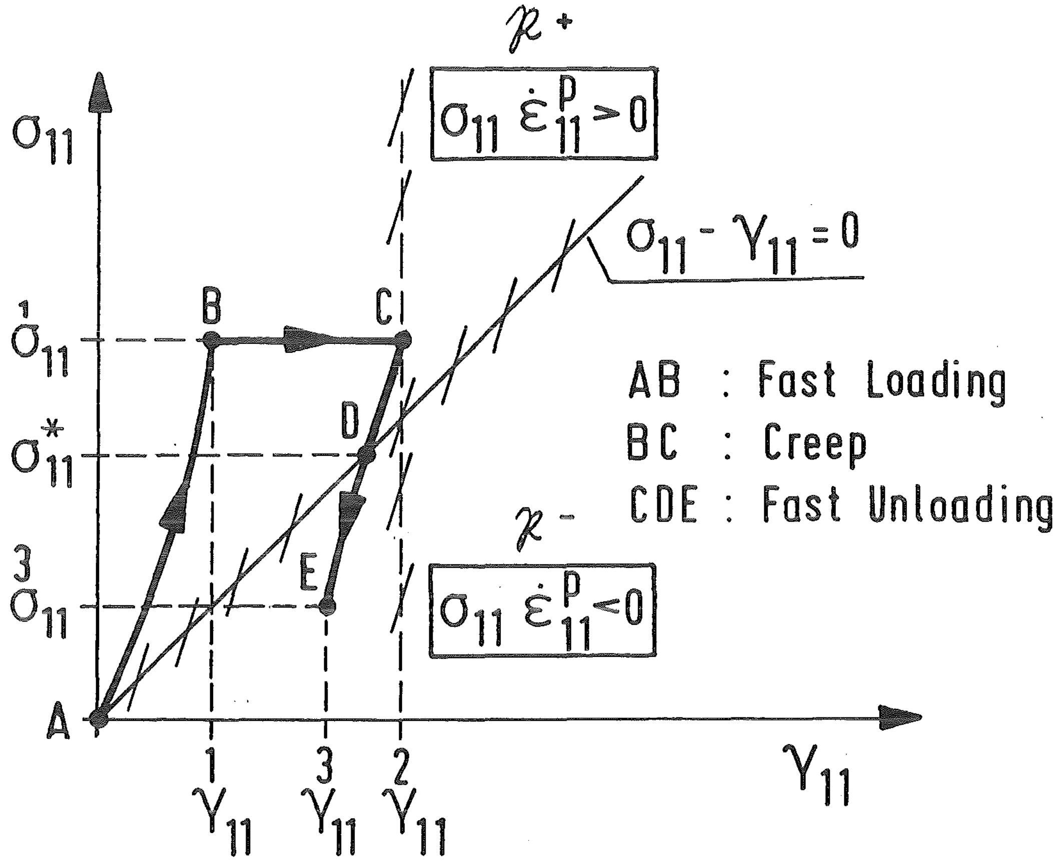

Fig. 3: Direction Field, Trajectories of Fast Loading, Creep and Fast Unloading Processes

$$\begin{aligned}\overset{1}{\gamma}_{11} &= \int_{t_0=0}^{t_1} \dot{\gamma}_{11}\, d\tau = \int_0^{t_1} \frac{3}{2} f_{11}\, d\tau + \int_0^{t_1} \frac{1+\nu}{E}\,\frac{2}{3}\,\psi(r)\,\dot{\sigma}\, d\tau \\ &= \int_0^{t_1} \frac{3}{2} f_{11}\, d\tau + \int_0^{\sigma_{11}} \frac{1+\nu}{E}\,\frac{2}{3}\,\psi[r]\, d\sigma .\end{aligned} \qquad (3.141)$$

For a sufficiently fast test the first term can be made as small as possible compared to the second since f_{11} is bounded and the time interval may be taken rather small. Thus $\dot{\gamma}_{11}$ is essentially controlled by the stress rate term during this test, i.e.

$$\dot{\gamma}_{11} \approx \frac{1+\nu}{E}\,\frac{2}{3}\,\psi[\Gamma]\,\dot{\sigma}_{11} \quad , \quad \dot{\sigma}_{11} > 0 \tag{3.142}$$

or

$$\frac{d\sigma_{11}}{d\gamma_{11}} \approx \frac{1}{C\,\psi[\Gamma]} > 0 \quad , \quad C := \frac{1+\nu}{E}\,\frac{2}{3} \tag{3.143}$$

since (3.142) is independent of the time scale. With (3.97) we have

$$\underset{\Gamma>0}{C\,\psi[\Gamma]} < C\,\psi[0] = \frac{1+\nu}{E}\,\frac{2}{3}\,\frac{\bar{E}}{E} < 1 \tag{3.144}$$

and therefore

$$\left(\frac{d\sigma_{11}}{d\gamma_{11}}\right)_{\Gamma>0} > \left(\frac{d\sigma_{11}}{d\gamma_{11}}\right)_{\Gamma=0} > 1 \tag{3.145}$$

so that the trajectory of this process transports the state in the domain above the boundary $\sigma_{11} - \gamma_{11} = 0$. We introduce the following substitution

$$\Gamma = |\sigma_{11} - \gamma_{11}| = \sigma_{11} - \gamma_{11} \geq 0 \tag{3.146}$$

which yields

$$\frac{d\Gamma}{d\sigma_{11}} = 1 - \frac{d\gamma_{11}}{d\sigma_{11}} = 1 - C\,\psi[\Gamma] > 0 \tag{3.147}$$

and consequently

$$0 < \int_0^{\overset{\prime}{\Gamma}} \frac{d\Gamma}{1 - C\,\psi[\Gamma]} = \int_0^{\overset{\prime}{\sigma}_{11}} d\sigma_{11} = \overset{\prime}{\sigma}_{11} \; ; \tag{3.148}$$

for a given function $\psi[\Gamma]$, subject to the conditions (3.97), and stress level $\overset{\prime}{\sigma}_{11}$, equ. (3.148) allows to determine $\overset{\prime}{\Gamma}$ and this yields with $\overset{\prime}{\Gamma} = \overset{\prime}{\sigma}_{11} - \overset{\prime}{\gamma}_{11}$ the equilibrium stress $\overset{\prime}{\gamma}_{11}$.

Starting from the state $(\overset{\prime}{\gamma}_{11}, \overset{\prime}{\sigma}_{11})$ a creep test at constant stess $\sigma_{11} = \overset{\prime}{\sigma}_{11}$ is performed and is terminated at the instant $t = t_2$ where the rate $\dot{\gamma}_{11}$ is still positive. This state is in the domain D+. At this state the equilibrium stress has increased to the value $\overset{2}{\gamma}_{11} > \overset{1}{\gamma}_{11}$. Then a *fast inloading* test is done lowering the stress to a positive value σ_{11}. Here again the contribution of f_{11} to the change in γ_{11} is negligible such that $(3.139)_2$ reads

$$\mathring{\gamma}_{11} = C'\,\psi[\Gamma]\,\mathring{\sigma}_{11} \qquad (3.149)$$

where now $\mathring{\sigma}_{11} < 0$ such that $\mathring{\gamma}_{11} < 0$. Then

$$\frac{d\sigma_{11}}{d\gamma_{11}} = \frac{1}{C'\,\psi[\Gamma]} > 0 \qquad (3.150)$$

where initially

$$\Gamma = |\sigma_{11} - \gamma_{11}| = \sigma_{11} - \gamma_{11} > 0. \qquad (3.151)$$

Equ. (3.150) represents an ordinary differential equation for the unknown function $\sigma_{11}\ (\gamma_{11})$ with initial value $(\overset{2}{\gamma}_{11}, \overset{2}{\sigma}_{11} = \overset{1}{\sigma}_{11})$. From the theory of O.D.E. we know that (3.150) admits a unique solution.

Equ. (3.150) defines the direction field of the trajectories of fast unloading (or fast loading!) processes in the $(\gamma_{11}, \sigma_{11})$-plane. Clearly, there is only one trajectory through each state. Along the domain boundary $(\sigma_{11} - \gamma_{11} = 0)$ we have $\Gamma = 0$. Thus, we see that all fast trajectories, which intersect the domain boundary, have a constant positive slope, somewhat larger than one, at the intersection.

If one sketches the direction field in the $(\gamma_{11}, \sigma_{11})$-plane (Fig. 3) one can imagine the trajectory of a process consisting of a fast loading, creep and fast unloading process. We now show that the unloading trajectory indeed *intersects* the domain boundary $(\sigma_{11} - \gamma_{11} = 0)$ at some positive stress value $\sigma_{11}{}^{*}$. For the first part CD of the unloading trajectory we find with (3.146) and (3.147)

$$\int_{\Gamma^{2}}^{\Gamma^{*}} \frac{d\Gamma}{1 - C'\,\psi[\Gamma]} = \int_{\overset{2}{\sigma}_{11} = \overset{1}{\sigma}_{11}}^{\sigma_{11}^{*}} d\sigma = \sigma_{11}^{*} - \overset{1}{\sigma}_{11}.$$

Along the unique unloading trajectory in the domain D^{+} we have with (3.146) and (3.147)

$$\int_{\Gamma^2}^{\Gamma} \frac{d\Gamma}{1-C\psi[\Gamma]} = \int_{\overset{2}{\sigma}_{11}=\overset{1}{\sigma}_{11}}^{\sigma_{11}} d\sigma = \sigma_{11} - \overset{1}{\sigma}_{11}$$

or

$$0 < \int_{\Gamma}^{\Gamma^2} \frac{d\Gamma}{1-C\psi[\Gamma]} = \overset{1}{\sigma}_{11} - \sigma_{11} \tag{3.152}$$

since (1 - $C\psi[\Gamma]$) > 0 and $\Gamma^2 - \Gamma > 0$. Equ. (3.152) represents a *unique* relation between Γ and σ_{11}. The required prove is furnished if it is shown that the choice $\Gamma = \Gamma^* = \sigma_{11}{}^* - \gamma_{11}{}^* = 0$ is associated with a positive value of the intersection stress $\sigma_{11}{}^*$.

We note that

$$\Gamma^2 = \overset{2}{\sigma}_{11} - \overset{1}{\sigma}_{11} = \overset{1}{\sigma}_{11} - \overset{2}{\gamma}_{11} < \overset{1}{\sigma}_{11} - \overset{1}{\gamma}_{11} = \Gamma^1. \tag{3.153}$$

Since $\Gamma^2 < \Gamma^1$, the comparison of the positive integrals in (3.152) and (3.148) reveals that the intersection stress $\sigma_{11}{}^*$ is a positive quantity in the interval

$$0 < \sigma_{11}^* < \overset{1}{\sigma}_{11}. \tag{3.154}$$

Thus, the unique fast unloading trajectory from the state C ($\gamma_{11}{}^2$, $\sigma_{11}{}^2$) leads into the domain D− with negative plastic stress power.

3.2.4 Conclusions

The results of section 3.2.2 show that *the thermoelastic strain-stress relation (Hooke's law) and the evolution equation for the equilibrium stress are in conflict with each other because of thermodynamic reasons: An appropriate Gibbs function does not exist.* Either one or the other of the constitutive relations need to be altered.

Also, if the equilibrium stress is not considered to be a true thermodynamic state variable, the model is in conflict with thermodynamics (section 3.2.3). According to the model, there exists states, which actually can be attained by a suitable process, where the plastic stress power is negative. However, under the above assumption the residual dissipation inequality requires the plastic stress power to be non-negative.

Before closing this section, we mention here an ad hoc modification of the model which promises consistence with thermodynamics. It appears natural not to change the thermoelastic strain-stress relation but to modify the evolution equation for the equilibrium stress.

The problematic term in (3.94) is the one proportional to the *total* strain rate, $\dot{e}_{kl}$. Of course, if this is changed to the plastic strain rate $\dot{\varepsilon}p_{kl}$, the existence of a Gibbs function is not a problem anymore.; but the residual dissipation inequality needs further analysis. However, then the modified model shows the unwanted responses [52]. A possible compromise is as follows. We introduce the strain partitioning (3.90). Then (3.94) takes the form

$$\dot{g}_{kl} = \frac{2}{3}\psi[\Gamma]\,\dot{\varepsilon}^{p}_{kl} - \frac{\dot{\Phi}}{b[\Gamma]}\left(g_{kl} - \frac{2}{3}E_t\,e_{kl}\right) + \frac{2}{3}\psi[\Gamma]\,\dot{e}^{e}_{kl}\,.$$

We note that

$$\frac{2}{3}\psi[\Gamma]\,\dot{e}^{e}_{kl} = \frac{2}{3}\psi[\Gamma]\,\frac{1+\nu}{E}\,\dot{s}_{kl}\,.$$

If the coefficient of the elastic strain rate or of the stress rate is not taken to be a function of Γ but is a strict constant K, the evolution equation is reduceable. This reduces the problem of thermodynamic consistence to that of satisfying the residual dissipation inequality. Then (3.94) changes to

$$\dot{g}_{kl} = \frac{2}{3}\psi[\Gamma]\,\dot{\varepsilon}^{p}_{kl} - \frac{\dot{\Phi}}{b[\Gamma]}\left(g_{kl} - \frac{2}{3}E_t\,e_{kl}\right) + K\dot{s}_{kl} \qquad (3.155)$$

where K is an additional material parameter, possibly the average value:

$$K = \frac{1}{\Gamma_{max}}\int_{0}^{\Gamma_{max}} \frac{2}{3}\psi[\Gamma]\,\frac{1+\nu}{E}\,d\Gamma \qquad (3.156)$$

if Γ_{max} can suitably be chosen. Of course, this modification needs further analysis.

3.3 Two Models Considered by Freed et al.

3.3.1 General Frame

Recently Freed, Chaboche and Walker [57, 82] presented thermodynamic considerations of kinematic hardening models of viscoplastic materials. We discuss here only those two models whose evolution equations for the backstress include a contribution that is linear in the stress rate. The purpose of the following discussion is to point out several aspects which have been overlooked so far. We will use the previous notation and indicate the corresponding notation used by the above authors if necessary.

The evolution law for the deviatoric viscoplastic strain ε^p_{ij} is assumed to be given by

$$\dot{\varepsilon}^p_{ij} = \frac{1}{2}\|\dot{\varepsilon}^p_{ij}\| \frac{S_{ij} - B_{ij}}{\|S_{ij} - B_{ij}\|} , \quad \dot{\varepsilon}_{ii} = 0 \tag{3.157}$$

with the norms

$$\left.\begin{aligned} \|S_{ij} - B_{ij}\| &= \left[\tfrac{1}{2}(S_{ij} - B_{ij})(S_{ij} - B_{ij})\right]^{1/2} \\ \|\dot{\varepsilon}^p_{ij}\| &= \left[2\, \dot{\varepsilon}^p_{ij}\, \dot{\varepsilon}^p_{ij}\right]^{1/2} = F(S_{mn}, B_{mn}, T, \ldots) \end{aligned}\right\} \tag{3.158}$$

and where S_{ij} and B_{ij} are the deviatoric stress and deviatoric back stress,

$$B_{ij} = 2H\beta_{ij} , \qquad H > 0 , \ \beta_{ii} = 0 \tag{3.159}$$

which accounts for the "kinematic hardening". The constant H has the dimension of stress; thus β_{ij} is dimensionless and is a deviatoric strainlike quantity. B_{ij} or alternatively β_{ij} are the internal variables.

To complete the description of (3.157) the norm $\|\dot{\varepsilon}^p_{ij}\|$ must be defined in terms of the stress, back stress, temperature, and possibly other quantities but for the following we do not need to specify the function F.

The internal variables β_{ij} are governed by an evolution law of the form

$$\dot{\beta}_{ij} = f_{ij}(S_{mn}, \beta_{mn}, T) + \check{E}\dot{S}_{ij} \tag{3.160}$$

$$\check{E} \geq 0$$

where $\check{E}$ is strictly a constant.

From a mathematical point of view the constant H is superfluous since (3.160) can also be written in terms of the back stress B_{ij} and H may be lumped with other quantities, i.e.,

$$\left.\begin{aligned} \dot{B}_{ij} &= 2H f_{ij} + 2H \check{E} \dot{S}_{ij} = \\ &= \tilde{f}_{ij}(S_{mn}, B_{mn}, T) + N \dot{S}_{ij} \\ \tilde{f}_{ij} &= 2H f_{ij} , \qquad N = 2H\check{E} . \end{aligned}\right\} \quad (3.161)$$

Further, it is assumed that the thermoelastic strain $\check{\varepsilon}^e_{ij}$

$$\begin{aligned} \varepsilon^{e}_{ij} &= \varepsilon_{ij} - \varepsilon^{p}_{ij} = \check{\varepsilon}^{e}_{ij}(\sigma_{mn}, T) \\ \varepsilon_{ij} &: \text{total strain} \end{aligned} \quad (3.162)$$

and the entropy $\check{\eta}$ are independent of the internal variables and are deriveable from the classical thermoelastic potential $\xi^e(\sigma_{mn}, T)$

$$\check{\varepsilon}^{e}_{ij} = -\varrho \frac{\partial \xi^{e}}{\partial \sigma_{ij}} , \qquad \check{\eta} = -\frac{\partial \xi}{\partial T} . \quad (3.163)$$

If F and ξ^e as well as f_{ij}, H and $\check{E}$ or alternatively $\tilde{f}_{ij}$ and $N \geqq 0$ are given, then the constitutive response is completely defined. Nevertheless, the thermodynamic consistence of these choices has to be proven.

This problem has been discussed in general terms in section 2.2 and 2.3. If the model is thermodynamically consistent, then a Gibbs function $\check{\xi}$ (σ_{mn}, β_{mn}, T) or alternatively $\tilde{\check{\xi}}$ (σ_{mn}, B_{mn}, T) should exist which satisfies the following restrictions (see equ. (2.232) and $(2.333)_1$)

$$\left.\begin{aligned} \check{\varepsilon}^{e}_{ij} &= -\varrho \frac{\partial \check{\xi}}{\partial \sigma_{ij}} - \varrho \frac{\partial \check{\xi}}{\partial \beta_{ij}} \check{E} \\ \check{\eta} &= -\frac{\partial \check{\xi}}{\partial T} \end{aligned}\right\} \quad (3.164)$$

and

$$\sigma_{kl} \dot{\varepsilon}^{p}_{kl} - \varrho \frac{\partial \check{\xi}}{\partial \beta_{ij}} f_{ij} \geq 0 . \quad (3.165)$$

We note that the coefficient of the rate term in (3.160) is strictly constant. Thus, according to the results of section 2.2, a transformation

$$\beta'_{ij} = h_{ij}(S_{mn}, \beta_{mn}) \tag{3.166}$$

exist which allows to reduce the evolution equation (3.160) to the standard form

$$\overset{\circ}{\beta}{}'_{ij} = f'_{ij}(S_{mn}, \beta'_{mn}, T) \tag{3.167}$$

(free of rates of external variables). With this conclusion and the assumption (3.163) the theorem on page 82 applies: A Gibbs function ξ (σ_{mn}, β_{mn}, T), depending on internal variables β_{mn} and satisfying the non-classical potential relations (3.164), exists but is required to have the form

$$\xi(\sigma_{mn}, \beta_{mn}, T) = \xi^{e}(\sigma_{mn}, T) + \xi^{\beta}(\sigma_{mn}, \beta_{mn}) \tag{3.168}$$

where ξ^β is restricted to be some function of only the transformed internal variables

$$\xi^{\beta}(\sigma_{mn}, \beta_{mn}) = \xi'^{\beta}(\beta'_{mn}). \tag{3.169}$$

Thus, the additional potential ξ^β must have a peculiar dependence on S_{mn} and β_{mn}.

With the results of section 2.2.3.2 a suitable transformation h_{ij} can now be easily determined. The function h_{ij} have to satisfy the overdetermined system of P.D.E.

$$\frac{\partial h_{ij}}{\partial S_{mn}} + \frac{\partial h_{ij}}{\partial \beta_{mn}} \check{E} = 0. \tag{3.170}$$

The existence of a common solution is assured since $\check{E}$ = const. With (2.169) a solution is

$$h_{ij} = A_{ijmn} S_{mn} + B_{ijmn} \beta_{mn}$$

where

$$A_{ijmn} = - B_{ijmn} \check{E}$$

Thus,

$$\beta'_{ij} = B_{ijmn} (\beta_{mn} - \check{E} S_{mn})$$

with B_{ijmn} arbitrary. The choice $B_{ijmn} = \delta_{im}\,\delta_{jn}$ yields

$$\beta'_{ij} = \beta_{ij} - \check{E}\, S_{ij} \tag{3.171}$$

such that

$$\dot{\beta}'_{ij} = f'_{ij}\left(S_{mn}, \beta'_{mn}, T\right) = f_{ij}\left(S_{mn}, \beta_{mn}, T\right). \tag{3.172}$$

Therefore, the Gibbs function ξ must have the following structure

$$\check{\xi}\left(\sigma_{mn}, \beta_{mn}, T\right) = \xi^{e}\left(\sigma_{mn}, T\right) + \xi^{\beta}\left(\beta_{mn} - \check{E} S_{mn}\right). \tag{3.173}$$

The simplest assumption for the additional potential is quadratic in the transformed internal variables β'_{mn}, i.e.,

$$\begin{aligned} \rho\, \xi^{\beta} &= \rho\, \xi'^{\beta}\left(\beta'_{mn}\right) = \frac{1}{2} M\, \beta'_{mn} \beta'_{mn} = \\ &= \rho\, \xi^{\beta}\left(S_{mn}, \beta_{mn}\right) = \frac{1}{2} M \left[\beta_{mn}\beta_{mn} - 2\check{E}\beta_{mn} S_{mn} + \check{E}^{2} S_{mn} S_{mn}\right] = \\ &= \frac{1}{2} M \left[\beta_{mn}\beta_{mn} + \check{E}\left(\check{E} S_{mn} S_{mn} - 2\beta_{mn} S_{mn}\right)\right] \end{aligned} \tag{3.174}$$

and in terms of the back stress

$$\begin{aligned} \rho\, \xi^{\beta} &= \frac{1}{2} M \left(\frac{1}{2H}\right)^{2} B'_{ij} B'_{ij} = \\ &= \frac{1}{2} M \left(\frac{1}{2H}\right)^{2} \left[B_{ij} B_{ij} + N\left(N S_{ij} S_{ij} - 2 B_{ij} S_{ij}\right)\right]. \end{aligned} \tag{3.175}$$

where

$$\left.\begin{aligned} B'_{ij} &= 2H\, \beta'_{ij} = B_{ij} - N S_{ij} \\ N &= \check{E}\, 2H . \end{aligned}\right\} \tag{3.176}$$

It is now enlightening to compare this with the assumptions of Freed et al. [82]. The Gibbs function is assumed to be given by

$$\rho\,\xi = \Psi = \rho\,\xi^{e}(\sigma_{mn},T) + \rho\,\xi^{\beta}_{F}(S_{mn},\beta_{mn}) \tag{3.177}$$

where ξ^e is the thermoelastic potential for an isotropic Hookean material and

$$\rho\,\xi^{\beta}_{F} = H\beta_{ij}\beta_{ij} + N\left(\frac{N}{4H}S_{ij}S_{ij} - \beta_{ij}S_{ij} + \Lambda(S_{uv},\beta_{uv})\right) \tag{3.178}$$

where Λ must satisfy*

$$\frac{\partial\Lambda}{\partial S_{ij}} + \frac{\partial\Lambda}{\partial\beta_{ij}}\,\frac{N}{2H} = 0. \tag{3.179}$$

Note that according to the nomenclature in ref. [82] we have

$$\check{E} = N/2H. \tag{3.180}$$

Thus, equ. (3.178) can be written as

$$\rho\,\xi^{\beta}_{F} = H\left\{\beta_{ij}\beta_{ij} + \check{E}\left(\check{E}S_{ij}S_{ij} - 2\beta_{ij}S_{ij} + 2\Lambda\right)\right\}. \tag{3.181}$$

As pointed out by Freed et al. the first three terms in (3.181) are introduced to remove unwanted cross products that would otherwise appear in the thermoelastic stress-strain relation. For similar reasons the additional undetermined function Λ, here called "the adjustment function", is restricted by (3.179). Except for the function Λ, the structures of (3.181) and (3.174) are equivalent. Thus (3.181) may be written as

$$\rho\,\xi^{\beta}_{F} = H\beta'_{ij}\beta'_{ij} + 2H\check{E}\Lambda(S_{mn},\beta_{mn})$$

or with (3.180)

$$\rho\,\xi^{\beta}_{F} = H\beta'_{ij}\beta'_{ij} + N\Lambda(S_{mn},\beta_{mn}). \tag{3.182}$$

We note that this expression contains the - actually superfluous - parameter H; because of this a definite physical significance is now attached to H. We further note that the first part in (3.182) satisfies the condition that ξ^{β}_{F} should be solely a function of β'_{ij} (or alternatively B'_{ij}). It is also seen that (3.182) does not contain a free parameter like M in relation (3.174). Instead, a yet undetermined function Λ is introduced whose purpose is to satisfy the intrinsic dissipation inequality. Aside

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* Note that this condition corresponds to $(2.237)_1$

from the fact that this is an artificial procedure one may question this for the following reason: From the previous discussion we know that Λ (S_{mn}, β_{mn}) must be *solely* a function of the transformed variables β'_{ij}. We come to this point later on.

In the following we discuss two models which differ in the structure of the function f_{ij} (S_{mn}, β_{mn}, T), the irreversible part of the evolution equation (3.160).

3.3.2 Case I (corresponds to model II in ref. [82])

According to [82], the irreversible part in (3.161) is given by

$$\tilde{f}_{ij}(S_{mn}, B_{mn}) = 2H f_{ij}(S_{mn},\beta_{mn}) = 2H\left(\dot{\varepsilon}^p_{ij} - \frac{B_{ij}-NS_{ij}}{2L}\|\dot{\varepsilon}^p_{ij}\|\right). \tag{3.183}$$

In terms of the internal variable β_{ij} we have for the irreversible part f_{ij}

$$\begin{aligned} f_{ij}(S_{mn},\beta_{mn}) &= \dot{\varepsilon}^p_{ij} - \frac{2H}{2L}\left(\beta_{ij} - \frac{N}{2H}S_{ij}\right)\|\dot{\varepsilon}^p_{uv}\| = \\ &= \dot{\varepsilon}^p_{ij} - \frac{2H}{2L}\underbrace{\left(\beta_{ij} - \check{E}S_{ij}\right)}_{\beta'_{ij}}\|\dot{\varepsilon}^p_{uv}\| = \\ &= f'_{ij}(S_{mn},\beta'_{mn}) \end{aligned} \tag{3.184}$$

and

$$\begin{aligned} \dot{\varepsilon}^p_{ij} &= \frac{1}{2}\|\dot{\varepsilon}^p_{uv}\|\frac{S_{ij}-B_{ij}}{\|S_{uv}-B_{uv}\|} = \frac{1}{2}\|\dot{\varepsilon}^p_{uv}\|\frac{S_{ij}-2H(\beta'_{ij}+\check{E}S_{ij})}{\|S_{uv}-2H(\beta'_{uv}+\check{E}S_{uv})\|} = \\ &= \frac{1}{2}\|\dot{\varepsilon}^p_{uv}\|\frac{(1-2H\check{E})}{|1-2H\check{E}|}\,\frac{S_{ij}-\frac{2H}{1-2H\check{E}}\beta'_{ij}}{\|S_{uv}-\frac{2H}{1-2H\check{E}}\beta'_{uv}\|}. \end{aligned} \tag{3.185}$$

In accordance with [82] we require*

$$(1-2H\check{E}) = (1-N) > 0 \implies N<1, \tag{3.186}$$

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[*] For a complete analysis of this model one should also discuss the implications of the case 1 - 2HE = (1 - N) ≤ 0; this will not be done here.

Thus

$$\dot{\varepsilon}^{p}_{ij} = \frac{1}{2}\|\dot{\varepsilon}^{p}_{uv}\| \frac{S_{ij} - \frac{2H}{1-2H\check{E}}\beta'_{ij}}{\|S_{uv} - \frac{2H}{1-2H\check{E}}\beta'_{uv}\|} . \tag{3.187}$$

If we introduce a new transformed back stress

$$B''_{ij} = \frac{1}{1-2H\check{E}} B'_{ij} = \frac{2H}{1-2H\check{E}}\beta'_{ij} = \frac{2H}{1-2H\check{E}}(\beta_{ij} - \check{E}S_{ij}) \tag{3.188}$$

which corresponds to X_{ij} in ref. [82]. Then (3.187) takes the form

$$\dot{\varepsilon}^{p}_{ij} = \frac{1}{2}\|\dot{\varepsilon}^{p}_{uv}\| \frac{S_{ij} - B''_{ij}}{\|S_{uv} - B''_{uv}\|} \tag{3.189}$$

and with (3.167), (3.184) and (3.188) the evolution equation for B''_{ij} is

$$\begin{aligned}\dot{B}''_{ij} &= \frac{2H}{1-2H\check{E}}\dot{\beta}'_{ij} = \frac{2H}{1-2H\check{E}} f'_{ij}(S_{mn},\beta'_{mn}) = \\ &= \frac{2H}{1-2H\check{E}}\left[\dot{\varepsilon}^{p}_{ij} - \frac{1-2H\check{E}}{2L} B''_{ij}\|\dot{\varepsilon}^{p}_{uv}\|\right] .\end{aligned} \tag{3.190}$$

In the following we use the admissible Gibbs function (3.173) with the free parameter M and we will try to find conditions on the material parameters - without introducing an auxiliary function Λ - such that the intrinsic dissipation inequality is satisfied for all states.

In terms of the transformed internal variables β'ij and with (3.173) and (3.174) the intrinsic dissipation inequality (3.165) reads

$$\gamma = S_{kl}\dot{\varepsilon}^{p}_{kl} - \rho\frac{\partial \xi^{\beta}}{\partial \beta'_{kl}}\dot{\beta}'_{kl} \geq 0 . \tag{3.191}$$

Since

$$\begin{aligned}\rho\frac{\partial \xi^{\beta}}{\partial \beta'_{kl}}\dot{\beta}'_{kl} &= M\beta'_{kl} f'_{kl}(S_{mn},\beta'_{mn}) = \\ &= M\beta'_{kl}\left(\dot{\varepsilon}^{p}_{kl} - \frac{2H}{2L}\beta'_{kl}\|\dot{\varepsilon}^{p}_{uv}\|\right)\end{aligned}$$

the dissipation γ is given by

$$\gamma = (S_{kl} - M\beta'_{kl})\,\dot{\varepsilon}^p_{kl} + \frac{2H}{2L} M \beta'_{kl}\beta'_{kl}\, \|\dot{\varepsilon}^p_{uv}\| . \qquad (3.192)$$

With (3.187) the first term is

$$(S_{kl} - M\beta'_{kl})\,\dot{\varepsilon}^p_{kl} = \frac{\frac{1}{2}\|\dot{\varepsilon}^p_{uv}\|}{\|S_{uv} - \frac{2H}{1-2H\check{E}}\beta'_{uv}\|}\left(S_{kl} - \frac{2H}{1-2H\check{E}}\beta'_{kl}\right)\left(S_{kl} - M\beta'_{kl}\right) =$$

$$= \frac{\frac{1}{2}\|\dot{\varepsilon}^p_{uv}\|}{\|S_{uv} - \frac{2H}{1-2H\check{E}}\beta'_{uv}\|}\Big[S_{kl}S_{kl} - \Big(M + \frac{2H}{1-2H\check{E}}\Big)\beta'_{kl}S_{kl} + \frac{2H}{1-2H\check{E}} M \beta'_{kl}\beta'_{kl}\Big].$$

With the definition

$$m := M\,\frac{1-2H\check{E}}{2H} > 0 \qquad (3.193)$$

we find

$$\gamma = \frac{\frac{1}{2}\|\dot{\varepsilon}^p_{uv}\|}{\|S_{uv} - \frac{2H}{1-2H\check{E}}\beta'_{uv}\|}\Big[S_{kl}S_{kl} - (m+1)\frac{2H}{1-2H\check{E}}\beta'_{kl}S_{kl} + m\Big(\frac{2H}{1-2H\check{E}}\Big)^2\beta'_{kl}\beta'_{kl}\Big] +$$

$$+ \frac{2H}{2L}\,\frac{2H}{1-2H\check{E}}\, m\, \beta'_{kl}\beta'_{kl}\, \|\dot{\varepsilon}^p_{uv}\| \qquad (3.194)_1$$

or alternatively in terms of the variable B''_{ij}

$$\gamma = \frac{\frac{1}{2}\|\dot{\varepsilon}^p_{uv}\|}{\|S_{uv} - B''_{uv}\|}\Big[S_{kl}S_{kl} - (m+1)B''_{kl}S_{kl} + m\, B''_{kl}S_{kl}\Big] +$$

$$+ \cdots$$

$$\ldots + \frac{2H}{2L}\,\frac{1-2H\check{E}}{2H}\, m\, B''_{kl}\, B''_{kl}\, \|\dot{\varepsilon}^{p}_{uv}\|. \qquad (3.194)_2$$

For the derivation of *necessary and sufficient conditions* on the material parameters it is now required to identify

$$\|\dot{\varepsilon}^{p}_{uv}\| = F(S_{mn}, \beta_{mn}) .$$

However, if one restricts the attention to *sufficient conditions* only, a solution can be obtained without this identification of F. Since the last term in (3.194) is non-negative, a sufficient conditions is derived by requiring that the [...] - bracket is non-negative. We introduce the following dimensionless quantities

$$x^2 = \frac{1}{H^2} S_{kl} S_{kl} > 0 , \quad y^2 = \frac{1}{H^2} (\sqrt{m}\, B''_{kl})(\sqrt{m}\, B''_{kl}) > 0 \qquad (3.195)$$

which are related to the second and simultaneous invariants of S_{kl} and $\sqrt{m}\,B''_{kl}$. Further

$$z := \frac{1}{H^2} (\sqrt{m}\, B''_{kl}) S_{kl} \quad \gtreqless 0 \qquad (3.196)$$

which corresponds to a simultaneous invariant of the tensor pair. Then the [...] - bracket in (3.194) reads

$$[\ldots] = H^2 \Big(x^2 - \frac{m+1}{\sqrt{m}} z + y^2 \Big) =: H^2 \omega . \qquad (3.197)$$

Since m > 0, the expression ω can become negative only for z > 0. For positive values the simultaneous invariant z is related to $x = +1/H \sqrt{S_{kl} S_{kl}}$ and $y = +1/H \cdot \sqrt{(\sqrt{m}\, B''_{kl}) S_{kl}}$ via the Schwarz inequality

$$z = \lambda x y \quad , \quad 0 < \lambda \leq 1 \qquad (3.198)$$

Thus, we obtain

$$\omega = x^2 - \frac{m+1}{\sqrt{m}} \lambda x y + y^2 . \qquad (3.199)$$

For fixed values x and y the most critical situation is obtained when

$$\lambda = 1 ,$$

a state which actually can be reached. Then the lower bound of ω is

$$\omega_{LB} = \omega(x, y, \lambda=1) = x^2 - \frac{m+1}{\sqrt{m}} xy + y^2 . \tag{3.200}$$

It is immediately obvious that, provided

$$m = 1 , \tag{3.201}$$

the lower bound (3.200) simplifies to

$$\omega_{LB} = (x - y)^2 \geq 0 ;$$

this is a quantity which is non-negative for all x, y. Thus, equ. (3.201) is just *one* sufficient condition on the material parameter to satisfy $\omega_{LB} \geqq 0$ which implies

$$\gamma \geq 0 .$$

We now analyse whether this is the only solution of the problem

$$\omega_{LB} \geq 0 , \qquad \forall x, y . \tag{3.202}$$

The equ. (3.200) represents a surface in the (x, y, ω_{LB})-space. We question under what conditions on the parameter

$$\tilde{m} := \frac{m+1}{\sqrt{m}} \tag{3.203}$$

this surface is nowhere in the negative half-space $\omega_{LB} < 0$. The surface intersects the (x, y)-plane along a curve defined by

$$\omega_{LB} = x^2 - \tilde{m} xy + y^2 = 0 , \quad x > 0, \; y > 0 .$$

This yields

$$y = x \left[\tfrac{1}{2} \tilde{m} \pm \sqrt{\left(\tfrac{1}{2} \tilde{m}\right)^2 - 1} \right] . \tag{3.204}$$

If

$$0 < \tfrac{1}{2} \tilde{m} < 1 , \tag{3.205}$$

the solution is complex; thus, there is no intersection of the ω_{LB}-surface with the (x, y)-plane. Since ω_{LB} is positive for x = 0 or y = 0, ω_{LB} is positive for all x>0, y>0. If

$$\tfrac{1}{2} \tilde{m} > 1 \tag{3.206}$$

then (3.204) defines two straight lines in the first quadrant of the (x, y)-plane (Fig. 4)

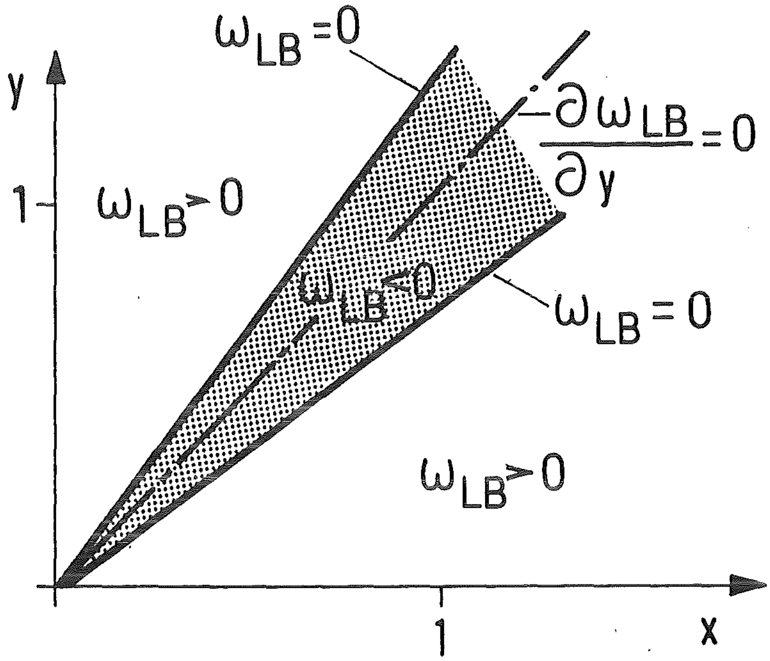

Fig. 4: Critical Domain of the Function ω_{LB}

Further

$$\frac{\partial \omega_{LB}}{\partial y} = -\tilde{m}\, x + 2y, \qquad \frac{\partial^2 \omega_{LB}}{\partial y^2} = +2,$$

the slope vanishes along the graph

$$y = \frac{1}{2}\tilde{m}\, x$$

along which ω_{LB} takes negative values everywhere (except x = 0)

$$(\omega_{LB})_{\frac{\partial \omega_{LB}}{\partial y}=0} = -\left(\frac{\tilde{m}}{2}x\right)^2.$$

Thus, since the ω_{LB}-surface is continuous and differentiable, the function ω_{LB} is negative within the wedge-shaped domain (Fig. 4) bounded by the straight lines (3.204).

Therefore, this domain represents a critical region in the (x, y)-space. Nonnegative values of ω_{LB} for all x and y are obtained only if (3.205) applies or if the critical domain vanishes, i.e., when the boundaries (3.204) join to form a single straight line which implies

$$\frac{\tilde{m}}{2} = 1. \tag{3.207}$$

Consequently, it is shown that the ω_{LB}-term is non-negative then and only then when

$$0 < \frac{1}{2}\tilde{m} \leq 1; \tag{3.208}$$

the value $\tilde{m} = 0$ is excluded from the discussion since $M>0$ is required. With (3.203) we have alternatively

$$0 < m \leq 1 \qquad \text{or} \qquad 0 < M \leq \frac{2H}{1-2H\check{E}}$$

or with (3.180) (3.209)

$$0 < \frac{M}{2H} \leq \frac{1}{1-N}.$$

These conditions are necessary and sufficient for ω_{LB} to be non-negative for all values of the stress S_{kl} and transformed back stress B''_{kl}. However, with regard to the complete dissipation inequality (3.194) they are only sufficient.

We will now compare these conclusions with the results by Freed et al. [82]. The viscoplastic contribution $\rho\xi^{B}{}_{F}$ of the Gibbs function is given by (3.178) or as shown above by (3.182). The function Λ (S_{mn}, β_{mn}) has been determined such that it cancels out those terms in the dissipation inequality that can become negative valued. This led to the condition

$$\frac{\partial \Lambda}{\partial S_{ij}} = \frac{N}{1-N}\left(\frac{N}{2H} S_{ij} - \beta_{ij}\right) \tag{3.210}$$

which, according to Freed et al., is the simplest of several possibilities. Observing (3.179), the authors obtained

$$\Lambda = \frac{N}{1-N}\left(\frac{N}{4H} S_{ij} S_{ij} - \beta_{ij} S_{ij} + \frac{H}{N}\beta_{ij}\beta_{ij}\right). \tag{3.211}$$

We note that with (3.171) and (3.180) this takes the form

$$\begin{aligned} \Lambda &= \frac{H}{1-N}\left(\beta_{ij}\beta_{ij} + \frac{N}{2H}\left(\frac{N}{2H} S_{ij} S_{ij} - 2\beta_{ij} S_{ij}\right)\right) = \\ &= \frac{H}{1-N}\beta'_{ij}\beta'_{ij}. \end{aligned} \tag{3.212}$$

This result indeed satisfies the condition that the "adjustment function" Λ should be solely a function of the transformed internal variable β_{ij}'. This representation is possible since the condition (3.210) can be written as

$$\frac{\partial \Lambda}{\partial S_{ij}} = -\frac{N}{1-N}\beta_{ij}' \, , \tag{3.213}$$

i.e., the r.h.s. is solely a function of β_{ij}' and does not explicitly depend on the stress S_{ij}. It appears that this a special property of this model.

If we combine (3.212) and $(3.182)_2$ we obtain

$$\begin{aligned} \rho f_F^{\beta} &= H\beta_{ij}'\beta_{ij}' + N\Lambda = \\ &= \left(H + H\frac{N}{1-N}\right)\beta_{ij}'\beta_{ij}' = \frac{H}{1-N}\beta_{ij}'\beta_{ij} . \end{aligned} \tag{3.214}$$

This result is a special case of the more general model (3.174) when the sufficient condition $(3.209)_3$ is accounted for: Obviously, the choice

$$M = M_F := 2\frac{H}{1-N} \tag{3.215}$$

makes the two Gibbs functions identical. The model of Freed et al. corresponds to the maximum admissible value of the parameter M.

3.3.3 Case II (corresponds to model III in ref. [82])

This second model regards the irreversible part f_{ij} (S_{mn}, β_{mn}) to be given by

$$f_{ij}(S_{mn}, \beta_{mn}) = \dot{\varepsilon}_{ij}^{p} - \frac{2H}{2L}\beta_{ij} \, \|\dot{\varepsilon}_{uv}^{p}\| . \tag{3.216}$$

This differs from (3.184) by the contribution of the stress tensor S_{ij} to the dynamic recovery term. Therefore, (3.160) and alternatively (3.161) read

$$\left.\begin{aligned} \dot{\beta}_{ij} &= \dot{\varepsilon}_{ij}^{p} - \frac{2H}{2L}\beta_{ij} \, \|\dot{\varepsilon}_{uv}^{p}\| + \check{E}\dot{S}_{ij} \\ &\text{or} \\ \dot{B}_{ij}'' &= 2H\left(\dot{\varepsilon}_{ij}^{p} - \frac{B_{ij}}{2L}\|\dot{\varepsilon}_{uv}^{p}\|\right) + N\dot{S}_{ij} . \end{aligned}\right\} \tag{3.217}$$

With the transformations (3.171), (3.176) and (3.188) the transformed evolution equations read

$$\left.\begin{aligned}\dot{\beta}'_{ij} &= \dot{\varepsilon}^p_{ij} - \frac{2H}{2L}\left(\beta'_{ij} + \check{E} S_{ij}\right)\|\dot{\varepsilon}^p_{uv}\| \\ &\text{or} \\ \dot{B}''_{ij} &= \frac{2H}{1-N}\left(\dot{\varepsilon}^p_{ij} - \frac{(1-N)B''_{ij} + N S_{ij}}{2L}\|\dot{\varepsilon}^p_{uv}\|\right)\end{aligned}\right\} \qquad (3.218)$$

with

$$B''_{ij} = \frac{2H}{1-2H\check{E}}\beta'_{ij} = \frac{2H}{1-N}\beta'_{ij}. \qquad (3.219)$$

The evolution equations for the viscoplastic strain rate (3.185) ÷ (3.189) apply as before. We note that the transformed back stress B_{ij}'' in (3.189) has the evolution equation $(3.218)_2$ which differs from (3.190) in the recovery term. The parameter $0 < N = 2H\check{E} < 1$ proportions the recovery term between the back stress B_{ij}'' and the applied stress S_{ij} [82].

Freed et al. proceed as follows. For the Gibbs function the assumption (3.178) is made as before but the "adjustment function" Λ is decomposed

$$\Lambda(S_{mn},\beta_{mn}) = \Lambda_1(S_{mn},\beta_{mn}) + \Lambda_2(S_{mn},\beta_{mn}) \qquad (3.220)$$

where Λ_1 is given by (3.211), i.e., Λ_1 is the Λ of Case I. The remaining function Λ_2 is used to cancel out those remaining terms in the dissipation inequality that can become negative valued. They obtain*

$$\frac{\partial\Lambda_2}{\partial\beta_{ij}} = -\frac{H}{L(1-N)} S_{kl}\beta_{kl}\left[\left(\frac{1}{2}\right)\frac{S_{ij}-2H\beta_{ij}}{\|S_{uv}-2H\beta_{uv}\|} - \frac{H}{L}\beta_{ij}\right]^{-1} =: G_{ij} \qquad (3.221)$$

and the constraint condition (3.179), i.e.,

$$\frac{\partial\Lambda_2}{\partial S_{ij}} = -\frac{N}{2H}\frac{\partial\Lambda_2}{\partial\beta_{ij}} =: H_{ij} = -\frac{N}{2H}G_{ij} \qquad (3.222)$$

---

* The expression in [82] needs a correction for the factor 1/2 in the [...]-bracket

This coupled system of P.D.E. has not been integrated and the existence of a common solution was taken for granted [57, 82]. However, the system (3.221) & (3.222) is clearly overdetermined. It is integrable then and only then when the r.h.s. satisfy the following conditions

$$\frac{\partial G_{ij}}{\partial \beta_{mn}} = \frac{\partial G_{mn}}{\partial \beta_{ij}}, \quad \frac{\partial G_{ij}}{\partial S_{mn}} = \frac{\partial H_{mn}}{\partial \beta_{ij}}, \quad \frac{\partial H_{ij}}{\partial S_{mn}} = \frac{\partial H_{mn}}{\partial S_{ij}}. \tag{3.223}$$

We will now show that a common solution does not exist. This then demonstrates that the approach of the "adjustment function", aside from beeing artificial and unphysical, does not necessarily work. This prove will not be performed with the use of the integrability conditions (3.223) but a less cumbersome approach is applied.

We choose the Gibbs function in the form

$$\rho\xi = \rho\,\xi^{e}(\sigma_{mn}, T) + \rho\,\xi^{\beta}(\overbrace{\beta_{mn} - \check{E}S_{mn}}^{\beta'_{mn}}) + A(S_{mn}, \beta_{mn}) \tag{3.224}$$

where $\rho\,\xi^\beta$ is given by (3.174) and A is an "adjustment function". If also

$$M = M_F = \frac{2H}{1-N}, \qquad A = N\Lambda_2, \tag{3.225}$$

then the choice (3.224) completely agrees with the choice of Freed et al. However, A or Λ_2 are required to be solely functions of the transformed internal variable β'_{ij}, i.e.,

$$A = A'(\beta'_{mn}) \qquad or \qquad \Lambda_2 = \Lambda'_2(\beta'_{mn}). \tag{3.226}$$

Without recourse to the results of section 2 we show this here again. Analogous to Case I the functions A or Λ_2 have to satisfy the constraint

$$\frac{\partial A}{\partial S_{ij}} + \check{E}\frac{\partial A}{\partial \beta_{ij}} = 0. \tag{3.227}$$

We introduce the new variable β'_{ij}. Then assume

$$A = A(S_{mn}, \beta_{mn}) = A'(S_{mn}, \beta'_{mn}). \tag{3.228}$$

Consequently,

$$\frac{\partial A}{\partial S_{ij}} = \frac{\partial A'}{\partial S_{ij}} + \frac{\partial A'}{\partial \beta'_{mn}} \frac{\partial \beta'_{mn}}{\partial S_{ij}} = \frac{\partial A'}{\partial S_{ij}} - \check{E} \frac{\partial A'}{\partial \beta'_{ij}}$$

$$\frac{\partial A}{\partial \beta_{ij}} = \frac{\partial A'}{\partial \beta'_{mn}} \frac{\partial \beta'_{mn}}{\partial \beta_{ij}} = \frac{\partial A'}{\partial \beta'_{ij}} .$$

Inserting this in (3.227) one gets

$$\frac{\partial A'}{\partial S_{ij}} = 0. \tag{3.229}$$

Thus, the general solution of (3.227) is given by (3.226).

The intrinsic dissipation takes the following form

$$\begin{aligned} \gamma &= S_{ij} \dot{\varepsilon}^{p}_{ij} - \varrho \frac{\partial \xi}{\partial \beta'_{ij}} \dot{\beta}'_{ij} = \\ &= S_{ij} \dot{\varepsilon}^{p}_{ij} - \left(M \beta'_{ij} + \frac{\partial A}{\partial \beta'_{ij}}\right)\left(\dot{\varepsilon}^{p}_{ij} - \frac{2H}{2L}(\beta'_{ij} + \check{E} S_{ij}) \| \dot{\varepsilon}^{p}_{uv} \|\right) = \\ &= (S_{ij} - M \beta'_{ij}) \dot{\varepsilon}^{p}_{ij} + M \frac{2H}{2L} \beta'_{ij} \beta'_{ij} \| \dot{\varepsilon}^{p}_{uv} \| + \\ &\quad + M \frac{2H}{2L} \check{E} \beta'_{ij} S_{ij} \| \dot{\varepsilon}^{p}_{uv} \| - \\ &\quad - \frac{\partial A}{\partial \beta'_{ij}} \left(\dot{\varepsilon}^{p}_{ij} - \frac{2H}{2L}(\beta'_{ij} + \check{E} S_{ij}) \| \dot{\varepsilon}^{p}_{uv} \|\right). \end{aligned} \tag{3.230}$$

Using the previous results (3.192) ÷ (3.194) we obtain

$$\begin{aligned} \gamma = \frac{\frac{1}{2} \| \dot{\varepsilon}^{p}_{uv} \|}{\left\| S_{uv} - \frac{2H}{1-2H\check{E}} \beta'_{ij} \right\|} \Big[& S_{kl} S_{kl} - (m+1) \frac{2H}{1-2H\check{E}} \beta'_{kl} S_{kl} \\ & + m \left(\frac{2H}{1-2H\check{E}}\right)^{2} \beta'_{kl} \beta'_{kl} \Big] + \\ + \dots \end{aligned}$$

$$\ldots + \frac{2H}{2L}\,\frac{2H}{1-2H\check{E}}\, m\, \beta'_{kl}\,\beta'_{kl}\, \|\dot{\varepsilon}^{p}_{uv}\| +$$

$$+ \frac{2H}{2L}\,\frac{2H}{1-2H\check{E}}\, \check{E}\, m\, \beta'_{kl}\, S_{kl}\, \|\dot{\varepsilon}^{p}_{uv}\| \tag{3.231}$$

$$- \frac{\partial A'}{\partial \beta'_{kl}} \left(\dot{\varepsilon}^{p}_{kl} - \frac{2H}{2L} \left(\beta'_{kl} + \check{E} S_{kl}\right) \|\dot{\varepsilon}^{p}_{uv}\| \right).$$

We impose the constraint (3.209)

$$0 < m \leq 1$$

which includes the choice m = 1 of Freed et al. Then all terms in (3.231) are non-negative except the last two rows. We now try to find a function A' such that these terms chancel, i.e.,

$$\frac{\partial A'}{\partial \beta'_{kl}} \left(\dot{\varepsilon}^{p}_{kl} - \frac{2H}{2L} \left(\beta'_{kl} + \check{E} S_{kl}\right) \|\dot{\varepsilon}^{p}_{uv}\| \right) =$$

$$= \frac{2H}{2L}\,\frac{2H}{1-2H\check{E}}\, \check{E}\, m\, \beta'_{kl}\, S_{kl}\, \|\dot{\varepsilon}^{p}_{uv}\|. \tag{3.232}$$

With (3.187) we get

$$\frac{\partial A'}{\partial \beta'_{kl}} \left(\frac{1}{2}\, \frac{S_{kl} - \frac{2H}{1-2H\check{E}}\beta'_{kl}}{\left\|S_{uv} - \frac{2H}{1-2H\check{E}}\beta'_{uv}\right\|} - \frac{2H}{2L}\left(\beta'_{kl} + \check{E} S_{kl}\right) \right) =$$

$$= \frac{2H}{2L}\,\frac{2H}{1-2H\check{E}}\, \check{E}\, m\, \beta'_{kl}\, S_{kl} \tag{3.233}$$

or

$$\frac{\partial A'}{\partial \beta'_{kl}} = \frac{H}{L}\,\frac{N}{1-N}\, m\,\beta'_{mn} S_{mn} \left(\frac{1}{2}\, \frac{S_{kl} - \frac{2H}{1-N}\beta'_{kl}}{\left\| S_{uv} - \frac{2H}{1-N}\beta'_{uv} \right\|} - \frac{H}{L}\left(\beta'_{kl} + \frac{N}{2H} S_{kl} \right) \right)^{-1} \tag{3.234}$$

or with $(3.225)_2$

$$\frac{\partial \Lambda'_2}{\partial \beta'_{kl}} = \frac{H}{L(1-N)}\, \beta'_{mn} S_{mn} \left(\frac{1}{2}\, \frac{S_{kl} - \frac{2H}{1-N}\beta'_{kl}}{\left\| S_{uv} - \frac{2H}{1-N}\beta'_{uv} \right\|} - \frac{H}{L}\left(\beta'_{kl} + \frac{N}{2H} S_{kl} \right) \right)^{-1}. \tag{3.235}$$

The left hand side is, for reasons of consistency, *solely* a function of the transformed internal variable β'_{kl}. However, the right hand side is depending on β'_{kl} and the applied stress S_{kl} as well and this variable cannot be eliminated. Therefore, (3.235) is inconsistent.To come more closely to the formulation of Freed et al. we make the choice (3.226) and write the r.h.s. of (3.231), using the internal variable β_{kl}, as follows

$$\begin{aligned} \gamma = {} & \frac{1}{2}\, \frac{\| \dot{\varepsilon}^{p}_{uv} \|}{\left\| S_{uv} - \frac{2H}{1-2H\breve{E}}\beta'_{uv} \right\|} \left[S_{kl} - \frac{2H}{1-2H\breve{E}}\beta'_{kl} \right]^{2} + \\ & + \frac{2H}{2L}\,\frac{2H}{1-2H\breve{E}} \left(\beta'_{kl}\beta'_{kl} + \breve{E}\beta'_{kl} S_{kl} \right) \| \dot{\varepsilon}^{p}_{uv} \| - \\ & - N\,\frac{\partial \Lambda'_2}{\partial \beta'_{kl}} \left(\dot{\varepsilon}^{p}_{kl} - \frac{2H}{2L}\beta_{kl} \| \dot{\varepsilon}^{p}_{uv} \| \right) = \\ = {} & \| \dot{\varepsilon}^{p}_{uv} \| \left\| S_{uv} - \frac{2H}{1-2H\breve{E}}\beta'_{uv} \right\| + \\ & + \frac{2H}{2L}\,\frac{2H}{1-2H\breve{E}} \left(\beta_{kl}\beta_{kl} - \underline{\breve{E}\beta_{kl} S_{kl}} \right) \| \dot{\varepsilon}^{p}_{uv} \| - \\ & \underline{- N\,\frac{\partial \Lambda'_2}{\partial \beta'_{kl}} \left(\dot{\varepsilon}^{p}_{kl} - \frac{2H}{2L}\beta_{kl} \| \dot{\varepsilon}^{p}_{uv} \| \right)}. \end{aligned} \tag{3.236}$$

Here the relation

$$\beta'_{kl}\beta'_{kl} + \check{E}\beta'_{kl}S_{kl} = \beta_{kl}\beta_{kl} - \check{E}\beta_{kl}S_{kl} \tag{3.237}$$

is used. We now chancel the terms which may become negative, i.e., the last row and the mixed term in the second row. We get with (3.157) and (3.159)

$$\frac{\partial\Lambda'_2}{\partial\beta'_{kl}}\left(\frac{1}{2}\,\frac{S_{kl}-2H\beta_{kl}}{\|S_{uv}-2H\beta_{uv}\|}-\frac{H}{L}\beta_{kl}\right) = -\frac{H}{L(1-N)}\beta_{kl}S_{kl}$$

or

$$\frac{\partial\Lambda'_2}{\partial\beta'_{kl}} = \frac{\partial\Lambda_2}{\partial\beta_{kl}} = -\frac{H}{L(1-N)}\beta_{mn}S_{mn}\left(\frac{1}{2}\,\frac{S_{kl}-2H\beta_{kl}}{\|S_{uv}-2H\beta_{uv}\|}-\frac{H}{L}\beta_{kl}\right)^{-1}. \tag{3.238}$$

Equ. (3.238) corresponds to (3.221). Further, with

$$\frac{\partial\Lambda'_2(\beta'_{mn})}{\partial S_{kl}} = \frac{\partial\Lambda_2(S_{mn},\beta_{mn})}{\partial S_{kl}} + \frac{\partial\Lambda_2}{\partial\beta_{mn}}\,\frac{\partial\beta_{mn}}{\partial S_{kl}} = \frac{\partial\Lambda_2}{\partial S_{kl}} + \frac{\partial\Lambda_2}{\partial\beta_{kl}}\check{E} = 0$$

we get

$$\frac{\partial\Lambda_2}{\partial S_{kl}} = -\check{E}\,\frac{\partial\Lambda_2}{\partial\beta_{kl}} = -\frac{N}{2H}\,\frac{\partial\Lambda_2}{\partial\beta_{kl}} \tag{3.239}$$

which corresponds to (3.222). We note that the transformation of the r.h.s. of (3.238) using the variable β'_{kl} instead of β_{kl} will not yield (3.235):

$$\begin{aligned}\frac{\partial\Lambda'_2}{\partial\beta'_{kl}} &= -\frac{H}{L(1-N)}\left(\beta'_{mn}S_{mn}+\frac{N}{2H}S_{mn}S_{mn}\right)\cdot\\ &\cdot\left(\frac{1}{2}\,\frac{S_{kl}-\frac{2H}{1-N}\beta'_{kl}}{\|S_{uv}-\frac{2H}{1-N}\beta'_{uv}\|}-\frac{H}{L}\left(\beta'_{kl}+\frac{N}{2H}S_{kl}\right)\right)^{-1}.\end{aligned} \tag{3.240}$$

This is so since the choice of the terms, which are required to chancel, is not unique. For example, if we add "zero" to the r.h.s. of (3.236) such that

$$\begin{aligned}\gamma = {} & \|\dot{\varepsilon}^{p}_{uv}\| \, \Big\| S_{uv} - \frac{2H}{1-2H\check{E}} \beta'_{uv} \Big\| + \\ & + \frac{2H}{2L} \frac{2H}{1-2H\check{E}} \beta'_{kl} \beta'_{kl} \, \|\dot{\varepsilon}^{p}_{uv}\| + \\ & + \frac{2H}{2L} \frac{2H}{1-2H\check{E}} \left(\beta_{kl}\beta_{kl} - \check{E}\beta_{kl} S_{kl} - \beta'_{kl}\beta'_{kl} \right) \|\dot{\varepsilon}^{p}_{uv}\| - \\ & - N \frac{\partial \Lambda'_2}{\partial \beta'_{kl}} \left(\dot{\varepsilon}^{p}_{kl} - \frac{2H}{2L} \beta_{kl} \, \|\dot{\varepsilon}^{p}_{uv}\| \right)\end{aligned} \qquad (3.241)$$

and if we require that the *last two rows chancel*, then we get, in terms of only the variables β'_{kl} and S_{kl}, exactly the condition (3.235). In any case the r.h.s. of (3.235) or (3.240) are not functions of only β'_{kl} which is required for thermodynamic consistency.

This example shows that the application of an "adjustment function, Λ" to obtain consistency with the dissipation inequality, is not only formal and also non-unique but may not yield a solution because of thermodynamic inconsistencies.

4. Thermodynamics on the Basis of an Extended Clausius-Duhem Entropy Inequality

4.1 Introduction

All thermodynamic restrictions derived so far are based on the evaluation of the classical Clausius-Duhem entropy inequality

$$\frac{d}{dt}\int_V \rho\eta \, dV - \int_V \rho s \, dV + \int_O \phi_k n_k \, dO \geq 0 \tag{4.1}$$

where ϕ_k is the entropy flux, a priori related to the heat flux q_k by

$$\phi_k = \frac{1}{T} q_k \, . \tag{4.2}$$

Note that

$\phi_k n_k < 0$: Flow of entropy *into* the volume V

(4.3)

$\phi_k n_k > 0$: Flow of entropy *out of* the volume V

where n_k is the external unit normal vector of the surface O. Further, s is the entropy supply due to heat sources given by

$$s = \frac{r}{T} \, . \tag{4.4}$$

The evaluation of the entropy inequality was based on the Coleman-Noll argument (section 2) which requires that body forces and heat supply are assigneable in any way such that the balance equations for linear momentum and energy are identically satisfied for arbitrary processes.

As described in section 1 this approach has been critizised for different reasons. Some of the arguments have been overcome by the new approach of Müller, e.g. [31], roughly scetched in section 1. Therefore, in the present context it appears to be worthwhile to study the thermodynamic restrictions of an elastic-viscoplastic material model, as described in section 2.3 by equ. (2.224) - (2.226), using the complete frame of Müller's approach; especially the entropy flux is put on a wider footing. The question is

(1) whether and under what conditions the same results as (2.232) and (2.233) are obtained and

(2) which preculiar problems arise in this development.

It is felt that agreement of the results of the two approaches would give further support to the earlier derivations in section 2.3.

4.2 Evaluation of the Extended Clausius-Duhem Entropy Inequality for Viscoplastic Material Models with Evolution Equations which Contain Rates of External Variables

The general characterization of the viscoplastic model is taken to be the same as in section (2.3), i.e., equ. (2.224) - (2.226). Thus,

$$\left.\begin{array}{lll} \text{thermoelastic strain} & \varepsilon^e_{kl} = \check{\varepsilon}^e_{kl}\,(\sigma_{mn}, T, \beta_\nu) = \check{\varepsilon}^e_{lk} \\ \text{internal energy} & \epsilon = \check{\epsilon}\,(\text{——}\;''\;\text{——}) \\ \text{entropy} & \eta = \check{\eta}\,(\text{——}\;''\;\text{——}) \\ \text{heat flux} & q_k = \check{q}_k\,(\text{——}\;''\;\text{——}, g_m) \end{array}\right\} \tag{4.5}$$

with evolution equations for the viscoplastic* strain ε^p_{kl}

$$\dot{\varepsilon}^p_{kl} = \check{h}_{kl}\,(\sigma_{mn}, T, \beta_\nu) = \check{h}_{lk} \tag{4.6}$$

and for the internal variables β_v

$$\begin{aligned} \dot{\beta}_\nu = {} & \check{f}_\nu\,(\sigma_{mn}, T, \beta_\nu) + \\ & + \check{E}_{\nu kl}\,(\text{——}\;''\;\text{——})\,\dot{\sigma}_{kl} + \\ & + \check{H}_\nu\,(\text{——}\;''\;\text{——})\,\dot{T}\,. \end{aligned} \tag{4.7}$$

The functions $\check{h}_{kl}$ and $\check{f}_v$ may change according to certain "switch conditions" which are controlled only by the state variables (σ_{mn}, T, β_v) but not by the rates of

---

* We recall here that the analysis is restricted to infinitesimal strains and rotations such that properly objective time derivatives are simply material derivatives.

the external variables. Note that only the heat flux depends on the temperature gradient

$$g_m = \frac{\partial T}{\partial x_m} \, . \tag{4.8}$$

The governing local balance equations are that of linear momentum and energy

$$\left.\begin{aligned} \rho\, \ddot{u}_k - \frac{\partial \sigma_{kl}}{\partial x_l} &= 0 \\ \rho\, \dot{\epsilon} - \sigma_{kl}\, \dot{\varepsilon}_{kl} + \frac{\partial q_k}{\partial x_k} &= 0 \end{aligned}\right\} \tag{4.9}$$

where the body force and heat supply are assumed to be absent.

We will now show that, with appropriate initial conditions, the constitutive equations $(4.5)_{1,2,4}$, (4.6) and (4.7) together with the balance equations define an initial value problem for the three displacements u_k, the temperature T, the viscoplastic strains and the internal variables.

We require that $(4.5)_1$ allows the inversion

$$\sigma_{kl} = \hat{\sigma}_{kl}(\varepsilon^e_{mn}, T, \beta_\nu) = \hat{\sigma}_{lk} \tag{4.10}$$

Then we have

$$\frac{\partial \hat{\sigma}_{kl}}{\partial \varepsilon^e_{mn}} \frac{\partial \check{\varepsilon}^e_{mn}}{\partial \sigma_{pq}} = \delta_{kp}\, \delta_{lq} \, . \tag{4.11}$$

If we understand that the derivatives, e.g., $\partial\hat{\sigma}_{kl}/\partial\varepsilon^e_{mn}$, are arranged in matrix form*, then the determinant of (4.11) is

$$\det\left(\frac{\partial \hat{\sigma}_{kl}}{\partial \varepsilon^e_{mn}}\right) \det\left(\frac{\partial \check{\varepsilon}^e_{mn}}{\partial \sigma_{pq}}\right) = 1$$

and thus

$$\det\left(\frac{\partial \hat{\sigma}_{kl}}{\partial \varepsilon^e_{mn}}\right) = \det\left(\frac{\partial \check{\varepsilon}^e_{mn}}{\partial \sigma_{pq}}\right)^{-1} \neq 0 \, . \tag{4.12}$$

---

* Consider the σ_{kl} and ε^e_{mn} arranged in column matrizes.

Here and in the following analysis we demand that the derivatives of (4.5) with respect to σ_{pq} are symmetric in the subscripts (pq), e.g.

$$\frac{\partial \check{\varepsilon}^e_{kl}}{\partial \sigma_{pq}} = \frac{\partial \check{\varepsilon}^e_{kl}}{\partial \sigma_{qp}} \quad , \quad \frac{\partial \hat{\sigma}_{kl}}{\partial \varepsilon^e_{pq}} = \frac{\partial \hat{\sigma}_{kl}}{\partial \varepsilon^e_{qp}} \; .$$

It is obvious that the variable "stress" can be eliminated from the constitutive functions in (4.5) - (4.7) such that, e.g., the internal energy is represented as a function of $\{\varepsilon^e_{mn}, T, \beta_v\}$

$$\epsilon = \hat{\epsilon}\left(\varepsilon^e_{mn}, T, \beta_v\right), \quad \text{etc.}$$

Further, the thermoelastic strain may be expressed in terms of total and viscoplastic strain

$$\left.\begin{aligned} \varepsilon^e_{mn} &= \varepsilon_{mn} - \varepsilon^p_{mn} \\ \text{where} \qquad & \\ \varepsilon_{mn} &= \frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}\right). \end{aligned}\right\} \qquad (4.13)$$

Thus, the problem can be formulated in terms of

the discplacements u_k \
the temperature T \
} their evolution governed by partial differential equations, i.e. the balance equations

the viscoplastic strains ε^p_{kl} \
the internal variables β_v . \
} the evolution governed by ordinary differential equation

We will not do this here but will keep to the original formulation.

The rate of internal energy is

$$\dot{\epsilon} = \frac{\partial \check{\epsilon}}{\partial \sigma_{mn}} \dot{\sigma}_{mn} + \frac{\partial \check{\epsilon}}{\partial T} \dot{T} + \frac{\partial \check{\epsilon}}{\partial \beta_v} \dot{\beta}_v \, .$$

With (4.7) we get

$$\dot{\epsilon} = \left(\frac{\partial\check{\epsilon}}{\partial\sigma_{mn}} + \frac{\partial\check{\epsilon}}{\partial\beta_\nu}\underset{\nu}{\check{E}}_{mn}\right)\dot{\sigma}_{mn} + \left(\frac{\partial\check{\epsilon}}{\partial T} + \frac{\partial\check{\epsilon}}{\partial\beta_\nu}\underset{\nu}{\check{H}}\right)\dot{T} + \frac{\partial\check{\epsilon}}{\partial\beta_\nu}\check{f}_\nu. \qquad (4.14)$$

The stress rate can be expressed in terms of more fundamental rate quantities just by time differentiation of (4.10)

$$\dot{\sigma}_{mn} = \frac{\partial\hat{\sigma}_{mn}}{\partial\varepsilon^e_{pq}}\dot{\varepsilon}^e_{pq} + \frac{\partial\hat{\sigma}_{mn}}{\partial T}\dot{T} + \frac{\partial\hat{\sigma}_{mn}}{\partial\beta_\nu}\dot{\beta}_\nu.$$

Observing (4.7) and (4.13)$_1$ we find

$$\dot{\sigma}_{mn} - \frac{\partial\hat{\sigma}_{mn}}{\partial\beta_\nu}\underset{\nu}{\check{E}}_{pq}\dot{\sigma}_{pq} = \left[\frac{1}{2}(\delta_{mp}\delta_{nq} + \delta_{np}\delta_{mq}) - \frac{\partial\hat{\sigma}_{mn}}{\partial\beta_\nu}\underset{\nu}{\check{E}}_{pq}\right]\dot{\sigma}_{pq} =$$

$$= \frac{\partial\hat{\sigma}_{mn}}{\partial\varepsilon^e_{pq}}\left(\dot{\varepsilon}_{pq} - \dot{\varepsilon}^p_{pq}\right) + \left(\frac{\partial\hat{\sigma}_{mn}}{\partial T} + \frac{\partial\hat{\sigma}_{mn}}{\partial\beta_\nu}\underset{\nu}{\check{H}}\right)\dot{T} + \frac{\partial\hat{\sigma}_{mn}}{\partial\beta_\nu}f_\nu. \qquad (4.15)$$

We put

$$B_{mnpq} := \frac{1}{2}(\delta_{mp}\delta_{nq} + \delta_{np}\delta_{mq}) - \frac{\partial\hat{\sigma}_{mn}}{\partial\beta_\nu}\underset{\nu}{\check{E}}_{pq} \qquad (4.16)$$

and require that the inverse $\overset{-1}{B}$ exists such that

$$\overset{-1}{B}_{klmn} B_{mnpq} = \frac{1}{2}(\delta_{kp}\delta_{lq} + \delta_{lp}\delta_{kq}) \qquad (4.17)$$

Therefore from (4.15) we derive

$$\dot{\sigma}_{kl} = \overset{-1}{B}_{klmn}\left\{\frac{\partial\hat{\sigma}_{mn}}{\partial\varepsilon^e_{pq}}\left(\dot{\varepsilon}_{pq} - \dot{\varepsilon}^p_{pq}\right) + \left(\frac{\partial\hat{\sigma}_{mn}}{\partial T} + \frac{\partial\hat{\sigma}_{mn}}{\partial\beta_\nu}\underset{\nu}{\check{H}}\right)\dot{T} + \frac{\partial\hat{\sigma}_{mn}}{\partial\beta_\nu}\check{f}_\nu\right\}. \qquad (4.18)$$

and the rate of internal energy (4.14) takes the following form

$$\dot{\epsilon} = \left(\frac{\partial \check{\epsilon}}{\partial \sigma_{kl}} + \frac{\partial \check{\epsilon}}{\partial \beta_\nu} \underset{v}{\check{E}}_{kl}\right) \overset{-1}{B}_{klmn} \frac{\partial \hat{\sigma}_{mn}}{\partial \varepsilon^{e}_{pq}} \left(\dot{\varepsilon}_{pq} - \dot{\varepsilon}^{p}_{pq}\right) +$$

$$+ \left\{\left(\frac{\partial \check{\epsilon}}{\partial T} + \frac{\partial \check{\epsilon}}{\partial \beta_\nu} \underset{v}{\check{H}}\right) + \left(\frac{\partial \check{\epsilon}}{\partial \sigma_{kl}} + \frac{\partial \check{\epsilon}}{\partial \beta_\nu} \underset{v}{\check{E}}_{kl}\right) \overset{-1}{B}_{klmn} \left(\frac{\partial \hat{\sigma}_{mn}}{\partial T} + \frac{\partial \hat{\sigma}_{mn}}{\partial \beta_\nu} \underset{v}{\check{H}}\right)\right\} \dot{T} +$$

$$+ \left\{\frac{\partial \check{\epsilon}}{\partial \beta_\nu} \check{j}_\nu + \left(\frac{\partial \check{\epsilon}}{\partial \sigma_{kl}} + \frac{\partial \check{\epsilon}}{\partial \beta_\nu} \underset{v}{\check{E}}_{kl}\right) \overset{-1}{B}_{klmn} \frac{\partial \hat{\sigma}_{mn}}{\partial \beta_\nu} \check{j}_\nu\right\}. \tag{4.19}$$

With this result all rate terms in (4.14) are expressed in terms of the rate of the external variables "total strain" and "temperature" and the viscoplastic strain rate which, however, is not an independent quantity since it is a function of the variables $\{\sigma_{mn}, T, \beta_\nu\}$.

Observing $(4.5)_4$, (4.10) and $(4.13)_1$ the divergence of the heat flux is

$$\frac{\partial q_k}{\partial X_k} = \frac{\partial \check{q}_k}{\partial \sigma_{mn}} \frac{\partial \hat{\sigma}_{mn}}{\partial X_k} + \frac{\partial \check{q}_k}{\partial T} g_k + \frac{\partial \check{q}_k}{\partial \beta_\nu} \frac{\partial \beta_\nu}{\partial X_k} =$$

$$= \frac{\partial \check{q}_k}{\partial \sigma_{mn}} \frac{\partial \hat{\sigma}_{mn}}{\partial \varepsilon^{e}_{pq}} \left(\frac{\partial \varepsilon_{pq}}{\partial X_k} - \frac{\partial \varepsilon^{p}_{pq}}{\partial X_k}\right) + \tag{4.20}$$

$$+ \left(\frac{\partial \check{q}_k}{\partial \sigma_{mn}} \frac{\partial \hat{\sigma}_{mn}}{\partial T} + \frac{\partial \check{q}_k}{\partial T}\right) g_k +$$

$$+ \left(\frac{\partial \check{q}_k}{\partial \sigma_{mn}} \frac{\partial \hat{\sigma}_{mn}}{\partial \beta_\nu} + \frac{\partial \check{q}_k}{\partial \beta_\nu}\right) \frac{\partial \beta_\nu}{\partial X_k} .$$

With (4.19) and (4.20) the energy balance equation $(4.9)_2$ reads

$$
\Big[\Big(\frac{\partial\check{\epsilon}}{\partial\sigma_{kl}}+\frac{\partial\check{\epsilon}}{\partial\beta_\nu}\underset{\nu}{\dot{E}}_{kl}\Big)B^{-1}_{klmn}\frac{\partial\hat{\sigma}_{mn}}{\partial\varepsilon^{e}_{pq}}-\hat{\sigma}_{pq}\Big]\big(\dot{\varepsilon}_{pq}-\dot{\varepsilon}^{p}_{pq}\big)\;-
$$

$$
-\Big[\Big(\frac{\partial\check{\epsilon}}{\partial T}+\frac{\partial\check{\epsilon}}{\partial\beta_\nu}\underset{\nu}{\check{H}}\Big)+\Big(\frac{\partial\check{\epsilon}}{\partial\sigma_{kl}}+\frac{\partial\check{\epsilon}}{\partial\beta_\nu}\underset{\nu}{\dot{E}}_{kl}\Big)B^{-1}_{klmn}\Big(\frac{\partial\hat{\sigma}_{mn}}{\partial T}+\frac{\partial\hat{\sigma}_{mn}}{\partial\beta_\nu}\underset{\nu}{\check{H}}\Big)\Big]\dot{T}\;+
$$

$$
+\Big\{\frac{\partial\check{\epsilon}}{\partial\beta_\nu}\check{j}_\nu+\Big(\frac{\partial\check{\epsilon}}{\partial\sigma_{kl}}+\frac{\partial\check{\epsilon}}{\partial\beta_\nu}\underset{\nu}{\dot{E}}_{kl}\Big)B^{-1}_{klmn}\frac{\partial\hat{\sigma}_{mn}}{\partial\beta_\nu}\check{j}_\nu\Big\}\;+
$$

$$
+\frac{\partial\check{q}_k}{\partial\sigma_{mn}}\frac{\partial\hat{\sigma}_{mn}}{\partial\varepsilon^{e}_{pq}}\Big(\frac{\partial\varepsilon_{pq}}{\partial x_k}-\frac{\partial\varepsilon^{p}_{pq}}{\partial x_k}\Big)\;+ \tag{4.21}
$$

$$
+\Big(\frac{\partial\check{q}_k}{\partial\sigma_{mn}}\frac{\partial\hat{\sigma}_{mn}}{\partial T}+\frac{\partial\check{q}_k}{\partial T}\Big)g_k\;+
$$

$$
+\Big(\frac{\partial\check{q}_k}{\partial\sigma_{mn}}\frac{\partial\hat{\sigma}_{mn}}{\partial\beta_\nu}+\frac{\partial\check{q}_k}{\partial\beta_\nu}\Big)\frac{\partial\beta_\nu}{\partial x_k}\;=\;0\,.
$$

Representing the divergence of the stress tensor explicitly, one gets for the momentum balance equation

$$
\rho\ddot{u}_k-\Big\{\frac{\partial\hat{\sigma}_{kl}}{\partial\varepsilon^{e}_{pq}}\Big(\frac{\partial\varepsilon_{pq}}{\partial x_l}-\frac{\partial\varepsilon^{p}_{pq}}{\partial x_l}\Big)+\frac{\partial\hat{\sigma}_{kl}}{\partial T}g_l+\frac{\partial\hat{\sigma}_{kl}}{\partial\beta_\nu}\frac{\partial\beta_\nu}{\partial x_l}\Big\}=0\,. \tag{4.22}
$$

The following initial conditions are prescribed in the body

initial displacements	$u_k(x_m,0)=\underset{0}{u}_k(x_m)$	
initial velocity	$\dot{u}_k(x_m,0)=\underset{0}{V}_k(x_m)$	
initial temperature	$T(x_m,0)=\underset{0}{T}(x_m)$	(4.23)
initial distribution of viscoplastic strain	$\varepsilon^{p}_{kl}(x_m,0)=\underset{0}{\varepsilon}^{p}_{kl}(x_m)$	
initial distribution of internal variables	$\beta_\nu(x_m,0)=\underset{0}{\beta}_\nu(x_m)$	

These data allow to calculate the initial strains, strain rates, strain and temperature gradients, and gradients of internal variables, i.e.,

$$\left.\begin{array}{ll}
\text{initial total strains} & \underset{0}{\varepsilon}_{kl} = \frac{1}{2}(\partial \underset{0}{u}_k/\partial x_l + \partial \underset{0}{u}_l/\partial x_k) \\
\text{initial strain rates} & \dot{\underset{0}{\varepsilon}}_{kl} = \frac{1}{2}(\partial \underset{0}{v}_k/\partial x_l + \partial \underset{0}{v}_l/\partial x_k) \\
\text{initial elastic strains} & \underset{0}{\varepsilon}^{e}_{kl} = \underset{0}{\varepsilon}_{kl} - \underset{0}{\varepsilon}^{p}_{kl} \\
\text{initial gradients of total strain} & \underset{0}{\varepsilon}_{kl,m} = \partial \underset{0}{\varepsilon}_{kl}/\partial x_l \\
\text{initial temperature gradients} & \underset{0}{g}_m = \partial \underset{0}{T}/\partial x_m \\
\text{initial gradients of viscoplastic strain} & \underset{0}{\varepsilon}^{p}_{kl,m} = \partial \underset{0}{\varepsilon}^{p}_{kl}/\partial x_m \\
\text{and internal variables} & \underset{0}{\beta}_{\nu,m} = \partial \underset{0}{\beta}_{\nu}/\partial x_m
\end{array}\right\} \quad (4.24)$$

and higher order spatial derivatives if necessary. With the constitutive equations and the stress-elastic strain relation (4.10) (inverse of (4.5)$_1$) the balance of energy equation determines the initial temperature rate $\dot{T}$ and the balance of linear momentum yields the initial acceleration $\ddot{u}_k$ at every point in the body.

It is clear that higher order time derivatives of the temperature and displacement at the initial instant can be obtained by time differentiation of the balance equations. For example, if the energy balance (4.21) is written as

$$A\dot{T} + B_{pq}\dot{\varepsilon}_{pq} + C = 0,$$

then time differentiation yields

$$A\ddot{T} + \dot{A}\dot{T} + B_{pq}\ddot{\varepsilon}_{pq} + \dot{B}_{pq}\dot{\varepsilon}_{pq} + \dot{C} = 0$$

and this determines $\ddot{T}$ at t = 0. This process of successive determination of the initial derivatives can be continued without limitations if the differential equations and initial conditions are analytic [70]. Then, the solution $u_k(x_m, t)$ and $T(x_m, t)$ can be expanded in a convergent power series with respect to time at t = 0 and they are unique [70]. Thus,

$$u_k(x_m,t) = \underset{0}{u}_k(x_m) + \underset{0}{v}(x_m)(t-t_0) + \frac{1}{2}\ddot{\underset{0}{u}}_k(x_m)(t-t_0)^2 + \dots$$

$$T(x_m,t) = T_0(x_m) + \dot{\underset{0}{T}}(x_m)(t-t_0) + \frac{1}{2}\ddot{\underset{0}{T}}(x_m)(t-t_0)^2 + \dots$$

Thus, this solution, representing a thermodynamic process, is uniquely determined by the initial conditions (4.23).

We now turn to the evaluation of the extended Clausius-Duhem entropy inequality. Here it is necessary to make an assumption about the entropy flux ϕ_k and the entropy source which are now constitutive assumptions in their own right and which go beyond the classical assumptions of the Coleman-Noll approach, i.e.,

$$\phi_k = \frac{1}{T} q_k \,, \qquad s = \frac{1}{T} r .$$

Above we assumed that the balance equations are source free. Therefore, we require that also

$$s \equiv 0.$$

For the entropy flux the following constitutive assumption is made

$$\phi_k = \frac{1}{T} q_k + \tilde{\varphi}\, g_k + k_k \tag{4.25}$$

where

$$\left.\begin{aligned} \tilde{\varphi} &= \check{\varphi}(\sigma_{mn}, T, \beta_\nu, g_m) \\ k_k &= \check{k}_k(\sigma_{mn}, T, \beta_\nu) \end{aligned}\right\} \tag{4.26}$$

The first term corresponds to classical expression (4.2) which is collinear with the heat flux. The second term is collinear with the temperature gradient. If the heat flux $(4.5)_4$ is also collinear with the temperature gradient, i.e.,

$$q_k = -\varkappa(\sigma_{mn}, T, \beta_\nu, g_m)\, g_k \,, \tag{4.27}$$

which will be assumed in the following, then the first two terms can be combined such that

$$\phi_k = \varphi\, q_k + k_k \tag{4.28}$$

where

$$\varphi = \check{\varphi}(\sigma_{mn}, T, \beta_\nu, g_m) = \frac{1}{T} - \frac{\tilde{\varphi}}{\varkappa} . \tag{4.29}$$

Then the entropy flux consists of a term collinear with the heat flux q_k and a term independent of the heat flux and the temperature gradient.

The first term is collinear with the heat flux q_k with a factor of proportionality which is not the inverse of the temperature. The second term in (4.28) is independent of any gradient since we do not consider the internal variables β_v to be gradient type variables. Note that the β_v's are scalar or first or second order tensors which need not to be specified at this level. The motivation for the choice of $(4.26)_2$ is not physical but mathematical: It is not related to any energy flux quantity appearing in the balance equations; the question arises whether the requirement of an identical satisfaction of the extended entropy inequality permits such an extra entropy flux.

It appears that an extra entropy flux would have physical support if the constitutive equations were assumed to depend on higher order spatial or temporal derivatives. For example, if the evolution equations (4.7) for the internal variables β_v involve a divergence term due to some flux quantity, then $\check{k}_l$ should depend on this flux quantity and thus also on derivatives of state variables, e.g. $\partial\beta_v/\partial x_k$. Then, of course, the evolution equations become partial differential equations; thus boundary conditions are required for the β_v's too which implies that the *β_v's become controllable quantities*: Thus they loose their character as internal variables. Such proposals have been made by, e.g., Aifantis [74], Bampi et al.[75], Bampi and Morro [76], Lehmann [64], Müller [31], Parry [78].

Nevertheless, we will not assume that the extra entropy flux k_l depends on the gradients $\partial\beta_v/\partial x_k$ since they do not appear in the assumed constitutive relation (4.5) - (4.7).

The evaluation of the entropy inequality (4.1) is based on the requirement that it is identically satisfied not for arbitrary fields of motion, temperature etc. but only for those which are solutions of the balance equations together with the constitutive relations. Since the body force and heat supply are prescribed quantities (they are supposed to vanish), the balance equations are not identically satisfied for an arbitrary choice of the displacement and temperature history $\{u_k (x_m, t), T(x_m, t)\}$ with initial conditions for the viscoplastic strain and internal variables. Therefore, the balance equations as well as the constitutive relations represent constraints on the fields which satisfy the entropy inequality. It is possible to account directly for these constraints by elimination of some of the quantities, e.g., expressing the temperature rate in terms of other quantities using the constraining energy balance equation. This may be a cumbersome procedure. Liu [33, 31] has shown that these constraints can be accounted for by the use of Lagrange multipliers. Then the local inequality takes the following form:

$$\varrho\dot{\eta} + \frac{\partial \phi_k}{\partial x_k} + \Lambda_{v_k}\left[\varrho \ddot{u}_k - \frac{\partial \sigma_{\ell k}}{\partial x_\ell}\right] + \Lambda_\epsilon\left[\varrho\dot{\epsilon} - \sigma_{k\ell}\underbrace{\left(\dot{\varepsilon}^e_{k\ell} + \dot{\varepsilon}^p_{k\ell}\right)}_{\dot{\varepsilon}_{k\ell}} + \frac{\partial q_k}{\partial x_k}\right] + \Lambda_{\beta_v}\left[\dot{\beta}_v - f_v - \underset{\vee}{\check{E}}_{mn}\dot{\sigma}_{mn} - \underset{\vee}{\check{H}}\dot{T}\right] \geq 0 \tag{4.30}$$

where the total strain rate $\dot{\varepsilon}_{kl}$ has been expressed in terms of the elastic and viscoplastic strain rate. Λ_{v_k}, Λ_ε and Λ_{β_v} are the Lagrange multipliers which may be functions of $\{\sigma_{mn}, T, \beta_v, g_m\}$. In (4.30) it is understood that the viscoplastic strain rate is expressed in terms of (4.6).

Time differentiation of $(4.5)_{1\text{-}3}$ yields*

$$\left.\begin{aligned}
\dot{\eta} &= \frac{\partial \check{\eta}}{\partial \sigma_{mn}}\dot{\sigma}_{mn} + \frac{\partial \check{\eta}}{\partial T}\dot{T} + \frac{\partial \check{\eta}}{\partial \beta_v}\dot{\beta}_v \\
\dot{\epsilon} &= \frac{\partial \check{\epsilon}}{\partial \sigma_{mn}}\dot{\sigma}_{mn} + \frac{\partial \check{\epsilon}}{\partial T}\dot{T} + \frac{\partial \check{\epsilon}}{\partial \beta_v}\dot{\beta}_v \\
\dot{\varepsilon}^e_{k\ell} &= \frac{\partial \check{\varepsilon}^e_{k\ell}}{\partial \sigma_{mn}}\dot{\sigma}_{mn} + \frac{\partial \check{\varepsilon}^e_{k\ell}}{\partial T}\dot{T} + \frac{\partial \check{\varepsilon}^e_{k\ell}}{\partial \beta_v}\dot{\beta}_v \,.
\end{aligned}\right\} \tag{4.31}$$

Collecting all terms involving the rates of the state variables σ_{mn}, T, β_v gives

$$\begin{aligned}
&\left[\varrho\frac{\partial \check{\eta}}{\partial \sigma_{mn}} + \Lambda_\epsilon\left(\varrho\frac{\partial \check{\epsilon}}{\partial \sigma_{mn}} - \sigma_{k\ell}\frac{\partial \check{\varepsilon}^e_{k\ell}}{\partial \sigma_{mn}}\right) - \Lambda_{\beta_v}\underset{\vee}{\check{E}}_{mn}\right]\dot{\sigma}_{mn} + \\
&+\left[\varrho\frac{\partial \check{\eta}}{\partial T} + \Lambda_\epsilon\left(\varrho\frac{\partial \check{\epsilon}}{\partial T} - \sigma_{k\ell}\frac{\partial \check{\varepsilon}^e_{k\ell}}{\partial T}\right) - \Lambda_{\beta_v}\underset{\vee}{\check{H}}\right]\dot{T} + \\
&+\left[\varrho\frac{\partial \check{\eta}}{\partial \beta_v} + \Lambda_\epsilon\left(\varrho\frac{\partial \check{\epsilon}}{\partial \beta_v} - \sigma_{k\ell}\frac{\partial \check{\varepsilon}^e_{k\ell}}{\partial \beta_v}\right) + \Lambda_{\beta_v}\right]\dot{\beta}_v + \ldots
\end{aligned} \tag{4.32}$$

---

* Note that all derivatives with respect to σ_{mn} are symmetric in (m, n).

$$\ldots + \Lambda_{v_k} \ddot{u}_k -$$

$$- \Lambda_{v_k} \frac{\partial \sigma_{kl}}{\partial x_l} + \frac{\partial \Phi_k}{\partial x_k} + \Lambda_\epsilon \frac{\partial q_k}{\partial x_k} -$$

$$- \Lambda_\epsilon \sigma_{kl} \dot{\varepsilon}^p_{kl} - \Lambda_{\beta_\nu} \check{f}_\nu \geq 0. \tag{4.32}$$

The stress rate can be represented in terms of the total strain rate and the temperature rate, equ. (4.18). Thus (4.32) takes the following form

$$\Big[\varrho \frac{\partial \check{\eta}}{\partial \sigma_{mn}} + \Lambda_\epsilon \Big(\varrho \frac{\partial \check{\epsilon}}{\partial \sigma_{mn}} - \sigma_{kl} \frac{\partial \check{\varepsilon}^e_{kl}}{\partial \sigma_{mn}} \Big) - \Lambda_{\beta_\nu} \check{E}_{\nu mn} \Big] \bar{B}^{-1}_{mnkl} \frac{\partial \hat{\sigma}_{kl}}{\partial \varepsilon^e_{pq}} \dot{\varepsilon}_{pq}$$

$$+ \Big\{ \Big[\varrho \frac{\partial \check{\eta}}{\partial T} + \Lambda_\epsilon \Big(\varrho \frac{\partial \check{\epsilon}}{\partial T} - \sigma_{kl} \frac{\partial \check{\varepsilon}^e_{kl}}{\partial T} \Big) - \Lambda_{\beta_\nu} \check{H}_\nu \Big]$$

$$+ \Big[\varrho \frac{\partial \check{\eta}}{\partial \sigma_{mn}} + \Lambda_\epsilon \Big(\varrho \frac{\partial \check{\epsilon}}{\partial \sigma_{mn}} - \sigma_{kl} \frac{\partial \check{\varepsilon}^e_{kl}}{\partial \sigma_{mn}} \Big) - \Lambda_{\beta_\nu} \check{E}_{\nu mn} \Big] \cdot$$

$$\cdot \bar{B}^{-1}_{mnkl} \Big(\frac{\partial \hat{\sigma}_{kl}}{\partial T} + \frac{\partial \hat{\sigma}_{kl}}{\partial \beta_\nu} \check{H}_\nu \Big) \Big\} \dot{T} +$$

$$+ \Big[\varrho \frac{\partial \check{\eta}}{\partial \beta_\nu} + \Lambda_\epsilon \Big(\varrho \frac{\partial \check{\epsilon}}{\partial \beta_\nu} - \sigma_{kl} \frac{\partial \check{\varepsilon}^e_{kl}}{\partial \beta_\nu} \Big) + \Lambda_{\beta_\nu} \Big] \dot{\beta}_\nu +$$

$$+ \Big[\varrho \frac{\partial \check{\eta}}{\partial \sigma_{mn}} + \Lambda_\epsilon \Big(\varrho \frac{\partial \check{\epsilon}}{\partial \sigma_{mn}} - \sigma_{kl} \frac{\partial \check{\varepsilon}^e_{kl}}{\partial \sigma_{mn}} \Big) - \Lambda_{\beta_\nu} \check{E}_{\nu mn} \Big] \cdot \tag{4.33}$$

$$\cdot \bar{B}^{-1}_{mnkl} \Big[\frac{\partial \hat{\sigma}_{kl}}{\partial \beta_\nu} \check{f}_\nu - \frac{\partial \hat{\sigma}_{kl}}{\partial \varepsilon^e_{pq}} \dot{\varepsilon}^p_{pq} \Big] +$$

$$+ \Lambda_{v_k} \ddot{u}_k$$

$$- \Lambda_{v_k} \frac{\partial \sigma_{kl}}{\partial x_l} + \frac{\partial \Phi_l}{\partial x_k} + \Lambda_\epsilon \frac{\partial q_k}{\partial x_k} + \ldots$$

$$\ldots - \Lambda_\epsilon \sigma_{kl} \dot{\varepsilon}^p_{kl} - \Lambda_{\beta_\nu} f_\nu \geq 0 . \tag{4.33}$$

In the next to the last row all terms involving spatial derivatives are collected. With (4.28) they are in explicit terms:

$$\begin{aligned}
&\frac{\partial \phi_k}{\partial x_k} + \Lambda_\epsilon \frac{\partial q_k}{\partial x_k} - \Lambda_{v_k} \frac{\partial \sigma_{kl}}{\partial x_l} = \\
&= \frac{\partial \varphi}{\partial x_k} \check{q}_k + (\check{\varphi} + \Lambda_\epsilon) \frac{\partial q_k}{\partial x_k} + \frac{\partial k_k}{\partial x_k} - \Lambda_{v_m} \frac{\partial \sigma_{mn}}{\partial x_n} = \\
&= \left[\frac{\partial \check{\varphi}}{\partial \sigma_{mn}} \check{q}_k + (\check{\varphi} + \Lambda_\epsilon) \frac{\partial \check{q}_k}{\partial \sigma_{mn}} + \frac{\partial \check{k}_k}{\partial \sigma_{mn}} - \Lambda_{v_m} \delta_{kn} \right] \frac{\partial \sigma_{mn}}{\partial x_k} + \\
&+ \left[\frac{\partial \check{\varphi}}{\partial T} \check{q}_k + (\check{\varphi} + \Lambda_\epsilon) \frac{\partial \check{q}_k}{\partial T} + \frac{\partial \check{k}_k}{\partial T} \right] \frac{\partial T}{\partial x_k} + \\
&+ \left[\frac{\partial \check{\varphi}}{\partial \beta_\nu} \check{q}_k + (\check{\varphi} + \Lambda_\epsilon) \frac{\partial \check{q}_k}{\partial \beta_\nu} + \frac{\partial \check{k}_k}{\partial \beta_\nu} \right] \frac{\partial \beta_\nu}{\partial x_k} + \\
&+ \left[\frac{\partial \check{\varphi}}{\partial g_m} \check{q}_k + (\check{\varphi} + \Lambda_\epsilon) \frac{\partial \check{q}_k}{\partial g_m} \right] \frac{\partial g_m}{\partial x_k} .
\end{aligned} \tag{4.34}$$

From (4.10) one derives

$$\begin{aligned}
\frac{\partial \sigma_{mn}}{\partial x_k} &= \frac{\partial \hat{\sigma}_{mn}}{\partial \varepsilon^e_{pq}} \frac{\partial \varepsilon^e_{pq}}{\partial x_k} + \frac{\partial \hat{\sigma}_{mn}}{\partial T} g_k + \frac{\partial \hat{\sigma}_{mn}}{\partial \beta_\nu} \frac{\partial \beta_\nu}{\partial x_k} \\
&= \frac{\partial \hat{\sigma}_{mn}}{\partial \varepsilon^e_{pq}} \left(\frac{\partial \varepsilon_{pq}}{\partial x_k} - \frac{\partial \varepsilon^p_{pq}}{\partial x_k} \right) + \frac{\partial \hat{\sigma}_{mn}}{\partial T} g_k + \frac{\partial \hat{\sigma}_{mn}}{\partial \beta_\nu} \frac{\partial \beta_\nu}{\partial x_k} .
\end{aligned} \tag{4.35}$$

Thus

$$\begin{aligned}
&\frac{\partial \Phi_k}{\partial x_k} + \Lambda_\epsilon \frac{\partial q_k}{\partial x_k} - \Lambda_{v_k} \frac{\partial \sigma_{kl}}{\partial x_l} = \\
&= \left[\frac{\partial \check{\varphi}}{\partial \sigma_{mn}} \check{q}_k + (\check{\varphi} + \Lambda_\epsilon) \frac{\partial \check{q}_k}{\partial \sigma_{mn}} + \frac{\partial \check{k}_k}{\partial \sigma_{mn}} - \Lambda_{v_m} \delta_{kn} \right] \cdot \\
&\qquad\qquad \cdot \frac{\partial \hat{\sigma}_{mn}}{\partial \dot{\varepsilon}_{pq}} \left(\frac{\partial \varepsilon_{pq}}{\partial x_k} - \frac{\partial \varepsilon^p_{pq}}{\partial x_k} \right) + \\
&+ \left\{ \left[\frac{\partial \check{\varphi}}{\partial T} \check{q}_k + (\check{\varphi} + \Lambda_\epsilon) \frac{\partial \check{q}_k}{\partial T} + \frac{\partial \check{k}_k}{\partial T} \right] + \right. \\
&\quad \left. + \left[\frac{\partial \check{\varphi}}{\partial \sigma_{mn}} \check{q}_k + (\check{\varphi} + \Lambda_\epsilon) \frac{\partial \check{q}_k}{\partial \sigma_{mn}} + \frac{\partial \check{k}_k}{\partial \sigma_{mn}} - \Lambda_{v_m} \delta_{kn} \right] \frac{\partial \hat{\sigma}_{mn}}{\partial T} \right\} g_k + \\
&+ \left\{ \left[\frac{\partial \check{\varphi}}{\partial \beta_v} \cdot \check{q}_k + (\check{\varphi} + \Lambda_\epsilon) \frac{\partial \check{q}_k}{\partial \beta_v} + \frac{\partial \check{k}_k}{\partial \beta_v} \right] + \right. \\
&\quad \left. + \left[\frac{\partial \check{\varphi}}{\partial \sigma_{mn}} \check{q}_k + (\check{\varphi} + \Lambda_\epsilon) \frac{\partial \check{q}_k}{\partial \sigma_{mn}} + \frac{\partial \check{k}_k}{\partial \sigma_{mn}} - \Lambda_{v_m} \delta_{kn} \right] \frac{\partial \hat{\sigma}_{mn}}{\partial \beta_v} \right\} \frac{\partial \beta_v}{\partial x_k} \\
&+ \left[\frac{\partial \check{\varphi}}{\partial g_m} \check{q}_k + (\check{\varphi} + \Lambda_\epsilon) \frac{\partial \check{q}_k}{\partial g_m} \right] \frac{\partial T}{\partial x_m \partial x_k} .
\end{aligned} \tag{4.36}$$

For given constitutive functions the entropy production (4.33) is a function of the state variables σ_{mn}, T, β_v and also ε^e_{kl} due to the presence of the inverse function $\hat{\sigma}_{mn}$ as well as the temporal and spatial derivatives

$$\dot{\varepsilon}_{pq}, \ \frac{\partial \varepsilon_{pq}}{\partial x_k}, \ \frac{\partial \varepsilon^p_{pq}}{\partial x_k}, \ \frac{\partial \beta_v}{\partial x_k}, \ \frac{\partial g_m}{\partial x_k} = \frac{\partial^2 T}{\partial x_m \partial x_k} . \tag{4.37}$$

The quantities $\ddot{u}_k$, $\dot{T}$, $\dot{\beta}_v$ appear only *linearly* in (4.33). Since the initial conditions (4.23) are arbitrary (smooth) functions of space, the first set may take locally (i.e. at a material point) arbitrary values independent of each other. The same is true for $\ddot{u}_k$, $\dot{T}$, $\dot{\beta}_v$, since the constraints on them were relaxed due to the Lagrange multipliers.

The arbitrary modulation of $\ddot{u}_k$, $\dot{T}$, $\dot{\beta}_v$ yields:

$$\left.\begin{aligned}
&\Big[\varrho \frac{\partial \check{\eta}}{\partial \sigma_{mn}} + \Lambda_\epsilon\Big(\varrho \frac{\partial \check{\epsilon}}{\partial \sigma_{mn}} - \sigma_{kl}\frac{\partial \check{\varepsilon}^e_{kl}}{\partial \sigma_{mn}}\Big) - \Lambda_{\beta_v} \underset{v}{\check{E}}_{mn}\Big] \cdot \\
&\qquad\qquad \cdot B^{-1}_{mnkl} \frac{\partial \hat{\sigma}_{kl}}{\partial \varepsilon^e_{pq}} \Big\}_{sym} = 0 \\[2ex]
&\Big\{\Big[\varrho \frac{\partial \check{\eta}}{\partial T} + \Lambda_\epsilon\Big(\varrho \frac{\partial \check{\epsilon}}{\partial T} - \sigma_{kl}\frac{\partial \check{\varepsilon}^e_{kl}}{\partial T}\Big) - \Lambda_{\beta_v} \underset{v}{\check{H}}\Big] + \\
&\quad + \Big[\varrho \frac{\partial \check{\eta}}{\partial \sigma_{mn}} + \Lambda_\epsilon\Big(\varrho \frac{\partial \check{\epsilon}}{\partial \sigma_{mn}} - \sigma_{kl}\frac{\partial \check{\varepsilon}^e}{\partial \sigma_{mn}}\Big) - \Lambda_{\beta_v} \underset{v}{\check{E}}_{mn}\Big] \cdot \\
&\qquad\qquad \cdot B^{-1}_{mnkl}\Big(\frac{\partial \hat{\sigma}_{kl}}{\partial T} + \frac{\partial \hat{\sigma}_{kl}}{\partial \beta_v} \underset{v}{\check{H}}\Big)\Big\} = 0 \\[2ex]
&\Big[\varrho \frac{\partial \check{\eta}}{\partial \beta_v} + \Lambda_\epsilon\Big(\varrho \frac{\partial \check{\epsilon}}{\partial \beta_v} - \sigma_{kl}\frac{\partial \check{\varepsilon}^e_{kl}}{\partial \beta_v}\Big) - \Lambda_{\beta_v}\Big] = 0
\end{aligned}\right\} \quad (4.38)$$

Observing (4.11), i.e.,

$$\frac{\partial \hat{\sigma}_{kl}}{\partial \varepsilon^e_{pq}} \frac{\partial \check{\varepsilon}^e_{pq}}{\partial \sigma_{rs}} = \delta_{kr}\,\delta_{ls} \qquad (4.39)$$

and (4.17), equ. (4.38)$_1$ simplifies to

$$\Big[\varrho \frac{\partial \check{\eta}}{\partial \sigma_{mn}} + \Lambda_\epsilon\Big(\varrho \frac{\partial \check{\epsilon}}{\partial \sigma_{mn}} - \sigma_{kl}\frac{\partial \check{\varepsilon}^e_{kl}}{\partial \sigma_{mn}}\Big) - \Lambda_{\beta_v} \underset{v}{\check{E}}_{mn}\Big] = 0. \qquad (4.40)$$

and therefore (4.38)$_2$ reduces to

$$\Big[\varrho \frac{\partial \check{\eta}}{\partial T} + \Lambda_\epsilon\Big(\varrho \frac{\partial \check{\epsilon}}{\partial T} - \sigma_{kl}\frac{\partial \check{\varepsilon}^e_{kl}}{\partial T}\Big) - \Lambda_{\beta_v} \underset{v}{\check{H}}\Big] = 0 \qquad (4.41)$$

Equ. (4.38)$_3$ yields an expression for the Lagrange multipliers Λ_{β_v}

$$\Lambda_{\beta_\nu} = -\varrho\frac{\partial\check{\eta}}{\partial\beta_\nu} - \Lambda_\epsilon\left(\varrho\frac{\partial\check{\epsilon}}{\partial\beta_\nu} - \sigma_{kl}\frac{\partial\check{\varepsilon}^e_{kl}}{\partial\beta_\nu}\right). \tag{4.42}$$

Combining (4.40), (4.41) and (4.42) gives

$$\left.\begin{aligned}
&\varrho\frac{\partial\check{\eta}}{\partial\sigma_{mn}} + \varrho\frac{\partial\check{\eta}}{\partial\beta_\nu}\underset{\nu}{\check{E}}_{mn} + \Lambda_\epsilon\left(\varrho\frac{\partial\check{\epsilon}}{\partial\sigma_{mn}} + \varrho\frac{\partial\check{\epsilon}}{\partial\beta_\nu}\underset{\nu}{\check{E}}_{mn}\right) - \\
&\qquad -\Lambda_\epsilon\left(\sigma_{kl}\frac{\partial\check{\varepsilon}^e_{kl}}{\partial\sigma_{mn}} + \sigma_{kl}\frac{\partial\check{\varepsilon}^e_{kl}}{\partial\beta_\nu}\underset{\nu}{\check{E}}_{mn}\right) = 0 \\
&-\varrho\frac{\partial\check{\eta}}{\partial T} + \varrho\frac{\partial\check{\eta}}{\partial\beta_\nu}\underset{\nu}{\check{H}} + \Lambda_\epsilon\left(\varrho\frac{\partial\check{\epsilon}}{\partial T} + \varrho\frac{\partial\check{\epsilon}}{\partial\beta_\nu}\underset{\nu}{\check{H}}\right) - \\
&\qquad -\Lambda_\epsilon\left(\sigma_{kl}\frac{\partial\check{\varepsilon}^e_{kl}}{\partial T} + \sigma_{kl}\frac{\partial\check{\varepsilon}^e_{kl}}{\partial\beta_\nu}\underset{\nu}{\check{H}}\right) = 0
\end{aligned}\right\} \tag{4.43}$$

which involve only a single yet undetermined multiplier. The arbitrary modulation of $\ddot{u}_k$, $\partial\varepsilon_{pq}/\partial x_k$, $\partial\varepsilon^P_{pq}/\partial x_k$, $\partial\beta_\nu/\partial x_k$ and $\partial g_m/\partial x_k = \partial T/\partial x_m\,\partial x_k$ in the inequality (4.33), observing (4.36), gives

$$\left.\begin{aligned}
&\Lambda_{V_k} = 0 \\
&\left[\frac{\partial\check{\varphi}}{\partial\sigma_{mn}}\check{q}_k + (\check{\varphi} + \Lambda_\epsilon)\frac{\partial\check{q}_k}{\partial\sigma_{mn}} + \frac{\partial\check{\dot{k}}_k}{\partial\sigma_{mn}} - \Lambda_{V_m}\delta_{kn}\right]\frac{\partial\hat{\sigma}_{mn}}{\partial\varepsilon^e_{pq}} = 0 \\
&\left\{\left[\frac{\partial\check{\varphi}}{\partial\beta_\nu}\dot{q}_k + (\check{\varphi} + \Lambda_\epsilon)\frac{\partial\check{q}_k}{\partial\beta_\nu} + \frac{\partial\check{\dot{k}}_k}{\partial\beta_\nu}\right] + \right. \\
&\left. + \left[\frac{\partial\check{\varphi}}{\partial\sigma_{mn}}\check{q}_k + (\check{\varphi} + \Lambda_\epsilon)\frac{\partial\check{q}_k}{\partial\sigma_{mn}} + \frac{\partial\check{\dot{k}}_k}{\partial\beta_\nu}\right]\frac{\partial\hat{\sigma}_{mn}}{\partial\beta_\nu}\right\} = 0 \\
&\left[\frac{\partial\check{\varphi}}{\partial g_m}\check{q}_k + (\check{\varphi} + \Lambda_\epsilon)\frac{\partial\check{q}_k}{\partial g_m}\right]_{Sym} = 0\,.
\end{aligned}\right\} \tag{4.44}$$

With (4.44)$_1$ and (4.11) we derive from (4.44)

$$\left.\begin{aligned} &\left[\frac{\partial\check{\varphi}}{\partial\sigma_{mn}}\, q_k + (\check{\varphi}+\Lambda_\varepsilon)\frac{\partial\check{q}_k}{\partial\sigma_{mn}} + \frac{\partial\check{h}_k}{\partial\sigma_{mn}}\right] &= 0\\ &\left[\frac{\partial\check{\varphi}}{\partial\beta_v}\, q_k + (\check{\varphi}+\Lambda_\varepsilon)\frac{\partial\check{q}_k}{\partial\beta_v} + \frac{\partial\check{h}_k}{\partial\beta_v}\right] &= 0\\ &\left[\frac{\partial\check{\varphi}}{\partial g_m}\, q_k + (\check{\varphi}+\Lambda_\varepsilon)\frac{\partial\check{q}_k}{\partial g_m}\right]_{sym} &= 0 \end{aligned}\right\} \quad (4.45)$$

Observing these results, the entropy inequality simplifies to

$$\left.\begin{aligned} &\left[\frac{\partial\check{\varphi}}{\partial T}\, q_k + (\check{\varphi}+\Lambda_\varepsilon)\frac{\partial\check{q}_k}{\partial T} + \frac{\partial\check{h}_k}{\partial T}\right] g_k -\\ &-\Lambda_\varepsilon\,\sigma_{kl}\,\dot{\varepsilon}^{p}_{kl} - \Lambda_{\beta_v}\check{f}_v \geq 0 \end{aligned}\right\} \quad (4.46)$$

where Λ_{β_v} is given by (4.42). With (4.27) condition (4.45)$_3$ reads

$$\begin{aligned} &\Big[\Big(\frac{\partial\check{\varphi}}{\partial g_m}\,\varkappa + (\check{\varphi}+\Lambda_\varepsilon)\frac{\partial\check{\varkappa}}{\partial g_m}\Big) g_k + \Big(\frac{\partial\check{\varphi}}{\partial g_k}\,\varkappa + (\check{\varphi}+\Lambda_\varepsilon)\frac{\partial\check{\varkappa}}{\partial g_k}\Big) g_m +\\ &+ (\check{\varphi}+\Lambda_\varepsilon)\,\check{\varkappa}\,\delta_{km} = 0 \end{aligned} \quad (4.47)$$

which must hold for all g$_k$ and also all σ_{mn}, β_v, T. For $g_k = 0$, $k = 1, 2, 3$, equ. (4.47) yields

$$\left[(\check{\varphi}+\Lambda_\varepsilon)\check{\varkappa}\right]_{g_k=0,\ k=1,2,3} = 0$$

and, provided

$$\check{\varkappa} \neq 0 \qquad \text{if } g_k = 0 \quad \forall k\,,$$

then

$$\check{\varphi}+\Lambda_\varepsilon = 0 \qquad \text{if } g_k = 0 \quad \forall k \qquad (4.48)$$

Further, if $k \neq m$

$$\left(\frac{\partial\check{\varphi}}{\partial g_m}\check{\varkappa} + (\check{\varphi}+\Lambda_\varepsilon)\frac{\partial\check{\varkappa}}{\partial g_m}\right)g_k + \left(\frac{\partial\check{\varphi}}{\partial g_k}\check{\varkappa} + (\check{\varphi}+\Lambda_\varepsilon)\frac{\partial\check{\varkappa}}{\partial g_k}\right)g_m = 0.$$

If $g_m = 0$, then

$$\left(\frac{\partial\check{\varphi}}{\partial g_m}\check{\varkappa} + (\check{\varphi}+\Lambda_\varepsilon)\frac{\partial\check{\varkappa}}{\partial g_m}\right)g_k = 0, \quad \forall g_k \neq 0.$$

Thus

$$\left[\frac{\partial\check{\varphi}}{\partial g_m}\check{\varkappa} + (\check{\varphi}+\Lambda_\varepsilon)\frac{\partial\check{\varkappa}}{\partial g_m}\right]_{g_m=0} = 0. \tag{4.49}$$

The results (4.48) and (4.49) are of a rather limited value since the functions $\check{\varphi}$ and $\check{\varkappa}$ are still of a general nature. If $\check{\varphi}$ and $\check{\varkappa}$ are *assumed* to be independent of the temperature gradient, then obviously

$$\varphi + \Lambda_\varepsilon = 0$$

and Λ_ε is independent of the temperature gradient. Such an assumption appears reasonable for $\check{\varkappa}$, since classical models have this property. But possibly it is too restrictive for $\check{\varphi}$.

We now apply the concept of material frame indifference. Under the change of the observer system (we consider only *proper* Euclidian transformations, i.e. translations and rotations) the temperature gradient, heat flux and entropy flux transform as objective vectors

$$g_m^* = Q_{mn}(t)\,g_n, \quad q_m^* = Q_{mn}(t)\,q_n, \quad \phi_m^* = Q_{mn}(t)\,\phi_n \tag{4.50}$$

where Q_{mn} is a proper orthogonal matrix,

$$\det(Q_{mn}) = +1$$

describing the time dependent rotational orientation between the stared and unstared observer system. The internal variables are assumed to be objective scalar vectors or tensors, but a definite specification is left open at this place.

Then in the starred observer system we have

$$\left.\begin{aligned} q_m^* &= -\check{\varkappa}^*(\sigma_{mn}^*, T^*, \beta_\nu^*, g_n^*)\, g_m^* \\ \phi_m^* &= \check{\varphi}^*(\;\text{——}\;,\;\text{——}\;)\, q_m^* + k_m^* . \end{aligned}\right\} \quad (4.51)$$

With $(4.50)_{1,2}$ we get

$$-\check{\varkappa}^*(\sigma_{mn}^*, T^*, \beta_\nu^*, g_n^*)\, Q_{mr}\, g_r = q_m^* = Q_{mn}\, q_n =$$
$$= -\check{\varkappa}(\sigma_{mn}, T, \beta_\nu, g_n)\, Q_{mn}\, g_n$$

which yields

$$\check{\varkappa}^*(\sigma_{mn}^*, T^*, \beta_\nu^*, g_n^*) = \check{\varkappa}(\sigma_{mn}, T, \beta_\nu, g_n) \quad (4.52)$$

Material frame indifference requires that the functions $\check{\varkappa}^*(\cdot, \cdot, \ldots)$ and $\check{\varkappa}(\cdot, \cdot, \ldots)$ are the same. With

$$\begin{aligned} \sigma_{mn}^* &= Q_{mk}\, Q_{nl}\, \sigma_{kl} \\ T^* &= T \end{aligned} \quad (4.53)$$

we get

$$\check{\varkappa}(Q_{mk}\, Q_{nl}\, \sigma_{kl}, T, \beta_\nu^*, Q_{nk}\, g_k) = \quad (4.54)$$
$$= \check{\varkappa}(\sigma_{mn}, T, \beta_\nu, g_n)$$

Thus, the function $\check{\varkappa}$ is an isotropic function under proper orthogonal transformations, and such a function is called hemitropic [79]. This means that $\check{\varkappa}$ cannot depend in an arbitrary manner on its arguments but only on combinations of them, invariant under proper orthogonal transformations. If the β_ν are objective scalars like T, then a proper list of irreducable invariants of the symmetric tensor σ_{kl} and the vector g_k can be derived from the representation given in [80]:

$$\sigma_{kk}, \quad \sigma_{kl}\,\sigma_{lk}, \quad \det(\sigma_{kl}), \quad g_m\, g_m$$

and the simultaneous invariants (4.55)

⋮

$$g_k\sigma_{kl}g_l\,,\quad g_k\sigma_{kl}\sigma_{lm}g_m\,,\quad e_{ijr}\,e_{jls}\,e_{mnt}\,g_r g_s g_t\,\sigma_{lm}\,\sigma_{np}\,\sigma_{pi}\quad\Big\}$$

where e_{klm} is the alternating tensor. Note that the last invariant drops from the list if also improper orthogonal transformations, i.e., the full group of orthogonal transformations, are considered.

From (4.51) we find with (4.50)

$$\Phi_k^* = \check{\varphi}^*(\sigma_{mn}^*, T^*, \beta_v^*, g_n^*)\, q_k^* + \check{k}_k^*(\sigma_{mn}^*, T^*, \beta_v^*) =$$

$$= \check{\varphi}^*(\text{——}\;''\;\text{——})\,Q_{kl}\,q_l + Q_{kl}\,\check{k}_l = Q_{kl}\,\Phi_l =$$

$$= \check{\varphi}(\sigma_{mn}, T, \beta_v, g_n)\,Q_{kl}\,q_l + Q_{kl}\,\check{k}_l$$

which yields

$$\left.\begin{aligned} \check{\varphi}^*(\sigma_{mn}^*, \dots) &= \check{\varphi}(\sigma_{mn}, \dots) \\ \check{k}_k^*(\sigma_{mn}^*, T^*, \beta_v^*) &= Q_{kl}\,\check{k}_l(\sigma_{mn}, T, \beta_v) \end{aligned}\right\}\quad (4.56)$$

and material frame indifference yields finally

$$\left.\begin{aligned} \check{\varphi}(Q_{mk}Q_{nl}\sigma_{kl}, \dots) &= \check{\varphi}(\sigma_{mn}, \dots) \\ \check{k}_k(Q_{mk}Q_{nl}\sigma_{kl}, \dots) &= Q_{kl}\,\check{k}_l(\sigma_{mn}, \dots) \end{aligned}\right\}\quad (4.57)$$

which shows that $\check{\varphi}$ and $\check{k}_k$ must be hemitropic scalar and vector valued functions.

Now we will make the simplest assumption with respect to the dependence of the functions $\check{\varkappa}$ and $\check{\varphi}$ on the temperature gradient: The influence is only via the magnitude

$$g = g_p g_p\,;$$

no coupling terms involving the stress tensor (according to the list of invariants (4.55)) should be present. Then condition (4.47) takes the following form

$$4\left(\frac{\partial\check{\varphi}}{\partial g}\,\check{\varkappa} + (\check{\varphi} + \lambda_\varepsilon)\frac{\partial\check{\varkappa}}{\partial g}\right) g_m g_n + (\check{\varphi} + \lambda_\varepsilon)\,\check{\varkappa}\,\delta_{km} = 0\,. \tag{4.58}$$

For $k \neq m$ and arbitrary gradients $g_m g_k \neq 0$ we obtain

$$\frac{\partial\check{\varphi}}{\partial g}\check{\varkappa} + (\check{\varphi}+\Lambda_\epsilon)\frac{\partial\check{\varkappa}}{\partial g} = 0; \tag{4.59}$$

then (4.58) gives

$$\check{\varphi}+\Lambda_\epsilon = 0. \tag{4.60}$$

This allows to conclude from (4.59)

$$\frac{\partial\check{\varphi}}{\partial g} = 0. \tag{4.61}$$

From $(4.45)_1$ and $(4.45)_2$ we get with (4.60) for arbitrary q_k, and observing that k_k is not depending on the heat flux q_k,

$$\frac{\partial\check{\varphi}}{\partial\sigma_{mn}} = 0, \quad \frac{\partial\check{\varphi}}{\partial\beta_\nu} = 0 \tag{4.62}$$

and

$$\frac{\partial\check{k}_k}{\partial\sigma_{mn}} = 0, \quad \frac{\partial\check{k}_k}{\partial\beta_\nu} = 0. \tag{4.63}$$

The results (4.61) and (4.62) imply that $\check{\varphi}$ is only a function of the temperature T

$$\check{\varphi} = \varphi(T) = -\Lambda_\epsilon(T). \tag{4.64}$$

and (4.63) requires the vector k_m to depend only on the temperature

$$k_m = \check{k}_m(T).$$

However, since k_m is objective, material frame indifference requires

$$\check{k}_m^*(T^*) = Q_{mn}\check{k}_n(T) = \check{k}_m(T^*) = \check{k}_m(T) \tag{4.65}$$

for arbitrary proper orthogonal Q_{mn}. Equ. (4.65) can only be satisfied if

$$\check{k}_m(T) \equiv 0. \tag{4.66}$$

Thus, within the frame of our assumptions *an extra entropy flux k_m, depending on σ_{mn}, T, and β_ν, does not exist.*

Whether these results are also obtained, if $\check{\varkappa}$ and $\check{\varphi}$ are assumed to depend also on the simultaneous invariants in (4.55), has not been analysed. However, in the study of thermoelastic materials a somewhat analogous problem exists and

Müller's results* [30, 31] support the expectation that the above results will also hold in this case.

With (4.64) the energetic Lagrange multiplier Λ_ε is given. It remains to characterize the constitutive function φ(T) and here again we follow Müller's approach [31].

We consider a control volume around the interface of two different materials (Fig. 3) which, however, belong to the same general class. Specifically this means that the balance equations and the entropy inequality have the same form. Here we have in mind the present viscoplastic model and the adjacent material is the classical thermoelastic material whose constitutve equations are well known.

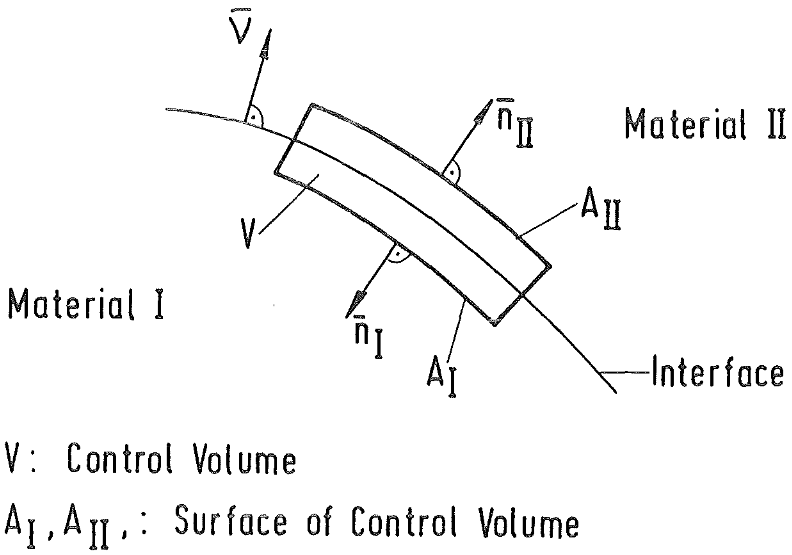

Fig. 5: Control Volume at an Interface

We apply the integral entropy inequality (4.1) with s ≡ 0, i.e.,

$$\frac{d}{dt}\int_V \rho\eta\, dV + \int_A \phi_k n_k\, dA \geq 0 \qquad (4.67)$$

to the control volume at the interface. With respect to the sign of the entropy flux we recall equ. (4.3).

For the interface we make the natural assumption that it does not possess energetic or entropic properties, i.e., there are no constitutive properties attached to it. Further, the density, specific entropy, and its rate as well as the en-

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[*] Unfortunately, a detailed derivation is not contained in [30, 31].

tropy flux vector ϕ_k in the two materials are *bounded* quantities. Deviding inequality (4.67) by the interface surface A_J gives

$$\frac{d}{dt}\int \rho\eta \frac{dV}{A_J} + \int_A \phi_k n_k \frac{dA}{A_J} \geq 0 . \tag{4.68}$$

We let the distance d of A_I and A_{II} shrink towards zero, then

$$\left.\begin{aligned} &A_I \longrightarrow A_J , \qquad A_{II} \longrightarrow A_J \\ &\bar{n}_I \longrightarrow \bar{\nu} , \qquad \bar{n}_{II} \longrightarrow -\bar{\nu} \\ &V \longrightarrow A_J d \end{aligned}\right\} \tag{4.69}$$

and (4.68) reduces in the limit d → 0 to

$$\left.\begin{aligned} &\phi_I + \phi_{II} \geq 0 \\ &\phi_I = \phi_{Ik}(-\nu_k) , \qquad \phi_{II} = \phi_{IIk}\nu_k \end{aligned}\right\} \tag{4.70}$$

or

$$(\phi_{IIk} - \phi_{Ik})\nu_k \geqq 0 . \tag{4.71}$$

This is a consequence of the fact that the volume integral in (4.68) is zero in the limit. In the imbalance equation (4.71) the equality applies and that is shown as follows. We introduce the specific entropy production rate σ

$$\sigma = \rho\dot{\eta} + \frac{\partial \phi_k}{\partial x_k}$$

such that (4.68) reads

$$\frac{d}{dt}\int_V \rho\eta \frac{dV}{A_J} + \int_A \phi_k n_k \frac{dA}{A_J} = \int_V \rho\sigma \frac{dV}{A_J} \geq 0$$

and σ can be expressed in terms of the assumed constitutive equations and functions. Since σ is assumed to be bounded, the integral entropy production rate vanishes in the limit. Thus

$$(\varphi q_k)_{II}\nu_k - (\varphi q_k)_I \nu_k = [\![\varphi q_k]\!]\nu_k = 0 \tag{4.72}$$

i.e., the normal entropy flux suffers no jump accross the interface.

The integral energy balance equation*

$$\frac{d}{dt}\int \rho e\, dV = \int_V \sigma_{kl}\,\dot{\varepsilon}_{kl}\, dV - \int_A q_k n_k\, dA$$

can be used in an analogous way to prove that the normal heat flux $q_k n_k$ is continuous across the interface, i.e.,

$$q_{II\,k}\,\nu_k - q_{I\,k}\,\nu_k = [\![q_k]\!]\,\nu_k = 0. \tag{4.73}$$

Combining (4.72) and (4.73) one gets finally

$$\varphi_{II}(T_{II}) - \varphi_{I}(T_{I}) = 0. \tag{4.74}$$

Assuming now the *continuity*** *of the temperature T*

$$T_{II} = T_{I} \tag{4.75}$$

the result (4.74) shows that the two constitutive functions φ_I and φ_{II} of the two adjacent media are the same. In the terminology of Müller $\varphi(T)$ is a *universal* function.

To identify the function $\varphi(T)$ we proceed as follows. In the classical thermoelastic case the entropy flux is usually assumed to be

$$\phi_k = \frac{1}{T}\, q_k .$$

Since $\varphi(T)$ is universal, this suggests to put

$$\varphi(T) = \frac{1}{T}$$

However, this is not a rigorous argument.

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* It is possible to start off also from a non-reduced energy balance equation involving the kinetic energy etc. [81].

** The continuity of the temperature may be motivated by considering the ideal interface as the limiting case of a thin interlayer without the capability of heat storage but following Fourier's law of heat conduction with a heat conductivity becomming very large. However, Müller [31, p. 457] has given an example where continuity of temperature does not apply.

We introduce the Gibbs function in the usual way

$$\xi = \check{\epsilon} - T\check{\eta} - \frac{1}{\rho}\sigma_{kl}\check{\varepsilon}^{e}_{kl} = \check{\xi}(\sigma_{mn}, T, \beta_v) \tag{4.76}$$

Then

$$\left.\begin{aligned}
\rho\frac{\partial\check{\xi}}{\partial\sigma_{mn}} &= \rho\frac{\partial\check{\epsilon}}{\partial\sigma_{mn}} - \rho T\frac{\partial\check{\eta}}{\partial\sigma_{mn}} - \check{\varepsilon}^{e}_{mn} - \sigma_{kl}\frac{\partial\check{\varepsilon}^{e}}{\partial\sigma_{mn}}\\
\rho\frac{\partial\check{\xi}}{\partial T} &= \rho\frac{\partial\check{\epsilon}}{\partial T} - \rho T\frac{\partial\check{\eta}}{\partial T} - \rho\check{\eta} - \sigma_{kl}\frac{\partial\check{\varepsilon}^{e}_{kl}}{\partial T}\\
\rho\frac{\partial\check{\xi}}{\partial\beta_v} &= \rho\frac{\partial\check{\epsilon}}{\partial\beta_v} - \rho T\frac{\partial\check{\eta}}{\partial\beta_v} - \sigma_{kl}\frac{\partial\check{\varepsilon}^{e}_{kl}}{\partial\beta_v}
\end{aligned}\right\} \tag{4.77}$$

Observing (4.64), the restrictions (4.43) take the form

$$\left.\begin{aligned}
&\rho\frac{\partial\check{\epsilon}}{\partial\sigma_{mn}} + \rho\frac{\partial\check{\epsilon}}{\partial\beta_v}\underset{v}{\check{E}}_{mn} - \sigma_{kl}\left(\frac{\partial\check{\varepsilon}^{e}_{kl}}{\partial\sigma_{mn}} - \frac{\partial\check{\varepsilon}^{e}_{kl}}{\partial\beta_v}\underset{v}{\check{E}}_{mn}\right) -\\
&\quad - \frac{1}{\varphi}\left(\rho\frac{\partial\check{\eta}}{\partial\sigma_{mn}} + \rho\frac{\partial\check{\eta}}{\partial\beta_v}\underset{v}{\check{E}}_{mn}\right) = 0\\
&\rho\frac{\partial\check{\epsilon}}{\partial T} + \rho\frac{\partial\check{\epsilon}}{\partial\beta_v}\underset{v}{\check{H}} - \sigma_{kl}\left(\frac{\partial\check{\varepsilon}^{e}_{kl}}{\partial T} - \frac{\partial\check{\varepsilon}^{e}_{kl}}{\partial\beta_v}\underset{v}{\check{H}}\right)\\
&\quad - \frac{1}{\varphi}\left(\rho\frac{\partial\check{\eta}}{\partial T} + \rho\frac{\partial\check{\eta}}{\partial\beta_v}\underset{v}{\check{H}}\right) = 0
\end{aligned}\right\} \tag{4.78}$$

and with (4.77) this reduces to

$$\left.\begin{aligned}
&\rho\frac{\partial\check{\xi}}{\partial\sigma_{mn}} + \rho\frac{\partial\check{\xi}}{\partial\beta_v}\underset{v}{\check{E}}_{mn} + \check{\varepsilon}^{e}_{mn} +\\
&+ \rho\left(T - \frac{1}{\varphi}\right)\frac{\partial\check{\eta}}{\partial\sigma_{mn}} + \varphi\left(T - \frac{1}{\varphi}\right)\frac{\partial\check{\eta}}{\partial\beta_v}\underset{v}{\check{E}}_{mn} = 0
\end{aligned}\right|$$

$$\left.\begin{aligned}&\rho\frac{\partial\check{\zeta}}{\partial T}+\rho\frac{\partial\check{\zeta}}{\partial\beta_\nu}\check{H}_\nu+\rho\check{\eta}+\\&+\rho\left(T-\frac{1}{\varphi}\right)\frac{\partial\check{\eta}}{\partial T}+\rho\left(T-\frac{1}{\varphi}\right)\frac{\partial\check{\eta}}{\partial\beta_\nu}\check{H}_\nu=0.\end{aligned}\right\}\quad(4.79)$$

This applies to the viscoplastic body as well as to the thermoelastic comparison body but, of course, the constitutive functions are different, except $\varphi(T)$ is universal.

For the thermoelastic body the conditions (4.79) read

$$\left.\begin{aligned}&\rho\frac{\partial\check{\zeta}_{/E}}{\partial\sigma_{mn}}+\check{\varepsilon}^{e}_{mn/E}+\rho\left(T-\frac{1}{\varphi}\right)\frac{\partial\check{\eta}_{/E}}{\partial\sigma_{mn}}=0\\&\rho\frac{\partial\check{\zeta}_{/E}}{\partial T}+\rho\check{\eta}_{/E}+\rho\left(T-\frac{1}{\varphi}\right)\frac{\partial\check{\eta}_{/E}}{\partial T}=0\end{aligned}\right\}\quad(4.80)$$

where the subscript $(\bullet)_{/E}$ denotes the thermoelastic body. However, in this case the thermoelastic potential relations

$$\left.\begin{aligned}&\rho\frac{\partial\check{\zeta}_{/E}}{\partial\sigma_{mn}}+\varepsilon^{e}_{mn/E}=0\\&\rho\frac{\partial\check{\zeta}_{/E}}{\partial T}+\rho\check{\eta}_{/E}=0\end{aligned}\right\}\quad(4.81)$$

apply and, therefore, (4.80) yields

$$\varphi(T)=\frac{1}{T}.\qquad(4.82)$$

In fact, this argument implies that we know the constitutive functions $\zeta_{/E}$, $\check{\varepsilon}^{e}_{/E}$, and $\check{\eta}_{/E}$. From an empirical point of view this is not a trivial matter; especially $\check{\eta}_{/E}$ and $\zeta_{/E}$ are not directly measureable.

Therefore, we follow here another approach which requires only the knowledge of $\check{\varepsilon}^e_{kl/E}$ and the specific internal energy $\check{\varepsilon}_{/E}$. The total differential of the entropy $\check{\eta}$ is

$$d\check{\eta} = \frac{\partial \check{\eta}}{\partial \sigma_{mn}} d\sigma_{mn} + \frac{\partial \check{\eta}}{\partial T} dT + \frac{\partial \check{\eta}}{\partial \beta_\nu} d\beta_\nu . \tag{4.83}$$

Here, of course, the integrability conditions

$$\frac{\partial^2 \check{\eta}}{\partial \sigma_{mn}\, \partial T} = \frac{\partial^2 \check{\eta}}{\partial T\, \partial \sigma_{mn}} \quad \text{etc.} \tag{4.84}$$

apply. From (4.78) we have

$$\left.\begin{aligned}
\frac{\partial \check{\eta}}{\partial \sigma_{mn}} &= \varphi\left(\frac{\partial \check{\epsilon}}{\partial \sigma_{mn}} + \frac{\partial \check{\epsilon}}{\partial \beta_\nu} \underset{\nu}{\check{E}}_{mn}\right) - \\
&\quad -\frac{1}{\varrho}\varphi\, \sigma_{kl}\left(\frac{\partial \check{\varepsilon}^e_{kl}}{\partial \sigma_{mn}} + \frac{\partial \check{\varepsilon}^e_{kl}}{\partial \beta_\nu} \underset{\nu}{\check{E}}_{mn}\right) - \\
&\quad - \frac{\partial \check{\eta}}{\partial \beta_\nu} \underset{\nu}{\check{E}}_{mn} \\
\frac{\partial \check{\eta}}{\partial T} &= \varphi\left(\frac{\partial \check{\epsilon}}{\partial T} + \frac{\partial \check{\epsilon}}{\partial \beta_\nu} \underset{\nu}{\check{H}}\right) - \\
&\quad -\frac{1}{\varrho}\varphi\, \sigma_{kl}\left(\frac{\partial \check{\varepsilon}^e_{kl}}{\partial T} + \frac{\partial \check{\varepsilon}^e_{kl}}{\partial \beta_\nu} \underset{\nu}{\check{H}}\right) - \\
&\quad - \frac{\partial \check{\eta}}{\partial \beta_\nu} \underset{\nu}{\check{H}} .
\end{aligned}\right\} \tag{4.85}$$

The requirement (4.84) then yields, after some of the terms have chancelled,

$$\left.\begin{aligned}
&\frac{d\varphi}{dT}\left[\frac{\partial \check{\epsilon}}{\partial \sigma_{mn}} + \frac{\partial \check{\epsilon}}{\partial \beta_\nu} \underset{\nu}{\check{E}}_{mn} - \frac{1}{\varrho}\sigma_{kl}\left(\frac{\partial \check{\varepsilon}^e_{kl}}{\partial \sigma_{mn}} + \frac{\partial \check{\varepsilon}^e_{kl}}{\partial \beta_\nu} \underset{\nu}{\check{E}}_{mn}\right)\right] + \\
&+ \varphi\left[\frac{\partial}{\partial T}\left(\frac{\partial \check{\epsilon}}{\partial \beta_\nu} \underset{\nu}{\check{E}}_{mn}\right) - \frac{1}{\varrho}\left(\sigma_{kl} \frac{\partial}{\partial T}\left(\frac{\partial \check{\varepsilon}^e_{kl}}{\partial \beta_\nu} \underset{\nu}{\check{E}}_{mn}\right)\right)\right] - \\
&- \frac{\partial}{\partial T}\left(\frac{\partial \check{\eta}}{\partial \beta_\nu} \underset{\nu}{\check{E}}_{mn}\right) = \cdots
\end{aligned}\right\} \tag{4.86}$$

$$\ldots = \varphi\Big[\frac{\partial}{\partial\sigma_{mn}}\Big(\frac{\partial\check{\epsilon}}{\partial\beta_\nu}\underset{\nu}{\check{E}}_{mn}\Big) - \frac{1}{\varrho}\Big(\frac{\partial\check{\varepsilon}^e_{mn}}{\partial T} + \frac{\partial\check{\varepsilon}^e_{mn}}{\partial\beta_\nu}\underset{\nu}{\check{H}} - \sigma_{kl}\frac{\partial}{\partial\sigma_{mn}}\Big(\frac{\partial\check{\varepsilon}^e_{kl}}{\partial\beta_\nu}\underset{\nu}{\check{H}}\Big)\Big)\Big]$$

$$- \frac{\partial}{\partial\sigma_{mn}}\Big(\frac{\partial\check{\eta}}{\partial\beta_\nu}\underset{\nu}{\check{H}}\Big). \qquad (4.86)$$

For the classical thermoelastic body this simplifies to

$$\frac{d\varphi}{dT}\Big[\frac{\partial\check{\epsilon}_{/E}}{\partial\sigma_{mn}} - \frac{1}{\varrho}\sigma_{kl}\frac{\partial\check{\varepsilon}^e_{kl}{}_{/E}}{\partial\sigma_{mn}}\Big] = -\varphi\frac{1}{\varrho}\frac{\partial\check{\varepsilon}^e_{mn}{}_{/E}}{\partial T}$$

or

$$\frac{1}{\varphi}\frac{d\varphi}{dT} = -\frac{1}{\varrho}\frac{\partial\check{\varepsilon}^e_{mn}{}_{/E}}{\partial T}\Big[\frac{\partial\check{\epsilon}_{/E}}{\partial\sigma_{mn}} - \frac{1}{\varrho}\sigma_{kl}\frac{\partial\check{\varepsilon}^e_{kl}{}_{/E}}{\partial\sigma_{mn}}\Big]^{-1}. \qquad (4.87)$$

The left hand side is solely a function of the temperature; this represents a severe restriction on the functions $\check{\varepsilon}^e_{mn}$ and $\check{\epsilon}_{/E}$. With the standard expressions

$$\varepsilon^e_{kl}{}_{/E} = \frac{1+\nu}{E}\sigma_{kl} - \frac{\nu}{E}\sigma_{mm}\delta_{kl} + A^{th}(T-T_0)\delta_{kl}$$

and

$$\epsilon_{/E} = \check{\zeta} - T\frac{\partial\check{\zeta}}{\partial T} + \sigma_{kl}\varepsilon^e_{kl}$$

where $\check{\zeta}$ is given by (4.76), the r.h.s. of (4.87) is (- 1/T); thus

$$\frac{d\ln\varphi/\varphi_0}{dT} = -\frac{1}{T}$$

which gives after integration

$$\varphi = \frac{C}{T} \qquad (4.88)$$

where C is a positive integration constant. That this can be chosen at will, e.g. C = 1, is shown as follows.

In the thermoelastic case the equations (4.85) simplify to

$$\left.\begin{aligned}\frac{\partial \check{\eta}}{\partial \sigma_{mn}} &= \varphi\left(\frac{\partial \check{\epsilon}}{\partial \sigma_{mn}} - \frac{1}{\rho}\,\sigma_{kl}\,\frac{\partial \check{\varepsilon}^{e}_{kl}}{\partial \sigma_{mn}}\right)\\ \frac{\partial \check{\eta}}{\partial T} &= \varphi\left(\frac{\partial \check{\epsilon}}{\partial T} - \frac{1}{\rho}\,\sigma_{kl}\,\frac{\partial \check{\varepsilon}^{e}_{kl}}{\partial T}\right)\end{aligned}\right\} \quad (4.89)$$

and (4.82) reads

$$d\eta = \frac{\partial \check{\eta}}{\partial \sigma_{mn}}\, d\sigma_{mn} + \frac{\partial \check{\eta}}{\partial T}\, dT\,;$$

here, $\check{\epsilon}$ and $\check{\varepsilon}^{e}_{kl}$ are assumed to be known.

With φ given by (4.88), which assures the integrability of (4.89), this system of partial differential equations can be integrated to yield the entropy $\check{\eta}$, and *clearly* $\check{\eta}$ *is proportional to the constant C.*

The Gibbs function is now introduced as follows

$$\begin{aligned}\check{\zeta}_{|E} &= \check{\epsilon}_{|E} - \frac{1}{\varphi}\,\check{\eta}_{|E} - \sigma_{kl}\,\check{\varepsilon}^{e}_{kl} =\\ &= \check{\epsilon}_{|E} - T\,\frac{\check{\eta}_{|E}}{C} - \sigma_{kl}\,\check{\varepsilon}^{e}_{kl}\,.\end{aligned} \quad (4.90)$$

This expression involves the ratio $\check{\eta}_E/C$ and therefore the Gibbs function, the determining potential for $\check{\varepsilon}^{e}_{kl}$, $\check{\epsilon}_{|E}$, and $\check{\eta}_{|E}$ is independent of the choice of C.

We are now ready to collect the main results of the section.

1. The *entropy flux* ϕ_k, initially assumed to be given by the general expression (4.25) with (4.26) but then specified to be

   $$\phi_k = \check{\varphi}(\sigma_{mn}, T, \beta_\nu, g)\, q_k + \check{k}_k(\sigma_{mn}, T, \beta_\nu) \quad (4.91)$$

   reduces to

   $$\phi_k = \varphi(T)\, q_k\,, \quad \varphi = \frac{C}{T}\,, \quad k_k \equiv 0 \quad (4.92)$$

   where $\varphi(T)$ is a "universal" function with an arbitrary positive constant C, conveniently chosen to be C = 1.

2. The *constitutive restrictions* (4.78) for the viscoplastic body in terms of the Gibbs function

$$\xi = \check{\xi}(\sigma_{mn}, T, \beta_\nu) = \check{\epsilon} - \frac{1}{\varphi}\check{\eta} - \sigma_{kl}\check{\varepsilon}^{e}_{kl} \tag{4.93}$$

are given by

$$\left.\begin{aligned} \check{\varepsilon}^{e}_{kl} &= -\varrho\frac{\partial\check{\xi}}{\partial\sigma_{kl}} - \varrho\frac{\partial\check{\xi}}{\partial\beta_\nu}\underset{\vee}{E}_{kl} \\ \check{\eta} &= -\frac{\partial\check{\xi}}{\partial T} - \frac{\partial\check{\xi}}{\partial\beta_\nu}\underset{\vee}{\check{H}}\,. \end{aligned}\right\} \tag{4.94}$$

If C = 1, then the Gibbs function (4.93) takes the classical form (4.76).

3. The *residual entropy inequality* (4.46), observing (4.64) and (4.66), simplifies to

$$\frac{\partial\varphi}{\partial T}\check{q}_k g_k + \varphi\,\sigma_{kl}\dot{\varepsilon}^{p}_{kl} - \Lambda_{\beta_\nu}\check{f}_\nu \geq 0 \tag{4.95}$$

where the Lagrange multiplier $\Lambda_{\beta\nu}$ is given by (4.42) or

$$\Lambda_{\beta_\nu} = \varphi\left\{\varrho\frac{\partial\check{\epsilon}}{\partial\beta_\nu} - \varrho\frac{1}{\varphi}\frac{\partial\check{\eta}}{\partial\beta_\nu} - \sigma_{kl}\frac{\partial\check{\varepsilon}^{e}_{kl}}{\partial\beta_\nu}\right\}$$

and in terms of the Gibbs function (4.93)

$$\Lambda_{\beta_\nu} = \varphi\,\varrho\,\frac{\partial\check{\xi}}{\partial\beta_\nu}\,. \tag{4.96}$$

Thus, (4.95) takes the form

$$\frac{\partial\varphi}{\partial T}\check{q}_k g_k + \varphi\left[\sigma_{kl}\dot{\varepsilon}^{p}_{kl} - \varrho\frac{\partial\check{\xi}}{\partial\beta_\nu}\check{f}_\nu\right] \geq 0$$

or with $(4.92)_2$ and $T > 0$

$$\sigma_{kl}\dot{\varepsilon}^{p}_{kl} - \varrho\frac{\partial\check{\xi}}{\partial\beta_\nu}\check{f}_\nu - \frac{1}{T}\check{q}_k g_k \geq 0 \tag{4.97}$$

which may be split in two seperate inequalities.

4.3 Conclusions

A comparison shows that the results of section 2.3, which are based on the classical Clausius-Duhem entropy inequality and the Coleman-Noll argument, agree with the main conclusions of section 4.2, that is the constitutive restrictions (2.232) and the residual entropy inequality (dissipation inequality) are confirmed.

Further, we note explicitely that the entropy flux ϕ_k, equ. (4.28) reduces to the classical expression $\phi_k = 1/T\ q_k$. This is due to the fact that the extra entropy flux k_k, assumed to depend only on σ_{mn}, T, β_v, vanishes identically and the constitutive function $\varphi = \check{\varphi}(\sigma_{mn}, T, \beta_v, g)$ is proven to be a universal function of only the temperature with the following structure: $\varphi = C/T$, $C > 0$ and arbitrary.

The analysis in section 4.2 demonstrated that Müller's approach is much more involved mathematically than the classical procedure. Its merits, amply proven for other constitutive classes, justify this inconvenience, and in the present case it has lent further support to the previous results of section 2.3. However, there are delicate points: The choice of independent variables, the assumptions about the entropy flux, and the assumed continuity of temperature at ideal interfaces of different materials. It may well be that a more complex constitutive structure will not allow to derive a complete set of conclusions from the entropy inequality if not additional specific assumptions about the model are made. Examples are: The anisotropic thermoelastic body which involves the temperature rate in the state equations for internal energy, etc., and possibly internal variable models involving gradients of the internal variables which are present in the evolution equations and which affect the entropy flux.

Last but not least we should mention that the present theoretical frame does not resolve the paradox of infinite propagation rates (e.g. heat waves). This requires a different choice of independent variables, constitutive equations and balance laws and is persued in "Extended Thermodynamics" [31, 77, 83].

5. Recapitulation and Final Remarks

The objective of this study is to derive and investigate thermodynamic restrictions for a class of internal variable models with evolution equations which contain the rates of external variables linearly. Thus, the evolution of the internal variables are governed by the usual irreversible contribution, depending only on the present state, and by a reversible but path dependent contribution.

In the first instance the thermodynamic analysis is based on the classical Clausius-Duhem entropy inequality and the Coleman-Noll argument. Also this analysis is restricted to infinitesimal strains and rotations.

It is proved that the classical potential relations for the stress and entropy are destroyed due to the presence of the reversible but path dependent contribution in the "mixed type" evolution equations. One obtains "non-classical potential relations". On the other hand the residual dissipation inequality is essentially determined by the irreversible part of the evolution equations as usual.

Following the previous work of Lubliner [63], explicit conditions are derived for the existence of a suitable transformation of the internal variables which permits the reduction of the evolution equations of "mixed type" to the purely irreversible type. If such a transformation exists, then the classical potential relations hold again. It is then shown that the reduceability of the evolution equations and path independence of the reversible part in the evolution equations are equivalent. This also explains the restoration of the classical potential relations if reduceability is possible.

Also evolution equations are studied which involve the stress as an independent external variable instead of the strain. Such a formulation is more closely related to viscoplastic material models. This implies the introduction of the Gibbs function or free enthalpy as a thermodynamic potential. However, the arbitrary choice of a stress field, which is one of the variables defining a thermomechanical process, may be problematic since the stress field may not be related to a compatible strain field. Conditions are indicated which assure that this variable choice is acceptable for the evaluation of the entropy inequality. Also the conditions for the reduceability of the evolution equations are given. For some special reversible contributions in the evolution equations explicit transformation functions are determined.

Further, the thermodynamic consistency of assumed state functions for the strain and entropy are studied to some extend. This question is also important for the analysis of the thermodynamic consistency of purely mechanical models. If the strain and entropy are prescribed as functions of the independent state variables, i.e., stress, temperature, and internal variables, then a Gibbs function satisfying the non-classical potential relations does not necessarily exist. This implies that such a prescribtion is thermodynamically inconsistent. Necessary conditions are derived for the existence of a Gibbs function depending on internal variables.

Eventually, the internal variable model is modified to represent a class of elastic-viscoplastic material models. Here the total strain is additively composed of a thermoelastic strain and a viscoplastic strain. The evolution equations for the viscoplastic strain are entirely controlled by an irreversible contribution. However, the evolution equations for the internal variables, which possibly control the nonlinear hardening, are of the "mixed type". Further, the irreversible part in the evolution equations may change according to certain switch conditions (e.g. a yield condition which defines an elastic range). These conditions, however, are assumed to depend solely on the independent state variables; no rates of the external variables are involved, i.e., they do not involve a distinction between loading and unloading. To this class of models the previous results are transferred.

The general results obtained are applied to several viscoplastic models proposed or discussed in the literature, where the evolution equations are all of the "mixed type".

The model of Robinson et al. [47] involves the temperature rate in the evolution equation for the drag stress, a proposal to better cope with non-isothermal processes. Originally, this model was not developed in a thermodynamic frame. It is shown that the evolution equation is reduceable and that a thermodynamically extended version can be set up which is formally consistent with the Clausius-Duhem entropy inequality. No restrictions for the parameters of the original model are obtained.

Secondly, a model of Krempl et al. [52] is analysed. This model uses an evolution equation for the internal variable - equilibrium stress - which involves the rate of total strain to describe the evolution of the hardening. Alternatively this may be written in a form which involves the stress rate. Up to now a thermodynamic analysis has not been performed. The analysis shows that the assumed thermoelastic strain-stress relation (Hooke's law) and the evolution equation for the equilibrium stress are in conflict with each other: An appropriate Gibbs function, depending on the internal variables, does not exist.

If the equilibrium stress is not considered to be a true thermodynamic state variable, then a Gibbs function independent of the equilibrium stress exists. However, then the model is conflict with the intrinsic dissipation inequality: There exists states, which actually can be reached by suitable processes, where the dissipation is negative.

Consequently, whatever choice is made, this model is thermodynamically inconsistent.

Finally, a supplementary analysis of a thermodynamic discussion by Freed et al. [82] for two models is performed. Both models involve the stress rate in the evolution equation for the internal variable whereas the thermoelastic strain-stress relation is given by Hooke's law. The purpose of this supplementary study is to analyse the somewhat artificial approach of introducing an "adjustment function" in the Gibbs function to satisfy the intrinsic dissipation inequality. For the first model this approach yields in fact a sufficient solution. However, an alternative direct choice of the Gibbs function, observing its thermodynamic restrictions, gives a more complete sufficient solution which contains the previous result. For the second model the approach of the "adjustment function" is shown to be thermodynamically inadmissible. This is also reflected in the fact that the determining equations for the "adjustment function" do not admit a solution. Therefore, this approach, aside from beeing artificial, is generally thermodynamically inadmissible.

These results generally demonstrate that the subsequent analysis of the thermodynamic consistency of material models not embedded in a thermodynamic frame may be a cumbersome excercise and may produce awkward surprises. Therefore, the models should be developed observing thermodynamical concepts from the beginning; this is especially so if the development of the model is outside the usual routes.

The study is closed with the evaluation of the extended Clausius-Duhem entropy inequality (concept of Müller) where the entropy flux is governed by an assumed constitutive equation in its own right; also the constraining balance equations are explicitly accounted for by the method of Lagrange multipliers (Liu's approach). This analysis is done for a viscoplastic material model with evolution equations of the "mixed type". It is shown that this approach is much more involved than the evaluation of the classical Clausius-Duhem entropy inequality with the Coleman-Noll argument. Its merits, proven for other constitutive classes than discussed here, justifies this inconvenience. In the present case it confirms and thus gives support for the thermodynamic restrictions derived with the classical procedure.

References

[1] T. Malmberg: Thermodynamics of a Viscoplastic Material Model with Internal Variables, Kernforschungszentrum Karlsruhe, KfK 4572, July 1990

[2] S. Nemat-Nasser ed.: On Nonequilibrium Thermodynamics in Continua, p. 94 in [3]

[3] S. Nemat-Nasser ed.: Mechanics Today, Vol. 2, Pergamon Press Inc., 1975

[4] G.A. Maugin, R. Drouot: Internal Variables and the Thermodynamics of Macromolecule Solutions, Int. J. Eng. Sc. 21, 7, pp. 705-724, 1983

[5] W. Muschik: Thermodynamische Theorien irreversibler Prozesse, Physik in unserer Zeit, 3. Jahrg., pp. 105-110, 1972

[6] K. Hutter: The Foundation of Thermodynamics, Its Basic Postulates and Implications, A Review of Modern Thermodynamics, Acta Mechanica 27, pp. 1-54, 1977

[7] B. Lavenda: Thermodynamics of Irreversible Processes, London and Basingstoke, The Macmillan Press, 1978

[8] C. Spinedi: Die Thermodynamischen Grundlagen der Kontinuumsmechanik (Vergleich zwischen verschiedenen Zugängen), J. Appl. Math. and Phys. 29, pp. 884-933, 1978

[9] G. Lebon, Ph. Mathieu: Etude Comparee de Diverses Theories de Thermodynamique du Non-Equilibre, Entropie 100, pp. 75-86, 1981

[10] W. Muschik: Thermodynamische Theorien, Überblick und Vergleich, ZAMM 61, T213-T219, 1981

[11] P. German, Q.S. Nguyen, P. Suquet: Continuum Thermodynamics, J. Appl. Mech., Trans. ASME 50, 4b, pp. 1010-1020, 1983

[12] W.Muschik: Thermodynamical Constitutive Laws - Outlines - , in: Constitutive Laws and Microstructure, D.R. Axelrod, W. Muschik (Eds.), Proc. of the Seminar Wissenschaftskolleg - Inst. for Advanced Study, Berlin, Feb. 23-24, 1987, Springer-Verlag, 1988

[13] B.D. Coleman, W. Noll: The Thermodynamics of Elastic Materials with Heat Conduction and Viscosity, Arch. Rat. Mech. & Anal. 13, pp. 167-178, 1963

[14] C. Truesdell: Rational Thermodynamics, 2nd Ed., New York, Berlin, Heidelberg, Tokyo, Springer Verlag, 1984

[15] W. Kern, G. Weiner, J. Meixner: Beziehungen zwischen der entropiefreien, der chemischen und der rationalen Thermodynamik der Vorgänge, Forschungsberichte des Landes Nordrhein-Westfalen Nr. 2449, Westdeutscher Verlag, 1974

[16] J. Meixner: Entropy einst und jetzt, Rheinisch-Westfälische Akademie der Wissenschaften, Vorträge N260, Opladen, Westdeutscher Verlag, 1976

[17] J. Keller: Über den Zweiten Hauptsatz der Thermodynamik Irreversibler Prozesse, Walter de Gruyter, 1976

[18] W.A. Day: The Thermodynamics of Simple Materials with Fading Memory, Springer Tracts in Natural Philosophy, vol. 22, Springer Verlag, 1972

[19] B.D. Coleman, D.R. Owen: A Mathematical Foundation for Thermodynamics, Arch. Rat. Mech. & Anal. 54, pp. 1-104, 1974

[20] M. Feinberg, R. Lavine: Foundations of the Clausius-Duhem Inequality, from [21]

[21] J. Serrin (Ed.): New Perspectives in Thermodynamics, Springer Verlag, 1986

[22] B.D. Coleman, V.J. Mizel: Thermodynamics and Departures from Fourier's Law of Heat Conduction, Arch. Rat. Mech. Anal. 13, pp. 245-261, 1963

[23] A.C. Eringen: A Unified Theory of Thermomechanical Materials, Int. J. Eng. Sci. 4, pp. 179-202, 1966

[24] M. Gurtin: Thermodynamics and the Possibility of Spatial Interaction in Elastic Materials, Arch. Rat. Mech. Anal. 19. pp. 339-352, 1965

[25] L.C. Woods: The Bogus Axioms of Continuum Mechanics, The Inst. of Math. and its Appl. 17, pp. 98-102, 1981

[26] L.C. Woods: More on the Bogus Axioms of Continuum Mechanics, The Inst. of Math. and its Appl. 18, pp. 64-65, 1982

[27] L.C. Woods: Thermodynamic Inequalities in Continuum Mechanics, IMA J. Appl. Math. 29, pp. 221-246, 1982

[28] A.E. Green: A Note on "Axioms of Continuum Mechanics", The Inst. of Math. and its Appl. 18, pp. 7-9, 1982

[29] I. Müller: Die Kältefunktion, eine universelle Funktion in der Thermodynamik viskoser, wärmeleitender Flüssigkeiten, Arch. Rat. Mech. & Anal. 40, pp. 1-36, 1971

[30] I. Müller: The Coldness, a Universal Function in Thermoelastic Bodies, Arch. Rat. Mech. & Anal. 41, pp. 319-332, 1971

[31] I. Müller: Thermodynamics, Interaction of Mechanics and Mathematics Series, Pitman Advanced Publ. Program, 1985

[32] T. Malmberg: On the Derivation of Thermodynamic Restrictions for Materials with Internal State Variables, Kernforschungszentrum Karlsruhe, KfK 4284, July 1987

[33] I.S. Liu: Method of Lagrange Multipliers for Exploitation of the Entropy Principle, Arch Rat. Mech. & Anal. 46, pp. 132-148, 1972

[34] J.E. Dunn, J. Serrin: On the Thermodynamics of Interstitial Working, Arch Rat. Mech. & Anal. 17, pp. 95-133, 1985

[35] J.E. Dunn: Interstitial Working and a Non-classical Continuum Thermodynamics, in [21]

[36] B.D. Coleman, M. Gurtin: Thermodynamics with Internal State Variables, The J. of Chem. Phys. 47, 2, pp. 597-613, 1967

[37] P. Perzyna, W. Wojno: Thermodynamics of a Rate Sensitive Plastic Material, Arch. Mech. Stos. 5, 20, pp. 499-511, 1968

[38] W. Olszak, P. Perzyna: Thermal Effects in Viscoplasticity, Proc. JUTAM Symp. East, Xilbridge, 1968, Springer Verlag, 1968

[39] P. Perzyna: Memory Effects and Internal Changes of a Material, Int. J. Non-Lin. Mech. 6, pp. 707-716, 1971

[40] P. Perzyna: Thermodynamics of Rheological Materials with Internal Changes, J. de Mecanique 10, 3, pp. 391-408, 1971

[41] J. Kratochvil, O.W. Dillon: Thermodynamics of Elastic-Plastic Materials as a Theory with Internal State Variables, J. Appl. Phys. 40, 8, pp. 3207-3218, 1969

[42] J. Kratochvil, O.W. Dillon: Thermodynamics of Crystalline Elastic-Viscoplastic Materials, J. Appl. Phys. 41, 4, pp. 1470-1479, 1970

[43] T. Malmberg: Thermodynamics of a Visco-Plastic Material Model with Internal Variables, Int. Conf. on Mechanics, Physics and Structure of Materials, Thessaloniki, Greece, Aug. 19-24, 1990

[44] T. Malmberg: On Thermodynamic Restrictions for Some Visco-Plastic Material Models, Transact. of the 11th Int. Conf. on Structural Mech. in Reactor Technology, vol. L, pp. 539-550, Toyko, Japan, Aug. 18-23, 1991

[45] D.N. Robinson: A Unified Creep-Plasticity Model for Structural Metals at High Temperature, ORNL/TM-5969, Oak Ridge Natl. Lab., 1978

[46] D.N. Robinson, R.W. Swindeman: Unified Creep - Plasticity Constitutive Equations for 2-$\frac{1}{4}$ Cr-1 Mo Steel at Elevated Temperature, ORNL/TM-8444, Oak Ridge Natl. Lab., 1982

[47] D.N. Robinson, P.A. Bartolotta: Viscoplastic Constitutive Relationships with Dependence on Thermomechanical History, NASA Contractor Report 174836, March 1985

[48] J.L. Chaboche: Viscoplastic Constitutive Equations for the Description of Cyclic and Anisotropic Behavior of Metals, Bull. de l'Acad. Polonaise Sci. / Ser. Sci. Tech. 25, pp. 33-39, 42-48, 1977

[49] K.P. Walker: Research and Development Program for Nonlinear Structural Modeling with Advanced Time-Temperature Dependent Constitutive Relationships, Performing Organ. Report PWA-5700-50, NASA CR-165533, 1981

[50] D.H. Allen: A Thermodynamic Framework for Comparison of Current Thermoviscoplastic Constitutive Models for Metals at Elevated Temperature, in: Constitutive Laws of Engineering Materials; Theory and Application, Proc. Int. Conf., Tucson, Ariz., USA, Jan. 10-14, 1983

[51] K.S. Chan, S.R. Bodner, K.P. Walker, U.S. Lindholm: A Survey of of Unified Constitutive Theories, Proc. 2nd Symp. on Nonlin. Const. Relations for High Temp. Appl., Cleveland, Ohio, 1984

[52] E. Krempl, J.J. McMahon, D. Jao: Viscoplasticity based on Overstress with a Differential Growth Law for the Equilibrium Stress, Mechanics of Materials 5, pp. 35-48, 1986

[53] M.B. Ruggles: Experimental Investigations of Rate-Dependent Behavior of AISI Type 304 Stainless Steel at Room and Elevated Temperatures, Thesis, Rensselear Polytechnic Inst., Troy, New York, 1987

[54] J.L. Chaboche: Constitutive Equations for Cyclic Plasticity and Cyclic Viscoplasticity, Int. J. of Plasticity 5, pp. 247-302, 1989

[55] V.G. Rameswamy, D.C. Stouffer, J.H. Laflen: A Unified Constitutive Model for the Inelastic Uniaxial Response of Rene'80 at Temperatures between 538 °C and 982 °C, J. Eng. Mater. and Techn. 112, pp. 280-286, 1990

[56] A.D. Freed, K.P. Walker: Model Development in Viscoplastic Ratchetting, NASA Techn. Mem. 102509, 1990

[57] A.D. Freed, J.L. Chaboche, K.P. Walker: On the Thermodynamics of Stress Rate in the Evolution of Back Stress in Viscoplasticity, NASA Techn. Mem. 103794, 1991

[58] A.D. Freed, J.L. Chaboche, K.P. Walker: A Viscoplastic Theory with Thermodynamic Considerations, Acta Mechanica 90, pp. 155-174, 1991

[59] K.D. Lee, E. Krempl: An Orthotropic Theory of Viscoplasticity Based on Overstress for Thermomechanical Deformations, Int. J. Sol. & Struct. 27, 11, pp. 1445-1459, 1991

[60] O.T. Bruhns, H. Hübel, U. Klein, J. Wordsworth: The Interatom Model, Proc. Int. Conf. on Constitutive Laws for Engineering Materials, Chongqing/China, Aug. 1989

[61] K.S. Chan, U.A. Lindholm: Inelastic Deformation under Non-Isothermal Loading, J. Eng. Mater. and Technol. 112, pp. 15-25, 1990

[62] J. Lubliner: On the Thermodynamic Foundations of Non-Linear Solid Mechanics, Int. J. Non-Lin. Mech. 7, pp. 237-254, 1972

[63] J. Lubliner: On the Structure of the Rate Equations of Materials with Internal Variables, Acta Mech. 17, pp. 109-119, 1973

[64] T. Lehmann: General Frame for the Definition of Constitutive Laws for Large Non-Isothermic Elastic-Plastic and Elastic-Visco-Plastic Deformations, in [65]

[65] T. Lehmann (Ed.): The Constitutive Law in Thermoplasticity, CISM Courses and Lectures No. 281, Springer Verlag, 1984

[66] C. Truesdell, W. Noll: The Nonlinear Field Theories of Mechanics, Encyclopedia of Physics, vol. III/3, Springer Verlag, 1965

[67] A. Duscheck: Vorlesungen über höhere Mathematik, vol. 3, Springer Verlag, Wien, 1953

[68] J. Kestin: A Course in Thermodynamics, vol. 1, McGraw-Hill Book. Comp., 1979

[69] E.A. Guggenheim: Thermodynamics, Classical and Statistical, in: "Principles of Thermodynamics and Statistics", Encyclopedia of Physics, vol. III/2, Springer Verlag, Berlin, Göttingen, Heidelberg, 1959

[70] R. Courant, D. Hilbert: Methoden der Mathematischen Physik II, Springer Verlag, Berlin, Heidelberg, New York, 1968

[71] R. Zurmühl, S. Falk: Matrizen und ihre Anwendungen, Part 1, Springer Verlag, 1984

[72] A.K. Miller (Ed.): Unified Constitutive Equations for Creep and Plasticity, Elsevier Applied Sc., London, New York, 1987

[73] B. Ilschner: Hochtemperatur-Plastizität, Reine und angewandte Metallkunde in Einzeldarstellungen, vol. 23, Springer Verlag, Berlin, Heidelberg, New York, 1973

[74] E.C. Aifantis: A Proposal for Continuum with Microstructure, Mech. Res. Comm. 5, pp. 139-245, 1978

[75] F. Bampi, A. Morro, D. Jou: Two Continuum Approaches to a Wavelength-Dependent Description of Heat Conduction, Physica 107A, pp. 393-403, 1981

[76] F. Bampi, A. Morro: Nonequilibrium Thermodynamics: A Hidden Variable Approach, aus [77]

[77] J. Casa-Vázques, D. Jon, G. Lebon: Recent Developments in Nonequilibrium Thermodynamics, Proceedings Barcelona, Spain, 1983, Lecture Notes in Physics No. 199, Springer Verlag, 1984

[78] G.F. Parry: On Internal Variable Models of Phase Transitions, J. of Elasticity 17, pp. 63-70, 1987

[79] C. Truesdell, W. Noll: The Non-Linear Field Theories of Mechanics, Encyclopedia of Physics, vol. III/3, Springer Verlag, Berlin, Heidelberg, New York, 1965

[80] A.J.M. Spencer: Theory of Invariants, in: Continuum Physics, A.C. Eringen (Ed.), vol. 1, Academic Press, New York & London, 1971

[81] T. Malmberg: Nichtisotherme Theorie großer Deformationen thermoelastischer Schalen
Teil 1: Auswertung von Energiebilanzbedingungen, KfK 4374
Teil 2: Auswertung von Entropiebilanzbedingungen, KfK 4375
Kernforschungszentrum Karlsruhe, Sept. 1988

[82] A.D. Freed, J.L. Chaboche, K.P. Walker: On the Thermodynamics of Stress Rate in the Evolution of Back Stress in Viscoplasticity, in: Anisotropy and Localization of Plastic Deformation, Proceedings of Plasticity '91: 3rd Int. Symp. on Plasticity and Its Current Applications, J.-P. Boehler, A.S. Khan, Elsevier Appl. Sc., 1991

[83] S. Sieniutycz, P. Salamon (Ed.): Extended Thermodynamic Systems, Advances in Thermodynamics, vol. 7, Taylor & Francis, 1992