Mechanisms for Two Phase Flow in Porous Media

G. Weber

Institut für Angewandte Informatik

Juli 1995

Mechanisms for Two Phase Flow

in Porous Media

G. Weber

Institut für Angewandte Informatik

A B S T R A C T

For a better understanding of transport mechanisms in soil for
a system with two phases of immiscible liquids the physics
of porous media gives again important contributions. In this
report, the considerations mainly concentrate on horizontal
transport. Our approach is based on the similarity solution of
the transport equation which reduces a given nonlinear partial
differential equation (PDE) to an ordinary differential equation
(ODE). It can be seen, how dimensionless similarity solutions
of the ODE depend, in addition to the similarity variable, on
two parameters:
- the capillary number Nc, giving the ratio of capillary forces
  and viscous forces, and
- the ratio of the viscosities of the two liquid phases.
It is shown, under which conditions different mechanisms of
transport are to be expected, such as
- a completely stable displacement or
- an unstable displacement, related to viscous fingering (DLA,
  Diffusion Limited Aggregation) or to capillary fingering
  (IP, Invasion Percolation).
These mechanisms are also strongly dependent on certain critical
exponents (charcteristic for DLA or IP). These relations are
discussed in our report. Again, for some regions of saturation,
mechanisms of displacement are either clearly dominated
- by imbibition (e.g. water pushing oil) or
- by drain (e.g. oil pushing water).
Some of the results are also transformed again from the
similarity solution of the ODE to a solution of the PDE (with
space- and time coordinates). It is seen, that even with this
somewhat simplifed approach, we obtain a considerable spectrum
of mechanisms.

Transportmechanismen für 2-Phasensysteme in Porösen Medien

K U R Z F A S S U N G

Zu einem besseren Verständnis der Transportmechanismen im Boden
für ein System aus zwei Phasen mit nichtmischbaren Flüssigkeiten
ist die Physik Poröser Medien auch hier von großer Bedeutung. Im
vorliegenden Bericht wird insbesondere horizontaler Transport
untersucht. Unser Vorgehen basiert auf einer Ähnlichkeitslösung
der Transportgleichung, wobei eine nichtlineare partielle Diffe-
rentialgleichung (PDG) zu einer gewöhnlichen Differential-
gleichung (DGL) reduziert wird. Es zeigt sich dabei, wie eine
dimensionslose Ähnlichkeitslösung der gewöhnlichen DGL, neben der
Ähnlichkeitsvariablen von zwei Parametern abhängt:
- der Kapillarzahl Nc, die das Verhältnis von kapillaren Kräften
  und viskosen Kräften angibt, und
- dem Verhältnis der Viskositäten der beiden flüssigen Phasen.
Es wird gezeigt, unter welchen Bedingungen verschiedene Transport-
mechanismen zu erwarten sind, nämlich
- eine völlig stabile Verdrängung oder
- eine instabile Verdrängung, die zum viskosen Fingern gehört,
  (DLA, Diffusion Limited Aggregation) oder die zum kapillaren
  Fingern gehört (IP, Invasions-Perkolation).
Diese Mechanismen hängen eng mit bestimmten kritischen Exponenten
zusammen (die für DLA oder IP und Perkolation charakteristisch
sind). Diese Beziehungen werden in unserem Bericht diskutiert.
Weiterhin gibt es in verschiedenen Bereichen der Sättigung
bestimmte Verdrängungsmechanismen, die entweder klar mit
- Imbibition (e.g. Wasser verdrängt Öl) oder
- Drainage (e.g. Öl verdrängt Wasser) zusammenhängen.
Einige unserer Resultate werden auch wieder von der Ähnlichkeits-
lösung der gewöhnlichen DGL zu einer normalen Lösung der PDG
zurücktransformiert (mit Raum- und Zeitkoordinaten). Man sieht
hier, daß bereits mit vereinfachten Modellen ein reiches Spektrum
von Mechanismen auftritt.

MECHANISMS FOR TWO PHASE FLOW IN POROUS MEDIA

CONTENTS

1. Transport of two Phases in a Porous Medium
===

1.1 Introduction

The main topic of this report is a better understanding of
the behavior of a system of two phases in a porous medium.
Since the knowledge of the behavior of such systems (e.g. of
oil and water in soil) is of importance for the ecology in a
country with fast increasing industry, this type of research
gives an important insight.
For the understanding of such a 2-phase system in soil, the
physics of porous media will give important contributions.
An important step to the understanding of such systems is the
scientific research on mechanisms of transport which will
be detailed now.

1.2 Modelling by Physics of Porous Media

By means of some basic concepts of the physics of porous media
a transport equation is obtained (nonlinear partial differential
equation (PDE)), with coefficients which are in general not
constant).
This equation describes the displacement of a liquid (defender)
by another liquid (invader) in a porous medium. It is known,
that frequently modelling by porous media is successful for
analyzing the transport of contaminants in soil.
For our purpose, it is useful, to divide the problem into two
subproblems:

(a) Simple porous media with a 1-dimensional displacement.
(b) More complex porous media (which are closer to reality but
    require a PDE with nonconstant coefficients and boundary
    conditions related to a 2- or 3-dimensional displacement.

We mainly concentrate on subproblrm (a) in this report.

1.3 Basic steps for a 1-dimensional displacement
----

(1) Choice of model

Using the transport equation mentioned for (a) (which is a PDE),
we describe a 1-dimensional transport. It will occur in this
case that the flow is directed in only one direction, without
significant contributions to transport in directions which are
orthogonal to the abszissa x.

(2) Method of solution

Now, the mathematical model (the PDE) will be tranformed to a
nondimensional representation. From this, a suitable similarity
transformation reduces the PDE to an ODE (ordinary differential
equation). As can be seen, this "similarity solution" will show
the main physical properties of the 1-dimensional displacement.

1.4 Interpretation of the solution

Having this similarity solution, already certain criteria can be
obtained, which closely can be related to regions of parameters
where certain mechanisms occur, i.e. mechanisms of displacement
in the porous medium. In our similarity solution, we have still
two parameters:
- the (reduced) ratio of viscosities
- the (reduced) ratio of viscous forces and capillary forces:
  the capillary number.

Criteria

Certain criteria regarding these parameters show, whether we
have
(2a) a completely stable displacemant,
(2b) a displacement which can be characterized as Invasion
     Percolation,
(2c) a displacement which can be characterized as Diffusion
     Limited Aggregation (DLA).

Note this:

Case (2a) produces a stable type of behavior.
Case (2b) produces an instable type of behavior, Invasion
Percolation, (having relations to percolation, but differences
from usual percolation).
Case (2c) will have a different type of istability (DLA) (also
having relations to fingering).
It has to be noted, that case (2a) is completely stable, but
cases (2b), (2c) are also related to some methods known from
the field of phase transitions (critical states, renormalization,
fixed scale transformation).

1.5 Structure of the report

We start in sect. 2 with some considerations of a Hele-Shaw Cell,
also introducing the capillary number Nc (i.e. ratio of the
capillary forces to viscous forces). Then different mechanisms for
transport are considered in sect. 3. In sect. 4 a PDE (based
on continuity equation and Darcy-law) is considered, using a
nondimensional representation and a similarity solution (reduced
saturation is obtained. Then in sect. 5 and 6 different characte-
ristics for transport are given. In sect. 7 from the similarity
solution a transformation from a similarity solution to a usual
solution (saturation) is obtained. Finally, in sect. 8, some
different versions of an ODE (also with the same type of similarity
solution) are discussed. In the Appendices, some problems are
discussed, such as more details on the similarity solution, and a
discussion of transitions between different mechanisms, with a
representation of parameters (such as permeability) by power laws,
and discussion of some computer programs used for this research.
Note that the main steps of this research are shown in Fig. 1-1.

```
+-------------------------------------------+
|         2 phase system in soil            |
|         =======================           |
| Displacement of a liquid (defender)       |
| by a different liquid     (invader)       |
+-------------------------------------------+
                    |
+-------------------------------------------+
|            Porous Medium                  |        (4.2-4.3)
|            =============                  |
| Model of the displacement by a            |
| Partial Differential Equation             |
+-------------------------------------------+
                    |
        +----------------------+
        |    dimensionless     |
        |    representation    |                (4.4)
        |       of PDE         |
        +----------------------+
                    |
        +----------------------+
        |    similarity        |
        |    transformation    |                (4-5)
        |    PDE ---> ODE       |
        +----------------------+
                    |                  (4.6)
        +----------------------+      +-------------------+
        |    similarity        |      | µ1/µ2 and         |
        |    solution          | <- - - - | Nc            |
        |    (ODE, formula)    |      | as parameters     |
        +----------------------+      +-------------------+
                    |            different exponents m,n
                    |                 (power laws)
        +-----------+-------------------+
        |           |                   |
+--------------+ +-------------+ +-------------+
| m=n=2        | | m=n > 1     | | m=n < 1     |
| Riccati-     | |             | |             |
| equation     | | sect.       | | sect.       |
| sect.4,5,6   | | 8.2         | | 8.3         |
+--------------+ +-------------+ +-------------+
        |
+--------------+
| inversion from|
| similarity    |
| aolution to   |
| normal        |
| solution (S(x,t)|
| sect. 7       |
+--------------+
```

Fig. 1-1 Main issues of the method for two phase systems

1.6 A few remarks on application

It seems quite clear from the various considerations in this
report, that the method can be applied for various systems of
two phases. Let us note two things: From experiments and from
the physics of porous media it follows, that for mechanisms
of displacement (such as for oil and water) capillary forces
and viscosities are responsible. Related to some experiments
in the physics of porous media and for different types of soils
a study is possible which can describe the behavior of some
liquids which are likely to give a two phase system in soil.
This understanding of mechanisms will contribute to planning
of ecology and also as a decision help for ecological
engineering.

2. 2-dimensional Flow

It may be noted, that a considerable number of phenomena, also
for two phases can be seen already by a significantly simplified
approach: The consideration of a 2-dimensional flow. For this
- experiments,
- simulations, and
- analytic evaluations
are possible for modelling.

2.1 A Model of the Hele-Shaw-Cell

For modelling a flow in 2 dimensions frequently a Hele-Shaw-Cell
can be used (see Fig. 2-1). It can be seen that this type of cell
is a pair of plain and parallel glass plates, with a narrow gap
between the plates, (e.g. of gap thickness of more or less 1 mm),
which has a considerable width and length (say, e.g. 10 cm), and
an input or output for liquids, which may according to different
requirements be at one end of the cell or in the middle of the
cell.
We assume, that along the top and bottom plate (Fig. 2-2) the
flow velocity v vanishes. With these boundary conditions, we
obtain

$$ v = \frac{3}{2} * \left(1 - \frac{4*z^2}{b^2} \right) * v_{||}(x,y) \tag{2-1} $$

with $v_{||}$ denoting the mean velocity having only components in
the x-y plane. We assume, that

$$ v = 0 \quad \text{for } z = \pm b/2, $$

i.e. at the top and bottom plate.
For a flow which has only negligible changes of $|v|$ in time,
$(d|v|/dt) = 0$, the general Navier-Stokes equation reduces to
the following expressions (2-2):

$$ \mu * \nabla^2 v = \text{grad } p \tag{2-2} $$

Substituting equ. (2-2) into (2-1) we obtain [Lamb]

$$ v_{||} = - \frac{b^2}{12*\mu} * \text{grad } p \tag{2-3a} $$

It should be noted, that (2-3a) is for the case of the Hele-Shaw model equivalent to Darcy's law In this case, Darcy's law is not only a phenomenological law, but could be derived from the basic equation of hydromechanics (2-2a) and a few assumptions which are reasonable. When we replace (b^2/12) by the permeability K we obtain the usual form of Darcy's law, which is definitely valid for a porous medium:

$$v \quad = - \; \frac{K}{\mu} \; *grad \; p \qquad\qquad (2\text{-}3b)$$

Now it is useful to introduce a dimensionless formalism, which will be closely related to a number of characteristic quantities as used for transport phenomena with two phases in porous media and for transport in soils.

2.2 Dimensionless quantities

We consider the two phase flow in a Hele-Shaw cell with a constant average velocity. We have a displacement of a viscous fluid by a non-viscous fluid (or a fluid with low viscosity), e.g. air. Inside the viscous fluid clearly Darcy's law holds. Thus the flow will be uniform, at least when sufficiently distant from the interface (between displacing fluid and displaced fluid). If the viscous fluid is incompressible, the simply have

$$\nabla^2 \; p = 0 \qquad\qquad (2\text{-}4)$$

In the x-direction we have

$$p \quad ---> \quad - \; \frac{12*\mu}{b^2} \; v_\infty \; *x \quad (for \; x \; ---> \; \infty) \qquad (2\text{-}5)$$

However, at the interface, conservation of fluids means that the phase boundary moves according to

$$n \; * \; \nabla \; p \quad = \quad \frac{12*\mu}{b^2} \; * \; vn \qquad\qquad (2\text{-}6)$$

where vn is the normal velocity of the interface. Moreover at the impermeable sidewalls of the cell,

$$y = \pm \; a \quad we \; have \quad \frac{\partial p}{\partial y} = 0 \; .$$

It is also important to note, that the pressure p is related to the surface tension and to the curvature as follows:
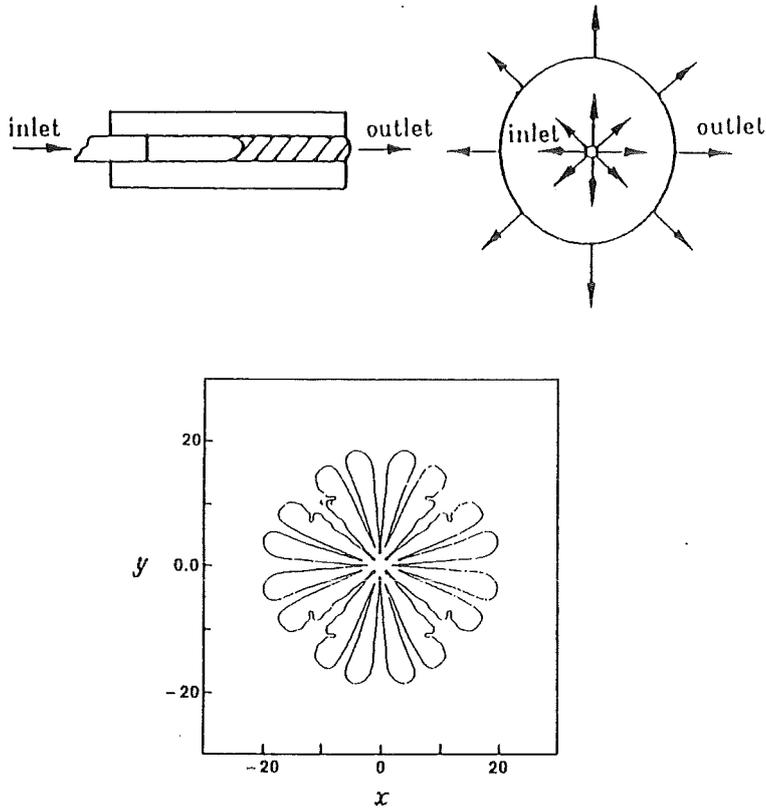
Fig. 2-1 Hele-Shaw-cell (schematic)

(a) inlet (left), outlet (right)
(b) inlet (center), outlet (outside)
(c) fingering in a Hele-Shaw-cell

$$p = p_{air} + \gamma * K \tag{2-7}$$

The following variables are dimensionless:

ϕ is called reduced pressure p,

$\overline{\gamma}$ is called reduced surface tension γ^2 .

We introduce these DIMENSIONLESS VARIABLES as follows:
=========================

$$\phi = - \frac{b^2}{12*\mu*v_{\infty}*a}*p \tag{2-8a}$$

$$\overline{\gamma} = \frac{1}{12*\mu*v_{\infty}}*(b/a)^2 \tag{2-8b}$$

Here
a   represents the half-channel width of the Hele-
          Shaw cell
b   gives the gap between top and bottom plate.
γ is the   surface tension
v_{∞} is the velocity at a place sufficiently distant
          from the interface.

It will be interesting to note the relation of the dimension-
less variable of (2-8b) to the capillary nunber Nc as shown
in section (2.3). Next we introduce a system of equations,
analogous to (2-4), (2-5), (2-6), referring to the uniform
flow (on the basis of the Saffman-Taylor model):

$$\overline{\nabla}^2 \phi = 0 \tag{2-9}$$

$$n*\overline{\nabla} \phi = vn \tag{2-10}$$

$$\phi (xint) = \overline{\gamma} * K \tag{2-11}$$

Here we have as boundary conditions

$$\phi \sim x \qquad \text{for large x}$$

and $$\frac{\partial \phi}{\partial y} = 0 \qquad \text{at } y = \pm 1 \tag{2-12}$$

This system the evolution equations for the Saffman-Taylor
model. Now it can be seen, that the pattern generation in
this model is due to morphological instabilities (which are
also of a certain similarity to instabilities as encountered
for solidification which occurs in dendritic growth (see
[Kessler]).

2.3 Steady state solutions and fingers

We note here that for the Suffman-Taylor model (equ. (2.9), (2-10). (2-11)) some analytic solutions of a stationary type exist. They describe the shape of the interface of a displacing liquid and a displaced liquid (e.g. air and water, or a liquid with low viscosity and a liquid with high viscosity). This is not the place to discuss the methods related to these solutions in detail (but see, e.g. [Kessler]). But let us note that from

$$n*v = n*\nabla \phi \qquad (2-13)$$

which is equivalent to (2-10) of the Suffman-Taylor model we can introduce a new parameter λ, where $1/\lambda$ is equal to the speed. Thus we obtain (in a suitable coordinate system)

$$\frac{y*n}{\lambda} = \frac{1}{\lambda} * \frac{\partial y}{\partial s} \qquad (2-14)$$

where s is the arc length (which is measured in the direction of the flow). Using a second function ψ (which has properties similar to ϕ) we can, using some formalism, obtain a solution which consists of a Fourier series:

$$x = \frac{2}{\pi *(1 - \lambda)} \sum_{n = 0}^{\infty} \frac{(-1)^n}{n} * \cos (n*\pi*y/\lambda) \qquad (2-15a)$$

This can be rewritten as follows:

$$x = \frac{1 - \lambda}{\pi} * \ln \frac{1 + \cos (\pi *y/\lambda)}{2} \qquad (2-15b)$$

A graphic representation of (2-15b) shows the interface between two liquids, which is a "Saffman-Taylor-finger" (See Fig. 2-2). It is important to note that comparing the shape of (2-15b) with an experimental finger (as shown in Fig. 2-3) produced by air displacing water (see [Kessler]) it can be seen that the two interfaces correspond quite closely.

Note: In some different cases solutions are only available by simulation or by rather involved analytical methods. The Saffman-Taylor model will be sufficient to give some insight to our problems.
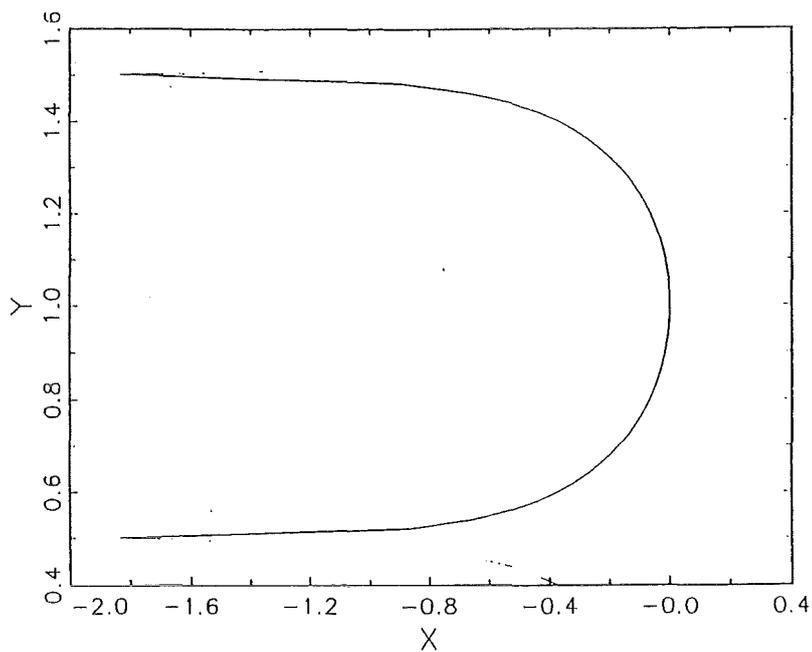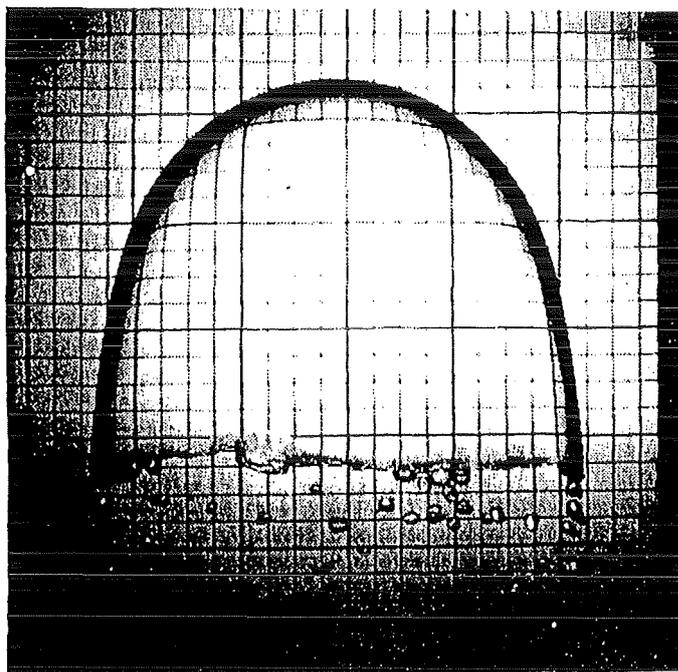
Fig. 2-2   Saffman-Taylor finger model

Fig. 2-3   Finger: Air displacing water

2.4 Capillary Number

Now the capillary number Nc will be introduced, which is one of
the most interesting characteristics for physics of porous media.
The capillary number is also widely used to describe the transport
of fluids in soils.
We consider here a disordered porous medium containing two viscous,
immiscible, incompressible Newtonian liquids which are in contact.
For instance, when a liquid with lower viscosity displaces a liquid
with higher viscosity (such as water displacing oil) it may be very
interesting to know which of two types of forces
- capillary forces or
- viscous forces
is stronger. As will be seen in this section, also for Hele-Shaw
cells a capillary number can be defined.
We assume, that the porous medium under consideration is two -
dimensional, and can be schematically modeled as shown in Fig. 2-5.
A porous medium can e.g. be characterized by two quantities;

R   pore radius and
r   typical radius of a throat,

where usually r << R holds (throats are usually smaller than
pores). Now we introduce on the basis of this model and a few
formulas from hydrodynamics these chararcteristics:

capillary pressure Pcap

$$\Delta \text{Pcap} \sim \frac{\gamma}{r} \qquad (2\text{-}16)$$

where

γ surface tension,
r   throat radius.

Based on the law of Hagen-Poiseuill for stationary flow we
obtain as viscous pressure drop Δ Pvisc

$$\Delta \text{Pvisc} \sim \frac{\mu * Q * R}{r^4} \qquad (2\text{-}17)$$

where

μ viscosity
Q   flow rate
R   radius of pore
r   throat radius.

It is possible to obtain the ratio Pcap/Pvisc to replace one
of the quantities, Q (flow rate) by introducing

    n1 = average number of pores/throat combinations per unit
        area

(e.g. n1 = 2).
We can write for the average velocity (Darcy-velocity)

$$v \sim n1*Q$$

and for the porosity Θ related to R:

$$\Theta \sim n1*R^2 .$$

Thus we can write for the ratio of viscous pressure and
capillary pressure:

$$\frac{\Delta \text{Pvisc}}{\Delta \text{Pcap}} \sim \frac{\mu*v}{\gamma} * \left(\frac{R}{r} \right)^3 * \frac{1}{\Theta} \qquad (2\text{-}18)$$

We can define the CAPILLARY NUMBER Nc which is clearly
related to the ratio of viscous pressure and capillary pressure:

$$\boxed{Nc = \frac{\mu*v}{\gamma}} \qquad (2\text{-}19)$$

We call also cg the geometry related constant:

$$cg = \quad * \left(\frac{r}{F} \right)^3$$

Thus we can write for Nc

$$Nc \sim cg* \frac{\Delta \text{Pvisc}}{\Delta \text{Pcap}} \qquad (2\text{-}20)$$

We can compare the capillary number Nc with the constant $1/\tilde{\gamma}$
as introduced in equ.(2-8b) for Hele-Shaw model. If we
separate from $1/\tilde{\gamma}$ the factors due to Hele-Shaw geometry we
immediately see that

$$\tilde{\gamma} = \frac{\gamma}{\mu*v} * \frac{(b/a)^2}{12}$$

$$= \frac{}{\mu*v} * CHS \qquad (2\text{-}21)$$

where CHS   refers to Hele-Shaw geometry.

Thus we can relate

$$Nc = \frac{CHS}{\overline{\gamma}}$$ (2-22)

This means (while μ and v   are constant):

Surface tension γ	Capillary Number Nc	$\overline{\gamma}$ - factor
INCREASING	DECREASING	INCREASING
DECREASING	INCREASING	DECREASING

Examples

In Fig.2-4 a number of simulations are shown for a Hele-Shaw
cell,which are related to stepwise decreasing surface tensions.
It can be seen that instability and ramification increases with
decreasing surface tension.

Relation to critical exponents

It is also possible to explain the behavior of the capillary
number as related to Fractal Dimensions and critical exponents,
as useful for percolation, invasion percolation and diffusion
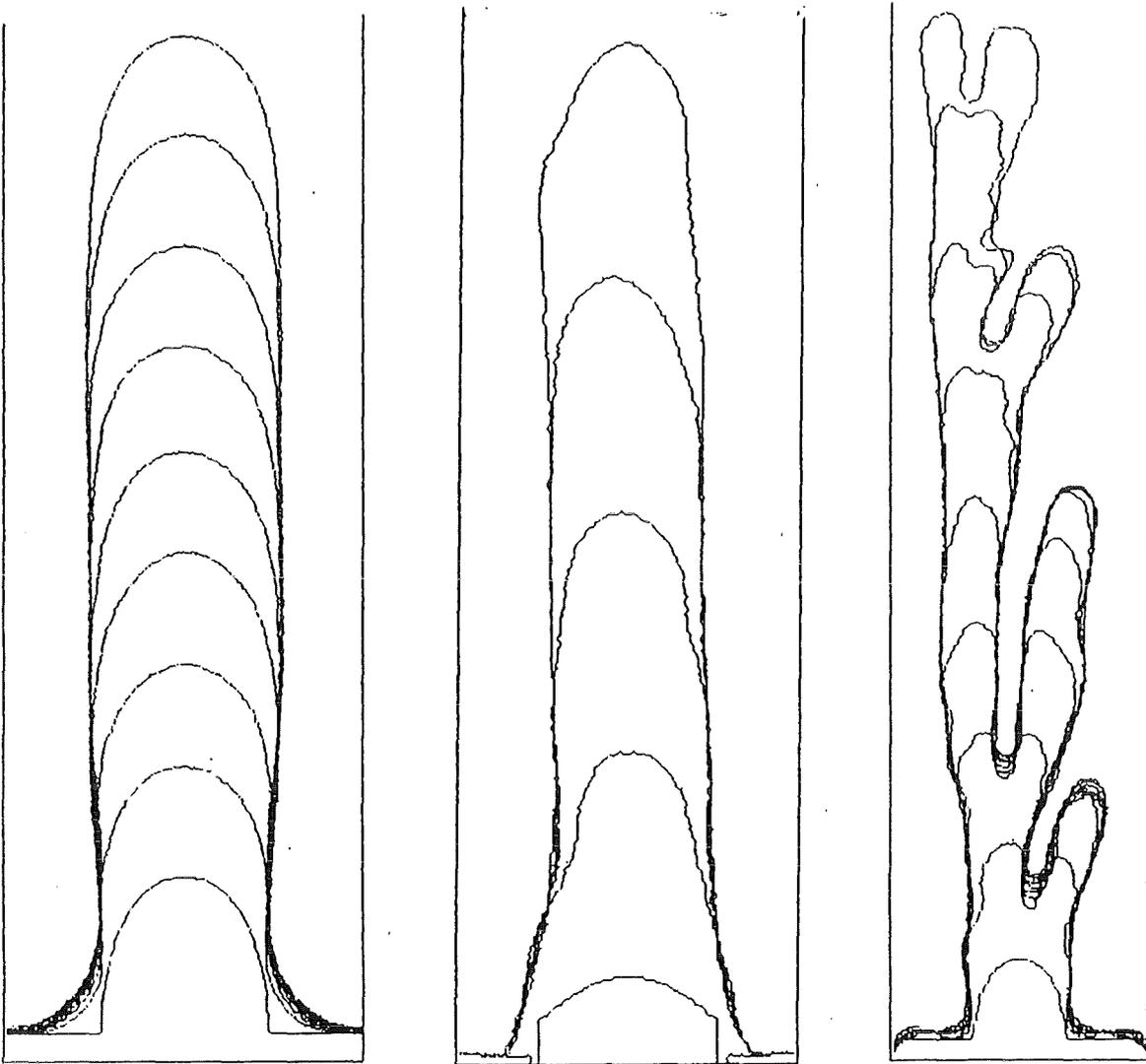limited aggregation (DLA, as dicsussed in Appendix A3 and A4).

Fig. 2-4 Simulations for a Hele-Shaw-cell

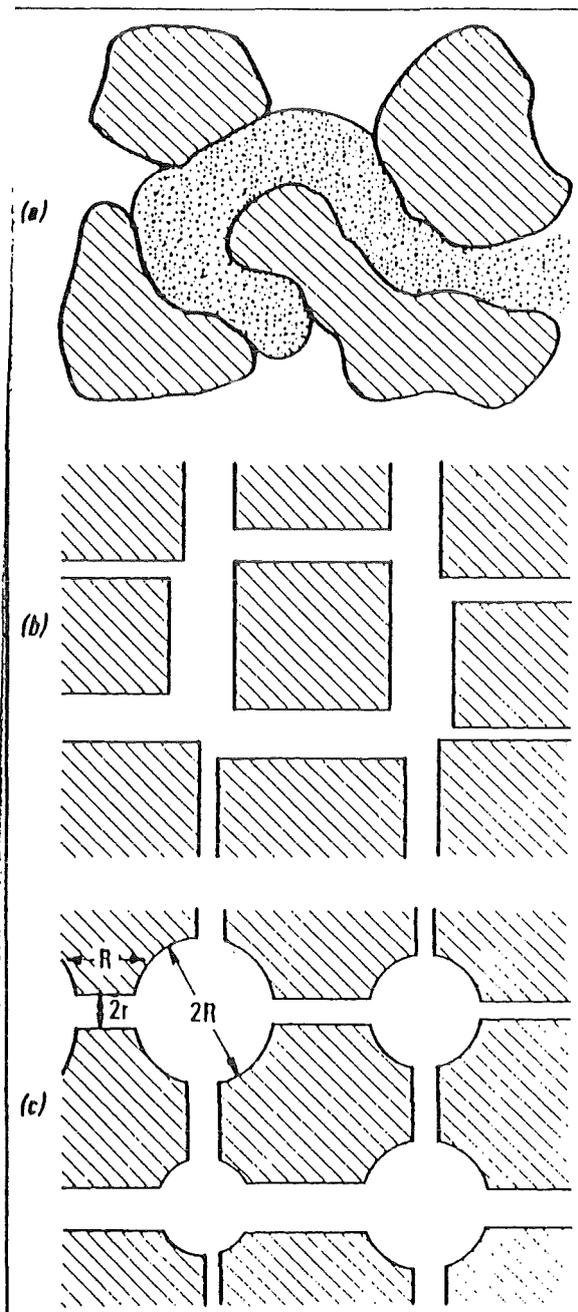(ramification of fingers for
decreasing surface tension)

Fig. 2-5   Modelling of a 2D porous medium

3. Different types of transport

Now this is the place to ask for different mechanisms. These
mechanisms will again be part of the considerations related
to the representation of different types of displacement in
section 4 (Differential equation).

3.1 Criteria for different mechanisms

It is frequently useful to obtain for 2-phase systems (e.g. of
of water and oil in porous media) a classification which relates
to
- viscous forces and to
- surface tension.
We obtain this way the following types of systems:

(1) Systems where usually viscous forces are LARGE compared
    to surface tensions. In those systems a viscous displacement
    occurs.

(2) Systems where usually viscous forces are SMALL compared
    to surface tensions. Here different mechanism may occur.
    One of them is Invasion percolation [Weber].

It is also important to know, wheter 2 phases are miscible or
immiscible. (See for more details [Sahimi]).

Viscous displacement

For viscous displacement two types occur.

(1a) Displacement of fluid B by injection of the more viscous
fluid A.
This displacement is stable. Here no strong deviations from
a flat surface will occure. Clearly, deviations due to an
inhomogeneous structure of pores still are possible.

(1b) Displacement of fluid B' by injection of the less viscous
fluid A'.
Instabilities largely depend on
- viscous forces
- the type of the porous medium
- the question, how far the liquids are miscible or not.

Here mechanisms, similar to DLA, but also certain mechanisms
of fingering and the occurrence of strongly splitted fingers
are possible.
It has to be noted, that here we are in a situation with an
clear analogy to the Hele-Shaw cell (sect. 2).

Displacement mainly due to capillary forces

We assume that fluids can be characterized as

- NON WETTING (NW)
- WETTING (W).

Note: Each fluid has a contact angle with the wall.

A fluid for which the contact angle is smallter than 90 degr.,
is called NON WETTING (NW, eg. oil is non-wetting).
A fluid for which the contact angle is larger thant 90 degr.,
is called WETTING. (W, eg. water is wetting).

Several types of Displacement

In this relation we consider several types of displacement:
Drainage, Imbibition.

DRAINAGE:

```
A displacement where a Non Wetting  fluid is pushing the
wetting fluid, is called DRAINAGE.
                  NW ----> W
```

IMBIBITION:

```
A displacement where a Wetting fluid is pushing the
Non Wetting fluid, is called IMBIBITION.
            W ----> NW
```


A table of essential flow mechanisms

Now a table of flow mechanisms is shown. We follow [Lenormand].
[Lenormand]. This table also refers to different types of
porous media, but also types of pores (which allow flow by
film etc.) Some of these types are characterized in figures.

	Type of flow	Large pores	Small pores
DRAINAGE	flow in ducts	invasion percolation	
IMBIBITION	no flow by film	invasion percolation	compact growth
	flow by film	bond percolation	compact cluster growth
VISCOUS FLOW	stable	flat interface between fluids	
	unstable	gradient governed growth (DLA, fingering)	

Table 3-1 Basic mechanisms
─────────

Note on equivalence of gradient governed growth and DLA
──
Recently, it has been shown by Koza, that modelling of a
diffusion limited agregation (DLA) by an algorithm and modelling
of the process of pushing an incompressible Newtonian fluid by
an   inviscid fluid   through a porous medium by an algorithm
leads to two equivalent models [Koza]. Here quantitative rela-
tions could be shown between

- hydrodynamics in random porous media and
- fractal theory of stochastic DLA-aggegrates.

Thus in a field which is close to our considerations of invasion
pecolation an equivalence has been established. This makes it
easier to relate the considerations for DLA, and for Invasion
Percolation.

Next we consider in more detail the conditions how for viscous
flow fingering arises.

3.2 Formation of viscous fingers

Now, a model for the formation of viscous fingers is described, which is useful for our consideration of mechanisms. It shows, under which conditions a viscous finger is likely to grow.

We assume that in a linear Hele-Shaw-cell the Darcy-law holds, see also [Lamb] and [Kessler].

$$v = - \frac{K}{\mu} * grad(p + \rho*g*x) = grad \, \phi \qquad (3-1)$$

Note that

$$\phi = p + \rho*g*x$$

is called velocity potential.

For the Hele-Shaw-cell we have for permeability

$$K = \frac{b^2}{12*\mu} \qquad (3-2)$$

which will also be written as KH (referring to Hele-Shaw-cell). Now we assume that a small perturbation exists giving a wavelike form to the plane interface between the two liquids (eg. for the oil-water interface). This can in our cell be represented as

$$x = a*exp(\, i*n*y + \delta*t \,) \qquad (3-3)$$

with

    a = amplitude
    i = imaginary unity
    n = wave number
    δ = constant.

Note, that the wavelength for this wavelike interface is

$$\lambda = \frac{2*\pi}{n} \qquad (3-4)$$

Now we assume to have two

- immiscible and
- incompressible liquids.

Thus the following equation (Laplace-equation) holds for velocity V and velocity-potential respectively.

$$\text{div } V \quad = \quad \nabla^2 \phi = 0 \qquad\qquad (3-5)$$

For two liquids (1), (2) we obtain, due to continuity of pressure

$$\frac{\partial \phi_1}{\partial x} = \frac{\partial \phi_2}{\partial x} \qquad\qquad (3-6)$$

or, explicitly introducing the periodic perturbation of the interface

$$\frac{\partial \phi_1}{\partial x} = V + a * \delta * \exp(i*n*y + \delta*t) \qquad\qquad (3-7)$$

where
the first term of the r.h.s. refers to velocity of the interface, and the second term of the r.h.s. refers to the perturbation.

Due to the fact that one liquid is on one side of the perturbed interface while the other liquid is on the other side of the interface, we can write as follows:

$$\phi_1 = V*x - (a*\delta/n)*\exp(i*n*y - n*x + \delta*t) \qquad (3-8a)$$

$$\phi_2 = V*x + (a*\delta/n)*\exp(i*n*y + n*x + \delta*t) \qquad (3-8b)$$

These are solutions of (3-6) for ϕ_1, ϕ_2.

Now we note, that the pressure in liquid 1 is

$$p1 = - (\mu1/KH)*\phi_1 - \rho1*g*x \qquad\qquad (3-9a)$$

 This formula also holds for liquid 2:

$$p2 = - (\mu2/KH)*\phi_2 - \rho2*g*x \qquad\qquad (3-9b)$$

Now we assume for the moment to be in a region where surface tension can be neglected. Thus we can relate the parameters (from (3-8), as follows:

$$\frac{\delta}{n} \left(\frac{\mu1 + \mu2}{KH} \right) = (\rho1 - \rho2)*g + \left(\frac{\mu1 - \mu2}{KH} \right)*V \qquad (3-10)$$

It can be seen, that if the r.h.s. of (3-10) is positive, then also the l.h.s. is positive. From (3-3) we know, that only for $\delta > 0$ a small perturbation can increase, but for $\delta < 0$ it will vanish. This leads to some conditions for increasing perturbations, that is for growing fingers.

First we note this:

(a) We regard, for simplicity, to have a Hele-Shaw-cell which allows only horizontal movements, i.e. the $(\rho_1 - \rho_2)*g$ term vanishes.

(b) We assume, that for a horizontal movement viscous growth always appears when the less viscous fluid (2) is injected to displace the more viscous fluid (2, defender), i.e. when $\mu_1 > \mu_2$.

(c) In addition to the pressure (3-9) we also add a term for capillary pressure pc.

Capillary pressure

We obtain the total pressure pt with

$$pt = p + \gamma * K \qquad (3-11)$$

with
γ = surface tension at interface,
K = curvature of interface

From (3-3) we obtain an upper limit for the curvature as follows:

$$x'' > \frac{x''}{(1 + (x')^2)^{3/2}} = K \qquad (3-12)$$

For x'' we can write

$$x'' = n^2 = \frac{(2*\pi)^2}{\lambda^2} \qquad (3-13)$$

With (3-10) and (3-13) we obtain

$$\frac{\delta * \lambda}{2*\pi} \left(\frac{\mu_1 + \mu_2}{KH} \right) = \left(\frac{\mu_1 - \mu_2}{KH} \right)*V - \frac{4*\pi^2}{\lambda^2} * \gamma \qquad (3-14)$$

Now we can state that $\delta > 0$ only if

$$0 < \left(\frac{\mu_1 - \mu_2}{KH} \right)*V + \frac{4*\pi^2}{\lambda^2} * \gamma \qquad (3-15)$$

Solving for λ, we obtain a critical value of λ, which is useful as a criterion. We recall, that KH = 12/b².

$$\lambda_c = 2*\pi*b*\left(\frac{\gamma}{12*(\mu_1 - \mu_2)*V} \right)^{1/2} \qquad (3-16)$$

The inequality

$$\lambda > \lambda_c \qquad (3-17)$$

is a necessary condition for instability within the horizontal
Hele-Shaw-cell. For $\lambda < \lambda c$, no instability will occur. It can
be seen, that the following quantities can contribute to an
instability (i.e. growth of fingers):

(a) decreasing of b (thickness of cell),
(b) decreasing of γ(surface tension) at interface between liquids,
(c) increasing of $(\mu 1 - \mu 2) > 0$, (difference of viscosities
    $\mu 1$, $\mu 2$)
(d) increasing of velocity V.

Note also this:
If we have a cell which is not horizontal, a more complicated
version of (3-16) would apply. These considerations referring to
instability can only a represent a limited range of phenomena,
see for more details, [Kessler] and for the distinction between
miscible and immiscible liquids [Sahimi].

General formulation of criteria for growth

From the definition of $\tilde{\gamma}$ (which had been used for dimension-
less variables for Hele-Shaw-cells, in sect. 2.1) we recall that

$$\tilde{\gamma} = \frac{\gamma}{12*\mu*v_\infty}*(b/a)^2 \qquad (3-18)$$

Moreover, also the relation

$$\tilde{\gamma} = 1/C$$

holds, where C is the capillary number.

Thus with (3-16) it can be said, that λ_c is a monotonus function
of $\tilde{\gamma}$ and of $1/C$.

This gives the following rules [R1], [R2] :

[R1]

> For DECREASING $\tilde{\gamma}$, we have DECREASING λc, thus an
> INCREASING INSTABILITY, an increasing tendency for
> fingers to grow.

And also,

[R2]

> For INCREASING C (CAPILLARY NUMBER) we have DECREASING λ_c, thus an INCREASING INSTABILITY, an increasing tendency for fingers to grow.

It will be seen in sect. 3.2 that these rules are are equivalent to a part of the considerations for 2D-porous media, and that they are also useful for interpretation of the parameters of the ODE in section 4.

3.3 A phase diagram showing different mechanisms

As can be seen in Table 3-1, we have for 2 phase flow basically
3 mechanisms (sect. 3.1):

- stable flow (also called anti-DLA)
- DLA (also related to fingering)
- Invasion Percolation

There are usually some considerations for different mechanisms:

(1) Experimental techniques, especially in 2 dimensions
    (Hele-Shaw-Cell, sect. 2, see [Lenormand]).

(2) Results from simulation or statistics
    (see [Lenormand]).

(3) Results related to statistical mechanics or renormalization,
    with critical states, percolation, invasion percolation,
    fractals, fractal dimensions ([Lenormand], [Pietronero],
    [Weber]). Some conditions related to invasion percolation in
    2- and 3-dimensions have been discussed applying fractal
    dimensions [Weber].

(4) Some considerations closely related to physical properties,
    such as viscosities and capillary numbers (see sect. 3.2).

(5) Finally also the analytic evaluation of two phase flow in a
    porous medium are shown in sects. 4 and 5 of this report.
    This gives rather interesting relations of flow to certain
    mechanisms.

    Now in Fig. 3-1 (Phase Diagram) conditions for the different
    mechanisms of flow are shown. Thus we obtain criteria, under
    which conditions for

    - capillary number C (given as ln C)   and

    - relation of viscosity ($\mu 2$) of injected fluid and
            viscosity ($\mu 1$) if displaced fluid

    $$M = \frac{\mu 2}{\mu 1}$$

    one of the three mechanisms occurs, and also where transition
    from one of the mechanisms to an other mechanisms is to be
    expected.
    It can also be seen more exactly, where the transitions are
    to be expected (see [Lenormand]). Also the discussion from
    sect. 3.2 gives some details for transitions, i.e. for
    stable ---> unstable.
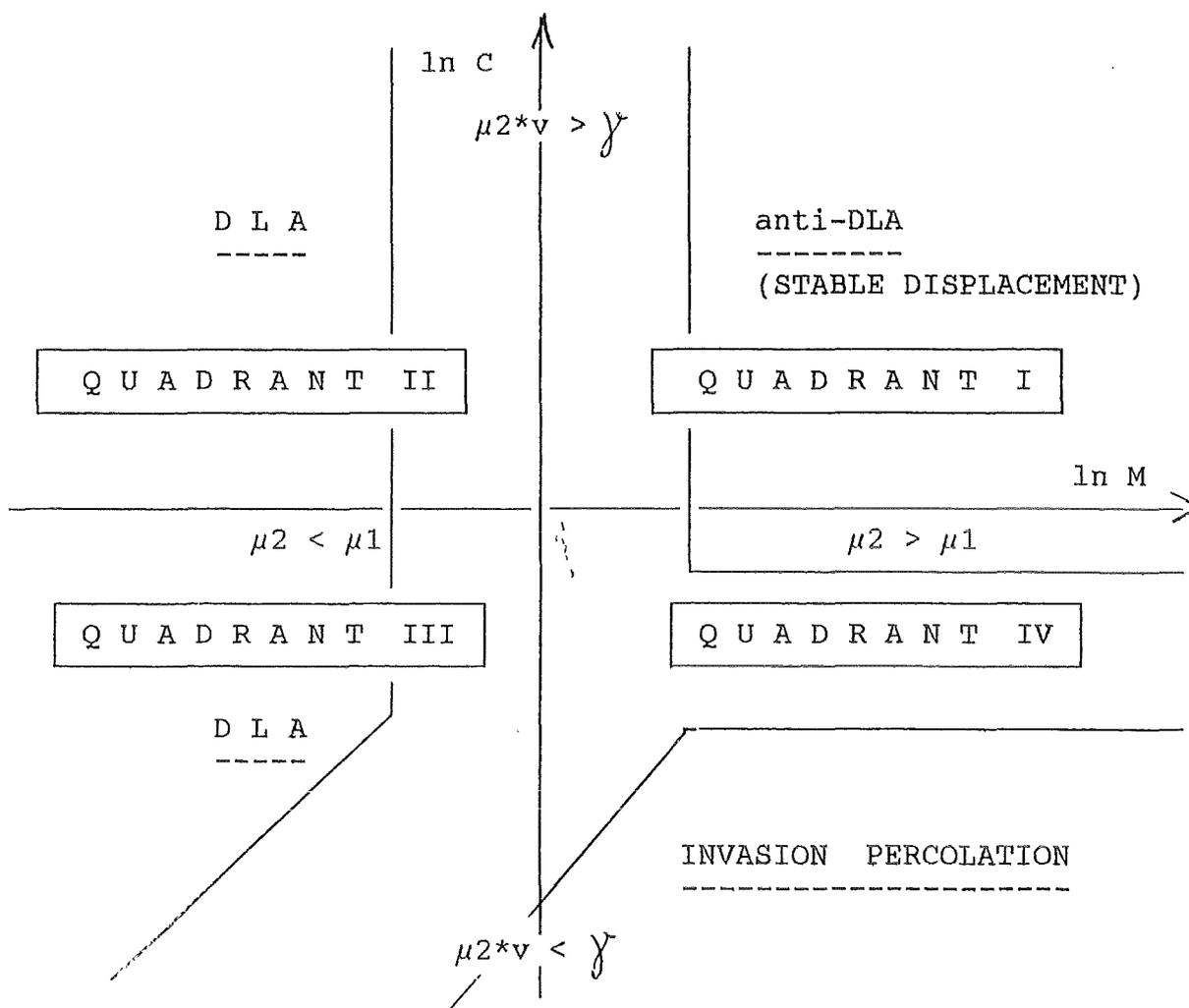
Fig.3-1 Phase-Diagram, showing 3 different mechanisms

For Fig. 3-1 the following notations are used:

$$C = \frac{\mu 2^* v}{\gamma} \quad \text{capillary number}$$

$$M = \frac{\mu 2}{\mu 1} \quad \mu i \text{ viscosity with} \quad \begin{array}{l} i = 2 \quad \text{injected fluid} \\ i = 1 \quad \text{displaced fluid} \end{array}$$

Note: It can be seen that the following MECHANISMS are in

QUADRANTS I, II, III, and IV:

- DLA is in QUADRANT II and III
- anti-DLA is in QUADRANT I and partly in QUADRANT IV
- INVASION PERCOLATION is in QUADRANT III and IV

(a) (b)

(c)

log M = -4   -2   -1   0   2
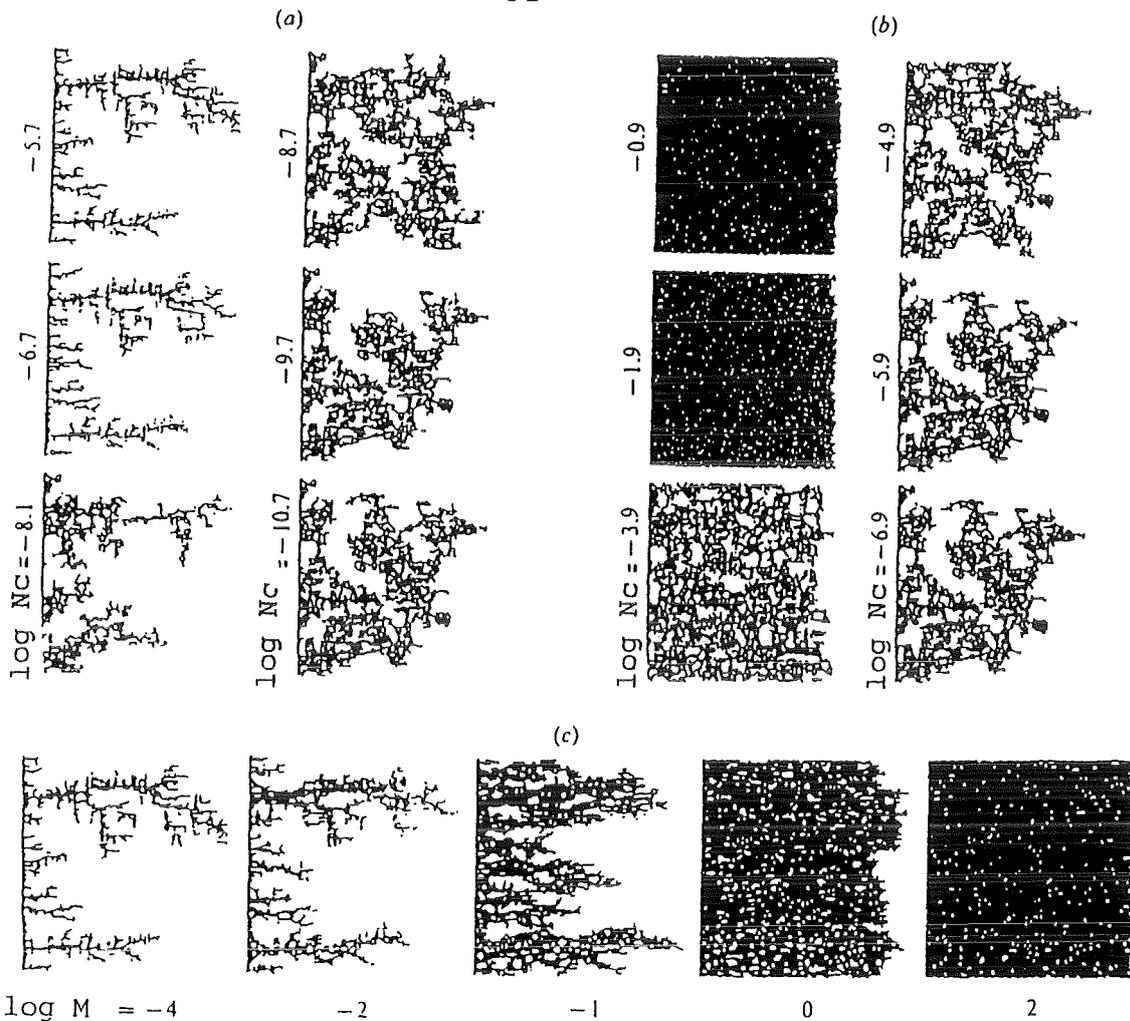
Fig. 3-2   Some transitions (see [Sahimi2])

(a) From viscous to capillary fingering (invasion percolation)

log M = -4.7

log Nc = -8.1   -6.7   -5.7   (viscous fingering)

log Nc = -10.7   -9.7   -8.7   (capillary fingering)

(b) From stable displacement to capillary fingering

log M =   1.9

log Nc = -3.9   -1.9   -0.9   (stable displacement)

log Nc = -6.9   -5.9   -4.9   (capillary fingering)

(c) From viscous fingering to stable displacement

log Nc = 0
                          transition
log M = -4   -2   -1   0   2

(viscosity ratio M = $\mu 2/\mu 1$, capillary number Nc)

4. Evaluation of two phase flow in a porous medium

4.1 Basic Equations

Introduction

Following some considerations mainly referring to multiphase flow phenomena, we now come to an analytical description of multiphase flow phenomena in a porous medium. We frequently will also make comparisons with the concepts of sections 2 and 3, which will be most helpful for an understandig of the mechanisms, eg. to decide where are
- stable flows (comparable to anti-DLA)
- fingering instabilities due to prevalent viscous forces (related to DLA)
- instabilities due to capillary forces (related to invasion percolation)

to be expected.

It is to be noted, that the following analytical solution bases on the similarity transformations of independent and dependent variables of a partial differential equation, which is discussed in Appendix A1. In this section, we are closely following the line of [Pistiner].

Darcy Equations

We first note the generalized Darcy-Equations, relating the flows of two phases, such as water (w) and a fuel (f), eg. oil, which are nonmiscible, qw, qf and the respective pressures, Pw, Pf:

$$qw = - \frac{k*Krw(Sw)}{\mu w} * \frac{\partial Pw}{\partial x} \qquad (4-1)$$

$$qf = - \frac{k*Krf(Sf)}{\mu f} * \frac{\partial Pf}{\partial x} \qquad (4-2)$$

where

- k   permeability of porous medium
- Krw(Sw), Krf(Sf) relative permeabilities
- Sw, Sf are the degrees of saturation of fluids, water and fuel (which can vary between 0 and 1),
- μw, μf is the viscosity of water (w) and a different fuel (f), eg. oil,
- Pw, Pf is the pressure, for the water and the fuel.

It has to be noted, that permeabilities Krw(Sw) and Krf(Sf) in the simplest cases are power-law dependent on Sw, Sf. This will be discussed in more detail in Appendix A4. Here we only note, that for this section, the power law can be approximated by some integer exponents.

Saturation

We note, that when water and liquid completely saturate the porous medium, the fluid saturations satisfy the following relations:

$$Sw + Sf = 1. \qquad (4-3)$$

Capillary pressure

For surfaces between water and fuel the following pressure arises, due to the difference of Pw, Pf:

$$Pc(Sf) = Pf - Pw \qquad (4-4)$$

For Pc the following relation exists:

$$\frac{d\ Pc}{d\ Sf} > 0 \qquad (4-5)$$

This means, that Pc is always a nondecreasing function of Sf.

4.2 Continuity equations

We can write conservation laws for each phase (having no chemical reaction or adsorption and any other mechanism which would require a different form of conservation laws).
We note, that the quantities qw, qf and the quantities Sw, Sf enter these conservation equations (conservation of qw and Sw):

$$\frac{\partial\ qw}{\partial\ x} + \theta * \frac{\partial\ Sw}{\partial\ t} + qwL = 0 \qquad (4-6)$$

$$\frac{\partial\ qf}{\partial\ x} + \theta * \frac{Sf}{t} + qfL = 0 \qquad (4-7)$$

where

- θ is the porosity of the porous medium

- qwL, qfL are contributions from a level of the soil, which is clearly a "less permeable part", having significantly lower permeability than the main section. These contributions can be neglected here.

This way, we have for the simplified consideration of the water-phase.

$$\frac{\partial\ q_w}{\partial\ x} + \theta * \frac{\partial\ S_w}{\partial\ t} = 0 \qquad (4\text{-}8)$$

And similarly also for qf, the fuel-phase.

4.3 Continuity and Darcy-Law

On the basis of saturation and fractional flow we can now
relate continuity and Darcy-Law in an a way quite analogous
to the convection equation for single phase problems.

We first introduce the fractional flow F(Sf), which is the
relation of

$$\frac{\text{flow of fuel}}{\text{flow of fuel + flow of water}} = \frac{\text{flow of fuel}}{\text{total flow}} \qquad (4\text{-}9)$$

With the generalized Darcy-law (4-1), (4-2) we obtain

$$F(Sf) = \frac{\dfrac{k*Krf}{\mu f} * \dfrac{\partial\ P}{\partial\ x}}{\dfrac{k*Krf}{\mu f} * \dfrac{\partial\ P}{\partial\ x} + \dfrac{k*Krw}{\mu w} * \dfrac{\partial\ P}{\partial\ x}} \qquad (4\text{-}10)$$

which is equivalent to

$$F(Sf) = \frac{1}{1 + (\mu w*Krf)/(\mu f*Krw)} \qquad (4\text{-}11)$$

In Appendix A4, equ. (A4-15) it is shown how Krf and Krw depend
on the saturation Sf by a power law. This power law is related
to the normal percolation, as is discussed in the Appendix
A3 and A4.

A quantity similar to the fractional flow F(Sf) is related to
the capillary forces, it is denoted by G(Sf). We have for the
capillary pressure, Pc, and its derivative Pc':

$$Pc'(Sf) = \frac{d\ Pc(Sf)}{d\ Sf} \qquad (4\text{-}12)$$

(See also equ. (4-5), stating that Pc' is positive). G(Sf)
is defined

$$G(Sf) = \frac{k*(Krw/\mu w)*Pc'(Sf)}{1 + (Krw*\mu f)/(Krf*\mu w)} \qquad (4\text{-}13)$$

as can be seen in Appendix A2.
This quantity is required to complete the convection equation
for two phases.

Convection equation

If we introduce a further quantity, the total flow

$$q(t) = qw + qf, \qquad (4-14)$$

we can write the following convection equation:

$$\theta * \frac{\partial Sf}{\partial t} = \frac{\partial}{\partial x}*(G* \frac{\partial Sf}{\partial x}) - q(t)* \frac{\partial F}{\partial x} \qquad (4-15)$$

Here we have

- θ Porosity
- Sf Saturation
- F fractional flow (see equ. (4-11))
- G a quantity related to capillary forces, and fractional
     flow (see (A2-5) and (4-13)),
-     q(t) specific flow.

The specific flow can also be written as a product:

$$q(t) = qo*qe(t) \qquad (4-15)$$

Note: The factor qo will be used for our dimensionless
approach.

Assumptions for Saturation

The fuel saturation consists of two contributions:

- continuous (or flowing) part Sfc
- discontinuous (or stagnant) part Sfd
  (due to pores or paths where no output is possible)

$$Sf = Sfc + Sfd \qquad (4-16)$$

We assume that saturations Sfc and Sfd are in   equilibrium,
thus it is possible to say that Sfd and Sfc are related by
a time-independent factor:

$$Sfd = ck*Sfc \qquad (4-17)$$

As a further simplification we assume here that the factor ck is of the order of 1.

Representation of fractional flow F

We already noted, that Krw, Krf depend on power laws (see Appendix A4). This way it will also be possible to approximate F(Sf) by a power law, writing

$$F(Sfc) = Fo*Sfc^n \qquad (4-18)$$

where
- n is an exponent
- Fo is a factor which depends on the ration ($\mu f/\mu w$) but is independent of Sf (see also section 2 and [Lenormand]).

Representation for Derivative of Pc(Sfc)

Also the derivative of Pc(Sfc) (capillary pressure, eq. (4-12), can be represented by a power law, if we multiply this quantity by the factor Krw(Sfc) :

$$Pc'(S)*Krw(Sfc) = \gamma fw*Pco*Kro*Sfc^{-m} \qquad (4-19)$$

where
- γ fw   fuel-water surface tension
- Pco, Kro are constants (>0), related to the geometry of the porous medium,
- m   is an exponent.

Representation of G-function

With (4-11), (4-12) as well as (4-18), (4-19) we also obtain

$$G(Sfc) = \frac{k*Kro}{\mu w} * Fo*Pco* \gamma fw*Sfc^{n-m} \qquad (4-20)$$

We note, that all the terms used here, are already defined in equations (4-18), (4-19).

Note: Now the subscript fc will be dropped for the respective quantities, we will write S instead of Sfc, since no confusion is possible.

4.4 Dimensionless considerations

For some considerations, e.g. for statistical physics of sedimentary rocks (see [Po-zen Wong]), it is convenient to use quantities as "charcteristic throat size" and "characteristic length". This is not to obtain precision; but note, that some

approaches considerably simplify, if it can be assumed that
one is in a region, where all lengths are large compared to
the characteristic length. Similarly, we also introduce two
characteristic quantities here, to see that we are in an
asymptotic region.

Characteristic length

$$L = k*Kro*Pco \qquad (4-21)$$

where
- k   permeability
- Kro, Pco factors which have been used for (4-19)

Characteristic time

$$T = \frac{L* \theta *(1+ck)}{qo*Fo} \qquad (4-22)$$

where
- L   (see (4-21), characteristic length)
- θ porosity
- ck   proportionality factor between Sfd, Sfc
- qo   proportionality factor (see (4-15a))
- Fo   factor dependent on ($\mu f/\mu w$) (see (4-18)).

Using (4-21), (4-22) we introduce the following dimensionless
quantities.

Dimensionless length

$$\hat{x} = x/L \qquad (4-23a)$$

Dimensionless time

$$\hat{t} = t/T \qquad (4-23b)$$

Dimensionless continuity equation

Now we can write a dimensionless continuity equation. But
first note the following: We assume that the less permeable
part (above and below the considered porous medium) is now
of a permeability which may be neglected. Thus we can write

$$\frac{\partial S}{\partial \hat{t}} + \frac{\partial Is}{\partial \hat{x}} = 0 \qquad (4-24)$$

with

$$Is = qe(\hat{t})*S^{n} - (1/Nc)*S^{n-m} * \frac{\partial S}{\partial \hat{x}} \qquad (4-25)$$

Here, Is has been called fuel flux. It is a function of qe
(flow) and of S (saturation), as well as of Nc (capillary
number). In sect. 6, a thorough discussion of the concept of
fuel flux is given.

Capillary number Nc

Note that Nc is the capillary number Nc:

$$Nc = \frac{\mu w^* qo}{\gamma fw}$$ (4-26)

where
- μw viscosity of water
- fw   surface tension
- qo   a factor related to q

Note: For more details on the capillary number see sect. 2, 3.

With this continuity equation a relation between transport
of fuel and water in a porous medium and limits of fractal
patterns ([Lenormand], [Kessler]) can be established.

Relations of properties to bondary conditions

Solution of (4-24), (4-25) (in the region $-\infty < \hat{x} < +\infty$) gives
Is (fuel flux in a porous medium). We note here that

$$Is((\hat{x}, \hat{t}) \longrightarrow 0 \quad \text{for} \quad |\hat{x}| \longrightarrow \infty$$ (4-27)

For $S(\hat{x}, \hat{t})$ we obtain a dimensionless fuel mass for the porous
medium as follows:

$$M(\hat{t}) = \int_{-\infty}^{+\infty} S(\hat{x}, \hat{t}) \, d\hat{t}$$ (4-28)

We note, that with some changes of the equations (4-27)-(4-28)
used here, also finite but small permeability of the lower
medium can be included in the calculation. We consider only
one level without interaction with upper or lower medium.

4.5 Similarity variables

Now we come to a representation, with transformed variables and
functions. It is possible, by this type of transform to obtain
from a partial differential equation an ordinary differential
equation. This is called "similarity method" and is discussed
in Appendix A1. We note here, that equivalent to equations
(A1-1a) and (A1-7b) we obtain the following transformations;

$$\xi = \hat{x} * \hat{t}^{-1/2}$$ (4-29)

$$Se = f(\xi) * \hat{t}^{-1/2}$$ (4-30)

As can be seen in Appendix A1 and in the literature (see e.g.
[Bluman], [Dresner]) this transformation for the independent
variables (x, t), and the dependent variable (Se), which come
from the continuity equation (4-24) and (A2-8). We obtain this
way f and the following intermediate quantity, g, which are
closely related to the continuityequation, but now on the
"level" of ordinary differential equations.

The corresponding ordinary differential equation

Now, due to a transformation of the continuity equation
the following intermediate quantity g(ξ) (related to fuel
flux) is obtained:

$$g(\xi) = f^n - \frac{1}{Nc}*f'$$
(4-31)

Thus we come to the ordinary differential equation which
corresponds to our continuity equation (24).

$$[g - b*\xi*f]' = 0$$
(4-32)

To this differential equation we introduce boudary conditions
which are clearly related to an asymptotic behavior of g and
flux Is. We have

$$g \longrightarrow 0 \quad for \quad |\xi| \longrightarrow \infty$$
(4-33a)

and also with (27)

$$f \longrightarrow 0 \quad for \quad |\xi| \longrightarrow \infty$$
(4-33b)

With these conditions the fuel flux Is can be rewritten as
in relation to g

$$Is = g(\xi)*\hat{t}^{-1}$$
(4-34)

An integration of (4-32) gives this:

$$g - b*\xi*f = C1$$
(4-35)

Note that C1 is an integration constant. Due to the
boundary conditions (4-33), (4-34) this will be

$$C1 = 0$$
(4-36)

With (4-31) and (4-32) we will have the following ordinary
differential equation for the transformed variable f, related
to saturation.

$$f' = Nc*(f^n - (1/2)*f*\xi)$$
(4-37)

It may be noted, that for

n = 2 this is the Riccati-equation (see e.g. [Davis]).

But also for other values of n, some analytical solutions
have been obtained (see e.g. [Pistiner]).
We see that in (4-37)
- the function f is defined, the transformed and nondimen-
  sioneal saturation of the fuel
- and that its variable is ξ , the transformed nondimen-
  sional coordinate, related to location.

We also note that Nc, the capillary number enters (4-37) as a
parameter.

4.6 Analytic solution

For the analytic solution we introduce first a dimensionless
reduced mass of the fuel Me

$$Me = \int_{-\infty}^{+\infty} f(\xi)\, d\xi \qquad (4\text{-}38)$$

We could have two ranges of parameter n

$$n > 1$$
$$n < 1 \qquad (4\text{-}39)$$

We deal here mainly with n > 1, and obtain the following
analytic solution:

$$f(\xi) = \frac{\exp(-(1/4)*Nc*\xi^2)}{[\, f^{-1}(0) - (\pi *Nc*(n-1))^{1/2} - erf((1/2)*(Nc)^{1/2}*\xi)\,)} \qquad (4\text{-}40)$$

Note that f(0) depends on the reduced mass Me. Now, for
n = 2, we have a Riccati-equation and the solution is

$$f(\xi) = \frac{\exp(-(1/4)*Nc*\xi^2)}{(\pi *Nc)^{-1/2}*(1/(tanh((1/2)*Nce)-erf((1/2)*(Nc)^{1/2}*\xi))} \qquad (4\text{-}41)$$

Note that the effective capillary number is

$$Nce = Me*Nc \qquad (4\text{-}42)$$

with the reduced fuel mass

$$Me = (1/Nc)*ln\left(\frac{1 + tanh((1/2)*Nce)}{1 - tanh((1/2)*Nce)}\right) \qquad (4\text{-}43)$$

Evaluation of (4-41)

In the following calculations it is possible to work with Mc = 1.
Now equation (4-41) has been used for a GAUSS-computer program
[GAUSS]. It is shown in Appendix A5. All results available from
this similarity solution are given in sect. 5.

4.7 Further results related to similarity transform

Until now, we discussed the following ODE, due to the similarity
transform:

$$f' = Nc*(f^n - (1/2*f* \zeta))$$ (4-37)

In the previous sections (4.5) and (4.6) we specialized on the
case n = 2, thus obtaining a Riccati-equation. In can be seen,
that this equation was obtained by using for (4-25)

$$Is = qe(t)*\hat{S}^n - (1/Nc)*S^{n-m} * \frac{\partial S}{\partial \hat{x}}$$ (4-25)

the following assumptions:

$$n - m = 0$$

$$n = 2$$ (4-44)

By a suitable generalization, now assuming

$$n - m < 0$$

or (4-45)

$$m + m > 0 ,$$

we obtain the following differential equation:

$$f' = Nc(f^n - b*\zeta *f^{(1-m+n)})$$ (4-46)

Equation (4-46) gives, due to different values for the exponents
m, n, further similarity solutions. The most interesting of
these solutions are discussed in sect. 8.

5. A first characteristic for transport

5.1 Introduction

It is now possible, following sect. 4, to show characteristics
for the transport in porous media, based on equation (4-41):

$$f(\xi) = \frac{\exp(-(1/4)*Nc*\xi^2)}{(\pi*Nc)^{1/2}*(1/\tanh((1/2)*Nce)-\mathrm{erf}((1/2)*(Nc)^{1/2}*\xi))} \qquad (4\text{-}41)$$

Here f(;Nc,Me) is the reduced saturation (see (4-30), it is
a function of the similarity variable,

$$\xi = \hat{x}*\hat{t}^{-1/2} \qquad (4\text{-}29)$$

and of the two parameters

Nc   capillary number (see (4-28)),
Me   the dimensionless reduced mass (see (4-42)).

We set in this section again Me = 1, thus we have only Nc as a
parameter. If some other values of Me are of interest, this will
be explicitly mentioned.
For the maximum of f(ξ) we can state with (4-37) the following
relation

$$f'(\xi) = Nc*(f^n - (1/2))*f*\xi = 0 \qquad (5\text{-}1)$$

Thus, this maximum is clearly related with Nc. We have only the
parameter Nc left, and get a family of f-curves (similarity
solutions).

5.2 Figures and Tables

Figures

The similarity solution (reduced saturation) f is shown as a
function of the similarity variable (see Fig. 5-1, to 5-6).

Tables

All maxima of the f-curves could be identified by a procedure,
giving the coordinates for fmax: (ξ,f)max. These are listed
in tables: Table 5-1 to 5-4.

LIST OF RESULTS

ranges for Nc	Figures	Tables
0.1 to 1.0	5-1 5-2	Table 5-1
1.0 to 10.0	5-3	
11.0 to 20.0	5-4	Table 5-2
20.0 to 30.5	5-5 5-6	
22.2 to 24.0	− .	Table 5-3
23.4 to 23.58	−	
10.0 to 100.0	−	Table 5-4
100.0 to 1000.0	−	

5.3 Interpretation of results

The results given in evaluations of f(xi;Nc) are shown in
the LIST OF RESULTS. We have here regions showing a different
behavior related to the Nc-value used for the reduced
saturation.

Region A:
We have for capillary numbers Nc, where

    0 < Nc < 23.44

the following behavior:

The maximum of f(xi;Nc) (similarity solution, red. saturation)
is INCREASING WITH Nc, and the xim -coordinate for the
maximum is slightly shifting to the right with increasing Nc.
(See Figs. 5-1 to 5-6, Tables 5-1 and 5-2).

Region B:

We have for capillary numbers Nc,   where

   Nc > 23.44

the following behavior:

The maximum of f(xi;Nc) (similarity solution, red. saturation)
is DECREASING WITH Nc, and the xim-coordinate for the maximum
is shifting to the left with increasing Nc.
(See Fig. 5-6 , Table 5-3 and 5-4).

Location of the highest value for f(xi;Nc) :

In Table 5-3 it can be seen, that for Nc = 23.44 + 0.02 the
location for the highest value for f(xim;Nc) occurs. This
is important to note for the interpretation of the similarity
solution (red. saturation) in relation to possible mechanisms.

5.4 Some 3-dimensional representations for f(xi;Nc)

In Figs. 5-6, 5-7, 5-8, 5-9 the following behavior of f(xi,Nc)
is shown with 3-dimensional representations.

Values for f(xi,Nc):

In Fig. f5-6, 5-7 the evaluation of f(xi,Nc) is shown:
Fig. 5-6, 5-7a with 3-dimensional representation of f(xi,Nc),
Fig. 5-7b Lines of constant f(xi,Nc), projected on xi,Nc-plane.

Partial derivative of f(xi,Nc) with regard to xi

In Fig. 5-8 the partial derivative ∂ f(xi,Nc) / ∂ xi
is shown, giving the slope of f(xi,Nc). Note that the partial
derivative of f(xi,Nc) with regard to xi has been evaluated
by symbolic calcuation using the MAPLE-system, as is given
in Appendix A6.

Partial derivative of f(xi,Nc) with regard to Nc

In Fig. 5-9 the partial derivative ∂ f(xi,Nc)/ ∂ Nc
is shown. It is interesting to note these results to obtain
further insights for the interpretation of the behavior of the
reduced saturation in relation to Nc-values. This partial
derivative is also obtained using the MAPLE-systen (see also
Appendix A6).
Also note the following: For the curves of section 5, a GAUSS
computer program has been used, which is given in Appendix A5.
For information on the program see [GAUSS].
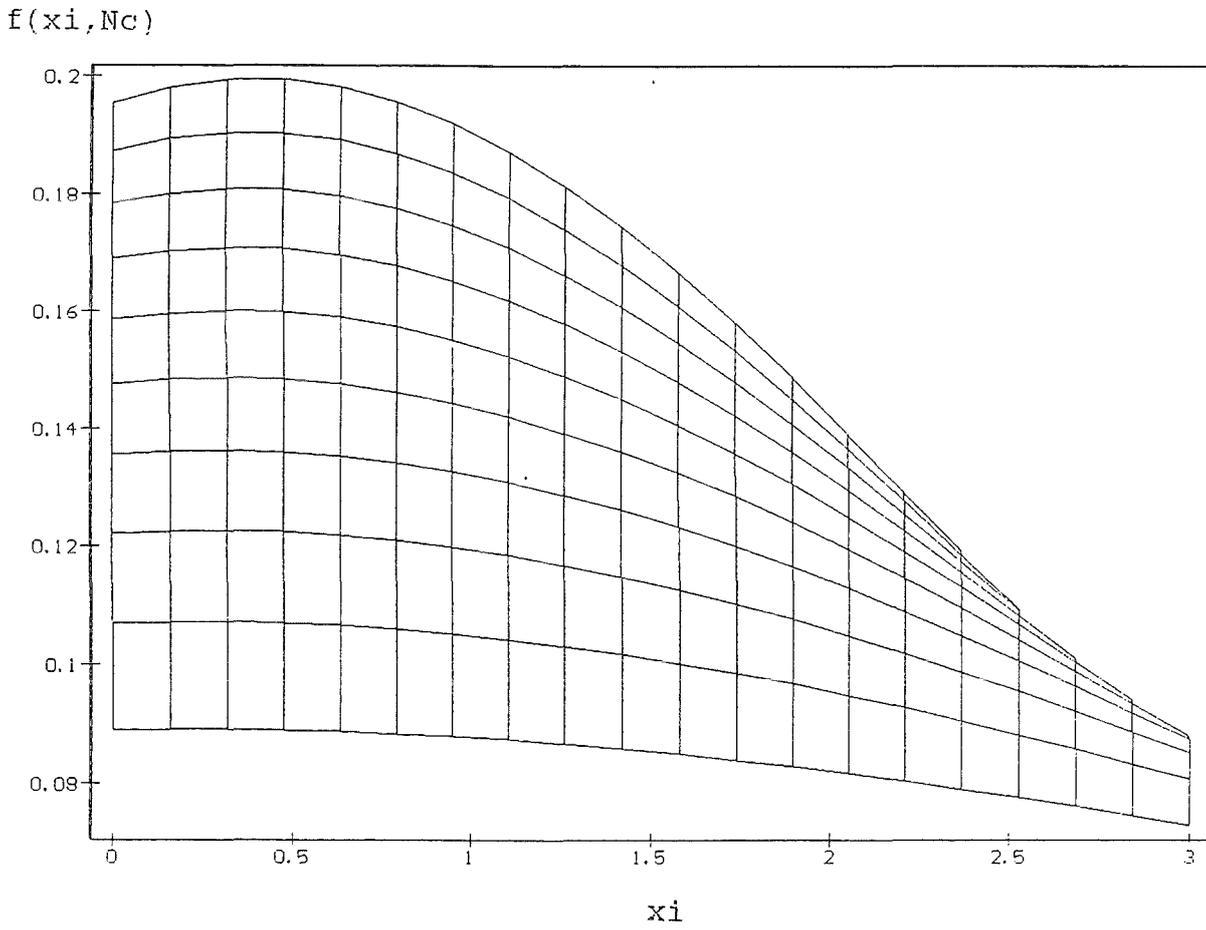
f(xi,Nc)

Fig. 5-1 Representation of f(xi, Nc), Nc = 0.1..0.5
----
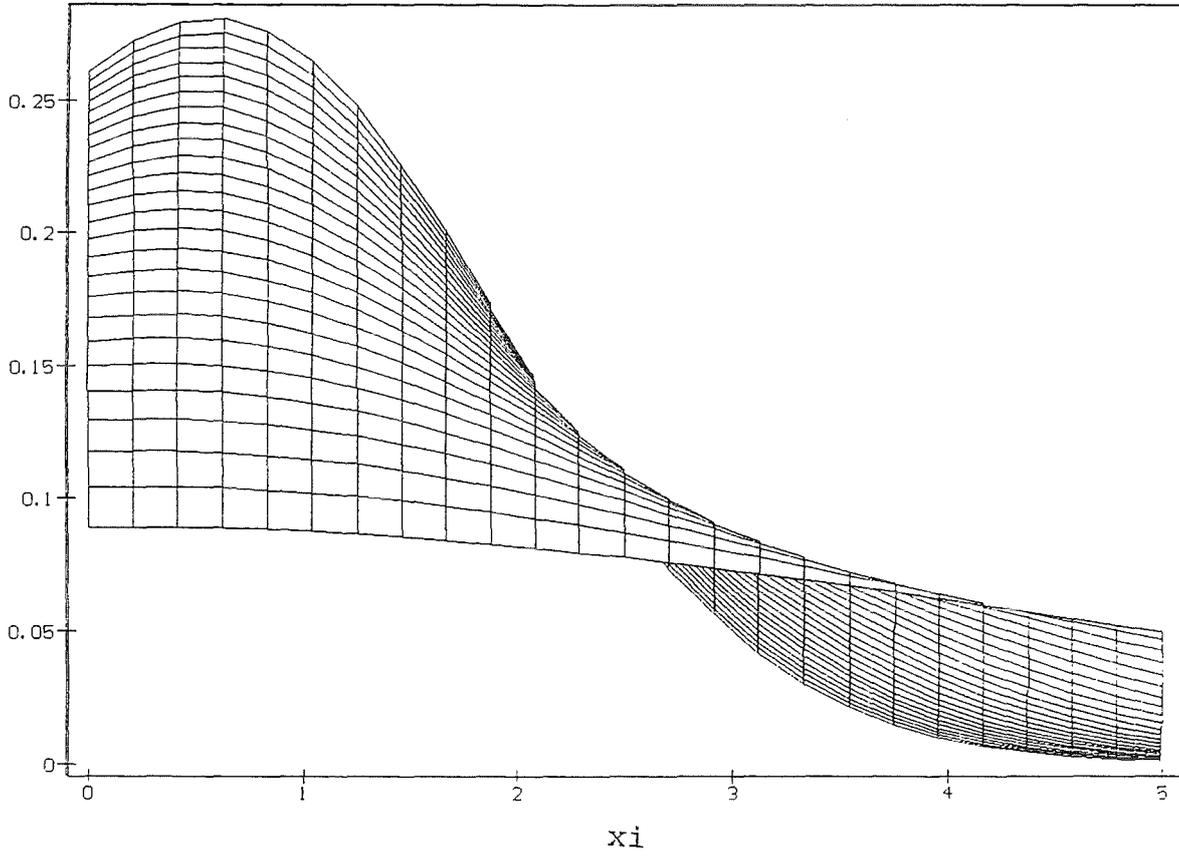
f(xi,Nc)

Fig. 5-2 Representation of f(xi, Nc), Nc = 0.5..1.0
----
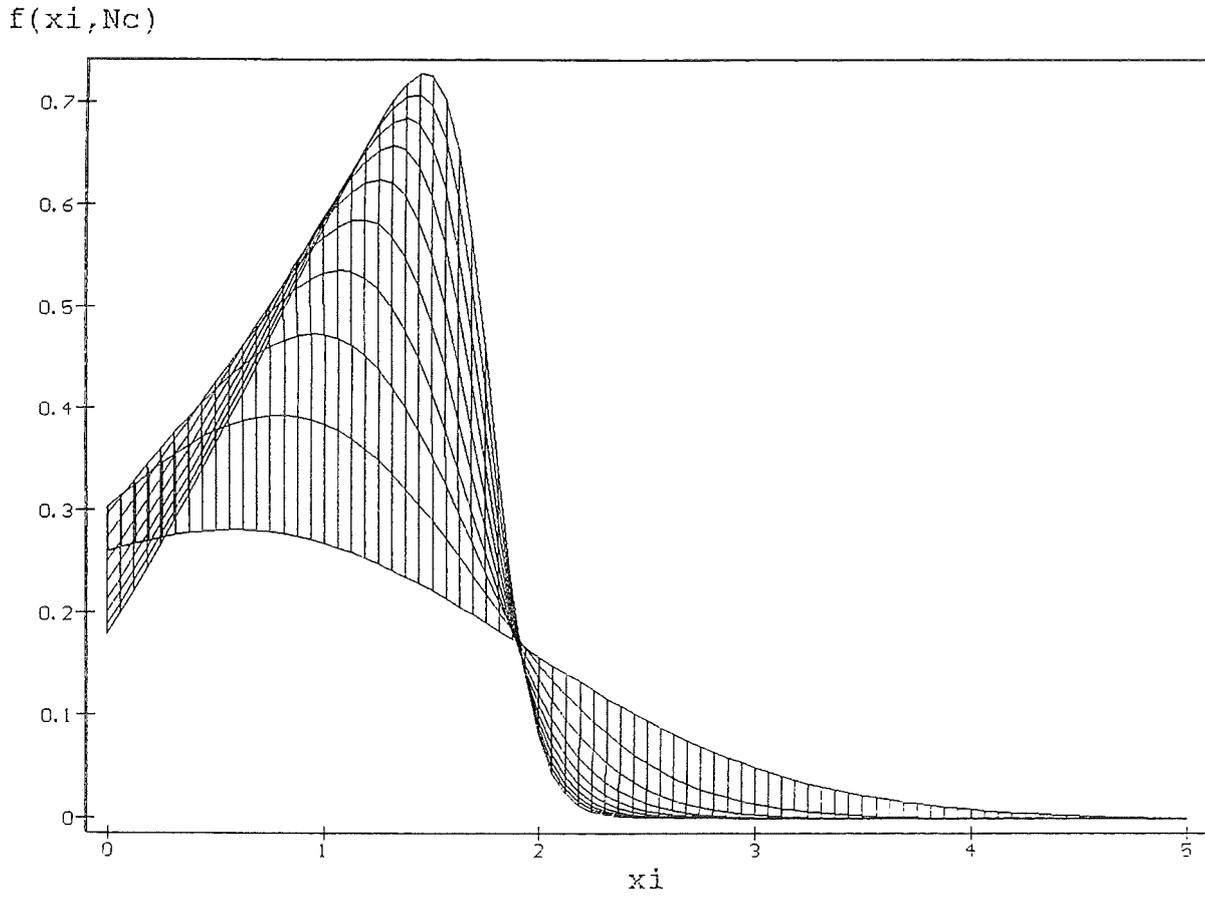
f(xi,Nc)

Fig. 5-3 Representation of f(xi, Nc), Nc = 1.0..10.0
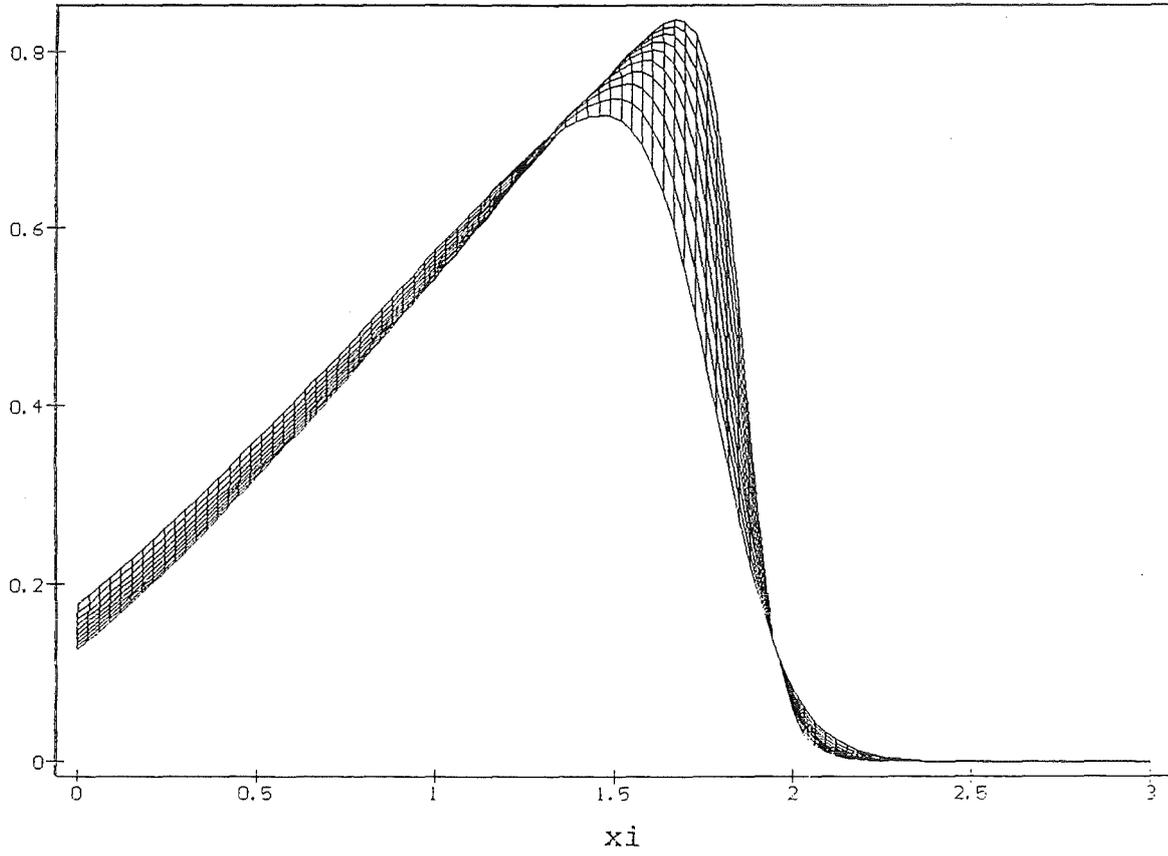
f(xi,Nc)

Fig. 5-4 Representation of f(xi, Nc), Nc = 10.0..20.0
----

f(xi,Nc)
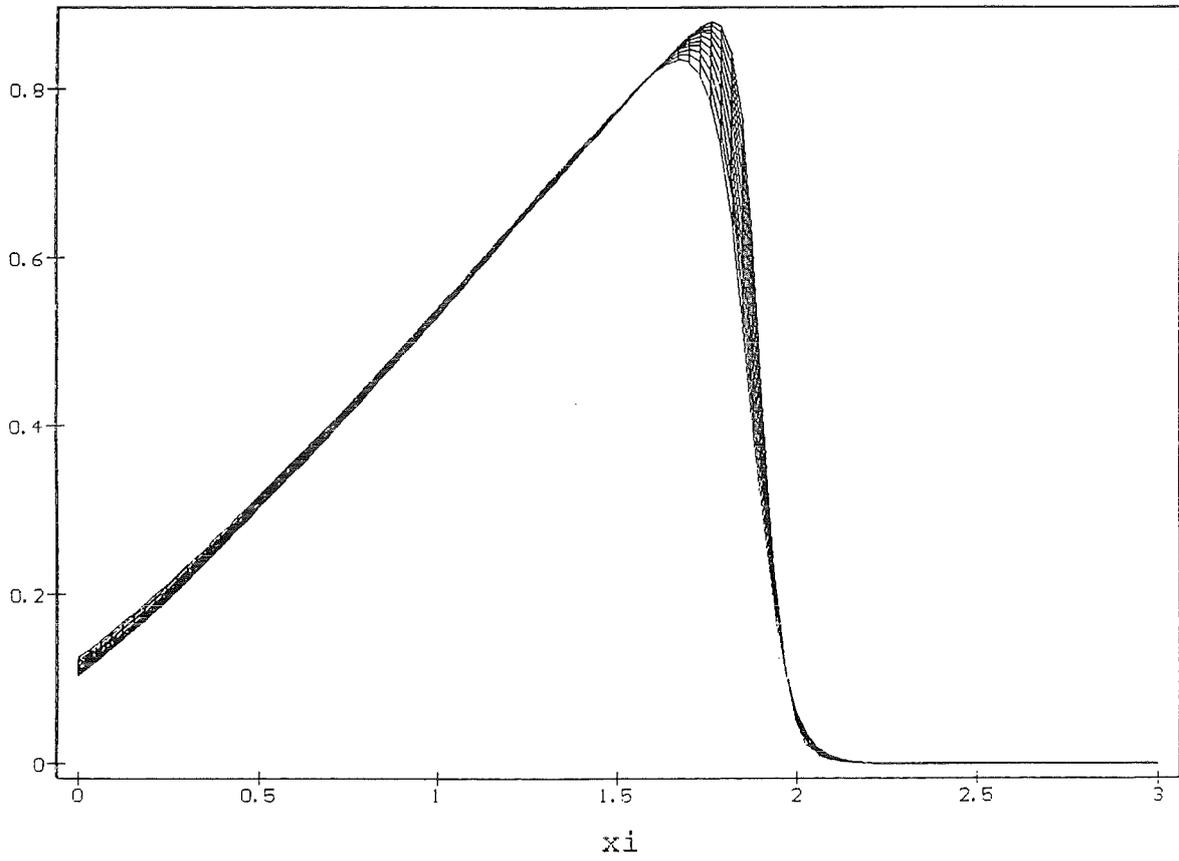
Fig. 5-5 Representation of f(xi, Nc), Nc = 20.0..30.0
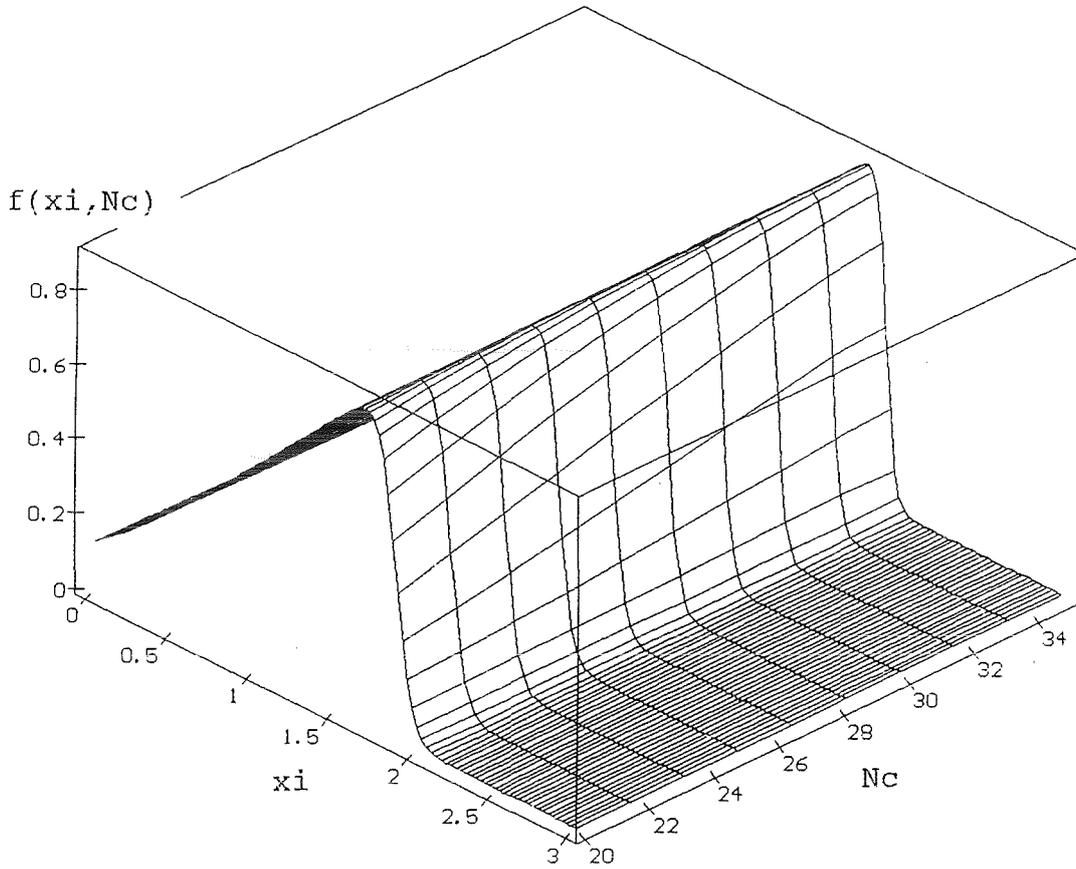
Fig. 5-6 Representation of f(xi, Nc), Nc = 20.0..35.0

        (3-D-representation)
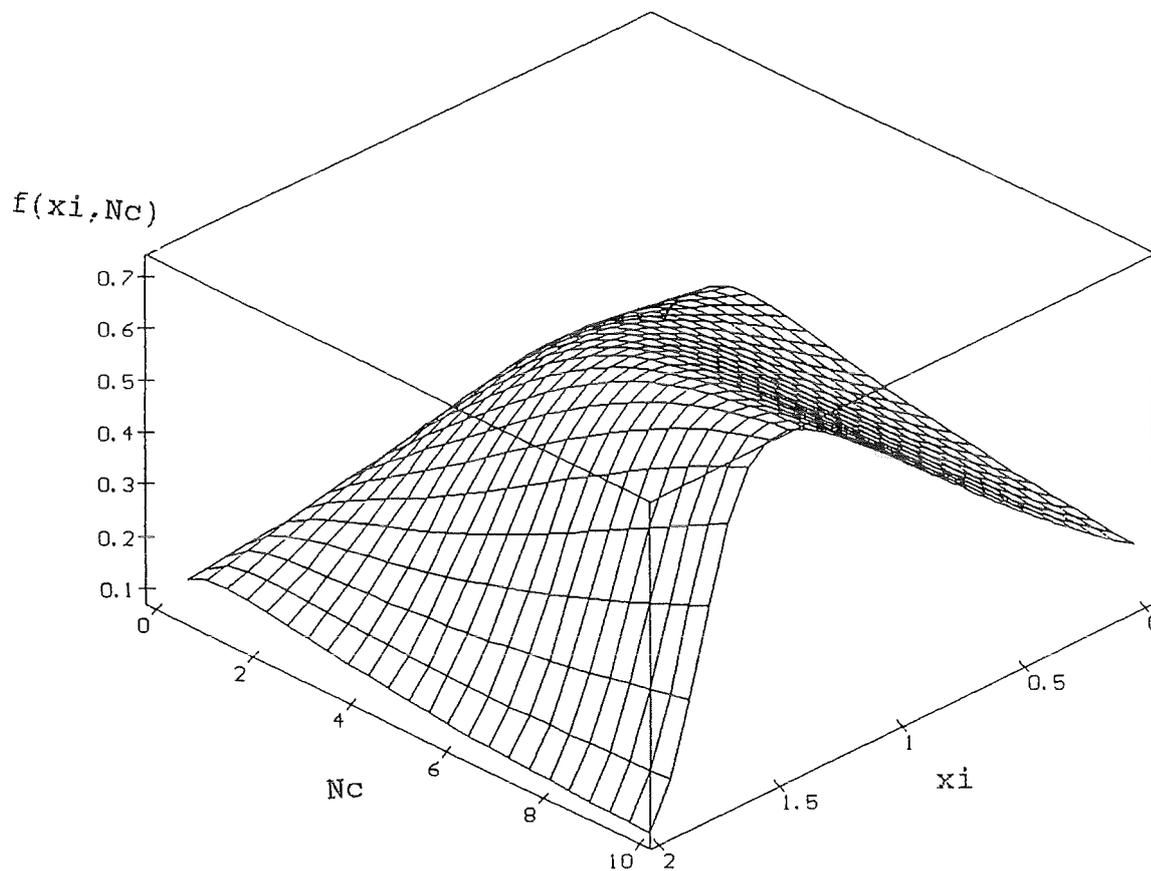
Fig. 5-7a   Representation of f(xi, Nc)   (3-Dim)
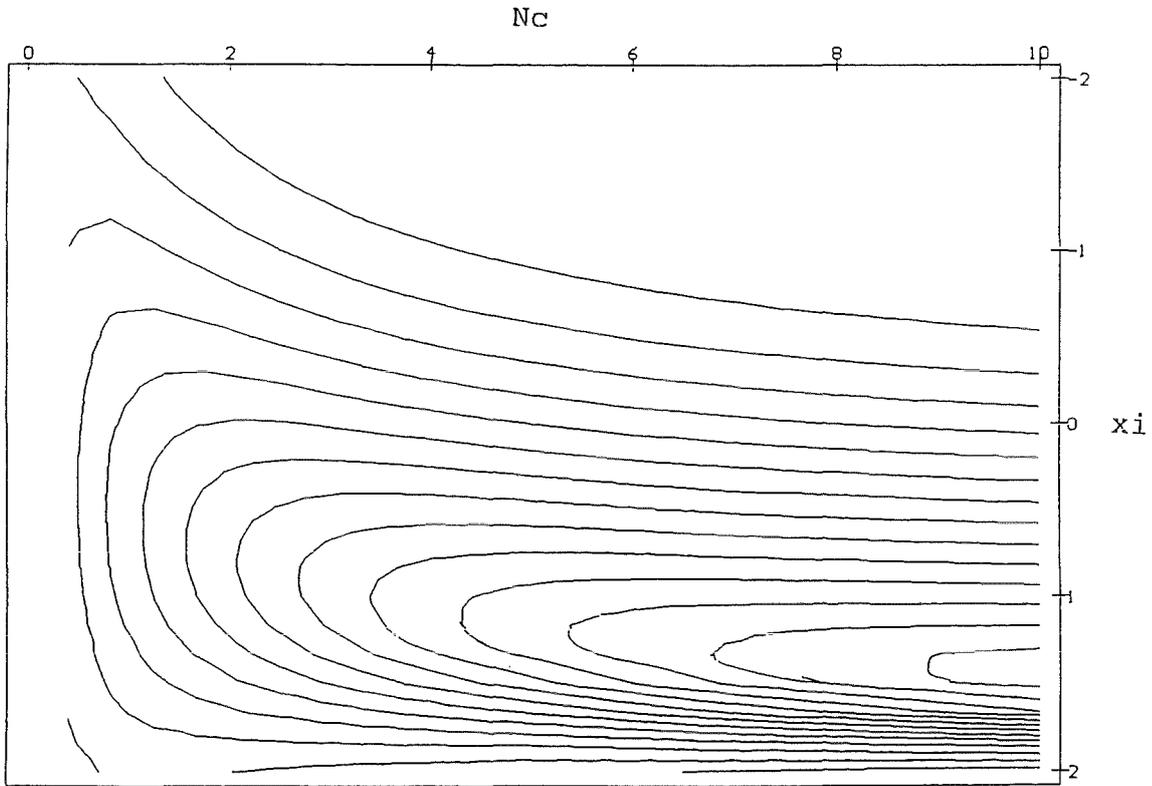
Fig. 5-7b Lines of constant f(xi,Nc) (projection)
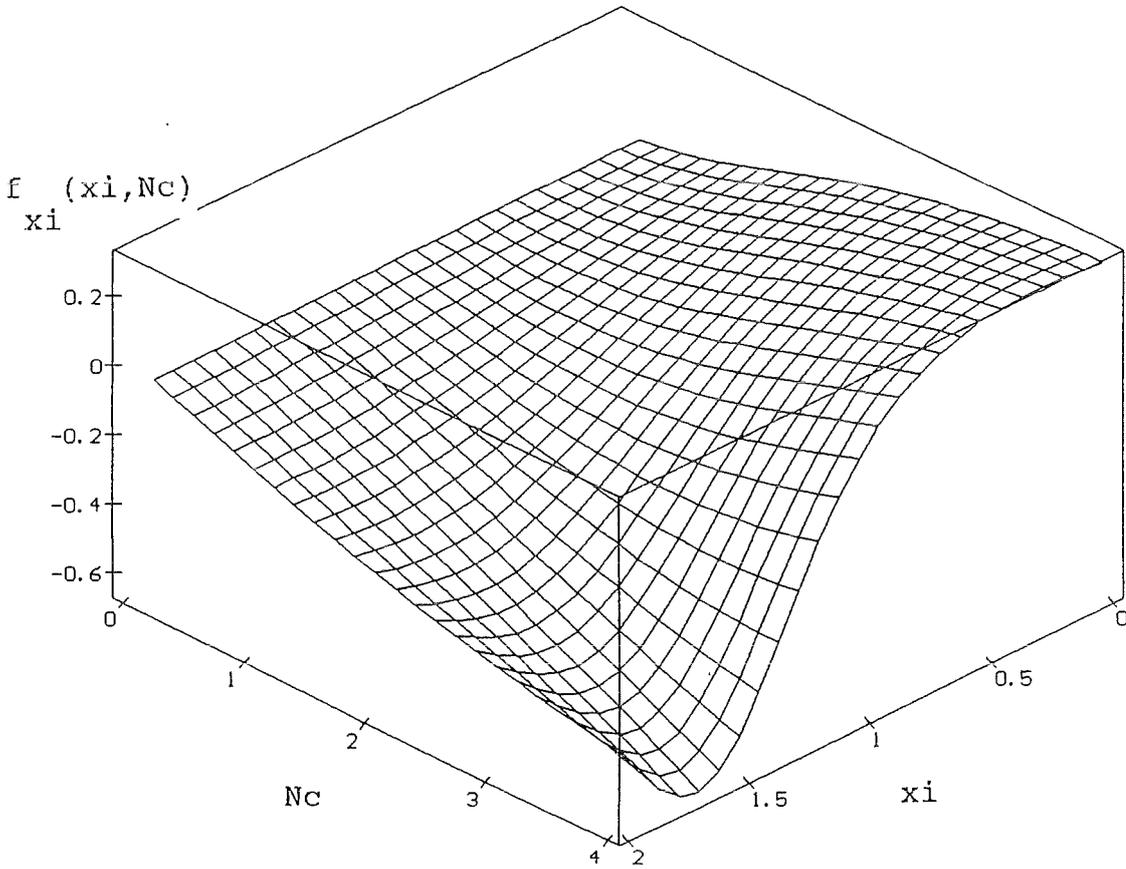
Fig. 5-8   Partial derivative $f_{xi}(xi,Nc) = \dfrac{\partial\, f(xi,Nc)}{\partial\, xi}$
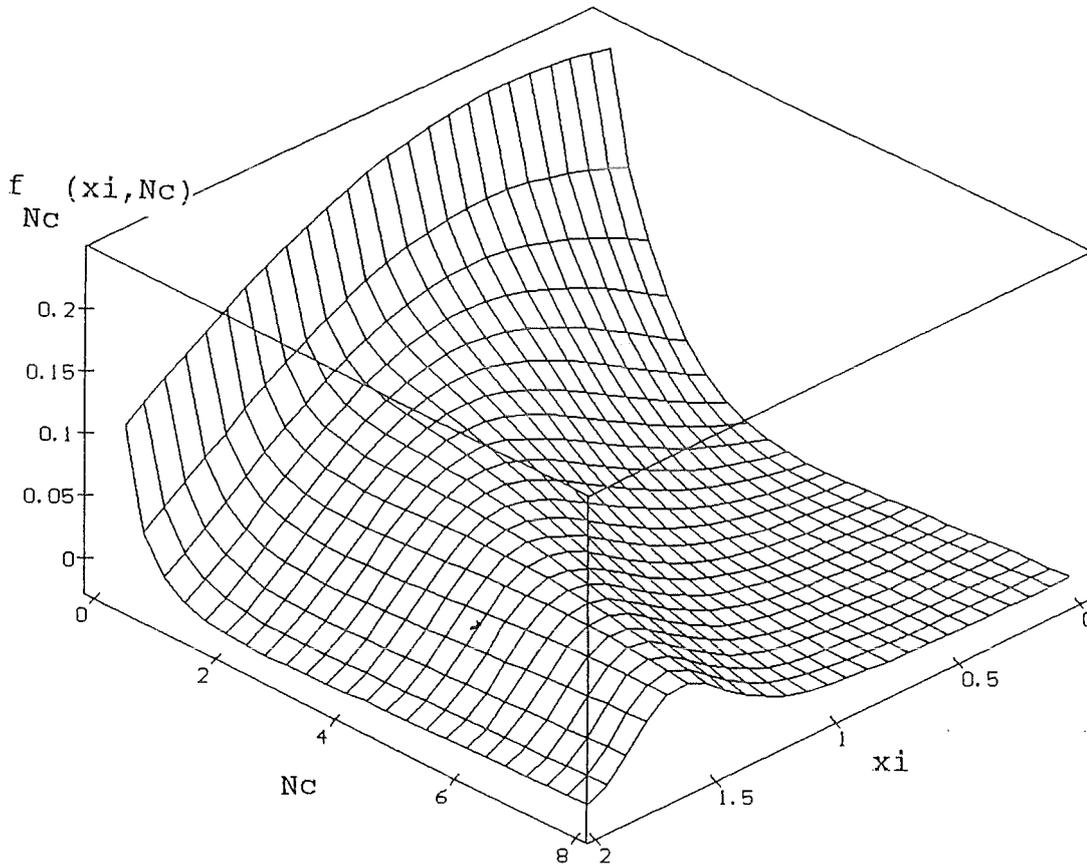
Fig. 5-9   Partial derivative $f_{Nc}(xi,Nc) = \dfrac{\partial f(xi, Nc)}{\partial Nc}$

RESULTS (range Nc = 0.10 to 1.00)

COORDINATES OF MAXIMUM (xim, f(xim))

Nc	xim	fm
0.10	0.20000000	0.08920210
0.20	0.30000000	0.12613731
0.30	0.35000000	0.15445750
0.40	0.40000000	0.17830521
0.50	0.45000000	0.19928119
0.60	0.50000000	0.21819741
0.70	0.50000000	0.23558067
0.80	0.55000000	0.25170955
0.90	0.55000000	0.26677945
1.00	0.60000000	0.28104848

==

RESULTS (range Nc = 1.00 to 10.00)

COORDINATES OF MAXIMUM (xim, f(xim))

Nc	xim	fm
1.00	0.6000000	0.28104848
2.00	0.8500000	0.39313070
3.00	1.0000000	0.47370813
4.00	1.1000000	0.53578587
5.00	1.2000000	0.58514608
6.00	1.3000000	0.62515881
7.00	1.3500000	0.65815782
8.00	1.4000000	0.68555944
9.00	1.4500000	0.70892873
10.00	1.5000000	0.72896388

==

Table 5-1   Maxima of f(xi;Nc)

RESULTS (range Nc = 11 to 20)

COORDINATES OF MAXIMUM (xim, f(xim))

Nc	xim	fm
11.00	1.5500000	0.74555946
12.00	1.5500000	0.76094193
13.00	1.6000000	0.77425242
14.00	1.6000000	0.78591824
15.00	1.6500000	0.79602475
16.00	1.6500000	0.80631099
17.00	1.6500000	0.81364963
18.00	1.7000000	0.82180120
19.00	1.7000000	0.82970929
20.00	1.7000000	0.83545798

==

RESULTS (range Nc = 20 to 29)

COORDINATES OF MAXIMUM (xim, f(xim))

Nc	xim	fm
20.00	1.7000000	0.83545798
21.00	1.7000000	0.83977318
22.00	1.7500000	0.84572892
23.00	1.7500000	0.84957135
24.00	1.7500000	0.84756914
25.00	1.7000000	0.84366005
26.00	1.7000000	0.83791725
27.00	1.6500000	0.81735071
28.00	1.6500000	0.80594792
29.00	1.6000000	0.78805031

==

Table 5-2   Maxima of f(xi;Nc)

```
RESULTS (range Nc = 22 to 24)

COORDINATES OF MAXIMUM (xim, f(xim))
```

Nc	xim	fm
22.20	1.7500000	0.84646251
22.40	1.7500000	0.84678552
22.60	1.7500000	0.84651433
22.80	1.7500000	0.84967250
23.00	1.7500000	0.84957135
23.20	1.7500000	0.84875620
23.40	1.7500000	0.84688471
23.60	1.7500000	0.85169245
23.80	1.7500000	0.85027077
24.00	1.7500000	0.84756914

```
======================================================
```

```
RESULTS (range Nc = 23.4 to 23.6)

COORDINATES OF MAXIMUM (xim, f(xim))
```

Nc	xim	fm	
23.40	1.7500000	0.84688471	
23.42	1.7500000	0.84662161	
23.44	1.7500000	0.85219411	(*)
23.46	1.7500000	0.85215596	
23.48	1.7500000	0.85211132	
23.50	1.7500000	0.85205996	
23.52	1.7500000	0.85200160	
23.54	1.7500000	0.85193597	
23.56	1.7500000	0.85186279	
23.58	1.7500000	0.85178176	

```
======================================================
```

Table 5-3   Maxima of f(xi;Nc)

(*) Note that Nc = 23.44 + 0.02 is the location
    which gives the highest value for f(xim, Nc).

RESULTS (range Nc = 10 to 100)

COORDINATES OF MAXIMUM (xim, f(xim))

Nc	xim	fm
10.00	1.5000000	0.72896388
20.00	1.7000000	0.83545798
30.00	1.6000000	0.77455142
40.00	1.3500000	0.67056218
50.00	1.2000000	0.59391256
60.00	1.1000000	0.54244269
70.00	1.0500000	0.51324423
80.00	0.9500000	0.46536509
90.00	0.9000000	0.44119881
100.00	0.8500000	0.41523894

==

RESULTS (range Nc = 100 to 1000)

COORDINATES OF MAXIMUM (xim, f(xim))

Nc	xim	fm
100.00	0.85000000	0.41523894
200.00	0.60000000	0.28810317
300.00	0.50000000	0.23635741
400.00	0.45000000	0.20994640
500.00	0.40000000	0.18491138
600.00	0.35000000	0.16002300
700.00	0.35000000	0.15872356
800.00	0.30000000	0.13448896
900.00	0.30000000	0.13392032
1000.00	0.30000000	0.13295212

==

Table 5-4   Maxima of f(xi;Nc)
----

6. A further characteristic for transport

6.1 Continuity equations and flux

Now, a further characteristic for the transport in porous media will be shown. We start from
- continuity equations and
- the reduced flow.
We recall, that in
- sect. 4.3 (Continuity and Darcy law) and
- sect. 4.4 (Dimensionless considerations)
the following dimensionless continuity equation was used:

$$\frac{\partial S}{\partial t} + \frac{\partial Js}{\partial x} = 0 \qquad (4\text{-}24)$$

with

$$Js = qe(\hat{t})*S^{n} - (1/Nc)*S^{n-m} * \frac{\partial S}{\partial \hat{x}} \qquad (4\text{-}25)$$

where Js has been called fuel flux. It has been noted that (4-24) is a simplified and dimensionless version of the continuity equations (4-6), (4-7) for flow and saturation S (of water and fluid). Using the concept of fractional flow (4-9)-(4-11) the two equations (4-6) and (4-7) for flow and saturation (of water and fluid) have been reduced to one equation.
Relations of fractional flow to a power law (see (4-18) to (4-20)), contribute to the explicit version of (4-25), as discussed in further detail in Appendix A4. However

$$\frac{\partial S}{\partial \hat{x}}$$

can be calculated with the analytic solution for saturation by means of similarity variables (sect. 4.6), see especially equs. (4-37)-(4-45) and Appendix A1.
Note that solution (4-43) applies only to a transport which is horizontal.

Non-horizontal flow

If we have to consider also gravity and some other influences, some further analytic solutions are available ([Pistiner1]). This leads to the following formula for Saturation flux (6-1) which is given here:

$$J = q(\hat{t})*S^{n} - (1/Nc)*S^{n-m} * \frac{\partial S}{\partial \hat{x}} - (NB/Nc)*S^{1} \qquad (6\text{-}1)$$

where

J	fuel saturation flow
S	Saturation
$\hat{x}, \hat{t}$	reduced coordinates
n, m	exponents related to fractional flow (cf.(4-18)-(4-20))
l	a further exponent
Nc	capillary number
$q_*(\hat{t})$	a time dependent funtion

We note that

$$NB = sign(q) \; \frac{*Kro(\rho w - \rho f)*g}{fw} \qquad (6\text{-}2a)$$

with

a factor related to ρw (density of water), ρf (density of fluid), and gravity force g.

Considering a horizontal flow only, we evidently set

$$NB = 0 \qquad (6\text{-}2b)$$

Assumptions

Assuming a horizontal flow and a relation to analytic solution of sect. 4.6 we obtain:

$NB = 0,$
$m = n = 1$
Moreover, we have for

$$u = \frac{2 - m}{2 + n - m} = 1/2 \qquad (6\text{-}3)$$

and for q(t) we obtain

$$q_*(\hat{t}) = \hat{t}^{-1/2} \qquad (6\text{-}4)$$

With these assumptions, the saturation flux J introduced in (6-1) can be rewritten

$$J = \hat{t}^{-1/2} *S - (1/Nc)* \frac{\partial S}{\partial \hat{x}} \qquad (6\text{-}5)$$

6.2 Interpretation of flux curve J(S;t,Nc)

For a detailed evaluation and interpretation of the J-curves
some considerations for the dimensionless continuity equation
(4-24) are useful. We recall the continuity equation:

$$\frac{\partial S}{\partial \hat{t}} + \frac{\partial J}{\partial \hat{x}} = 0 \qquad (4\text{-}24)$$

Substitution in the continuity equation

For this equation we make the following substitution:

$$\frac{\partial S}{\partial \hat{t}} + \frac{\partial J}{\partial S} * \frac{\partial S}{\partial \hat{x}} = 0 \qquad (6\text{-}6)$$

The term $\partial J / \partial S$ is of special interest for the interpretation
of the J-curve given in sect 6.3:

$$W = \frac{\partial J}{\partial S} \qquad (6\text{-}7)$$

With (6-7) equ. (6-6) can be written:

$$\frac{\partial S}{\partial \hat{t}} + W * \frac{\partial S}{\partial x} = 0 \qquad (6\text{-}8)$$

Interpretation of W

(a) It is to be noted that W is also called "group velocity".
    [Pistiner2]. The concept of group velocity can evidently
    be related to electromagnetic waves.

(b) Note that W especially describes the propagation of
    "iso-saturation lines" (lines in space where saturation
    remains constant). An example for iso-saturation lined
    is shown in Fig. 6-1: Lines for constant S on x,t-plane.

(c) There is also a relation to the dimensionless quantities
    used for the Hele-Shaw cell (sect.2.2). In (2-8a) we
    introduced the reduced pressure, ϕ . Here (e.g. in case
    of an air/water interface) in the Hele-Shaw cell the
    condition for constant ϕ on the interface can be
    written:

$$\frac{\partial \phi}{\partial t} + v*\nabla \phi = 0 \qquad (6-9)$$

These are three relations useful for the discussion of W.
This way it can be said that iso-saturation lines

Different behavior of flow of two phases

It can be stated that, in this model the following types of
movements of two phases occur. We say that fuel has

(1) a movement against the water flow, if W > 0,

(2) a movement along the water flow, if W < 0.

Now from (6-8) we have

$$W = \frac{\partial J}{\partial s}\bigg|_{\hat{t}} = - \frac{\dfrac{\partial s}{\partial \hat{t}}\bigg|_{\hat{x}}}{\dfrac{\partial s}{\partial \hat{x}}\bigg|_{\hat{t}}} \qquad (6-10)$$

Note that the partial derivatives are for fixed values of $\hat{t}, \hat{x}$
respectively. As will be seen in sect, 6.3, this relation
gives interesting information on the J-curves.

Relation of J to functions f, g (4-35), (4-37)

The relation

$$J = \frac{s*\hat{x}}{2*\hat{t}} \qquad (6-11)$$

can be shown using eqs(4-29),(4-39) (see also Appendix A1):

$$\hat{x} = *\hat{t}^{1/2} \qquad (4-29)$$

$$s = f*\hat{t}^{-1/2} \qquad (4-30)$$

Multiplying we obtain

$$s*\hat{x} = \xi *f \qquad (6-12)$$

or evidently (6-11).
With (4-25) we also have

$$g = b* \xi *f \qquad (4-35)$$

and with b = 1/2, we are at (4-34):

$$J = \frac{g}{\hat{t}} \qquad (4-34)$$

These are important relations used for the GAUSS-program (see Appendix A5) for evaluation of the $J(S;\hat{t},Nc)$-curve.

Additional relations for W (group velocity)

A second expression for W can be obtained, differentiating formula (6-11), which gives

$$W = Us + S * \frac{\partial Us}{\partial S}\bigg|_{\hat{t}} \qquad (6-13)$$

Here we introduce

$$Us = \frac{\hat{x}}{2*\hat{t}} \qquad (6-14)$$

Note that the partial derivative is for a fixed value of $\hat{t}$, and that Us is called "phase velocity" (or wave velocity ([Pistiner1]).
It is also to be noted that (6-13) has interesting connections to electromagnetic waves. Eq. (6-13) is formally the same expression als the equation showing the "group velocity" as a function of "phase velocity".

6.3 Some details of Js-flux curve

A few details of interpretation of flux curevs are given here.
It is to be noted, that some of statements, such as properties
of group velocity W are based on sects. 6.1 and 6.2.

Description of curve

In Fig. 6-2 we can see that this curve can be divided in the
following sections:
We start with A, then divide the curve in 4 sections:
A-B, B-C, D-C, A-C.
It can also be seen that for the points
B, C and D there is a horizontal and vertical tangent.
This will be related to the derivative $\partial J / \partial S$ which is
equivalent to W.

Group velocity W

Recall the definition of W (6-10):

$$W = \left. \frac{\partial J}{\partial S} \right|_{\hat{t}} = \frac{\left. \frac{\partial S}{\partial \hat{t}} \right|_{\hat{x}}}{\left. \frac{\partial S}{\partial \hat{x}} \right|_{\hat{t}}} \qquad (6\text{-}10)$$

With the r.h.s. of (6-10) the following statements can be
made:

(a) The J-curve has a horizontal tangent (W = 0) if

$$\left. \frac{\partial S}{\partial \hat{t}} \right|_{\hat{x}} = 0 \qquad (6\text{-}15a)$$

(b) The J-curve has a vertical tangent (W very large) if

$$\left. \frac{\partial S}{\partial \hat{x}} \right|_{\hat{t}} = 0 \qquad (6\text{-}15b)$$

(c) Sign of W
it can be seen from the definition of W (6-10) that we have
W > 0 if the conditions

$$\frac{\partial S}{\partial \hat{t}} > 0 \quad \text{and} \quad \frac{\partial S}{\partial \hat{x}} < 0 \qquad (6\text{-}16a)$$

or if the conditions

$$\frac{\partial S}{\partial \hat{t}} < 0 \quad \text{and} \quad \frac{\partial S}{\partial \hat{x}} > 0 \qquad\qquad (6\text{-}16b)$$

hold. Similarly, we have W < 0 if the conditions (6-16c)

$$\frac{\partial S}{\partial \hat{t}} > 0 \quad \text{and} \quad \frac{\partial S}{\partial \hat{x}} > 0 \qquad\qquad (6\text{-}16c)$$

or if the conditions (6-16d)

$$\frac{\partial S}{\partial \hat{t}} < 0 \quad \text{and} \quad \frac{\partial S}{\partial \hat{x}} < 0 \qquad\qquad (6\text{-}16d)$$

hold. As will be confirmed in sect. 7, by some 3-dimensional representations, there exist pairs of ($\partial S/\partial \hat{t}$ and $\partial S/\partial \hat{x}$), which cause in various ways W > 0 or W < 0.

6.4 Interpretation of various conditions of flow

Flow direction

In (6-10) to (6-16) we see that W (group velocity) depends on the quantities

$$\frac{\partial S}{\partial \hat{x}} \quad \text{and} \quad \frac{\partial S}{\partial \hat{t}}$$

Assume now a saturation curve S which presents the injection of a fuel at a permanent point source at xo with a water flow in the porous medium in the x-direction only (from left to right, at a fixed time $\hat{t}1$.
Then we have, under the condition $\partial S/\partial \hat{t} < 0$, as is shown in Fig. 6-3, the following two cases:

(a) For the fuel which is flowing against the water flow, i.e. on the left of the point-source, xo, we have

$$\left. \frac{\partial S}{\partial \hat{x}} \right|_{\hat{t}1} > 0 \qquad\qquad (6\text{-}17a)$$

(b) For the fuel which is flowing in the same direction as the water flow, i.e. on the right of the point source, xo, we have

$$\left. \frac{\partial S}{\partial \hat{x}} \right|_{\hat{t}1} < 0 \qquad\qquad (6\text{-}17b)$$

This evidently leads to a increasing and decreasing W-curve.
Note:
Similar considerations can also be made for other conditions,
leading to various pairs of ($\partial S/\partial \hat{t}$, $\partial S/\partial \hat{x}$),   with the
3-dimensional representation of $S(\hat{x},\hat{t})$ in sect. 7.

Relations to drainage and to imbibition

As can be seen from the literature [Pristiner1], the sections
of of the J - curve are related with
  - drainage   (oil pushes water)
  - imbibition   (water pushes oil),
see sect. 3.1. It is also interesting to make the following
consideration. It has to be noted, that the following is only
an illustration, but not a derivation of this type of an
interpretation.
As can be seen in Fig. 6-4, there is a relation between
saturation S and capillary pressure, also based on experiments
[Sahimi]. Similar considerations apply to relations of the
saturation and relative permeability Kr (see Appendix A4). It
can be seen, that for different values of saturation S various
mechanisms occur, such as drainage and imbibition. In Fig. 6-4
we also note, that different mechanisms, such as
  - drainage
  - spontaneous imbibition
  - forced imbibiton
occur in sequence.
Since also flux J is again related to the relative permeability
Kr, this gives us a plausible interpretation of the mechanism
related to various parts of our J-curve.

Different entries of Table 6-1

Now the collums of Table 6-1 will be listed, to be considered
with the curve in Fig.6-2. All this is required for the
interpretation of the results given in Figs. 6-5.

(1) collumn 1: parts of J-scurve are indicated (see Fig. 6-2)
(2) collumn 2: describes curve (increasing, decreasing, tangents)
(3) collumn 3: gives group velocity W, defined in (6-10)
(4) collumn 4: describes iso-saturation lines (6-9a), (6-9b)
(5) collumn 5: gives $\partial S/\partial \hat{t}$ (6-10); (6-15), (6-16)
(6) collumn 6: gives $\partial S/\partial \hat{x}$ (6-10); (6-15), (6-16)
(7) collumn 7: states   if water/oil is in same direction or not
(8) collumn 8: states   if oil pushes water (drainage) or
                        if water pushes oil (imbibition).

Table 6.1 referring to Js-flux curve
----

(1)	(2)	(3)	(4)	(5) $\dfrac{\partial S}{\partial \hat{t}}$	(6) $\dfrac{\partial S}{\partial \hat{x}}$	(7)	(8)
parts of Js curve	Js flux-curve	W group-vel.	move of iso-satur. lines			water/ oil	imbib. or drain
A	incr. and decr.	> 0 and < 0	–	not 0	not 0	–	–
A–B	incr.	> 0	against water-flow	< 0 - - - or > 0	> 0 - - - < 0	——> <——	oil pushes water (drain)
B	horiz.	= 0	–	= 0	not 0	–	–
B–C	decr.	< 0	along water-flow	< 0 - - - or > 0	< 0 - - - > 0	——> ——>	water pushes oil (imbib)
C	vert.	= ∞ (*)	–	not 0	= 0	–	–
D–C	incr.	> 0	against water-flow	< 0 - - - or > 0	> 0 - - - < 0	——> <——	water pushes oil (imbib)
D	horiz.	= 0	–	= 0	not 0	–	–
A–D	decr.	< 0	along water-flow	< 0 - - - or > 0	< 0 - - - > 0	——> ——>	oil pushes water (drain)

(*) Note:
W = ∞ just is due to the vanishing of the denominator in (6-10).

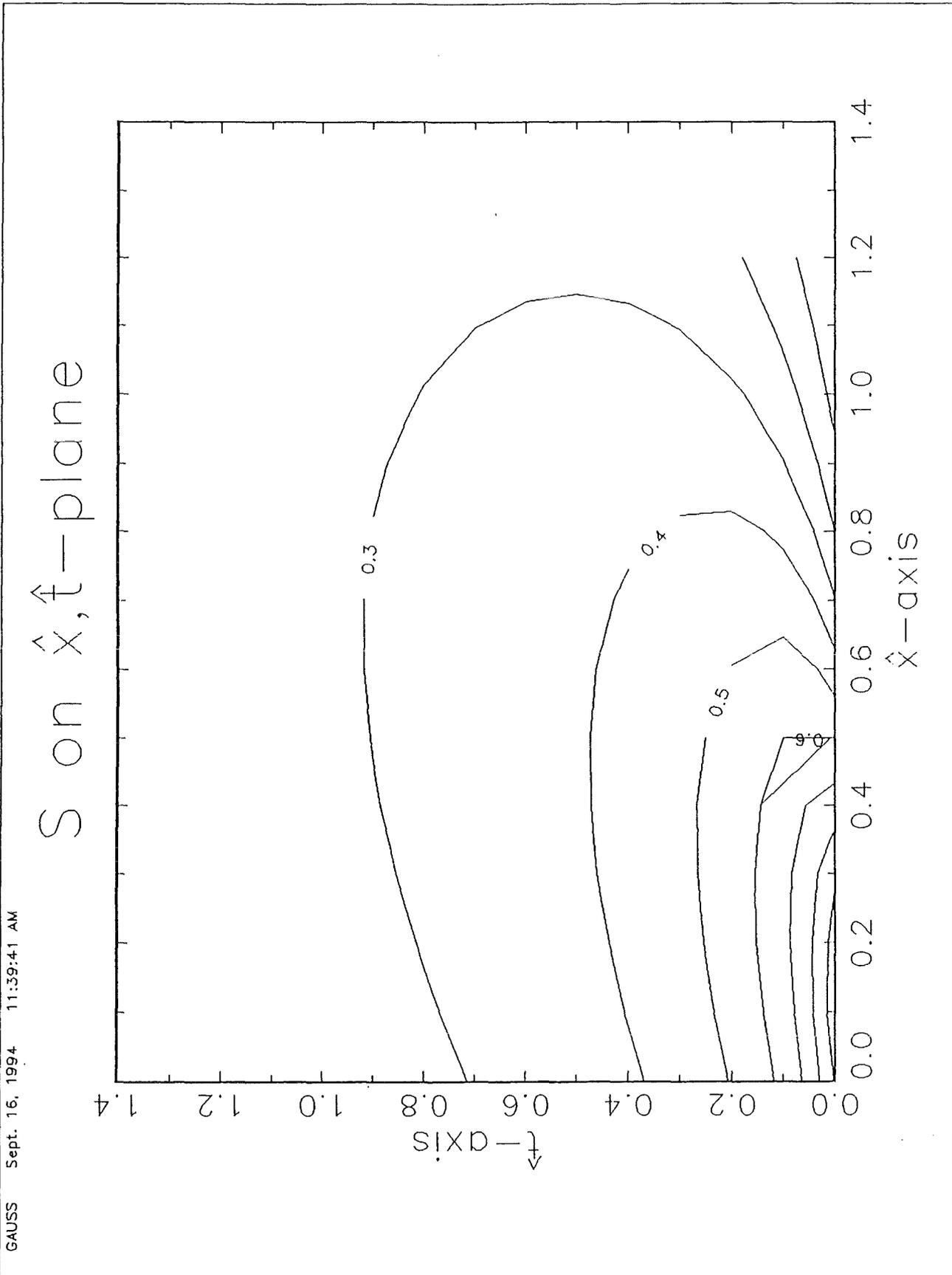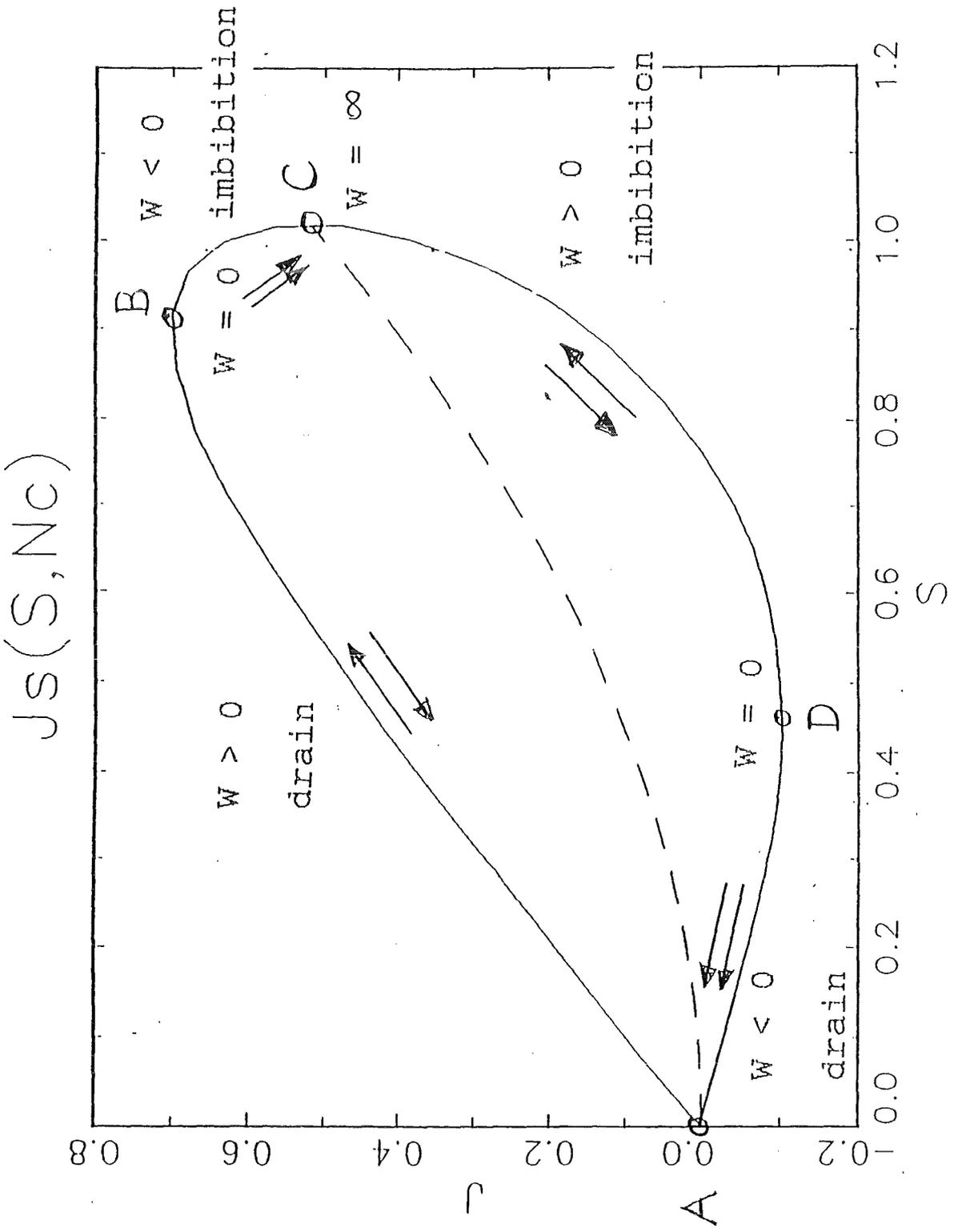Fig. 6-1   Isosaturation lines (in $\hat{x}$-$\hat{t}$ plane)

Fig. 6-2   Mechanisms related to parts of flux curve
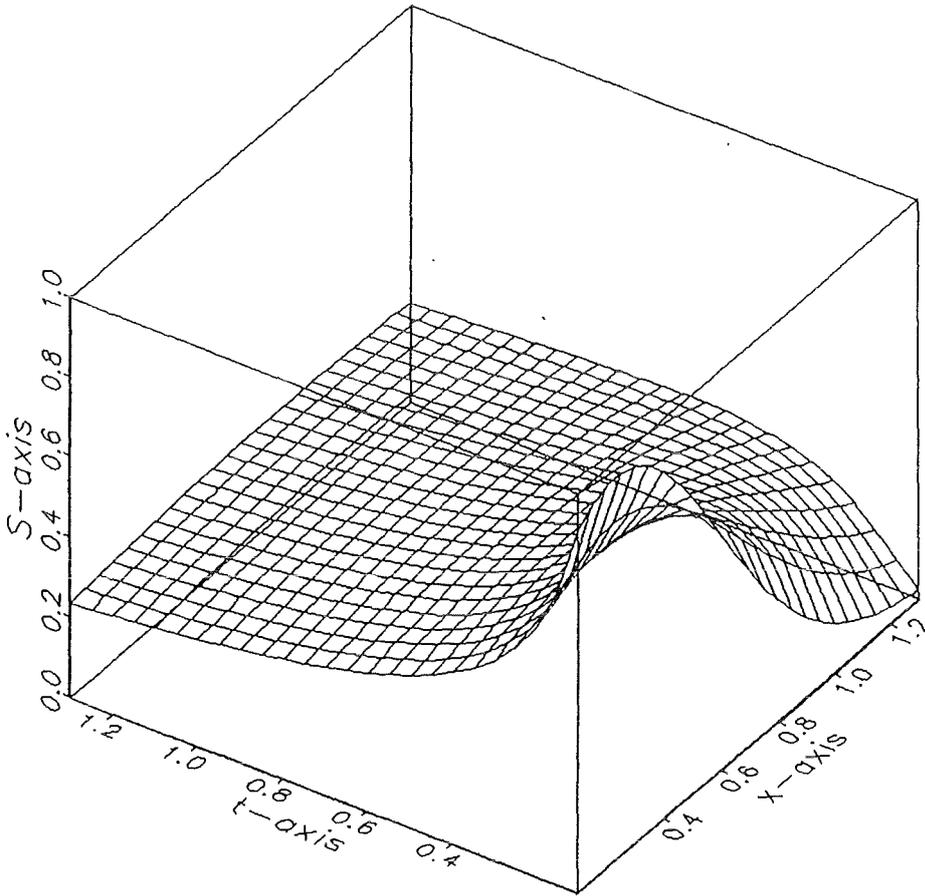
Saturation S

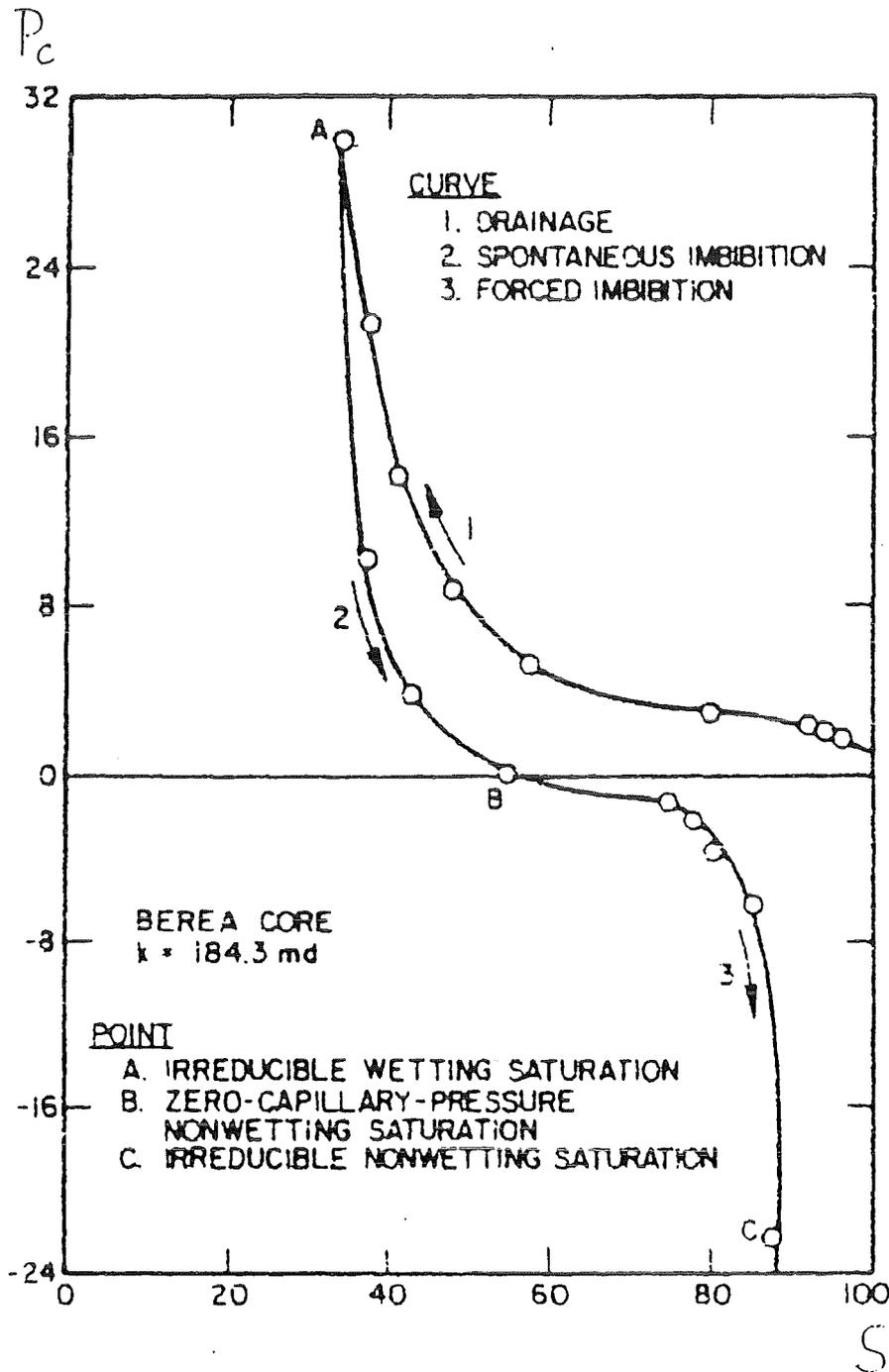Fig. 6-3　Increase and decrease of Saturation $S(\hat{x}, \hat{t})$

Fig. 6-4 An example for the following relations:
          (a) water saturation S (in percent) and
          (b) capillary pressure Pc

6.5 Figures and Tables

Here results for the fuel saturation flux J are shown. This gives some examples for the function (6-5):

$$J = \hat{t}^{-1/2} * S - (1/Nc)* \frac{\partial S}{\partial \hat{x}} \qquad (6-5)$$

For a number of different values of Nc and t we show results in Figures and Tables. Note also that in Fig. 6-5 a series of $f(\xi)$-curves ist presented.

RESULTS

	$\hat{t}$ = 1, 2, 3, 4, 5, 6	
Nc = 0.5	Fig. 6-6	
1	Fig. 6-7	Table 6-2
2	Fig. 6-8	
4	Fig. 6-9	
6	Fig. 6-10	Table 6-3
8	Fig. 6-11	
10	Fig. 6-12	
20	Fig. 6-13	Table 6-4
30	Fig. 6-14	

6.6 Interpretation of the results

We note from the figures and tables the following:

(1) If Nc is constant, and t increases, the dimensions of the curve decrease, including the maxima, see Figs. 6-6 to 6-14.

(2) If t is constant, and Nc increases, the dimensions of the curve increase, including the maxima. This is closely related to the results of sect. 5, see Figs. 6-6 to 6-14.

(3) At the same time also the shapes of the curve change (all Figures of sect. 6).

(4) For horizontal/vertical tangents
    we have a change in the mechanisms
    (a) from drain to imbibition (see Fig. 6-2 point B)
    (b) from imbibition to drain (see Fig. 6-2 point D)
    (c) there are also changes of the direction of the
        movement for iso-saturation lines (see Fig.6-2,
        and Table 6-1).

(5) It can be seen (Figs. 6-6 to 6-14), that with increasing Nc,
    close, i.e. points B and D become increasingly close (i.e.
    the coordinates for change in the mechanisms).

Note: For the curves shown in sect. 6 and the coordinates given
in Tables of sect. 6, a GAUSS computer program was used, which
is given in Appendix A5. For information on the program see
[GAUSS].
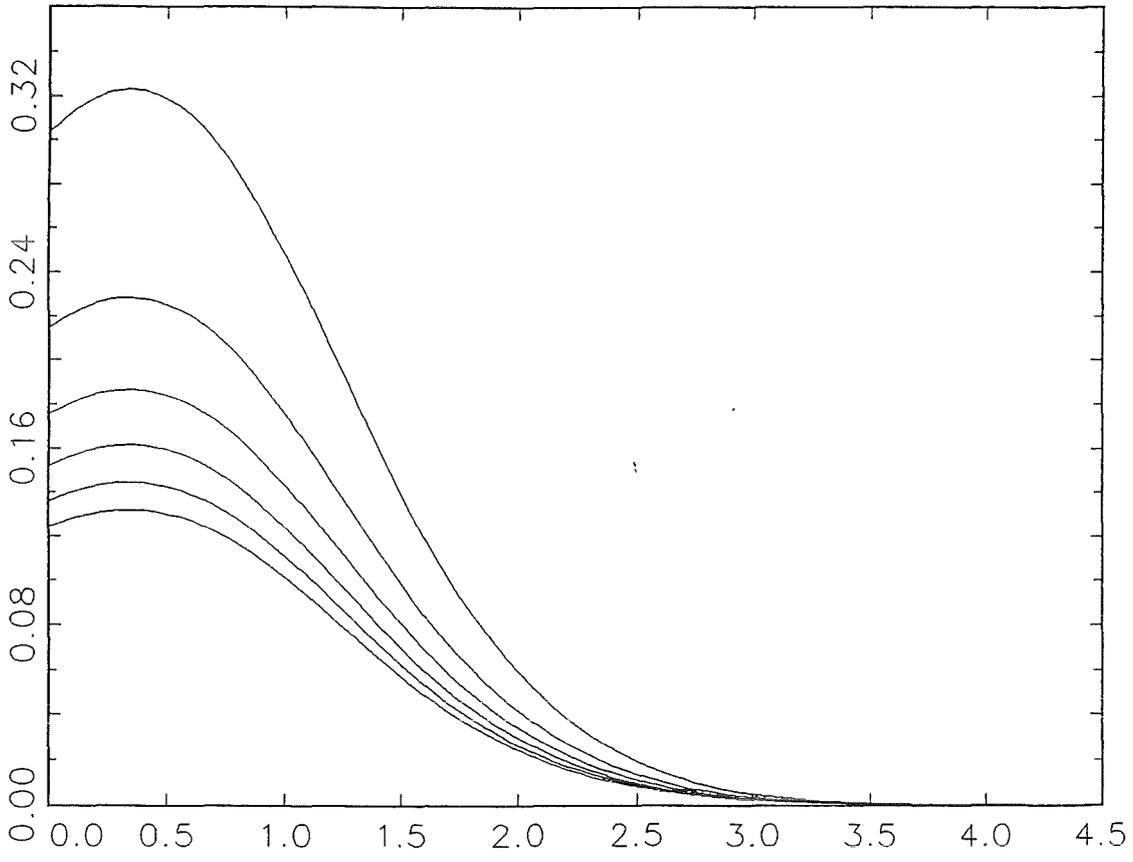
Fig. 6-5 Saturation $S(\hat{x}; Nc)$ dependent on Nc

$^{-76-}$Is(Se)
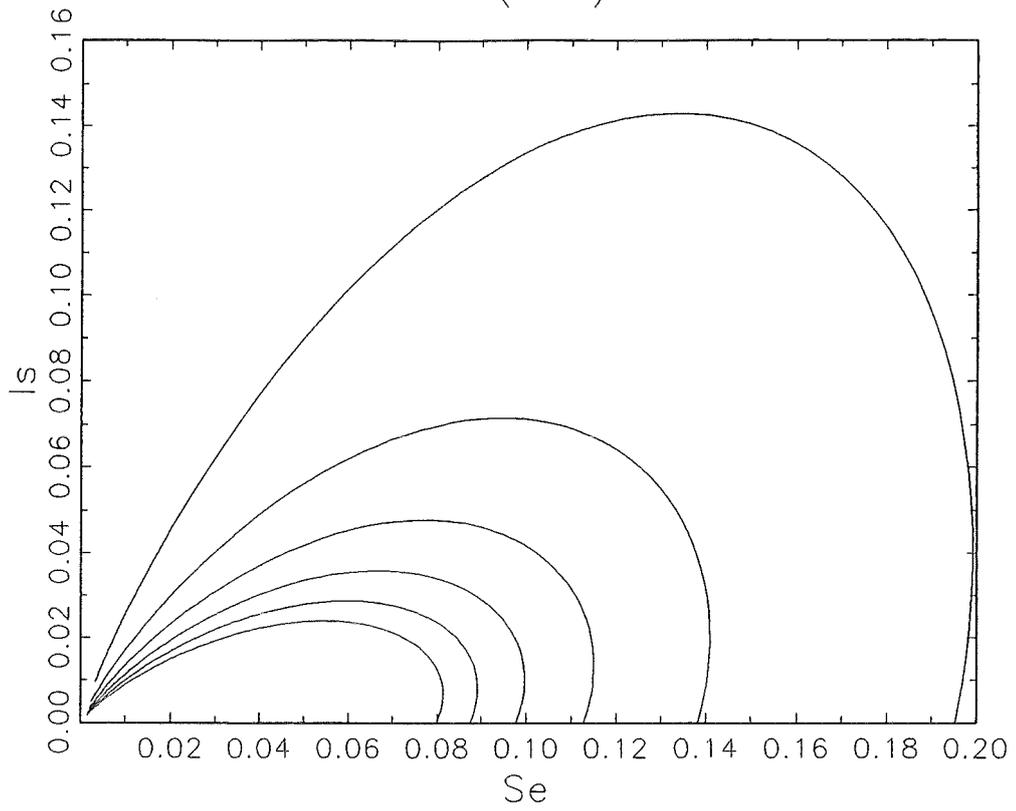
Fig. 6-6   Flux Is(Se), Nc = 0.5, t = 1,2,...,6
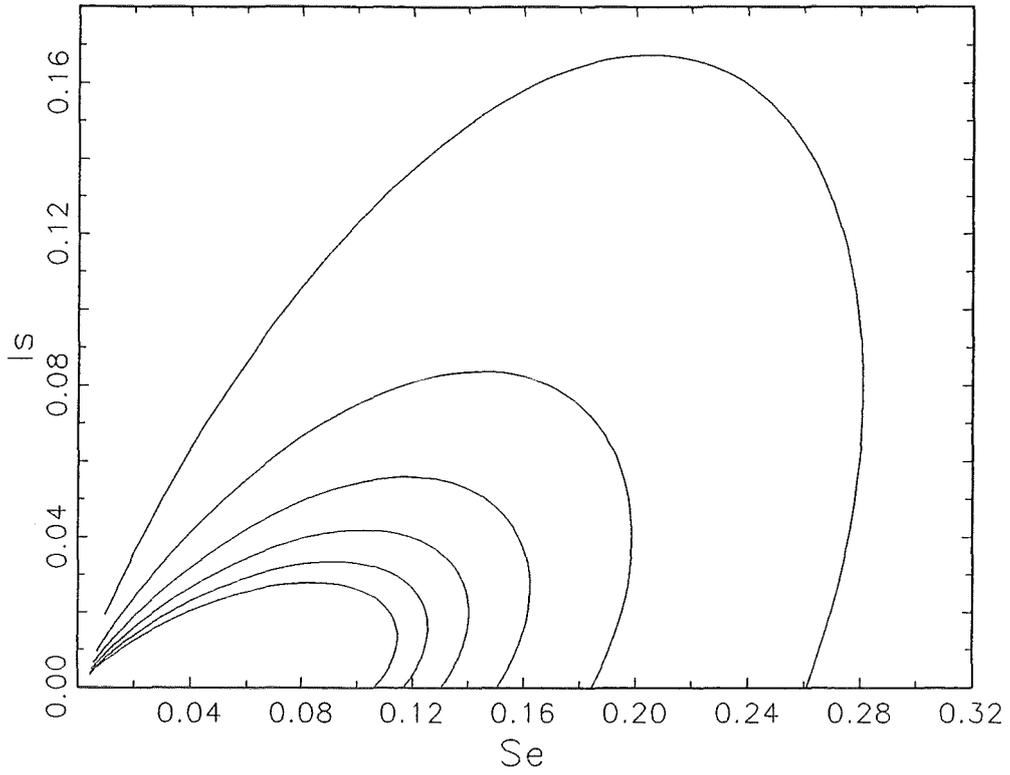
Fig. 6-7 Flux Is(Se), Nc = 1, t = 1,2,...,6
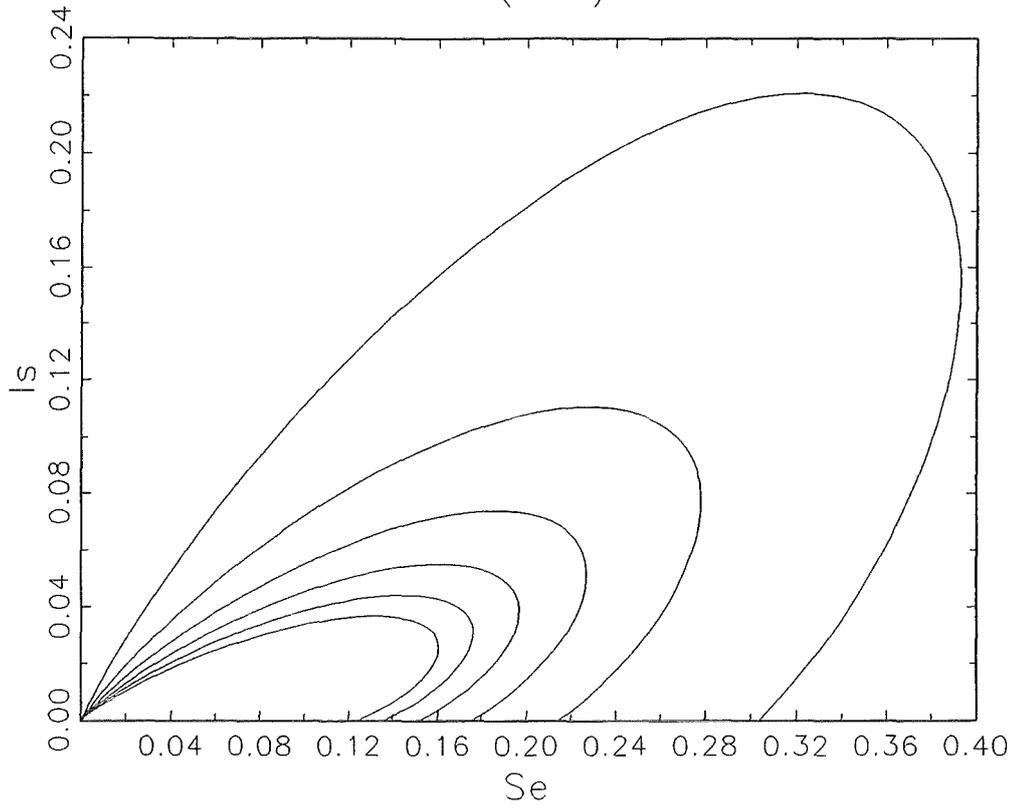
Is(Se)

Fig. 6-8   Flux Is(Se), Nc = 2.0, t = 1,2,...,6

Is(Se)
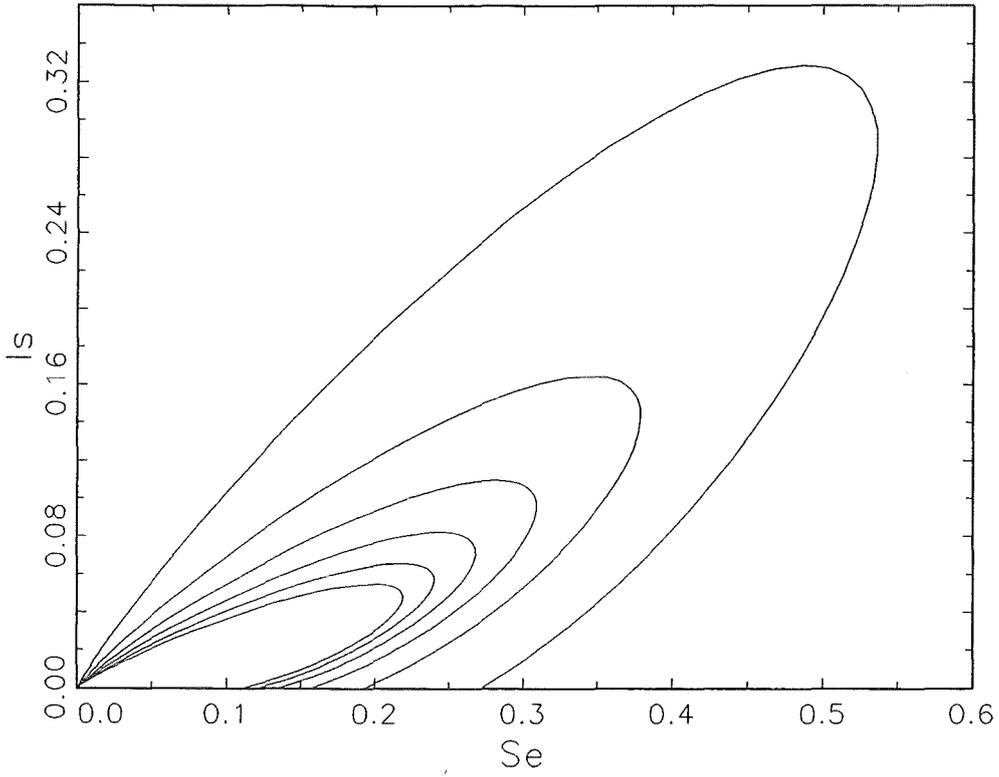
Fig. 6-9   Flux Is(Se), Nc = 4.0, t = 1,2,...,6
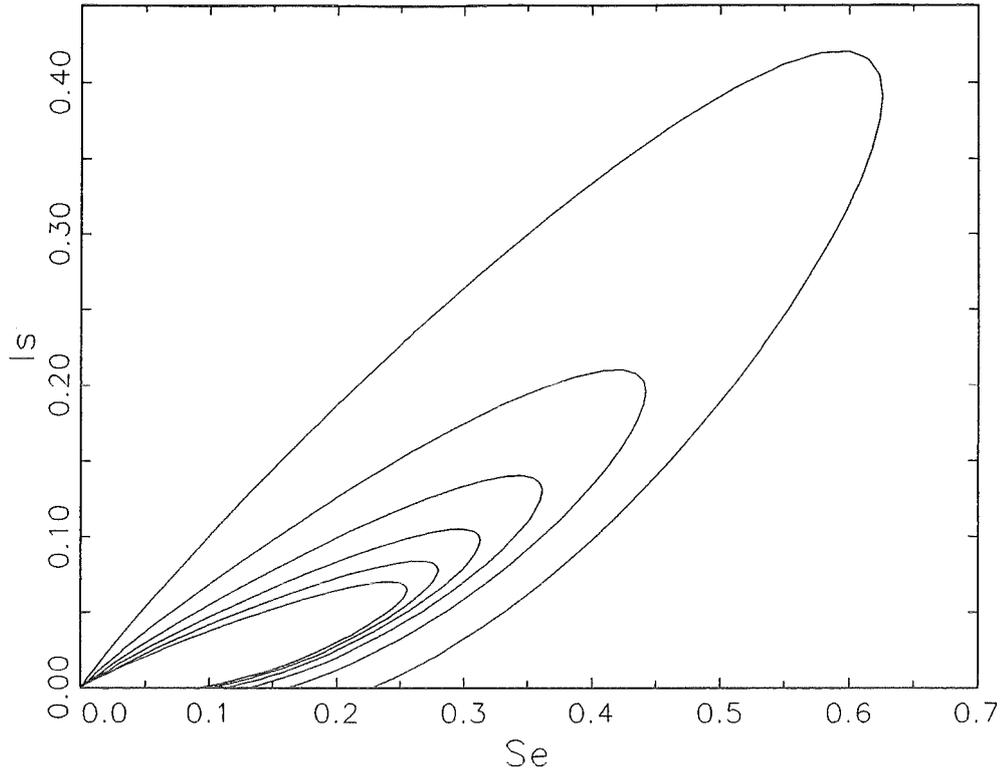
Is(Se)

Fig. 6-10   Flux Is(Se), Nc = 6.0, t = 1,2,...,6
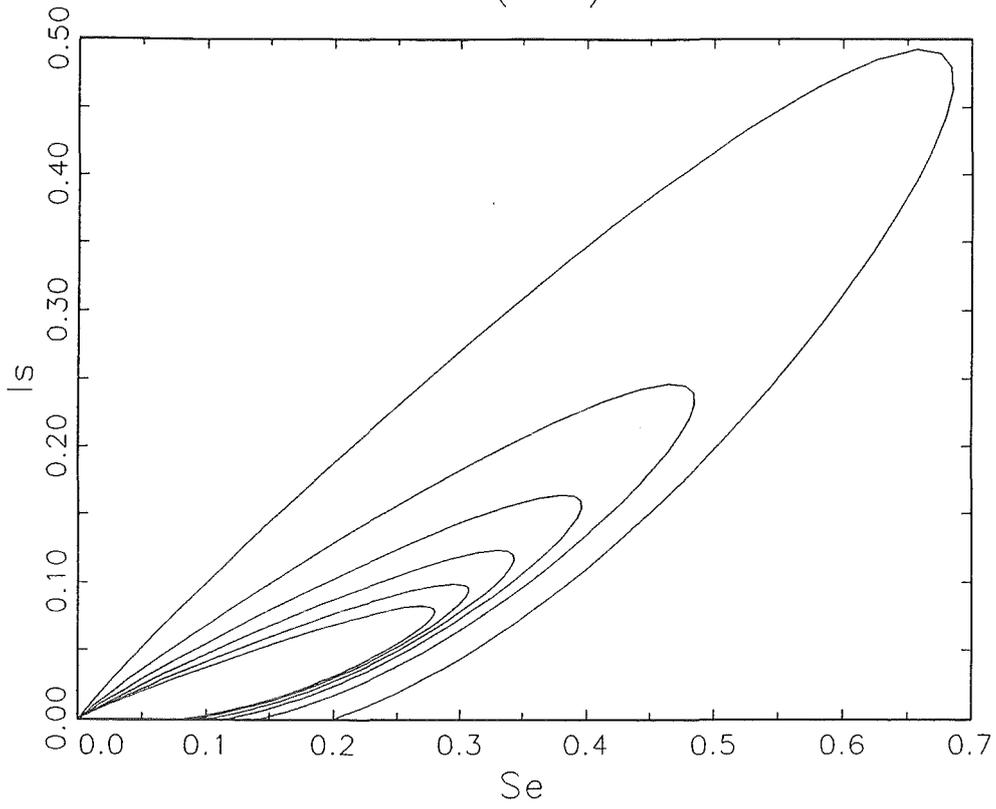
Fig. 6-11   Flux Is(Se), Nc = 8.0, t = 1,2,...,6
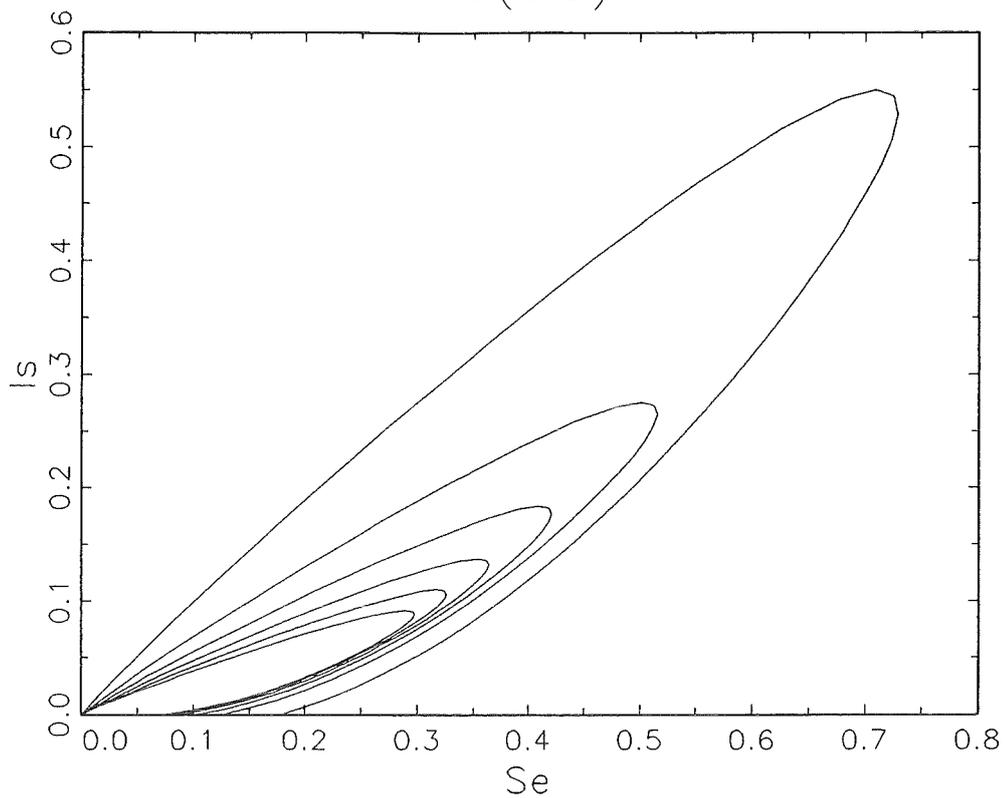
Fig. 6-12   Flux Is(Se), Nc = 10.0, t = 1,2,...,6

```
EXTREMA OF Is-CURVE
Nc   = 0.500
-----------------------------------
time t      max(Se)        Is(Se)

1.00        0.199          0.0397
2.00        0.141          0.0199
3.00        0.115          0.0132
4.00        0.010          0.0099
5.00        0.089          0.0079
6.00        0.081          0.0066
===================================


EXTREMA OF Is-CURVE
Nc   = 1.00
-----------------------------------
time t      max(Se)        Is(Se)

1.00        0.281          0.0790
2.00        0.199          0.0395
3.00        0.162          0.0263
4.00        0.141          0.0197
5.00        0.126          0.0158
6.00        0.115          0.0132
===================================


EXTREMA OF Is-CURVE
Nc   = 2.00
-----------------------------------
time t      max(Se)        Is(Se)

1.00        0.393          0.155
2.00        0.278          0.0773
3.00        0.227          0.0515
4.00        0.197          0.0387
5.00        0.176          0.0309
6.00        0.161          0.0258
===================================
```

Table 6-2   Extrema of Is-curves

```
EXTREMA OF Is-CURVE
Nc  = 4.00
-------------------------------------
time t       max(Se)        Is(Se)

1.00         0.536          0.287
2.00         0.379          0.143
3.00         0.309          0.0956
4.00         0.268          0.0717
5.00         0.240          0.0574
6.00         0.219          0.0478
=====================================


EXTREMA OF Is-CURVE
Nc  = 6.00
-------------------------------------
time t       max(Se)        Is(Se)

1.00         0.625          0.391
2.00         0.442          0.195
3.00         0.361          0.130
4.00         0.313          0.0977
5.00         0.280          0.0782
6.00         0.255          0.0652
=====================================


EXTREMA OF Is-CURVE
Nc  = 8.00
-------------------------------------
time t       max(Se)        Is(Se)

1.00         0.685          0.469
2.00         0.484          0.235
3.00         0.396          0.156
4.00         0.343          0.117
5.00         0.306          0.0939
6.00         0.280          0.0782
=====================================
```

Table 6-3 Extrema of Is-curves

```
EXTREMA OF Is-CURVE
Nc  = 10.0
------------------------------------
time t      max(Se)         Is(Se)

1.00        0.729           0.531
2.00        0.515           0.266
3.00        0.421           0.177
4.00        0.364           0.133
5.00        0.326           0.106
6.00        0.298           0.0885
====================================


EXTREMA OF Is-CURVE
Nc  = 20.0
------------------------------------
time t      max(Se)         Is(Se)

1.00        0.834           0.696
2.00        0.590           0.348
3.00        0.482           0.232
4.00        0.417           0.174
5.00        0.373           0.139
6.00        0.341           0.116
====================================


EXTREMA OF Is-CURVE
Nc  = 30.0
------------------------------------
time t      max(Se)         Is(Se)

1.00        0.782           0.611
2.00        0.553           0.306
3.00        0.451           0.204
4.00        0.391           0.153
5.00        0.350           0.122
6.00        0.319           0.102
====================================
```

Table 6-4   Extrema of Is-curves

7. Change from similarity solution to saturation

7.1 Introduction

We have seen a number of physical properties for two phase flow in porous media by means of the similarity solution (see sect. 4, 5, 6). Here we show explicitly the saturation S. On the basis of $f(\xi)$, the similiarity solution (sect. 4.6), we obtain by a suitable transformation $S(\hat{x}, \hat{t})$, a function of $\hat{x}, \hat{t}$. We note that we use, for simplicity (if not otherwise stated) $\hat{x}, \hat{t}$, the dimensionless coordinates.

7,2 The transformation of the similarity solution

Based on the similarity transformations as given in equations (4-29), (4-30) and the similiarity solution, as given in equations (4-37)-(4-41) the saturation can be evaluated as follows.

$$= \hat{x} * \hat{t}^{-1/2} \qquad (7-1)$$

with

ξ similarity variable

$\hat{x}, \hat{t}$ dimensionless coordinates.

And also

$$S(\hat{x}, \hat{t}) = \hat{t}^{-1/2} * f(\xi) \qquad (7-2)$$

with

$S(\hat{x}, \hat{t})$ saturation

$f(\xi)$ similarity solution.

Now we substitute the dimensionless variables $\hat{x}, \hat{t}$ into the similarity solution obtained from (4-41) and obtain with (7-2) (7-3)

$$f(\xi) = \frac{\exp\left(-(1/4)*Nc*\xi^2\right)}{(\Upsilon *Nc)^{1/2} * \left[(1/(\tanh((1/2)*Nce) - erf((Nc/2)^{1/2}\xi))\right]} \qquad (7-3)$$

Using (7-1) and (7-2) we obtain $S(\hat{x}, \hat{t})$.

$$S(\hat{x}, \hat{t}) =$$

$$\hat{t}^{-1/2} * \frac{\exp\left(-(1/4)*Nc* \hat{x}^2 *(\hat{t}^{-1})\right)}{(\pi *Nc)^{1/2} * \left[(1/\tanh((1/2)*Nce) - \mathrm{erf}((Nc/2)^{1/2} *\hat{x}*\hat{t}^{-1/2}))\right]} \qquad (7-4)$$

As can be seen in Appendix A5, this transform is introduced in the GAUSS-program, which gives explicitly the values for $S(\hat{x},\hat{t})$. For saturation $S(\hat{x},\hat{t})$ various diagrams are given in sect. 7.3.

7.3 Figures and Tables for saturation S(x, t)

We present the following diagrams, also noting the parameters.
Here two things should be noted;
(1) All coordinates $\hat{x}$, $\hat{t}$ are dimensionless coordinates. Thus
S-values differ by a constant factor from calculations which
could be made with dimensioned variables such as x, t.
(2) As can be seen, also parameters like Nc have been changed.
Clearly, here variations, similar to the variations shown in
sect. 4, 5, 6 will occur. See also sect. 7.4 regarding the
short time behavior of the similarity solution.

We have the following types of figures:

(a)   Here $S(\hat{x}, \hat{t})$ is shown in 3-D diagrams.

(b)   Here the projection of lines for constant S(x, t) are
      shown in the $\hat{x},\hat{t}$-plane (isosaturation lines)

(c) $S(\hat{x})$ for varying $\hat{x}$ and fixed $\hat{t}$

                       $S(\hat{t})$ for varying $\hat{t}$ and fixed $\hat{x}$

Here   (i,j) = (1,1)   denotes $\hat{x}$ low, $\hat{t}$ low,
and   (i,j) = (25,25) denotes $\hat{x}$ high, $\hat{t}$ high.

Note:
It can be seen, that the curves $S(\hat{x})$, $S(\hat{t})$ are obtained
from the 3-D diagrams, fixing $\hat{t}$ and fixing $\hat{x}$ respectively.

Now the Figures, showing the results are listed in more
detail.

	(a)	(b)	(c)	
Nc	3D	proj.	$\hat{x}$-axis	$\hat{t}$-axis
0.5	Fig.7-1	Fig.7-2	Fig.7-3 Fig.7-5	Fig.7-4 Fig.7-6
1.0	: Fig.7-7	Fig.7-8	Fig.9-9 Fig.7-11	Fig.7-10 Fig.7-12
2.0	Fig.7-13	Fig.7-14	Fig.7-15 Fig.7-17	Fig.7-16 Fig.7-18
4.0	Fig.7-19	Fig.7-20	Fig.7-21 Fig.7-23	Fig.7-22 Fig.7-24

Table 7-1   Figures for different types for $S(\hat{x}, \hat{t})$

Note:
In Appendix A5 the program used for the calculations is
shown in detail, see also [GAUSS] for information on the
pragram system.

7.4 Limits of similarity solution (short time behavior)

It has to be noted as can be seen from (7-4), that for
t ---> 0 the solution S(x, t) has a singularity. It can be
seen solution (7-4) does not generally represent the behavior
of the two-phase system (see also [Pistiner1]). For longer
times the solution (7-4) approaches the real solution of the
problem. It may be noted, that for some exponents, m, n,
(introduced in (4-18) and (4-20), if not m = n holds, this
singularity does not exist. It is out of the scope of this
research to discuss the type of convergence of the usual
similarity solution.
In figures of section 7 the singularity is not graphically
shown, which would give a serious distorsion of the long-
time behavior.

Saturation S
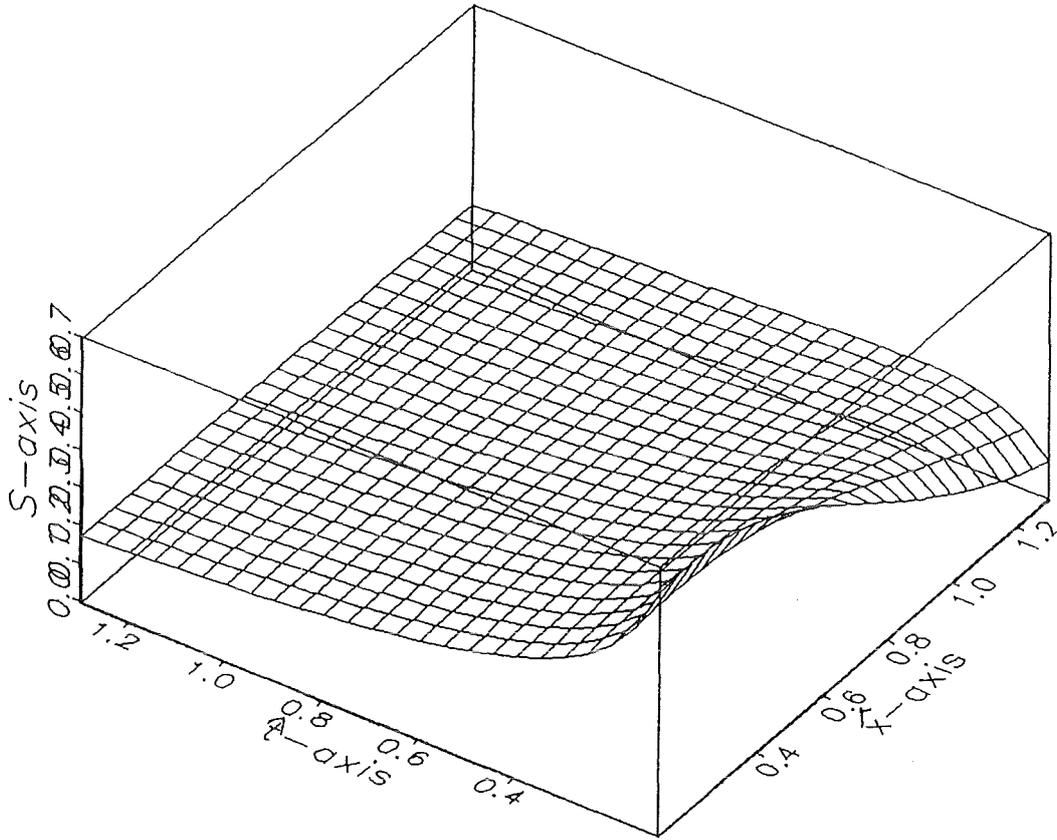
Fig. 7-1 Saturation S($\hat{x}$, $\hat{t}$) in 3-D, Nc = 0.5

S on $\hat{x},\hat{t}$-plane
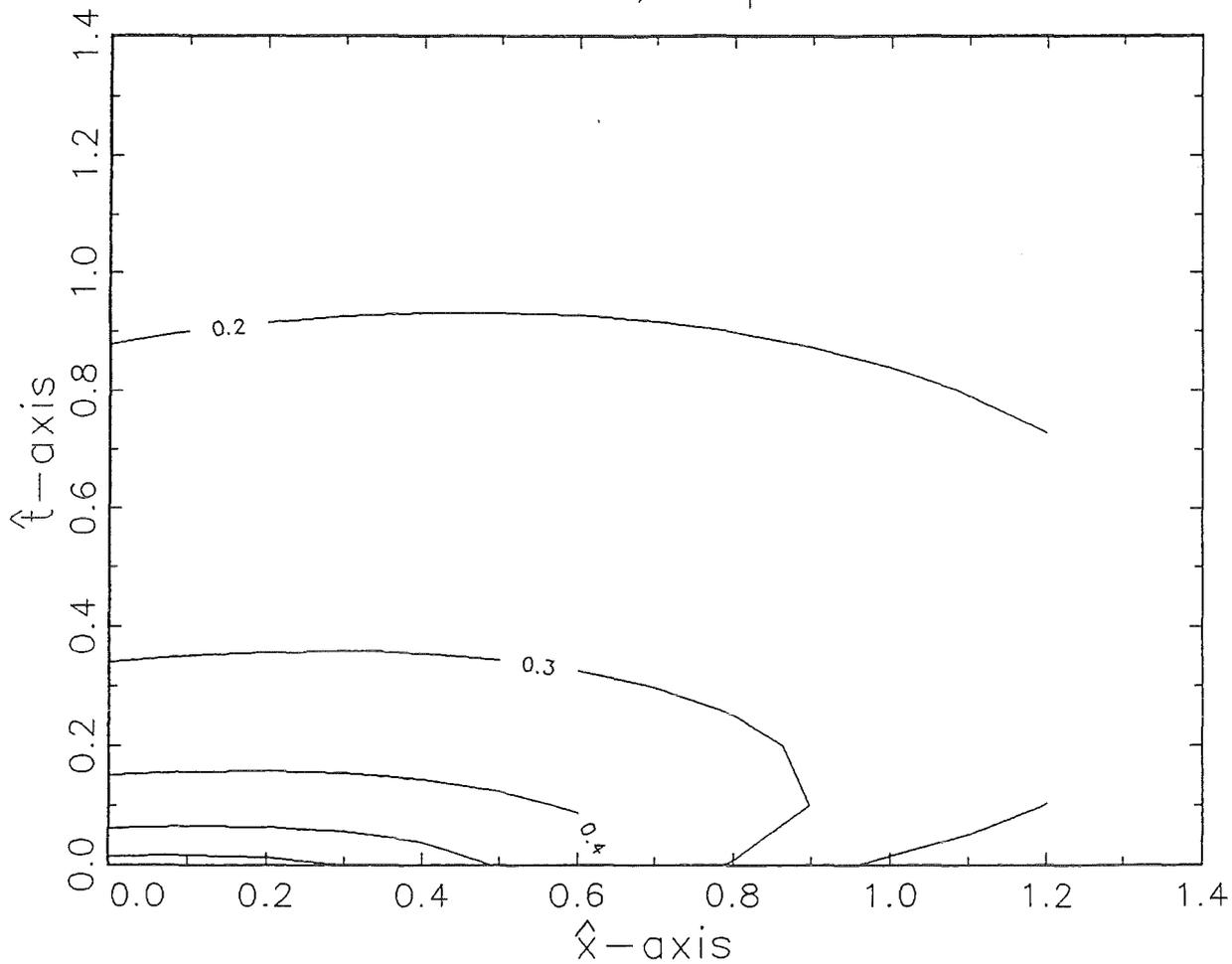
Fig. 7-2   Projection of $S(\hat{x}, \hat{t})$ on $\hat{x}$-$\hat{t}$-plane

Fig. 7-3 $S(\hat{x}, \hat{t})$ on $\hat{x}$-axis

Fig. 7-4 $S(\hat{x}, \hat{t})$ on $\hat{t}$-axis

Nc   = 0.5
(i, j) = (1, 1)

$S(\hat{x})$

$S(\hat{t})$

Fig. 7-5 $S(\hat{x}, \hat{t})$ on $\hat{x}$-axis

Fig. 7-6 $S(\hat{x}, \hat{t})$ on $\hat{t}$-axis

Nc = 0.5
(i, j) = (25,25)

Saturation S

Fig. 7-7   Saturation S($\hat{x}$, $\hat{t}$) in 3-D, Nc = 1.0

S on x̂,t̂-plane
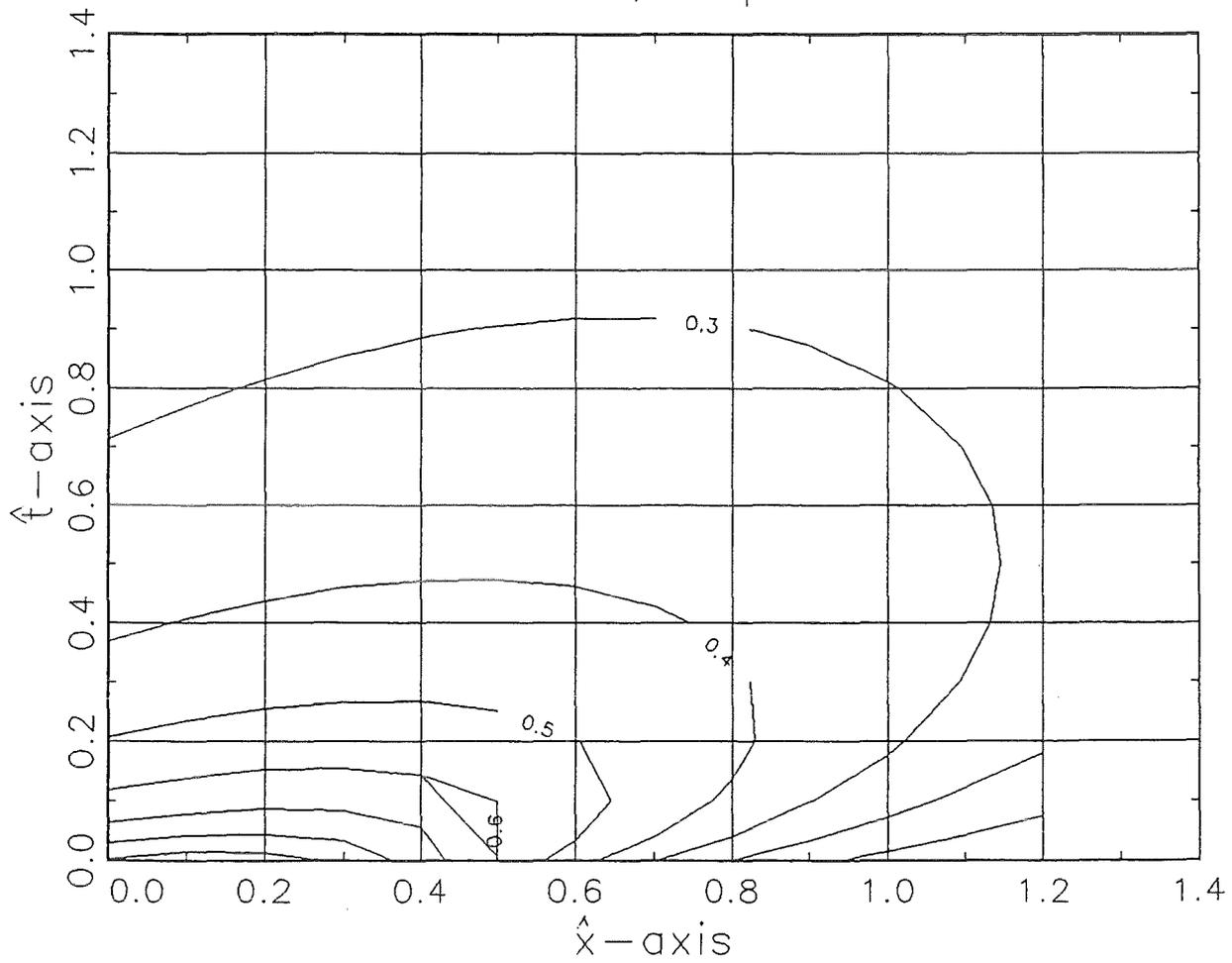
Fig. 7-8   Projection of S(x̂, t̂) on x̂-t̂-plane

Fig. 7-9   S($\hat{x}$, $\hat{t}$) on $\hat{x}$-axis | Nc   = 1.0
                                                  | (i, j) = (1,1)
Fig. 7-10 S($\hat{x}$, $\hat{t}$) on $\hat{t}$-axis

GAUSS   Oct. 11, 1994   12:43:45 PM

$$S(\hat{x})$$

GAUSS   Oct. 11, 1994   12:45:35 PM

$$S(\hat{t})$$
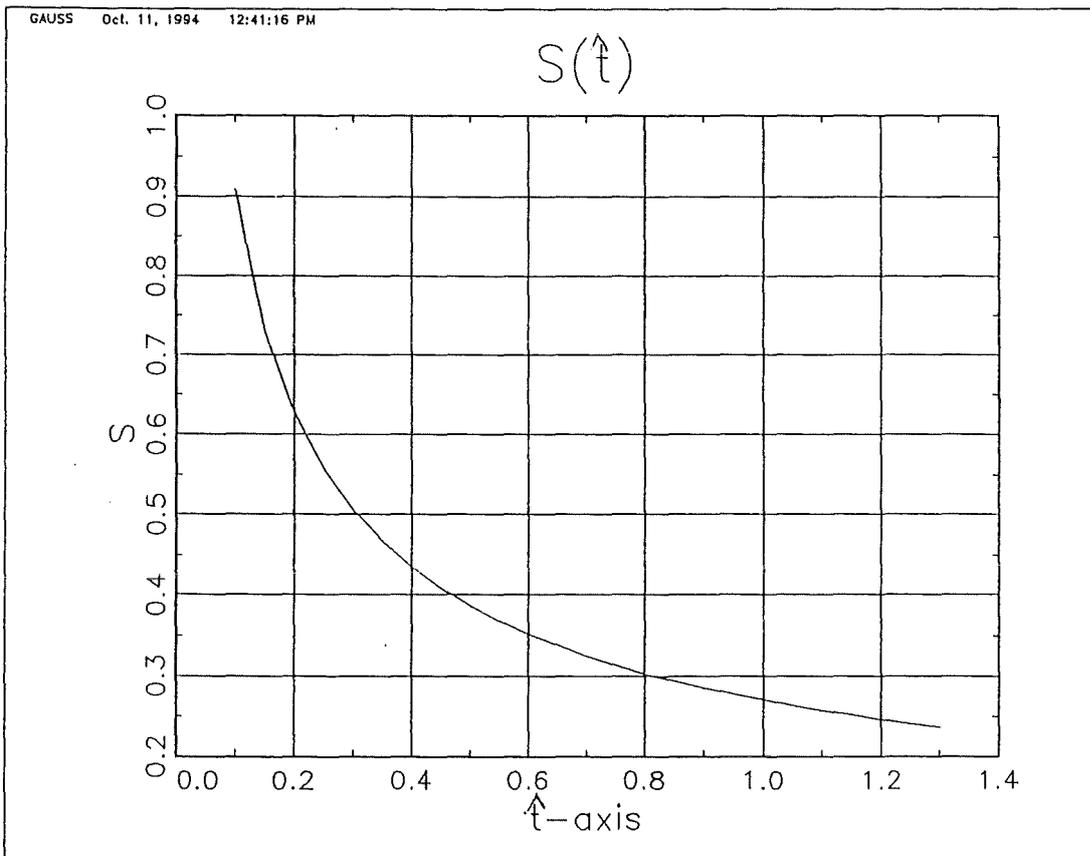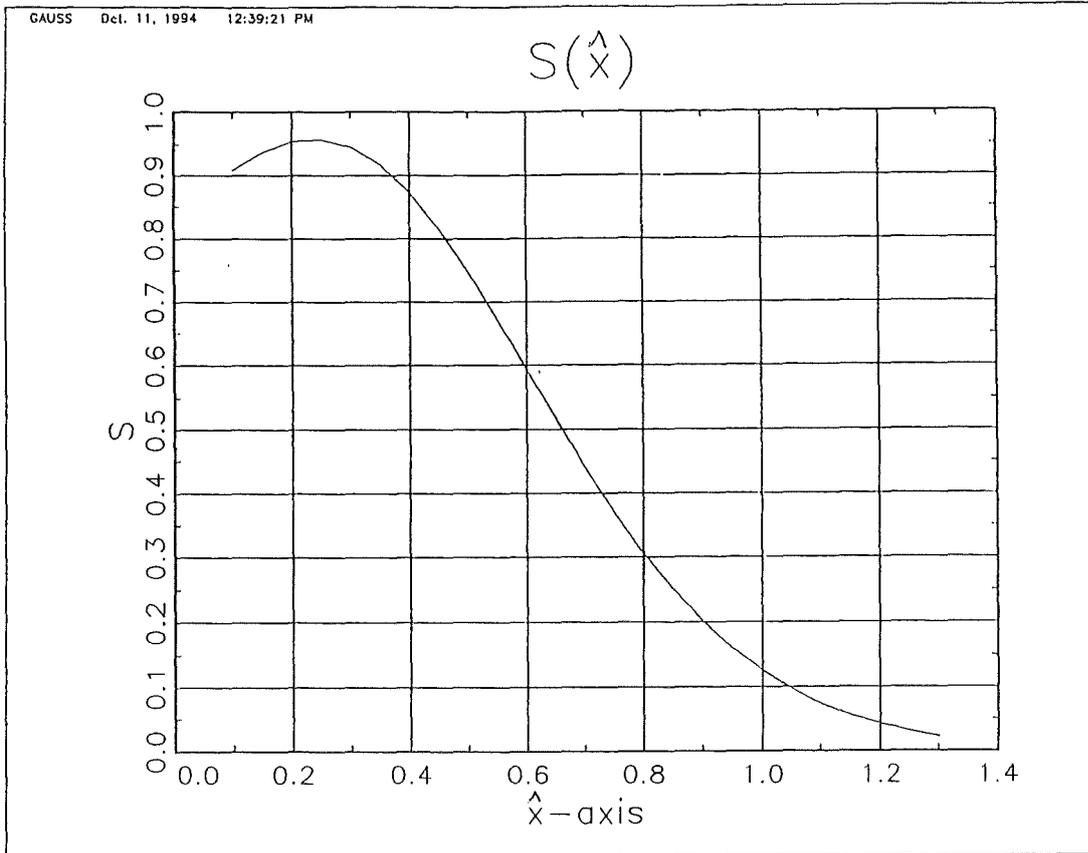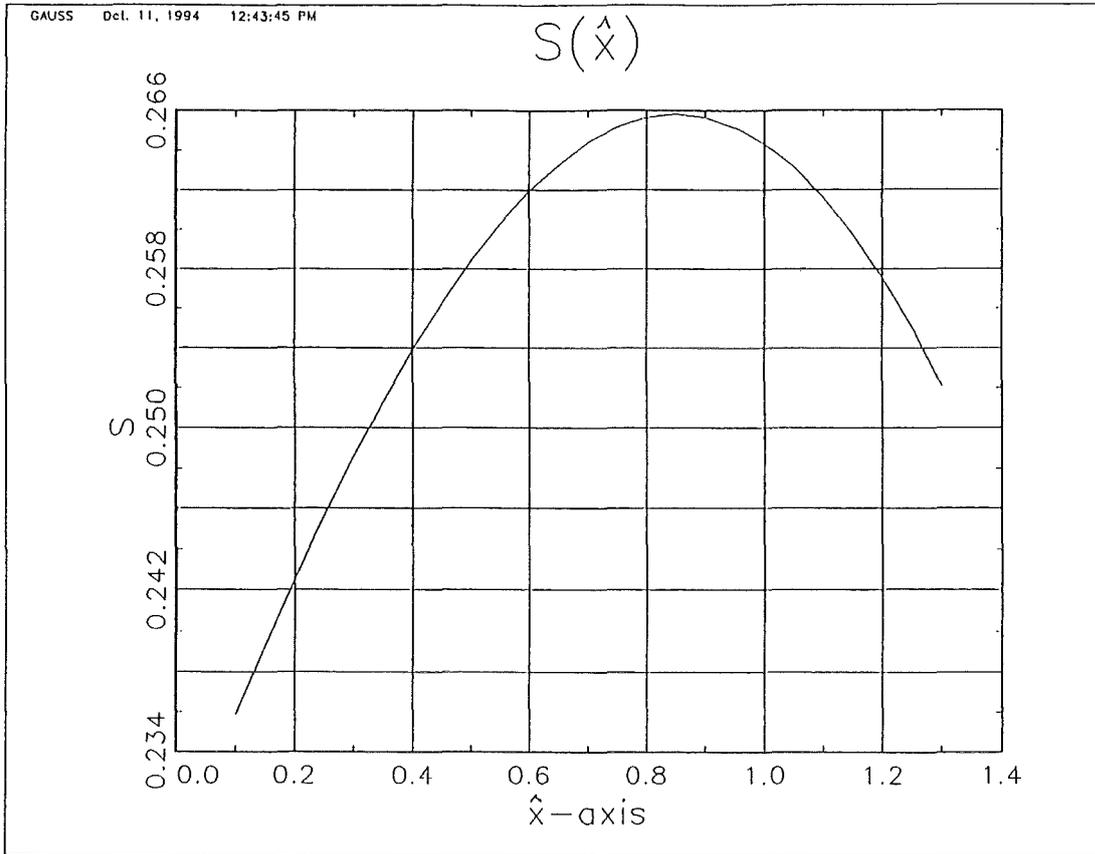
Fig. 7-11 $S(\hat{x}, \hat{t})$ on $\hat{x}$-axis | Nc = 1.0
_____ (i, j) = (25,25)
Fig. 7-12 $S(\hat{x}, \hat{t})$ on $\hat{t}$-axis
_____

Saturation S
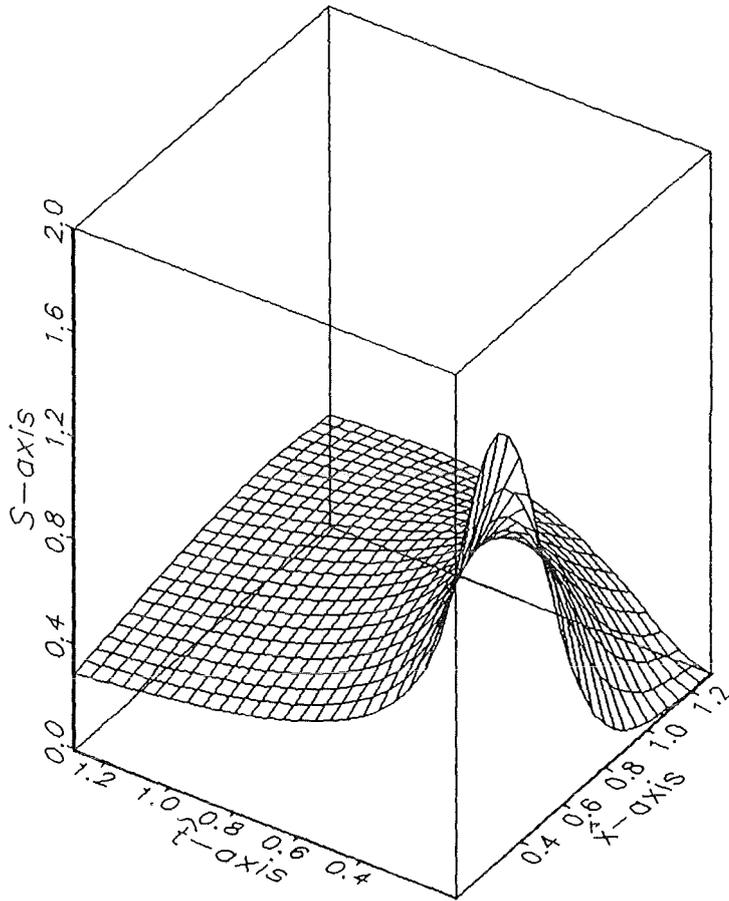
Fig. 7-13   Saturation $S(\hat{x}, \hat{t})$ in 3-D, Nc = 2.0

S on $\hat{x},\hat{t}$-plane
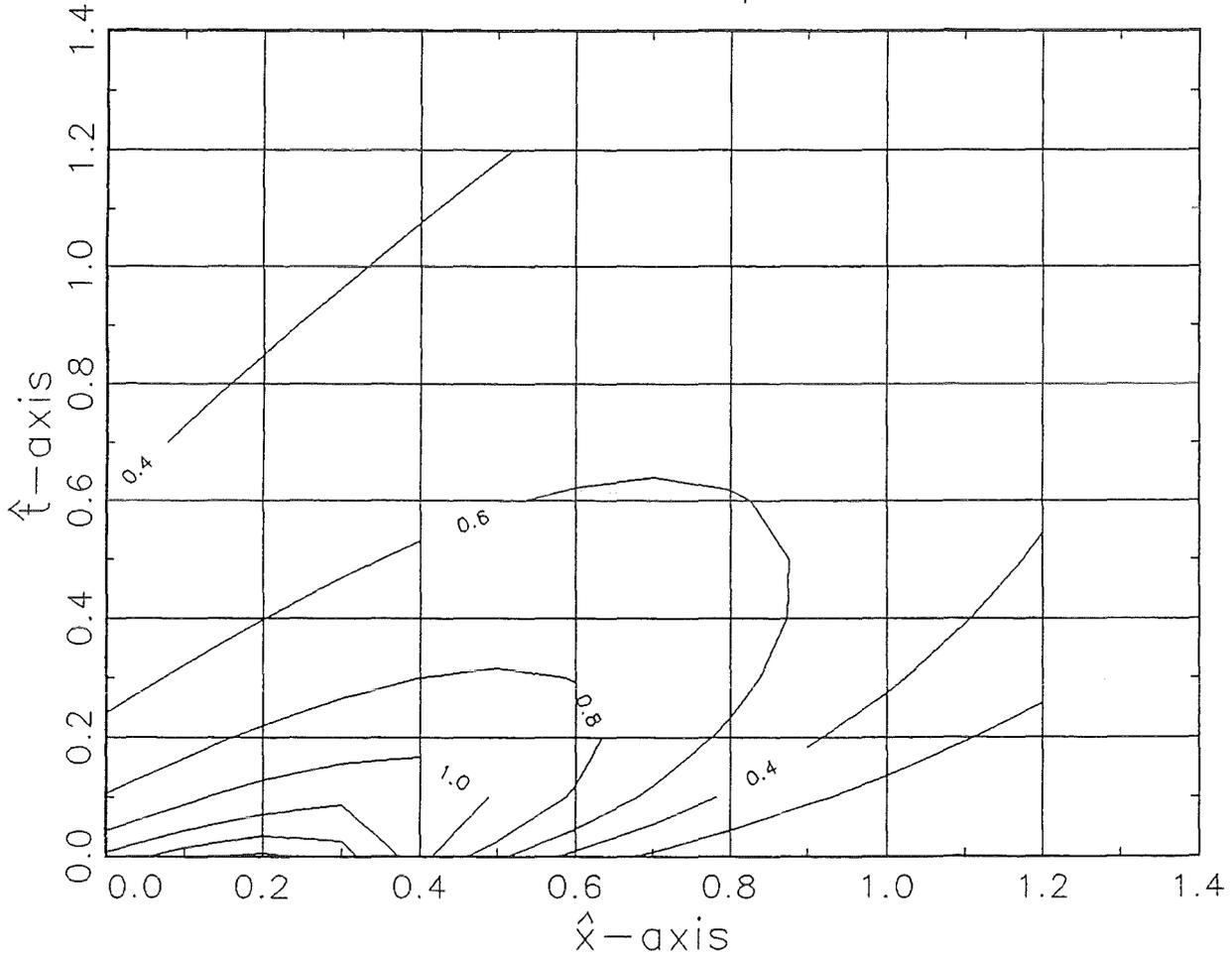
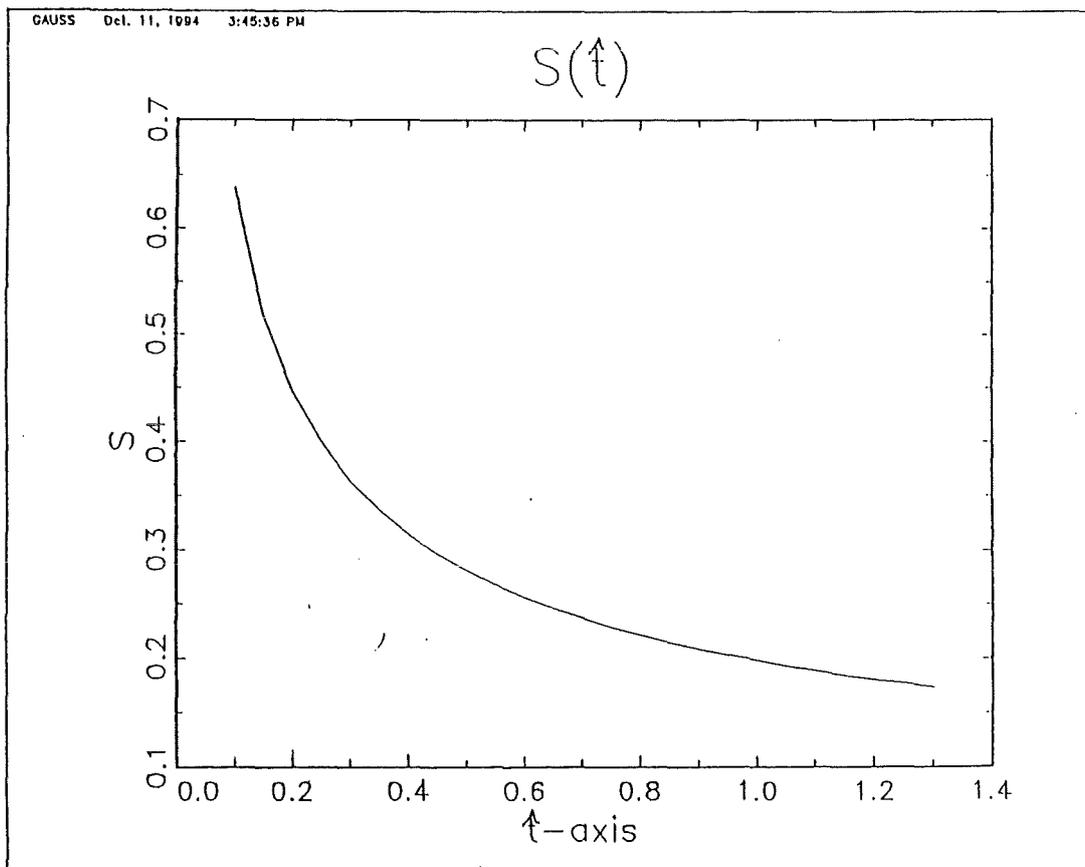Fig. 7-14   Projection of $S(\hat{x}, \hat{t})$ on $\hat{x}$-$\hat{t}$-plane
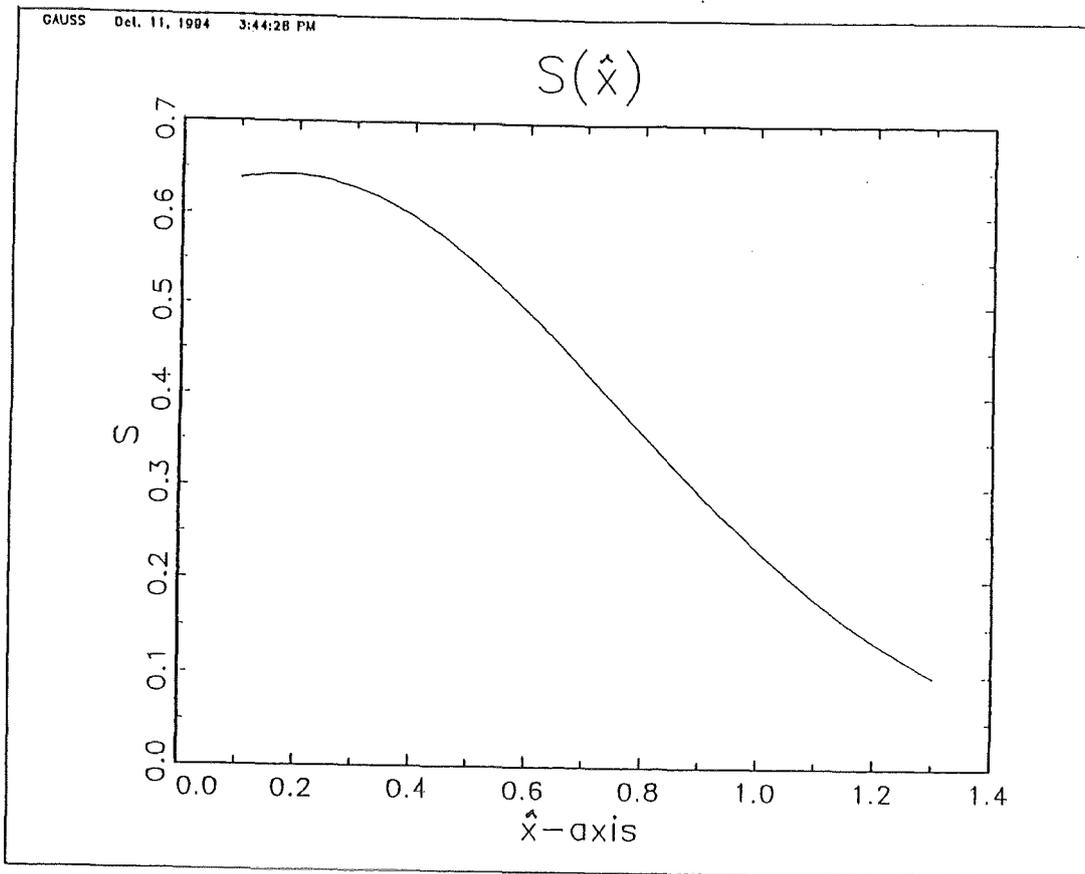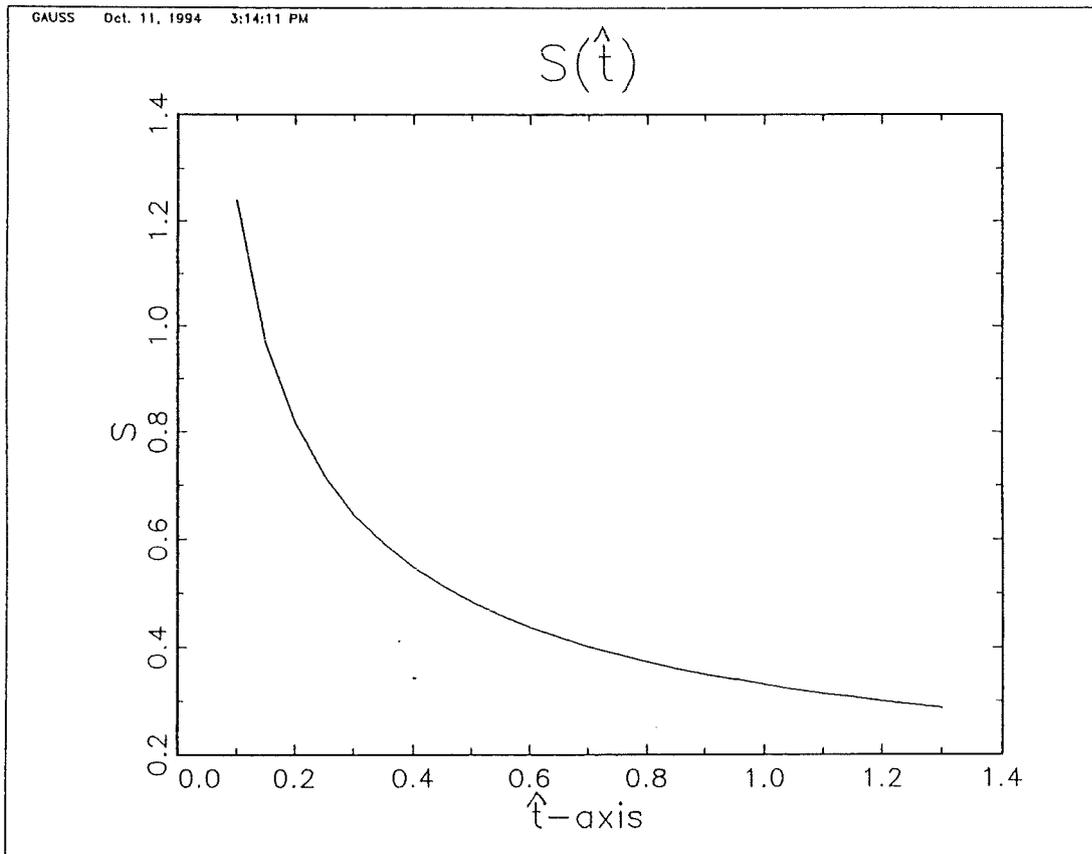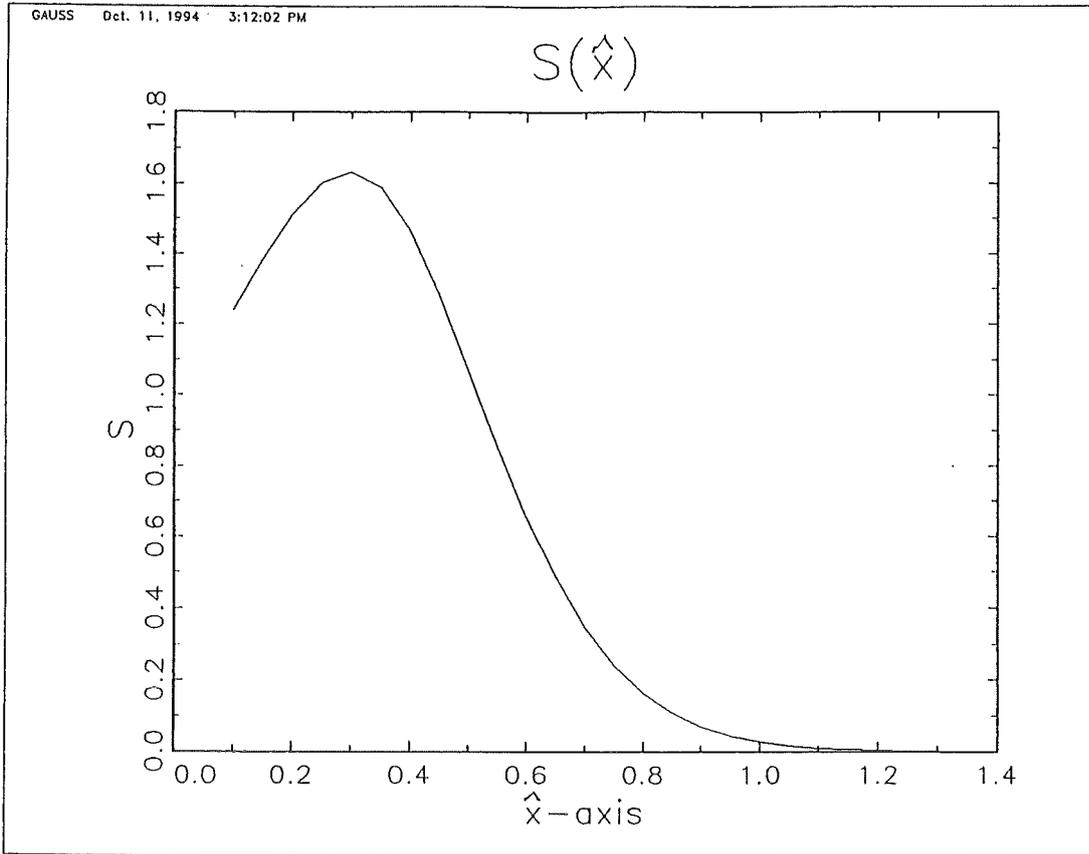
Fig. 7-15   S($\hat{x}$, $\hat{t}$) on $\hat{x}$-axis

Fig. 7-16   S($\hat{x}$, $\hat{t}$) on $\hat{t}$-axis
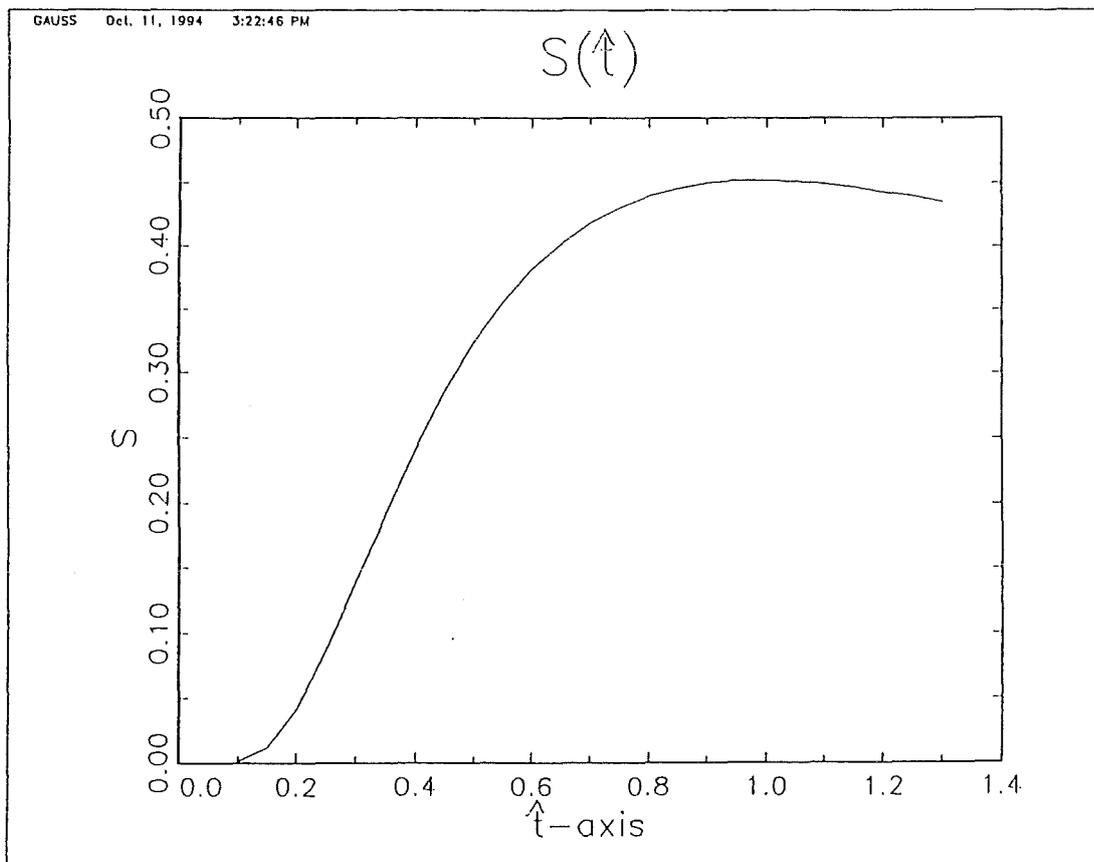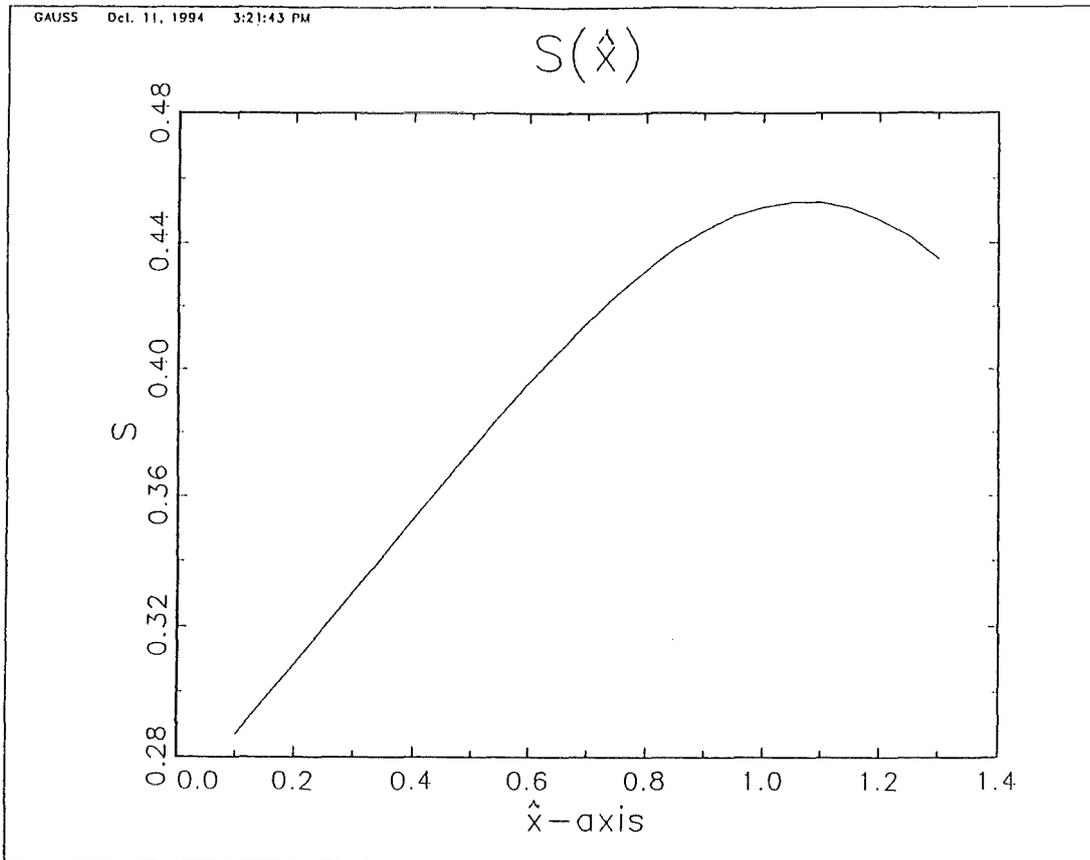
Nc   = 2.0
(i, j) = (1,1)

Fig. 7-17   S($\hat{x}$, $\hat{t}$) on $\hat{x}$-axis

Fig. 7-18   S($\hat{x}$, $\hat{t}$) on $\hat{t}$-axis

Nc   = 2,0
(i, j) = (25, 25)
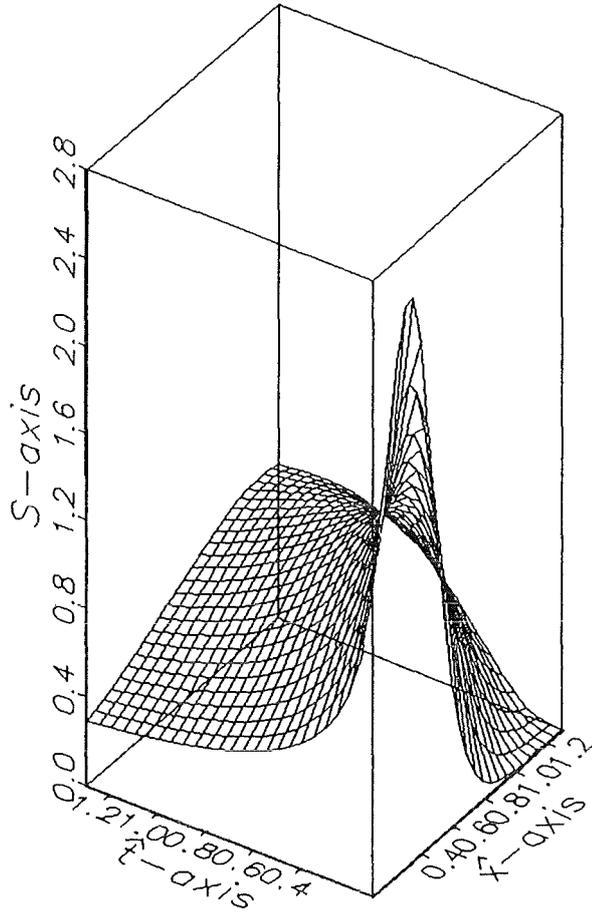
Saturation S

Fig. 7-19 Saturation S($\hat{x}$, $\hat{t}$) in 3-D, Nc = 4.0

S on $\hat{x}, \hat{t}$-plane
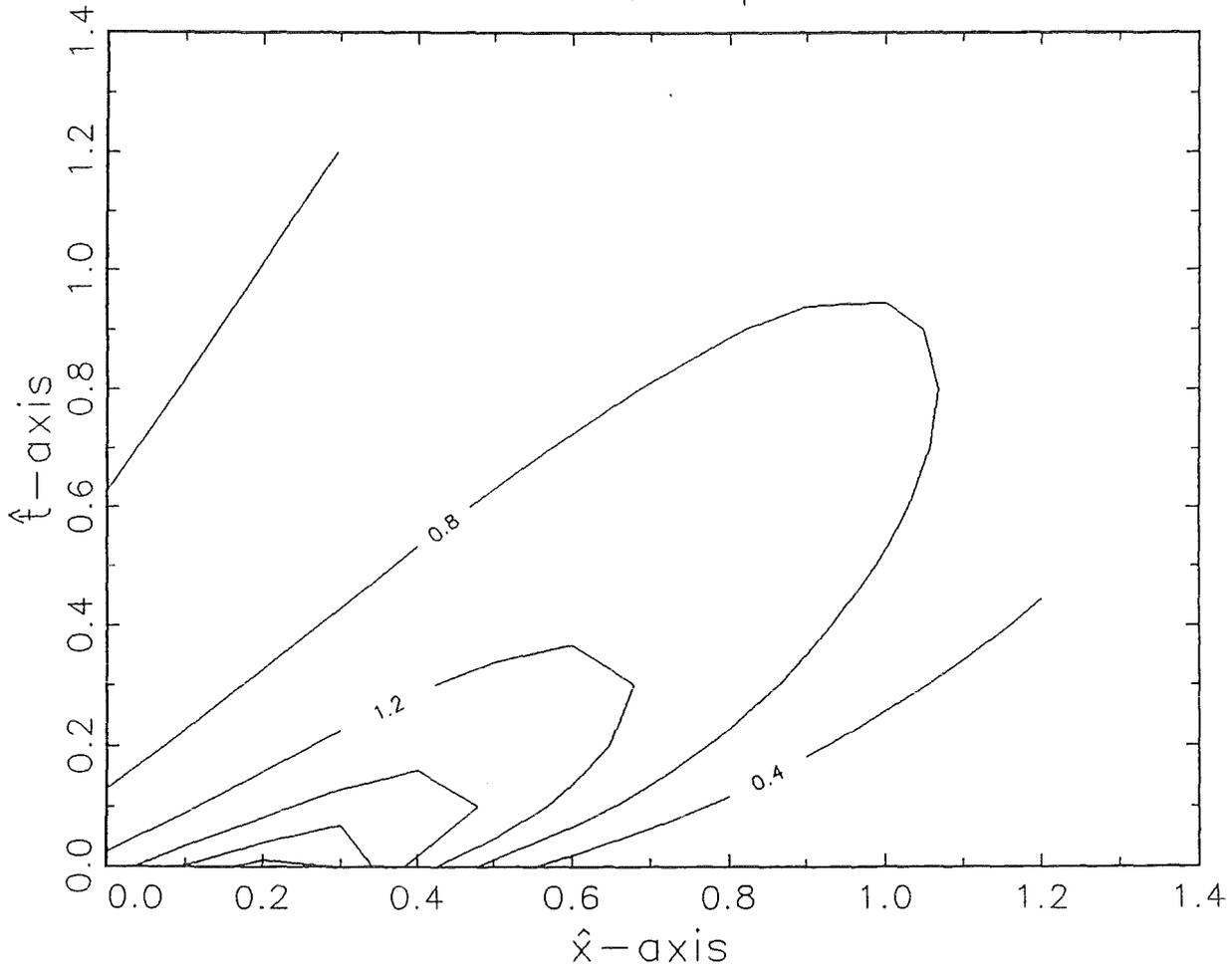
Fig. 7-20   Projection of $S(\hat{x}, \hat{t})$ on $\hat{x}$-$\hat{t}$-pane

$S\left(\hat{x}\right)$

$S\left(\hat{t}\right)$
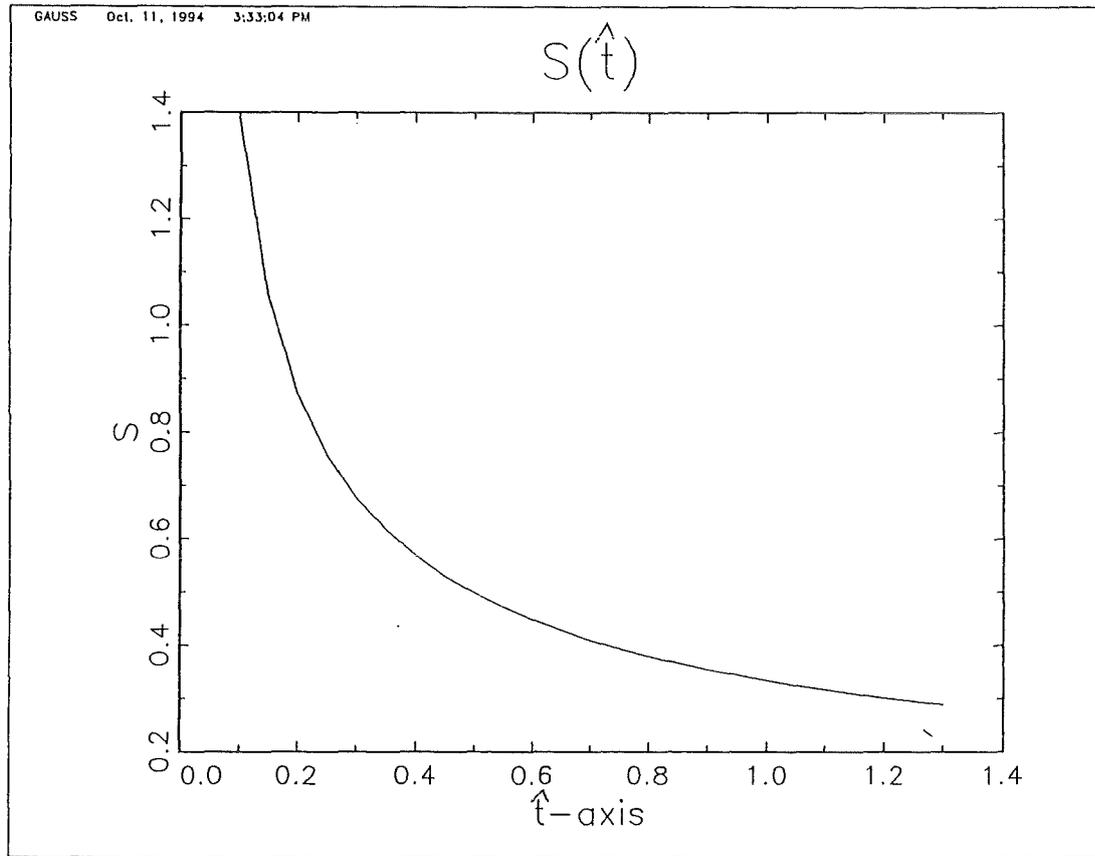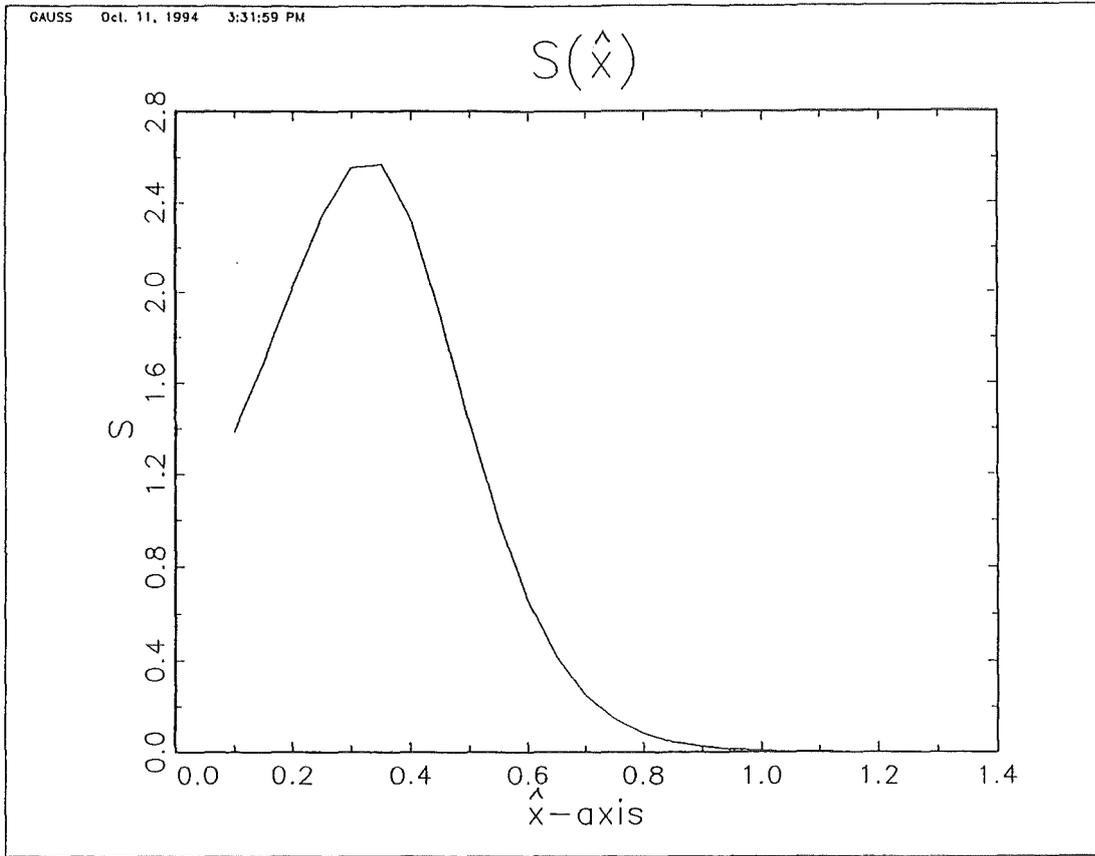
Fig. 7-21 $S(\hat{x}, \hat{t})$ on $\hat{x}$-axis | Nc   = 4.0
--------   (i, j) = (1, 1)
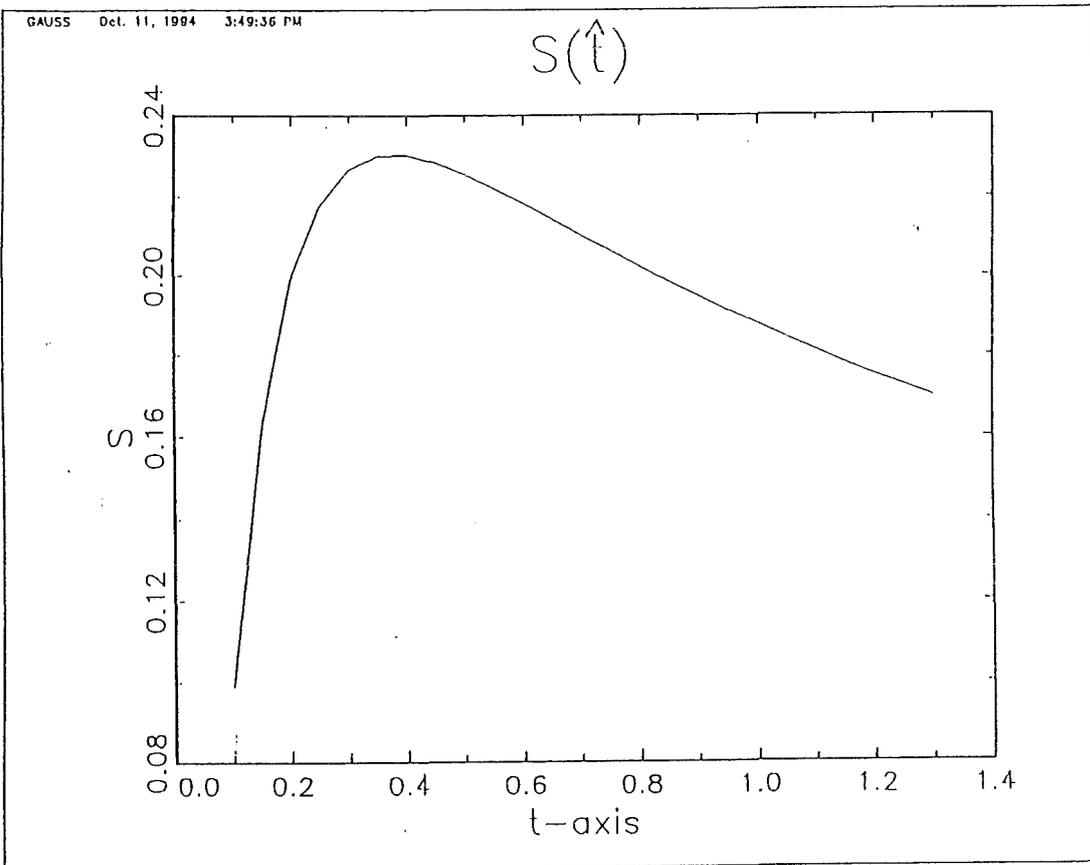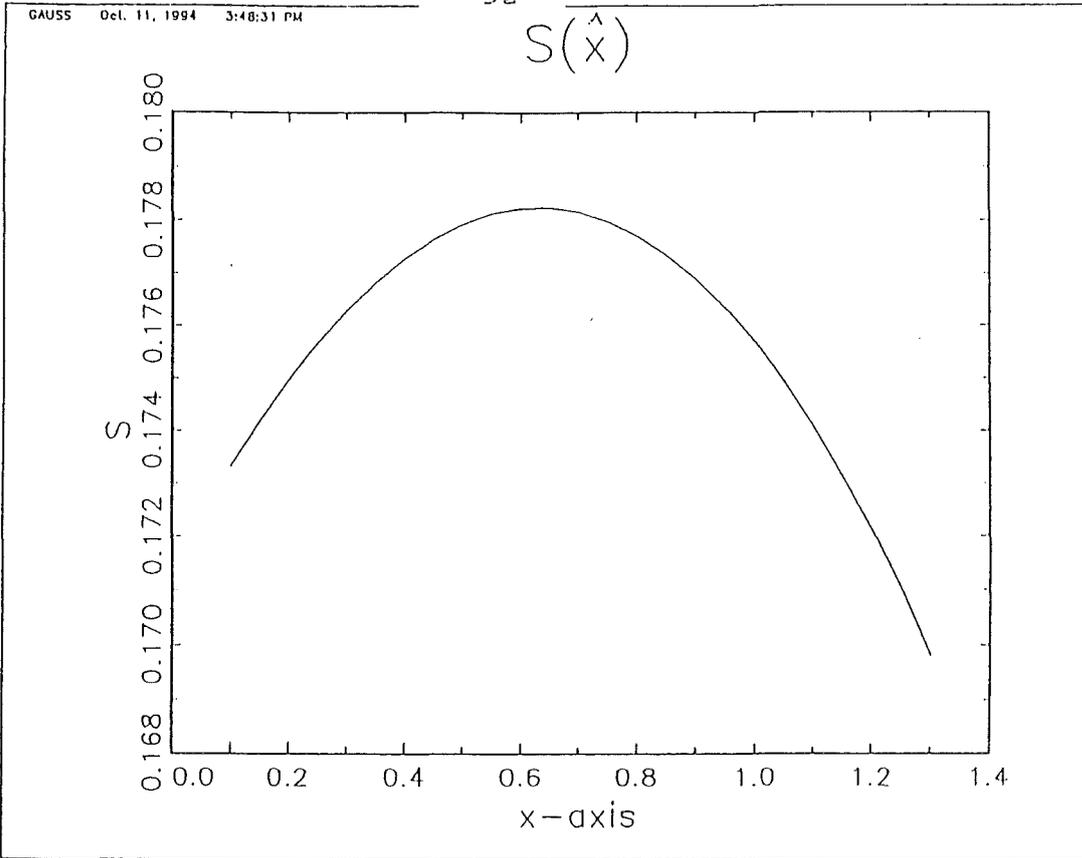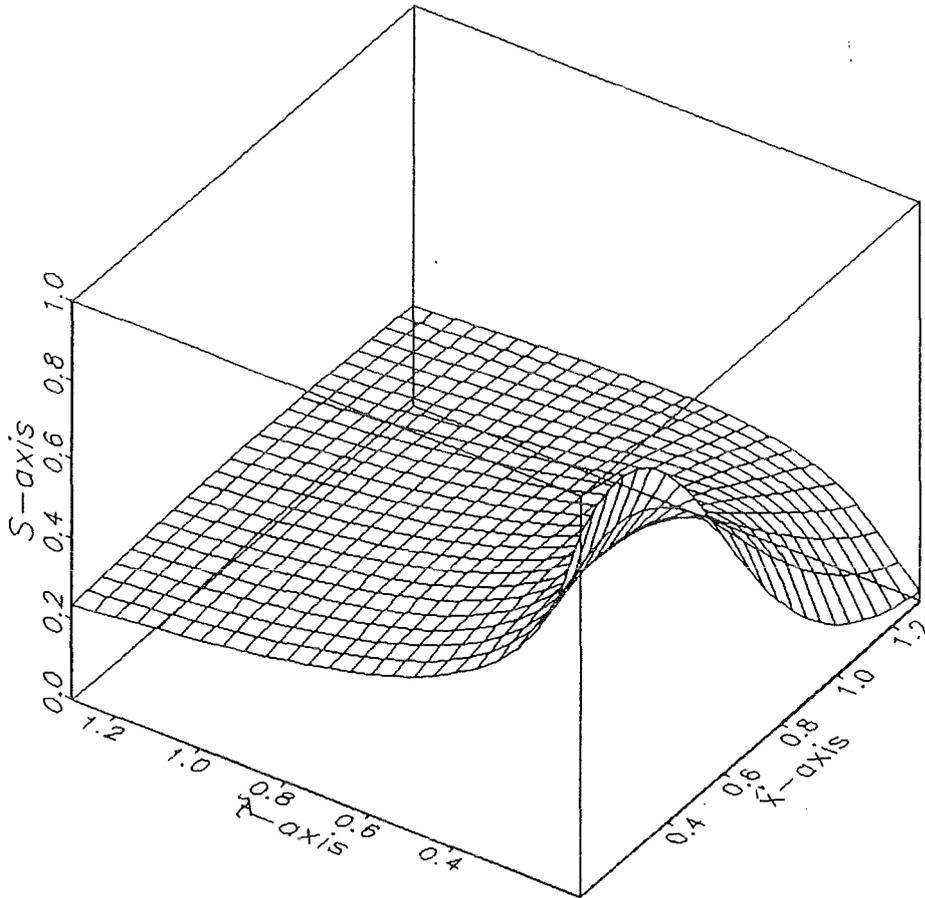Fig. 7-22 $S(\hat{x}, \hat{t})$ on $\hat{t}$-axis |

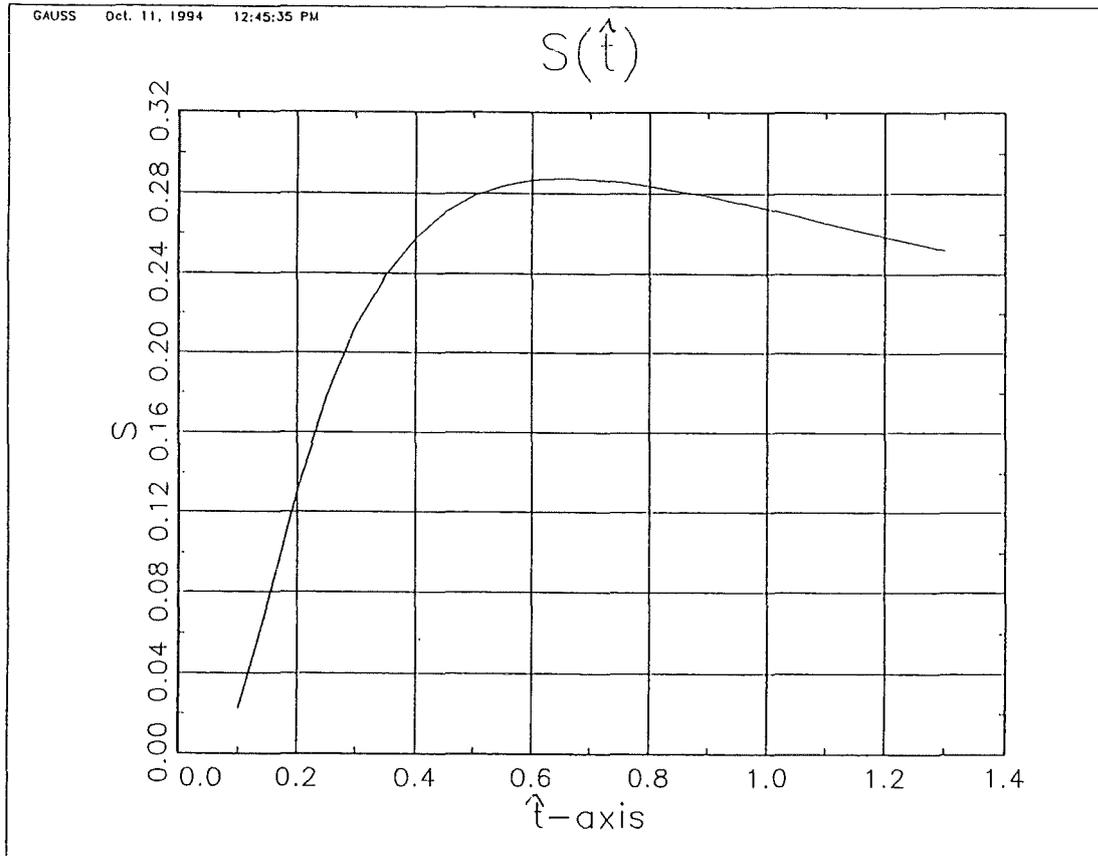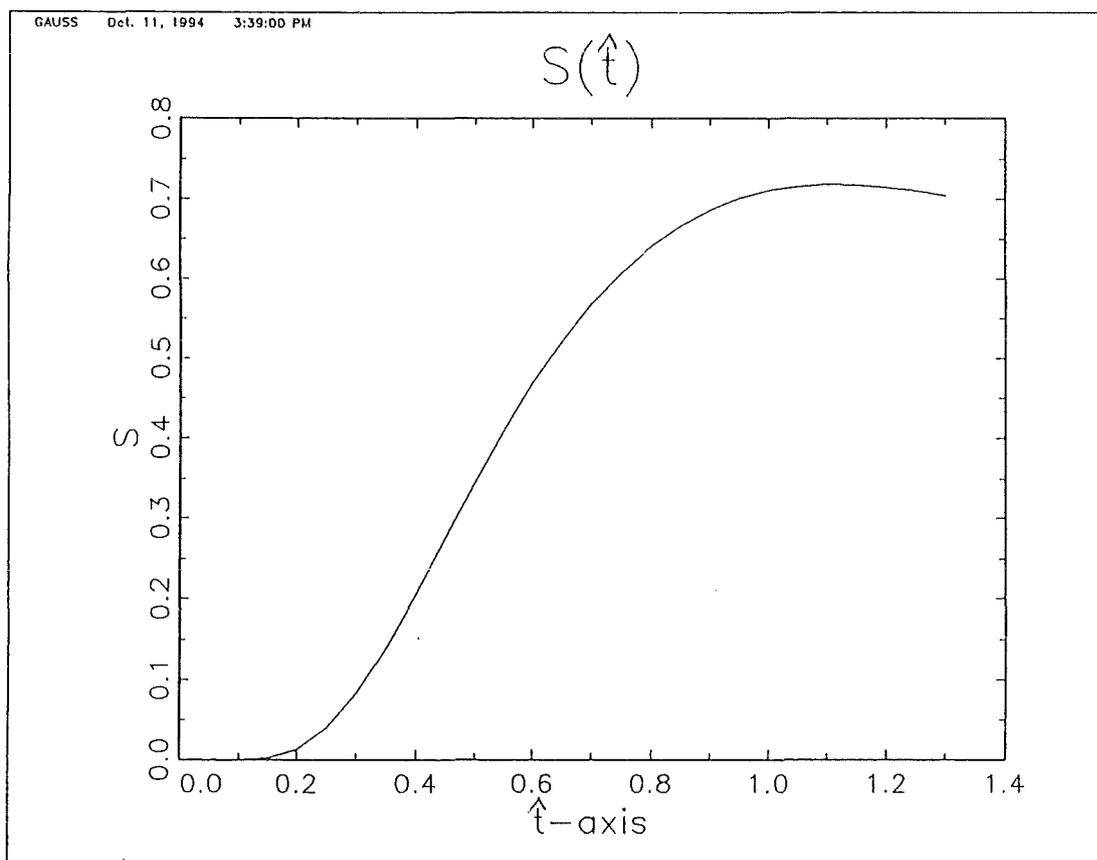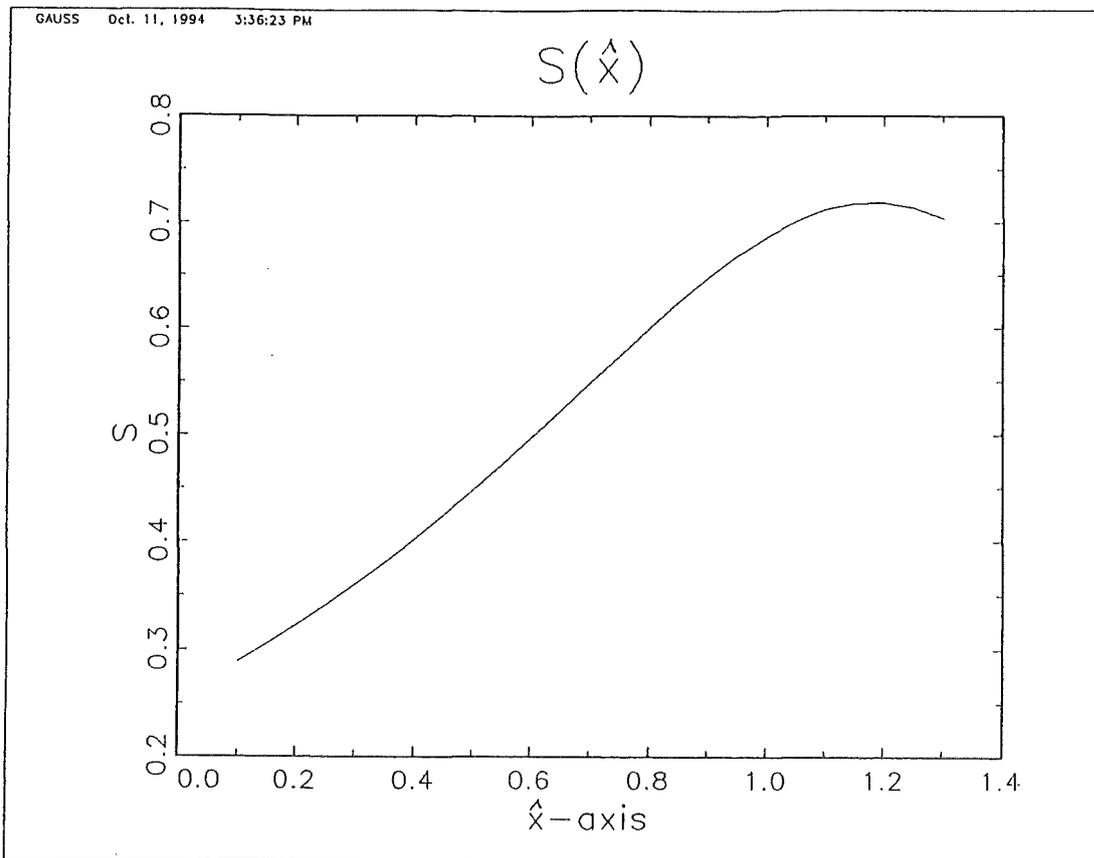
Fig. 7-23 $S(\hat{x}, \hat{t})$ on $\hat{x}$-axis Nc = 4.0
---------- $(i, j) = (25, 25)$
Fig. 7-24 $S(\hat{x}, \hat{t})$ on $\hat{t}$-axis

Nc	x	Smax(x)
0.10	0.250000	0.0892738165
0.20	0.350000	0.1265386591
0.30	0.425000	0.1555608087
0.40	0.500000	0.1805685617
0.50	0.550000	0.2032302626
0.60	0.600000	0.2244267202
0.70	0.650000	0.2446936341
0.80	0.675000	0.2643973328
0.90	0.725000	0.2837949587
1.00	0.750000	0.3031003858
2.00	0.950000	0.5162286029
3.00	1.025000	0.8200895435
4.00	1.100000	1.2917354470
5.00	1.150000	2.0387377665
6.00	1.175000	3.2371430294
7.00	1.200000	5.1612333725
8.00	1.225000	8.2536617260
9.00	1.225000	13.2579576751
10.0	1.250000	21.3586296377

INPUT VALUES

```
Initial value xo    = 0.100
initial value to    = 1.000
length of steps     = 0.025
Number of steps   n = 80.00
```

Table 7-2 Maxima for S(x) with different Nc

8. Solutions for different parameters

8.1 Introduction

As has been shown in sect. 4.7 an ODE is the final step of the similarity solution, based on the transport equation (convection equation) introduced in sect. 4.3. We recall this ODE here:

$$f' = Nc*(f^n - b* \xi *g^{(1+m-n)}) \qquad (4-46)$$

This equation gives, due to different values for exponents m, n, different solutions. The most interesting of them will be given in the following sections.

8.2 Exponents m, n larger than 1

Here we have the conditions

$$m = n \qquad \text{and} \qquad m, n > 1 \qquad (8-1)$$

It may be noted that the case

$$m = n \qquad \text{and} \qquad m = 2, n = 2 \qquad (8-2)$$

which is related to the Riccatti-equation has been discussed in detail in sects. 4 to 7.
For conditions (8-1) we have the following similarity solution:

$$f(\xi) = \frac{\exp(- (Nc/4)* \xi^2)}{[f^{(1-n)}(0) - \sqrt{(\pi *Nc*(n-1))}*erf((1/2)* (Nc*(n-1))*\xi]^{1/(n-1)}} \qquad (8-3)$$

Examples of (8-3) are shown in the following Tables and Figures:

In Tables 8-1, 8-2, 8-3

we show
- the maximum of $f(\xi)$ (similarity solution for saturation) for various exponents n
- the coordinates of horizontal and vertical tangent for the flux Js, (see also "group velocity", sect. 6.2, interpretation of flux curve)

In Figs. 8-1 to 8-6

we show
- the similarity solution $f(\xi)$ (transformed saturation)
- the shape of the flux Js.

Results
----

	n = 2	4	6	8
Nc =				
1	Fig. 8-1	8-2	-	-
2	Fig. 8-3	8-4	8-5	8-6

Interpretation

(a) $f(\xi)$: For increasing Nc there is for the curves shown here, an increase of the maximum of $f(\xi)$, similar to the property of the solutions shown for the case n = 2.

(b) $f(\xi)$: For increasing exponent n there is a decrease of the maximum of $f(\xi)$.

(c) Js: For increasing n there is a considerable change in the shape of Js (flux).

(d) Js: For increasing n there is a considerable increase in the maximum of Js.

It has to be noted, that for some regions of Nc where Nc << 1, the same relation clearly holds (increasing max. with increasing Nc). But there are also regions, where Nc >> 4, where the relation becomes more complicated, where no monotonicity exists.

8.3 Exponents m, n smaller than 1

Here we have for (4-46) also the following conditions

$$m = n \quad \text{and } m < 1, \, n < 1 \qquad (8-4)$$

It can be shown that the results for region (8-4) are different from the results of sect. 8.2:

$$f(\xi) = (f(0)*\exp(-\eta^2) + 2*\sqrt{(1-n)} * D(\eta)) \qquad (8-5)$$

We note the following conventions:

$$\eta = \frac{Nc*(1-n)}{2} * \xi \tag{8-6}$$

where again ξ is the similarity variable. We also note, that $D(\eta)$, which is usually called Dawsons Integral (see[Abramowitz]).

$$D(\eta) = \exp(-\eta^2) * \int_0^\eta \exp(\eta'^2)*d\eta' \tag{8-7}$$

$D(\eta)$ is closely related to the error function, multiplied with an term which will cause that $D(\eta)$ ---> 0 for η ---> ∞.

In Table 8-4 and Figs. 8-7 ff.

we show some properties of the similarity solution $f(\xi)$.

Results

	n = -4	-1	0	0.5
Nc = 1,2,3,4, 5,6,7,8	Fig. 8-7	8-8	8-9	8-10

Interpretation

(a) $f(\xi)$: The maximum of $f(\xi)$ remains almost constant with changing Nc.

(b) $f(\xi)$: For increasing Nc the abszissa of the maximum is is shifting to the left.

(c) $F(\xi)$: For increasing n the Maximum of $f(\xi)$ is decreasing.

8.4 Considerations for parameters

We do not discuss here other possibilities for the parameters, e.g. cases where m > n, or m < n holds.
For more details on the relations between exponents and the type of percolation, see Appendix A4.

fc
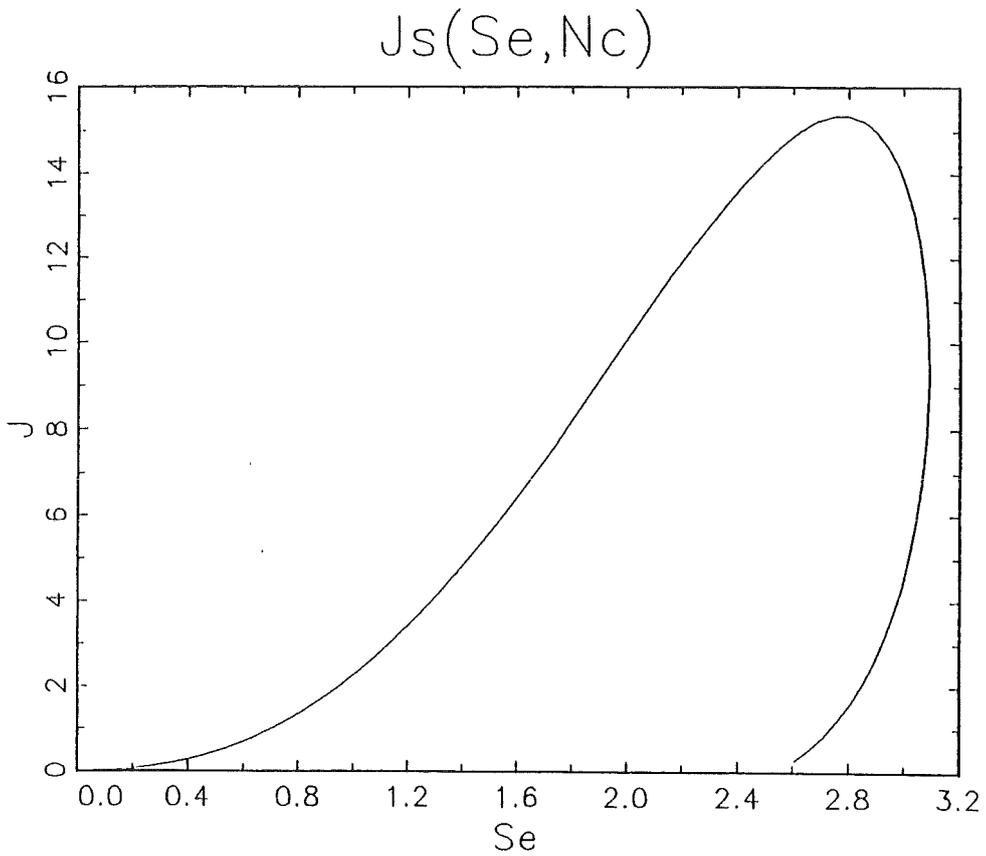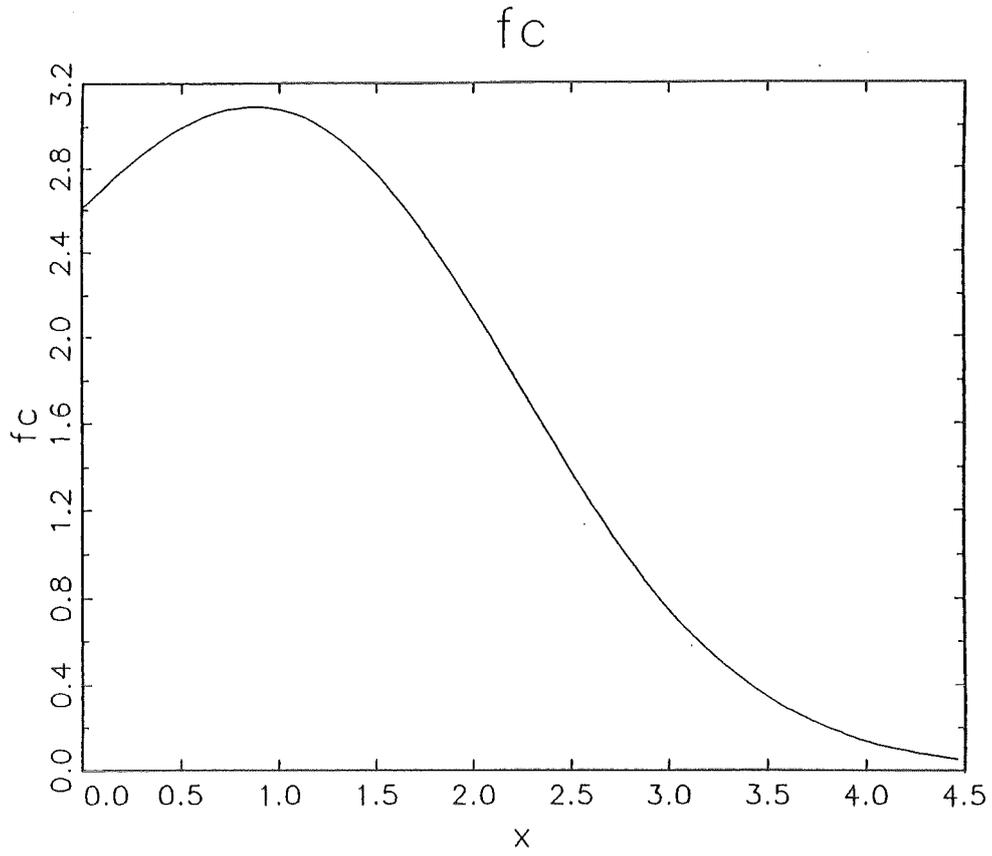
Js(Se,Nc)

Fig. 8-1   For given n: (a) f(ξ), reduced saturation

(b) saturation flux

fc

Js(Se,Nc)

Fig. 8-2   For given n: (a) f(ξ), reduced saturation

(b) saturation flux

fc

Js(Se,Nc)

Fig. 8-3 For given n: (a) f(ξ), reduced saturation

(b) saturation flux

fc

Js(Se,Nc)

Fig. 8-4   For given n: (a) f(ζ), reduced saturation

(b) saturation flux

fc

Js(Se,Nc)

Fig. 8-5   For given n: (a) f(ξ), reduced saturation

(b) saturation flux

fc

Js(Se,Nc)

Fig. 8-6 For given n: (a) $f(\xi)$, reduced saturation
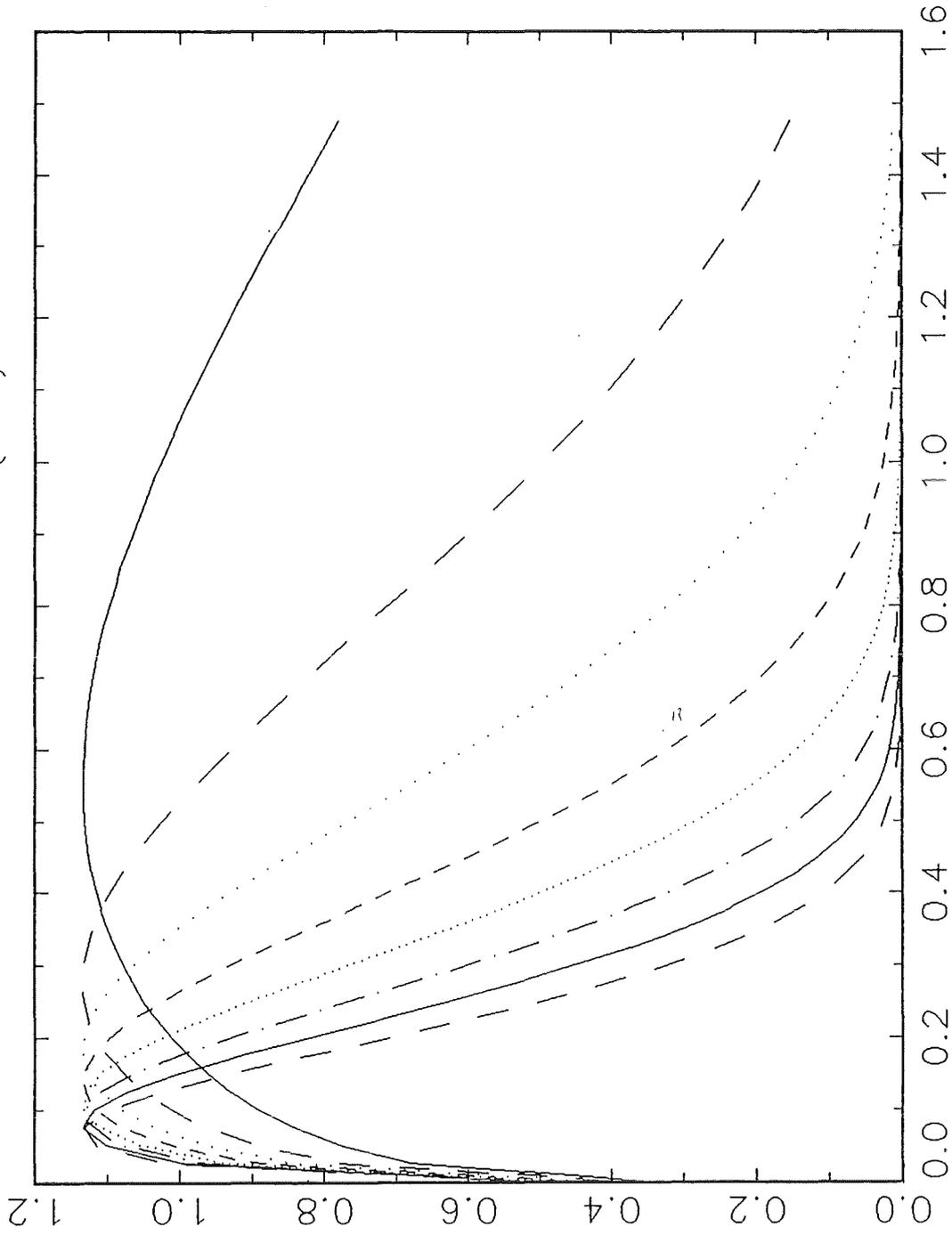
(b) saturation flux

Fig. 8-7   For given n < 1:

f(ξ), reduced saturation
(n = - 4.0)
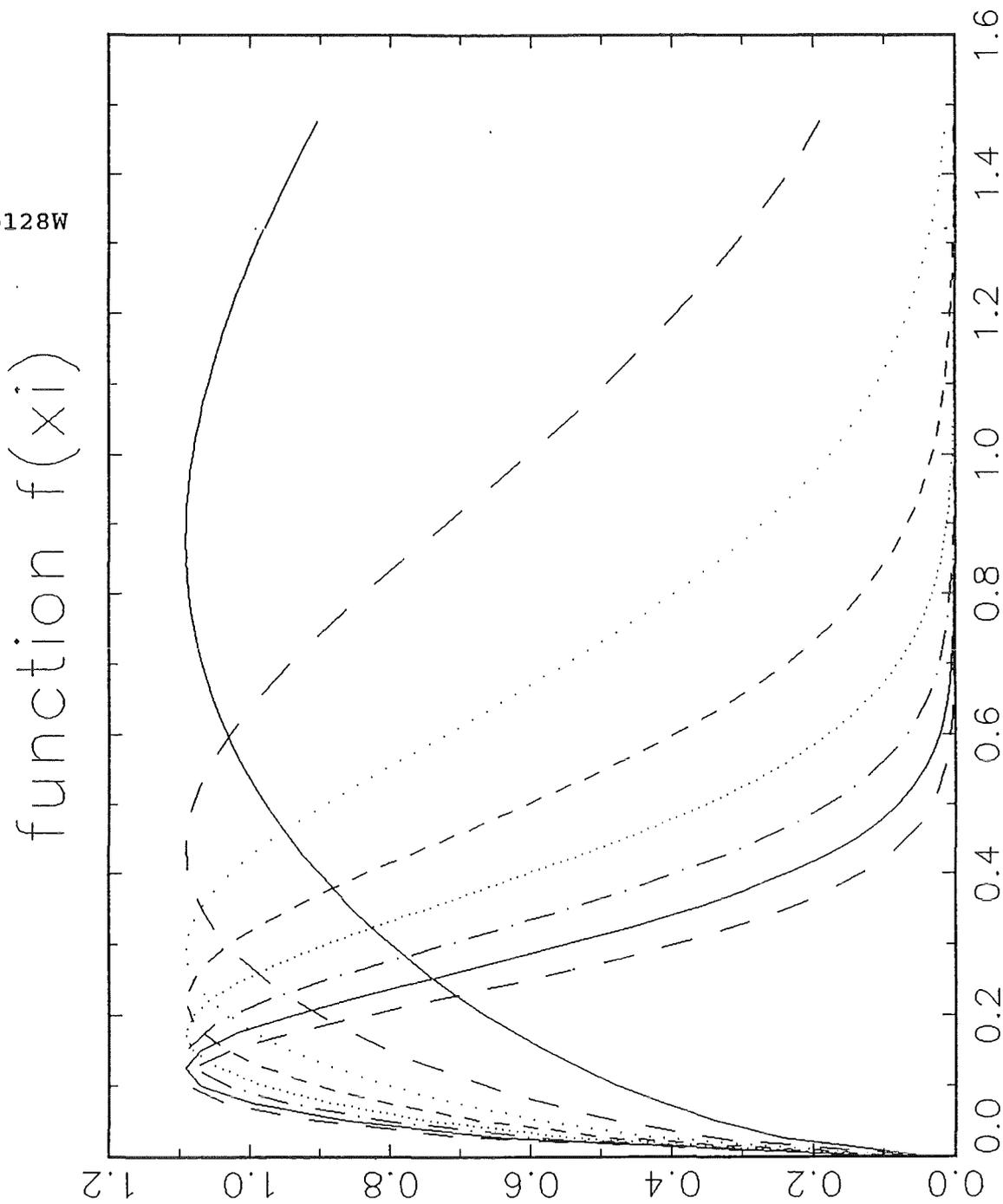
Fig. 8-8   For given n < 1:
----

f(ξ), reduced saturation
(n = - 1)
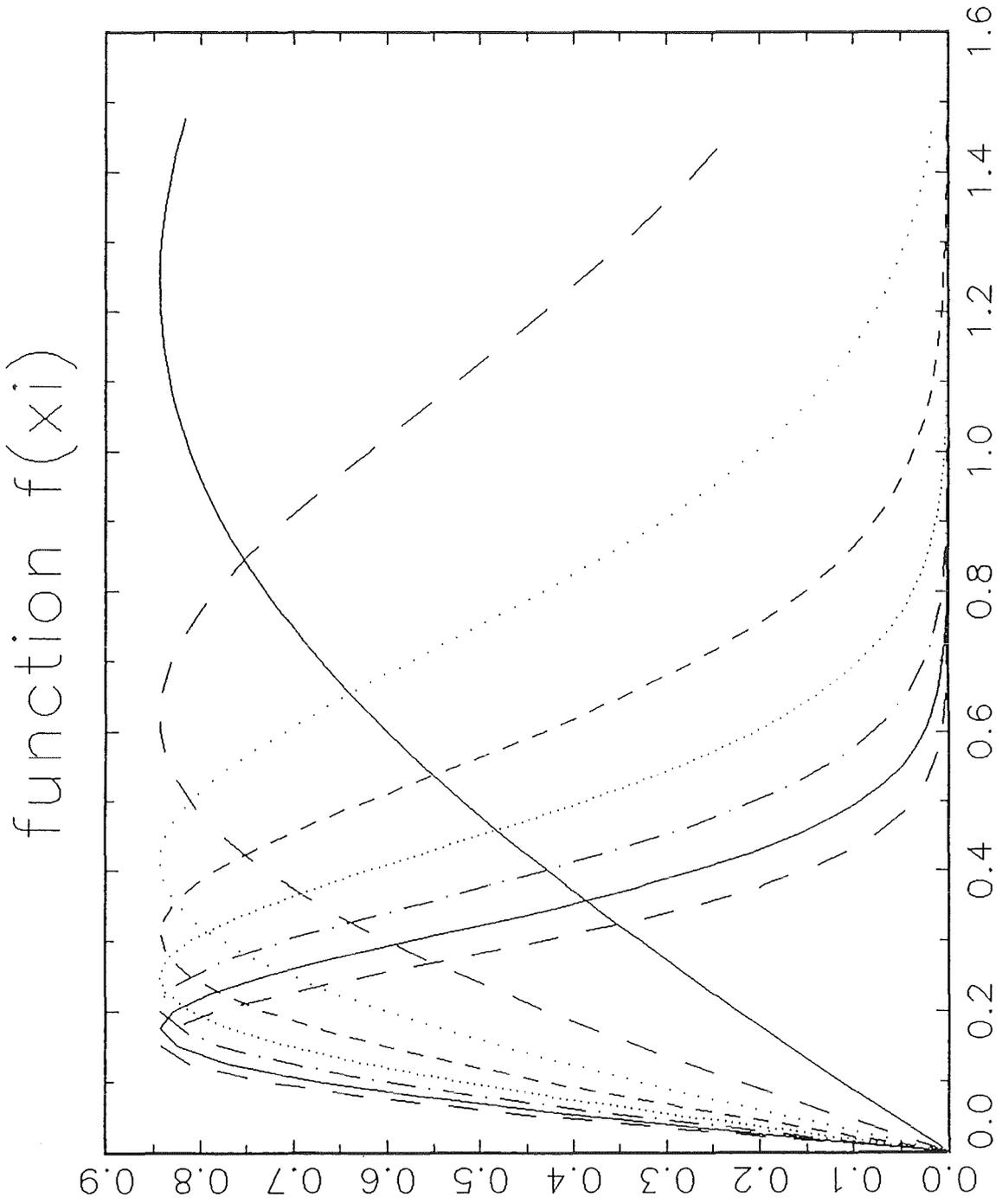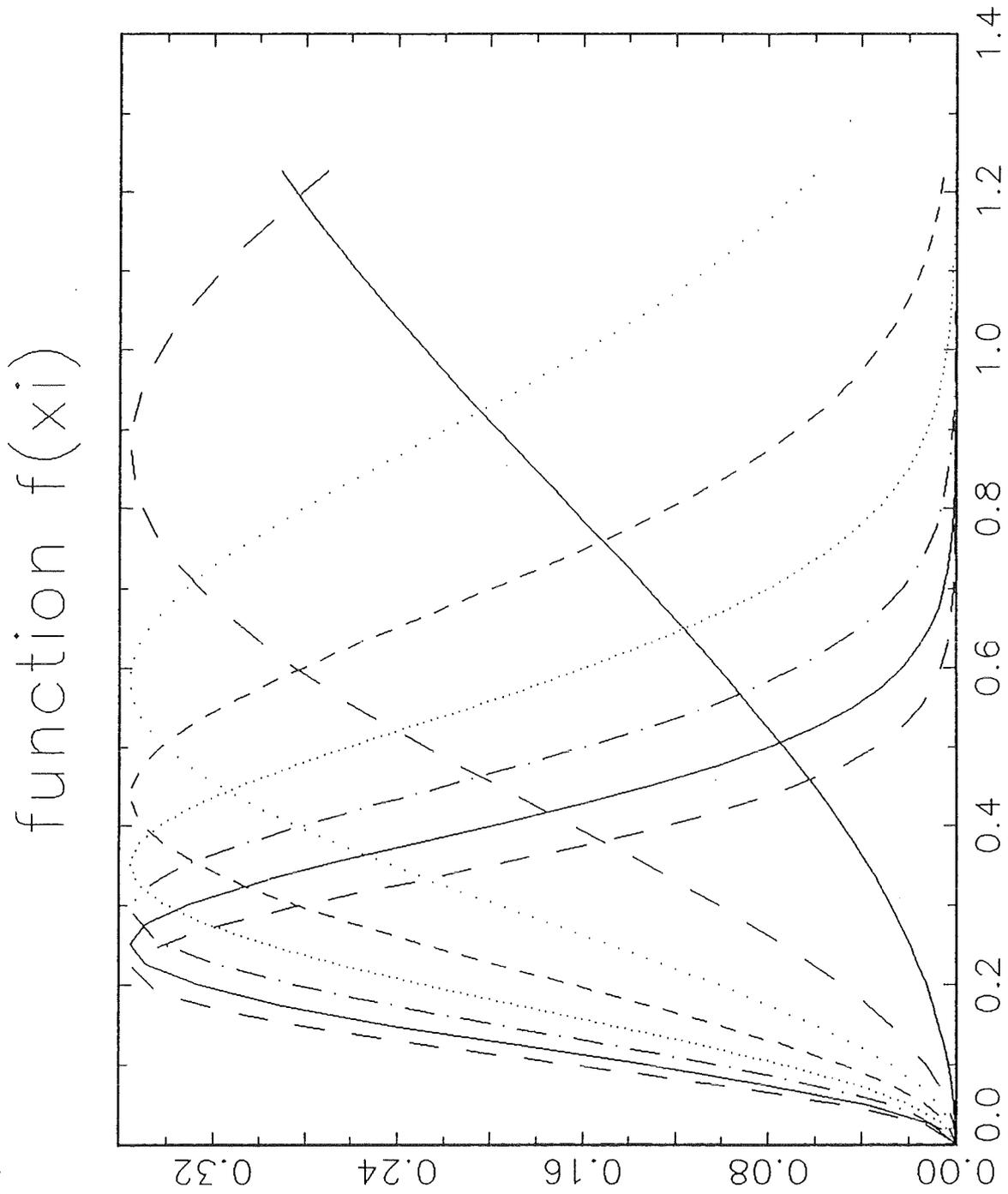
Fig. 8-9   For given n < 1:
----

f(ξ), reduced saturation
(n = 0.0)

Fig. 8-10   For given n < 1:
----

f(ξ), reduced saturation
(n = 0.5)

EXPONENT	RESULTS
n = 2.0 m = n	Nc = 1.0

	EXTREME VALUE
	
	(A) for f(xi) (reduced saturation)
	
	     max of f(xi) = 3.0895
	
	(B) for Is(Se) (saturation flux)
	
	     max of Is(Se) = 15.3427
	
	     max of Se = 3.0895
	     (same number as for (A))

n = 4.0 m = n	Nc = 1.0

	EXTREME VALUE
	
	(A) for f(xi) (reduced saturation)
	
	     max of f(xi) = 2.6137
	
	(B) for Is(Se) (saturation flux)
	
	    max of Is(Se) = 60.1139
	
	     max of Se = 2.6137
	     (same number as for (A))

Table 8-1 Solution (8-3): Reduced saturation and saturation flux for different parameters n (See also Figs. 8-1 and 8-2)

EXPONENT	RESULTS
n = 2.0 m = n	Nc = 2.0

	EXTREME VALUE
	
	(A) for f(xi) (reduced saturation)
	
	      max of f(xi) = 4.4033
	
	(B) for Is(Se) (saturation flux)
	
	      max of Is(Se) = 50.7146
	
	      max of Se = 4.4033
	      (same number as for (A))

n = 4.0 m = n	Nc = 2.0

	EXTREME VALUE
	
	(A) for f(xi) (reduced saturation)
	
	      max of f(xi) = 2.6202
	
	(B) for Is(Se) (saturation flux)
	
	    max of Is(Se) = 69.9632
	
	      max of Se = 2.6202
	      (same number as for (A))

Table 8-2 Solution (8-3): Reduced saturation and
saturation flux for different parameters n
(See also Figs. 8-3 and 8-4)

EXPONENT	RESULTS
n = 6.0 m = n	Nc = 2.0

	EXTREME VALUE
	
	(A) for f(xi) (reduced saturation)
	
	      max of f(xi) = 2.6072
	
	(B) for Is(Se) (saturation flux)
	
	      max of Is(Se) = 425.9975
	
	      max of Se = 2.6072
	      (same number as for (A))

n = 8.0 m = n	Nc = 2.0

	EXTREME VALUE
	
	(A) for f(xi) (reduced saturation)
	
	      max of f(xi) = 2.6072
	
	(B) for Is(Se) (saturation flux)
	
	    max of Is(Se) = 2749.1319
	
	      max of Se = 2.6072
	      (same number as for (A))

Table 8-3 Solution (8-3): Reduced saturation and

saturation flux for different parameters n
(See also Figs. 8-5 and 8-6)

EXPONENT   RESULTS

n = - 4.0	Nc	Max. for f(xi;Nc)
m = n < 1		
	1.0	1.1353
	2.0	1.1353
	3.0	1.1348
	4.0	1.1336
	5.0	1.1335
	6.0	1.1335
	7.0	1.1350
	8.0	1.1333
	first, last xm value	
	0.5750 0.1000	
n = -1.0		
m = n < 1		
	1.0	1.0922
	2.0	1.0917
	3.0	1.0917
	4.0	1.0917
	5.0	1.0922
	6.0	1.0916
	7.0	1.0921
	8.0	1.0888
	first, last xm value	
	0.9000 0.1250	

Table 8-4 Solutions from (8-5) (exponents n < 1)

Similarity solution (red.saturation for
different n, Nc (see Fig. 8-7 and 8-8).

CONCLUSION

The main topic of the research presented here, was a better
understanding of the behavior of a system of two immiscible
phases in a porous medium. A model, based on Darcy's law as
used for two phases and on the corresponding continuity
equation, leading to a transport equation (PDE) has been used
for this study. Applying a similarity transformation reducing
the nonlinear PDE to an ODE was the main step in our analytic
approach. It is important to recall some of our results which
are closely related to mechanisms of displacement and flow
(see Table 1 of this conclusion). It can be seen, that the
following conditions are of importance to these mechanisms:

(a) Different regions of the saturation, a quantity defining
    the percentage of one phase in relation to another phase.
(b) Ratio of viscosities of two phases.
(c) Nc, the capillary number, characterizing the ratio of
    capillary forces and viscous forces.
(d) Different types of power laws, describing mechanisms which
    are typical for different types of percolation.
Now these influences are shown in Table 1:

```
                        ┌─────────────────┐
                        │  similarity     │
                        │  solution       │
                        │                 │
                        │ (ODE , formula) │
                        └────────┬────────┘
                                 │
                        ┌────────┴────────┐
                        │  different      │
                        │  conditions     │
                        └─────────────────┘
           ┌─────────────────┼─────────────────┐
┌──────────────────┐  ┌──────────────────┐  ┌──────────────────┐
│ different         │  │ Nc (capillary no.)│  │ m = n            │
│ regions of        │  │ μ2/μ1 (ratio of  │  │ power laws for   │
│ saturation S      │  │     viscosities) │  │ permeabilities   │
└──────────────────┘  └──────────────────┘  └──────────────────┘
         │                     │                     │
┌──────────────────┐  ┌──────────────────┐  ┌──────────────────┐
│ different         │  │ different         │  │ different        │
│ mechanisms:       │  │ mechanisms:       │  │ types of simil.  │
│ - imbibition      │  │ - stable          │  │ solutions:       │
│ - drain           │  │ - unstable        │  │ different        │
│ - isosaturation   │  │   displacement    │  │ types of         │
│   lines           │  │   (fingers)       │  │ flow             │
└──────────────────┘  └──────────────────┘  └──────────────────┘
```
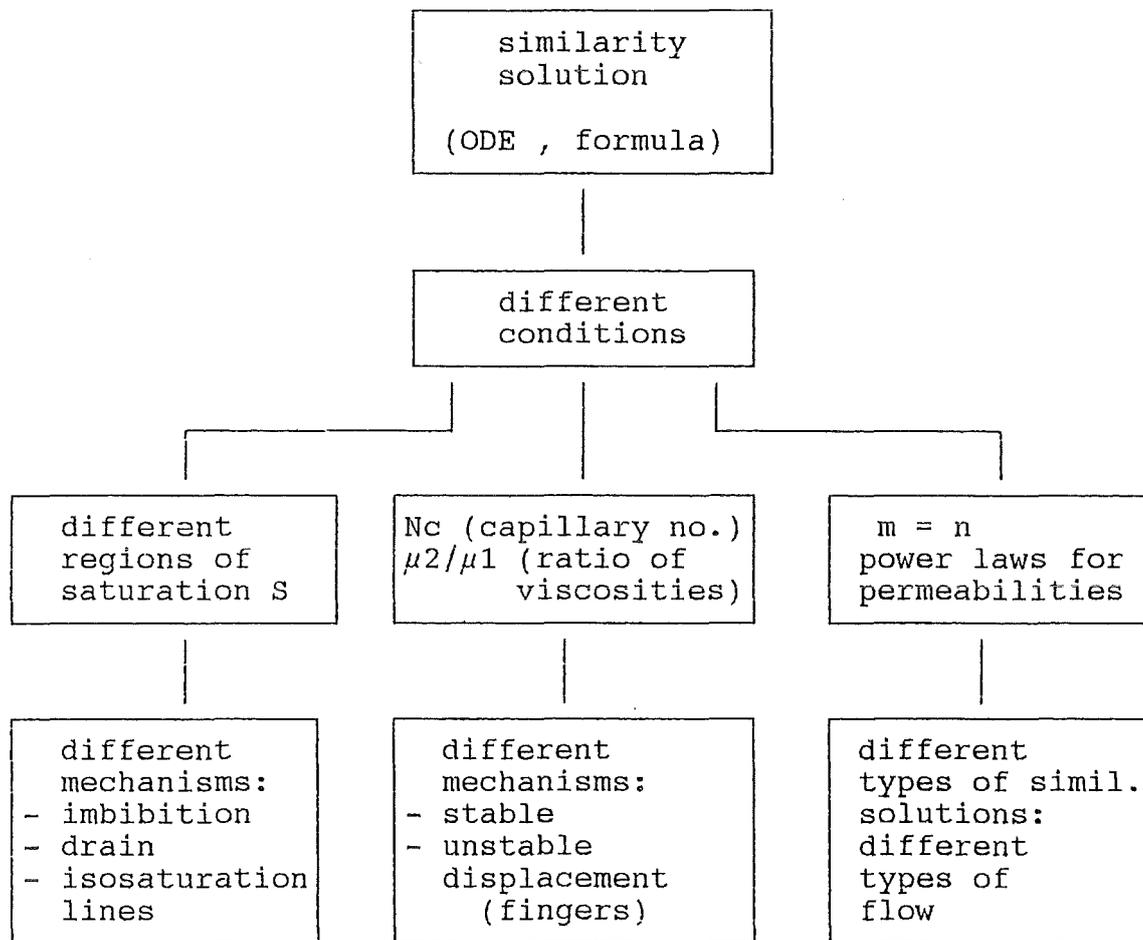
Table   Different conditions for similarity solution

As can be seen, these conditions are relevant for different
levels (such as large scale/small scale behavior).
It alsoshould be noted, that these influences are not always
independent. But it seems to be out of the scope of this
research, to study this type of dependencies in more detail.
We also note, that a more general approach such as
- a discussion of more combinations of the parameters m, n
  relating different power laws to permeabilities,
- a generalization to a movement which is no longer horizontal,
- or the use of a 2- or 3-dimensional model
will be of considerable interest. Clearly, as a first step,
the approach taken here is unavoidable.

A1 Similarity Methods for Partial Differential Equations

A1.1 Introduction

It is possible to obtain for partial differential equations
(PDE), by some methods a solution, or simplification:
- It is possible, to apply Fourier- or Laplace- transforms
- it is possible to simplify PDE by Bäcklund-transforms
  [Rogers].
- it is useful, to apply a transform of a PDE into a
  ODE, while still holding certain invariant properties,
  as is done with the "similarity solutions" (see [Bluman],
  [Dresner], [Barenblatt]).
Using similarity methods it is either possible
- to find for ODE an integrating factor,
- to reduce a PDE of 2nd order (or of higher order) by one
  variable. This is our problem here.

A1.2 Similarity method

We assume a partial differential equation (PDE), e.g. with
two independent variables x, t, and one dependent variable, u.

$$u_t = (x^m * u^n * u_x)_x \qquad (A1-1)$$

For a given PDE. Now transformations for (A1-1) are applied,
where a group of transformations with group invariants is
obtained which reduces PDE (A1-1) to an ODE. This type of a
solution is also called "similarity solution", because here
the following property exists: The profiles of u (dependent
variable) versus z (independent variable) at various values
of t (independent variable) are geometrically similar, and
can be obtained by suitably stretching the abszissa and the
ordinate. This similarity allows, to answer some physical
questions already on the level of the similarity solution,
as is shown in detail in sects. 4, 5, 6.

Boundary conditions

Boundary conditions must also be invariant to the group
introduced for similarity solutions, which is discussed
in more detail in sect. 4.

A1.3 Steps for similarity solution

We consider now a linear diffusion equation to make the steps
for similarity solution evident.

$$u_t - u_{zz} = 0 \qquad (A1-2)$$

It can be shown that (A1-2) is invariant under the following
group of transformations:

$$u^* = \lambda^* u$$
$$t^* = \lambda^2 * t \qquad\qquad (A1-3)$$
$$z^* = \lambda * z$$

where $0 < \quad < \quad$,
and   is any real number.
Now, any solution for (A1-2), u = f(z,t), is also a solution
of the PDE which has been transformed by (A1-3). If this
solution is the same, then the solution is called invariant
with regard to (A1-3). To obtain this, for the parameters in
(A1-3) the following requirements are necessary:

$$\lambda^2 * f(z,t) = f(*z, \lambda^2 *t) \qquad\qquad (A1-4)$$

Note;
Equ.(A1-4) is equivalent to saying that in a point (z,t,u)
for an integral surface S, i.e. where u = f(z,t) holds, there
also exists an image (z',t',u'), which also lies on the
integral surface S, i.e. where u' = f(z',t') holds. Then from
(A1-3) we have

$$\lambda^\alpha * u = f(\lambda * z, \lambda^2 * t).$$

Now replacing z by f(z,t) we obtain (A1-4). Differentiating
(A1-4) with respect to λ, and also setting $\lambda = 1$, we obtain
the auxiliary equation:

$$z * f_z \quad 2 * t * f_t \quad = \alpha * f \qquad\qquad (A1-5)$$

Here the following characteristic equations associated to
(A1-5) can be obtained (see e.g. [Bluman]):

$$\frac{dz}{z} = \frac{dt}{2*t} = \frac{df}{\alpha * f} \qquad\qquad (A1-6)$$

From (A1-6) two independent integrals are obtained,

$$I1 = z * t^{-1/2} \qquad\qquad (A1-7a)$$

$$I2 = f * t^{-/2} \qquad\qquad (A1-7b)$$

From this, the most general form, f can take under the
invariance conditions is (based on (A1-7a,b), if

$$\zeta = z*t^{-1/2} \quad,$$

$$f = t^{\alpha/2}*y(\zeta) \tag{A1-8}$$

Since this function is invariant, from (A1-8) we obtain, especially using the function y from (A1-8) the following substitutions

$$u_t = f_t =$$
$$t^{(\alpha/2)-1}*(-\frac{\alpha}{2}*y - \frac{1}{2}*z*t^{-1/2}*\frac{dy}{dx}) \tag{A1-9a}$$

$$u_z = t^{(\alpha/2-1)/2}*\frac{dy}{dx} \tag{A1-9b}$$

$$u_{zz} = t^{(\alpha/2)-1}*\frac{d^2y}{dx^2} \tag{A1-9c}$$

and with the assumption that $x = z*t^{-1/2} = \zeta$, we can re-write the differential equation (A1-2):

$$\frac{d^2y}{dx^2} = (\alpha/2)*y - (1/2)*y*\frac{dy}{dx} \tag{A1-10}$$

This is the basic way to obtain the similarity solution. Note: This method is to a large extent also typical for many cases of nonlinear PDEs. It also is useful for the diffusion equation (with nonlinearity) as shown in A2.

A2 From continuity equation to similarity solution

A2.1 Introduction

Based on some results of A1 (Methods for nonlinear partial
differential equations) we discuss here in more detail the
steps from a generalized interpretation of fractional flow
(A2.2) over a continuity equation (A2.3) to an ordinary
differential equation (A2.4), which is part of the similarity
solution (4.5).

A2.2 A generalized interpretation of fractional flow

We recall, the concept of fractional flow which is used for
2-phase flow (sect. 4.3):

$$\text{fractional flow of fuel} \quad = \quad \frac{\text{flow of fuel}}{\text{total flow}} \qquad \text{(A2-1)}$$

We write (using the generalized Darcy-law (4-10), (4-11):

$$F(Sf) \quad = \quad \frac{Krf/\mu f}{Krf/\mu f \; + \; Krw/\mu w} \qquad \text{(A2-2)}$$

Here Sf is the saturation of fuel. This formula is without
capillary pressure. But in the presence of capillary pressure,
Pc we obtain

$$Pc \quad = \quad Pf + Pw \qquad \text{(A2-3)}$$

where

Pf pressure of fuel
Pw pressure of water.

For pressures Pf, Pw, we can write (analogous to the Darcy
formulas) for the velocities Uw, Uf of the water phase, and
fuel phase:

$$Uw \quad = \quad - \; \frac{k*Krw}{\mu w} \; * \; \frac{\partial Pw}{\partial x} \qquad \text{(A2-4a)}$$

$$Uf \quad = \quad - \; \frac{k*Krf}{\mu f} \; * \; \frac{\partial Pf}{\partial x} \qquad \text{(A2-4b)}$$

It can be shown, that a type of fractional flow G(Sf) is also
obtained dependent on capillary pressures alone:

$$G(Sf) = \frac{k*Krf*Kro}{\phi*(Krw*\mu f + Krf*\mu w)} * \frac{\partial Pc}{\partial Sf} \qquad (A2-5)$$

With (A2-1)-(A2-5) we define the fractional flow Fm, which applies for vicsous and capillary forces :

$$Fm(Sf) = F(Sf) + (A*\phi/Q)*G(Sf)* \frac{\partial Sf}{\partial x} \qquad (A2-6)$$

Now Fm(Sf) can be expressed using the following concepts:

- dimensionless variables, such as introduced for x, t (4-23),
- definition of F(Sf), G(Sf) by polynomial expressions (4-18), (4-20).

This way we obtain the following quantity, also called in the following "Saturation Flux Is":

$$Is = qe(t)*S^n + (1/Nc)*S^{n-m} * \frac{\partial S}{\partial \hat{x}} \qquad (A2-7)$$

Note that the saturation flux Is a fractional flow (based on viscous and capillary forces, taking into account dimensionless variables, and explicit use of polynomial expressions (A4).

A2.3 Transformation of the continuity equation

We recall the continuity equation (4-24):

$$\frac{\partial S}{\partial \hat{t}} + \frac{\partial Is}{\partial \hat{x}} = 0 \qquad (A2-8)$$

where Is has been defined as a polynomial (4-25) and (A2-7). Using a new variable, , (A2-8) may be rewritten:

$$\frac{\partial S}{\partial \xi} * \frac{\partial \xi}{\partial \hat{t}} + \frac{\partial Is}{\partial \xi} * \frac{\partial \xi}{\partial \hat{x}} = 0 \qquad (A2-9)$$

From A1 we see how the similarity variable and similarity solution f( ) can be related to x and t :

$$\xi = \hat{x}*\hat{t}^{-1/2} \qquad (A2-10a)$$

$$S = f(\xi)*\hat{t}^{-1/2} \qquad (A2-10b)$$

Noting, in addition to (A2-10a,b) also the definition of saturation flux Is:

$$Is = g(\xi)*t^{\wedge-1} \tag{A2-11}$$

where $f(\xi)$ and $g(\xi)$ are functions of ξ, to be determined in the following differential equation:

$$(g(\xi) - b*\xi*f(\xi))' = 0 \tag{A2-12}$$

Here ' denotes differentiation with regard to variable ξ. On the basis of (A2-7), the polynomial representation of Fs and G (see (4-18), (4-20)) we rewrite (A2-12):

$$g(\xi) = f^n - \frac{1}{Nc} * f' \tag{A2-13}$$

This leads us to the following differential equation for f:

$$f' = Nc*(f^n - (1/2)*f*\xi) \tag{A2-14}$$

From continuity equation (A2-8) we obtained a differential equation for f which clearly is equivalent to the similarity solution (4-32), (4-37) and (4-41).

A3 Transitions between different mechanisms

A3.1 Basic considerations for transitions

For different mechanisms it was important to define the respective models and parameters. In relation to a phase diagram (sect.3.3) different mechanisms have been located:
- DLA (diffusion limited aggregation)
- IP (invasion percolation)
- anti-DLA (stable displacement).

It may be also interesting to see how these transitions can be characterized quantitatively. Let us first introduce some concepts from percolation. It is known (e.g. from [Wilkinson], [Pietronero], [Weber]) that invasion percolation is at least to some extent related to classical percolation. We first introduce notations used for our phase diagram (sect. 3.3). Then we recall a few concepts from percolation theory and then make some considerations regarding the transitions between mechanisms.

A3.2 A classification of transitions

As is shown in Fig. A3-1, there are the following boundaries:

Boundaries for invasion percolation domain

(a1) injecting the more viscous fluid ($\mu2 > \mu1$), boundary characterized by Ca*

(a2) injecting the less viscous fluid ($\mu2 < \mu1$), boundary characterized by Ca*

Boundaries for DLA domain

(b1) viscous limit, boundary characterized by M*

(b2) capillary limit, boundary characterized by Ca*

Boundaries for anti-DLA domain

(c1) capillary limit, boundary characterized by Ca*

(c2) viscous limit, boundary characterized by M*

All these transitions are shown in Fig. A3-1. But quantitative descriptions of the transitions will be given in sects. A3.3 to A3-5.
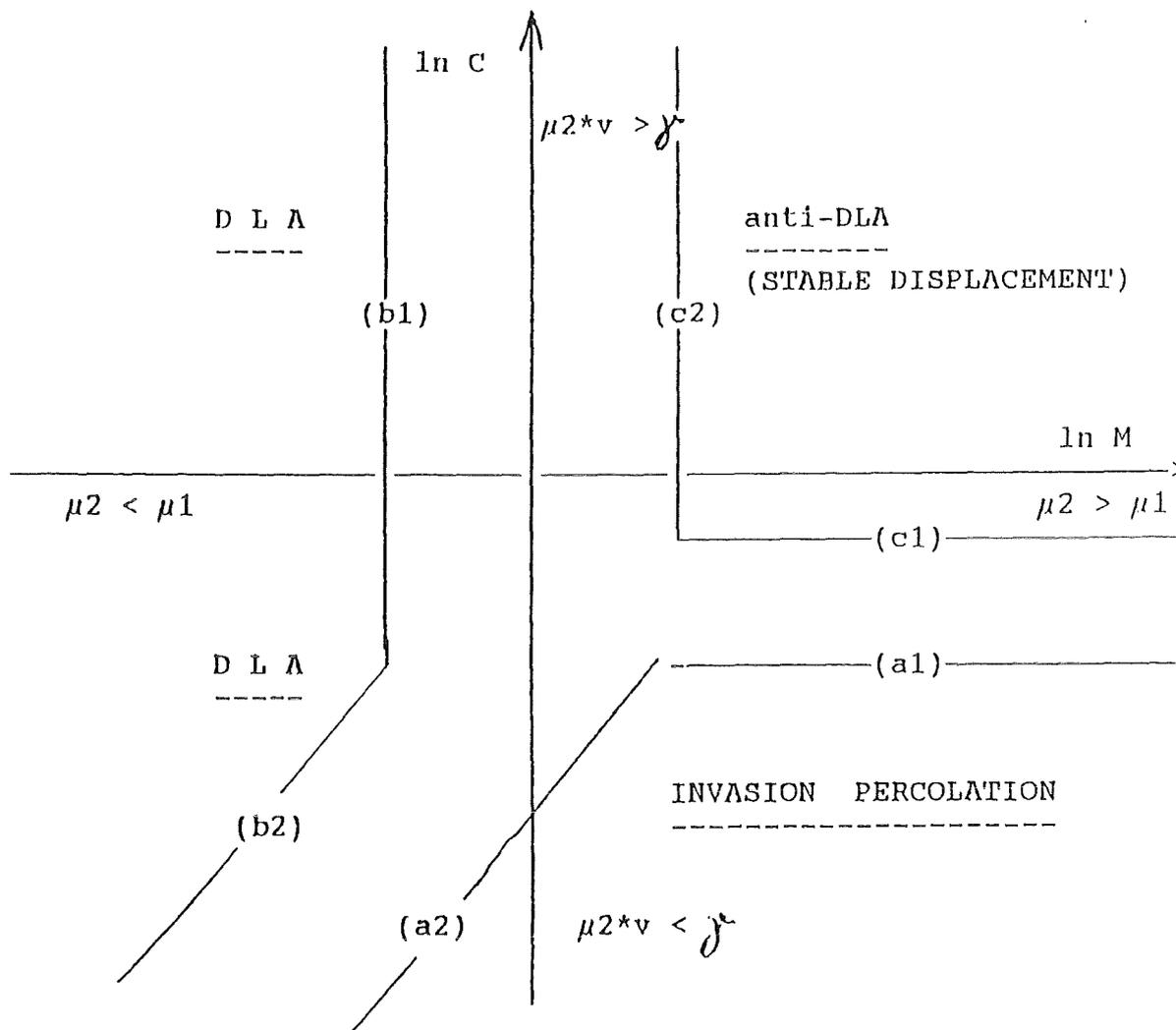
Fig.A3-1 Phase-Diagram, showing 3 different mechanisms

For Fig. 3-1 the following notations are used:

C capillary number

$$M = \frac{\mu 2}{\mu 1} \quad \text{where}$$

2 injected fluid

1 displaced fluid

A3.3 Critical exponents for percolation

We note that the following considerations are related to some
concepts and properties of invasion percolation as well as
conventional percolation [Wilkinson], [Lenormand].
The occupation probability p (of sites or bonds) in percolation
theory has a role comparable to temperature in thermal critical
phenomena. For percolation there exists a critical concentration
pc such that

(a) for p < pc only finite clusters exist, with "size" (i.e.
    number of sites), whereas
(b) for p > pc an infinite cluster exists, but also finite
    clusters.

Here we define ns(p) as the average number of finite sites with
size s. We note, that usually ns(p) is related to random physical
quantities. Here we have a scaling theory (see e.g. [Nakayama])
which states that there exist parameters, characteristic for this
system, which diverge at p = pc. This means that we can make for
ns(p) the following Ansatz, with a parameter s(p):

$$ns(p) \quad = \quad s^{-\tau} \quad F(\; s/s(p) \;) \qquad\qquad (A3-1)$$

We note that F(x) is an unknow function. But we will come to
certain regions where this function is greatly simplified, if s(p)
is close to pc. Here we can write as is well known the following
relation:

$$s(p) \qquad | \; p - pc \; |^{-(1/\quad)} \qquad\qquad (A3-2)$$

This is usually called a power law. We will have, for p ---> pc,
the limit s ---> ∞ (s is diverging). Close to this critical point,
we can write:

$$na(p) \quad = \quad s^{-\tau} \quad F(\; (p - pc){*}s^{\eta} \;) \qquad\qquad (A3-3)$$

where s(p) is replaced by s^{η} . This is also called a scaling form,
where quantities like this can be obtained for various physical
quantities in regions close to the critical value, pc.
This will be applied to several quantities of interest, such as
the correlation length.

A3.4 Introduction of correlation length

We now illustrate the concept of correlation length ξ as used for
a percolation network (Fig. A3-2a, b). Here two types of a network
are shown:

(a) Case of p < pc
Here only finite clusters exist. In Fig. A3-2a the following
things are shown:

(a)

(b)
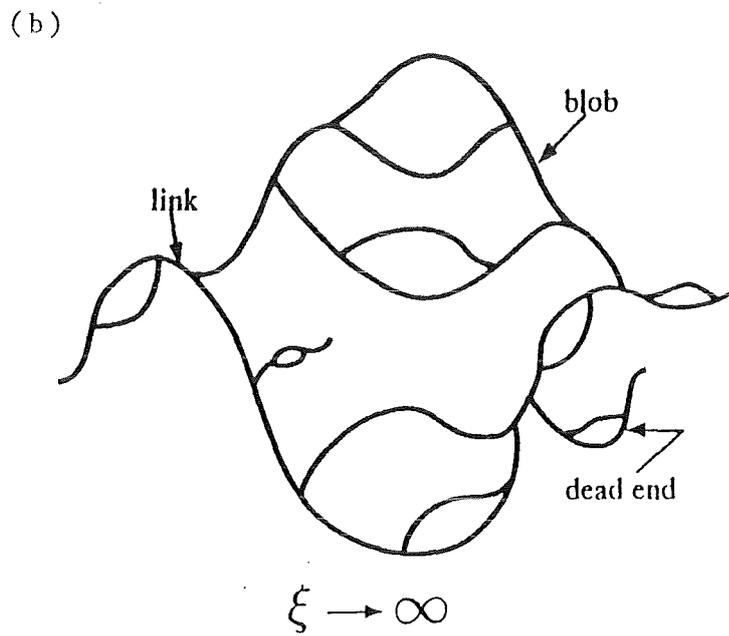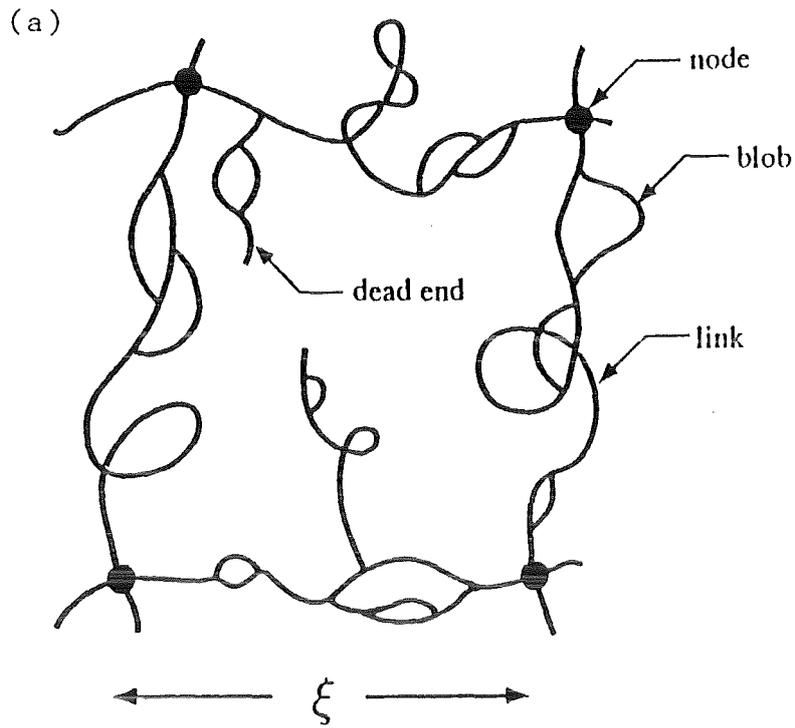
<u>Fig. A3-2</u>   (a) Network with nodes   (ξ finite)

(b) Network with links   ($\xi \to \infty$)

- nodes (shown as heavy circles),
- links (if a link is cut, no flow between two nodes is possible),
- blobs (a blob contains several parallel paths for flow),
- dead ends (dead ends can be cut without any consequence for flow).

Note that the distance between thwo nodes is of the order of ξ (the basic elements of correlation length). but the type of evaluation is discussed in A3.5.

(b) Case of p > pc
Here an infinite cluster (leading from input to output) is available (ξ ---> ∞). Then correlation length diverges.

A3.5 Calculation of correlation length

The diameter of finite clusters below pc is characterized by (p) and called correlation length. (p) is defined as the root mean square distance between two sites i and j (nodes) in the same cluster, averaged over all finite clusters. Thus we can at first for a given cluster of size s write the average distance between two sites i and j:

$$Rs^2 = \frac{1}{2*s^2} \sum_{i,j} | ri - rj |^2 \qquad (A3-4)$$

Now the number of ways connecting two sites in a given cluster of size s is s^2, thus s^2*ns becomes the number of ways connecting two sites in clusters of the same size s. Thus the correlation length (p) is Rs, averaged by the probability

$$\frac{s^2 ns}{s^2 ns}$$

This means

$$\xi^2 (p) = \frac{\sum Rs^2 *s^2 *ns}{\sum s^2 *ns} \qquad (A3-5)$$

Now we can assume that the same scaling function is possible as already mentioned in (A3-3), and obtain

$$Rs = s^{w'} H((p - pc)*s^{\eta}) \qquad (A3-6)$$

and we can write

$$\xi (p) = Xo*| p - pc |^{-\nu} \qquad (A3-7)$$

where Xo is a constant factor which can be evaluated only in some
cases, mostly approximately. We have $\gamma = \omega/\eta$. We note here,
that the correlation length represents

- the characteristic size of the clusters (for p < pc) and
- the characteristic size of the voids (for p > pc).

A3.6 A line which includes the cluster, the hull

Next an exponent is introduced which will be very useful for some
of the following considerations for the transition between some
mechanisms. Assume a finite cluster of percolation as shown in
Fig. A3-3a. In Fig. A3-3b the cluster is shown with a "hull" (heavy
curve). As can be seen, the following empty sites are connected
by a line which follow these conditions:

(a) they are adjacent to cluster sites and
(b) can be connected to infinity via a chain of empty sites,
     using next neighbors and nearest to next neighbors only.

This type of line encloses the cluster (see Fig. A3-3b). There
exists a measure for the "length" of this hull, Lh:

$$Lh \sim a^{dh} \tag{A3-8}$$

where a is the average size of the cluster, and dh the hull
exponent (see [Isichenko]).
It could be shown, that for dh the following holds:

$$dh = \frac{\gamma + 1}{\gamma} = 7/4 \tag{A3-9}$$

if = 4/3 and d = 2 (2-dimensional model of percolation).
This could be estimated using simulation (see [Sapoval] and could
be demonstrated using rigorous results of statistical mechanics
[Saleur]. We also note, that the percolation hull exponents could
be used for different purposes, especially for modelling the
fronts of diffusion limited aggregation (DLA) as has been shown
by Meakin and Family (see [Meakin]).
It is important to note, that (A3-8), (A3-9) also hold for many
finite systems (see [Isichenko]).

A3.7 Some considerations of porous media

We need a few concepts for an easy formulation of boundaries
to certain mechanisms.

Saturation as a function of M and Ca

Assume we have the fraction of invading fluid (saturation S) at
breakthrough as a function of M (shown in network simulations
of the three processes (see [Lenormand]). This function is shown
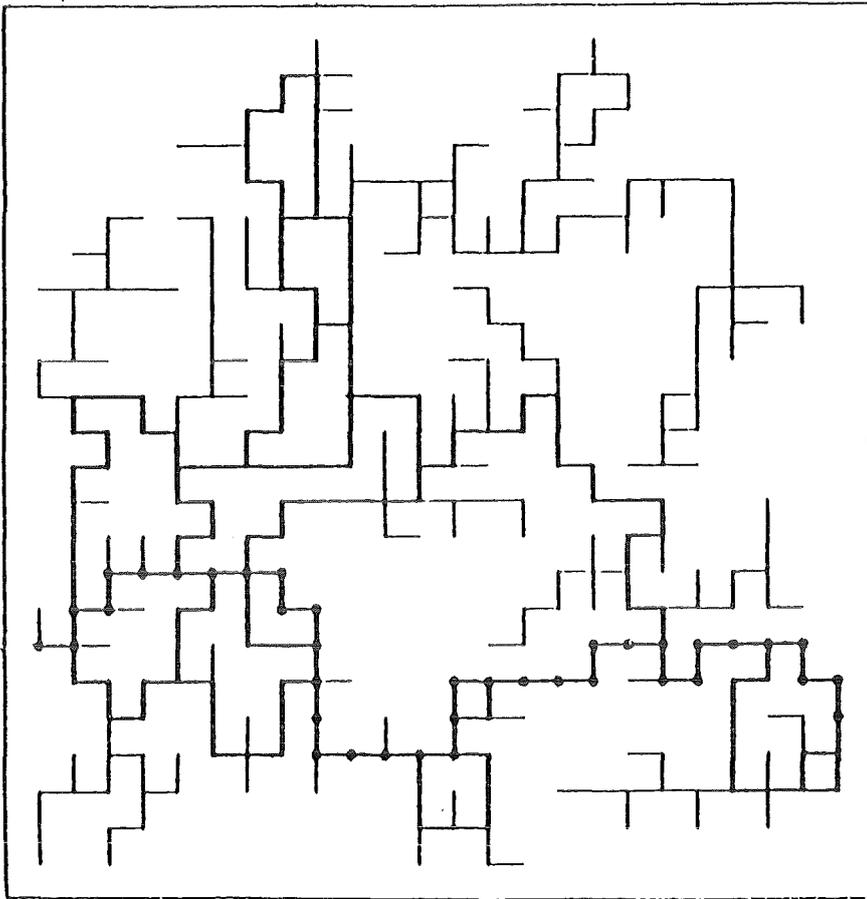
Fig. A3-3a   Cluster
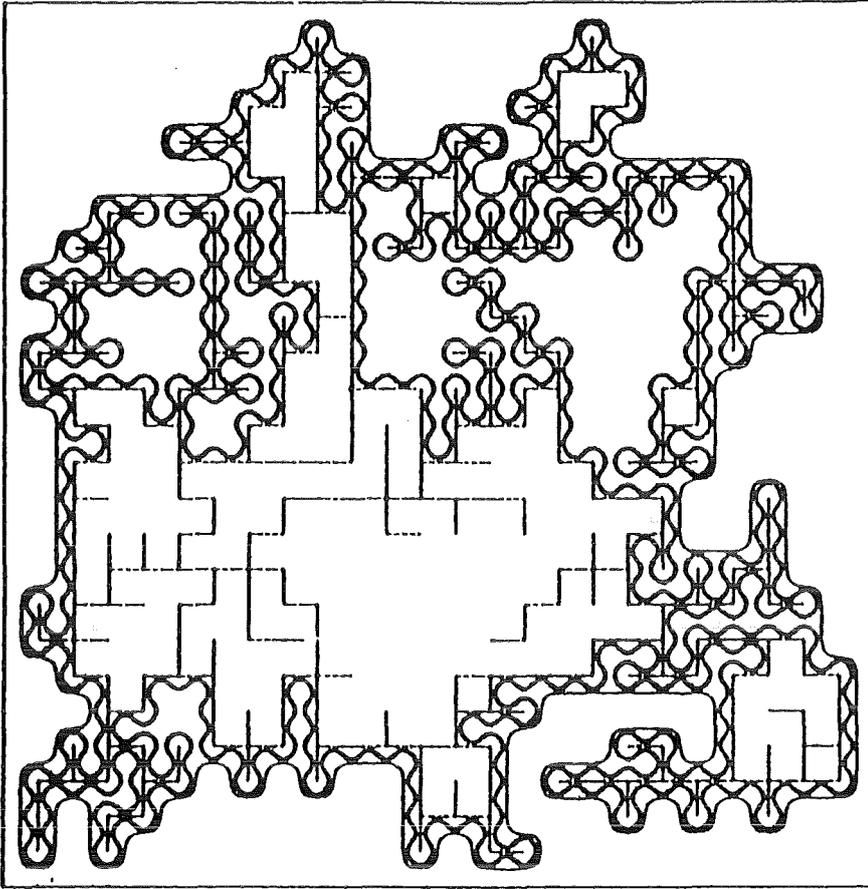
Fig A3-3b   Same cluster, with "hull"

in Fig. A3-4. In this process S can reach a maximum (called also
plateau). Already at a somwhat lower value of M, called M* ,
the limit of M will be reached, characterizing the change of
mechanisms. The distance

$$| \ S(M*) - S(plateau) \ | \ = \ \mathcal{E} \qquad\qquad (A3-10)$$

will be called $\mathcal{E}$. This quantity can be estimated from Simulation
[Lenormand].
A similar consideration also applies for S as function of Ca (where
Ca* is the limit value of the capillary number Ca). This will not
be derived in detail, but see [Lenormand].

Geometrical properties of a porous medium

The porous medium is a two dimensional network, interconnected
with capillaries (Fig. A3-5). The capillaries have a radius R,
which varies in the following interval:

$$[\ (1 - \mathcal{S})*Ro, \ (1 + \mathcal{S} \ )*Ro \ ] \qquad\qquad (A3-11)$$

We note that the size of the network is L x L, the mesh size is a.

Relation to Darcys law

For a flow where only one phase is present, we use the following
version of Darcys law:

$$\frac{q}{\Sigma} \ = \ \frac{k}{\mu} \ * \ \frac{\triangle P}{x} \qquad\qquad (A3-12)$$

q	flow
Σ	area for flow
k	permeability
μ	viscosity

$$\frac{\triangle P}{x} \qquad \text{approximation for pressure gradient} \qquad (A3-13)$$

We also note that for two phases a more general form for Darcys
law is required.
For resistance to flow we can write the mean conductance go of a
single channel, based on Poiseuilles law):

$$go \ = \ (1/8)* \ \pi \ * \ \frac{Ro^4}{a*\mu} \qquad\qquad (A3-14)$$

and also, using $= a^2$, x = a we can write again

$$k \ = \ (1/8)* \ \pi \ * \ \frac{Ro^2}{a} \qquad\qquad (A3-15)$$
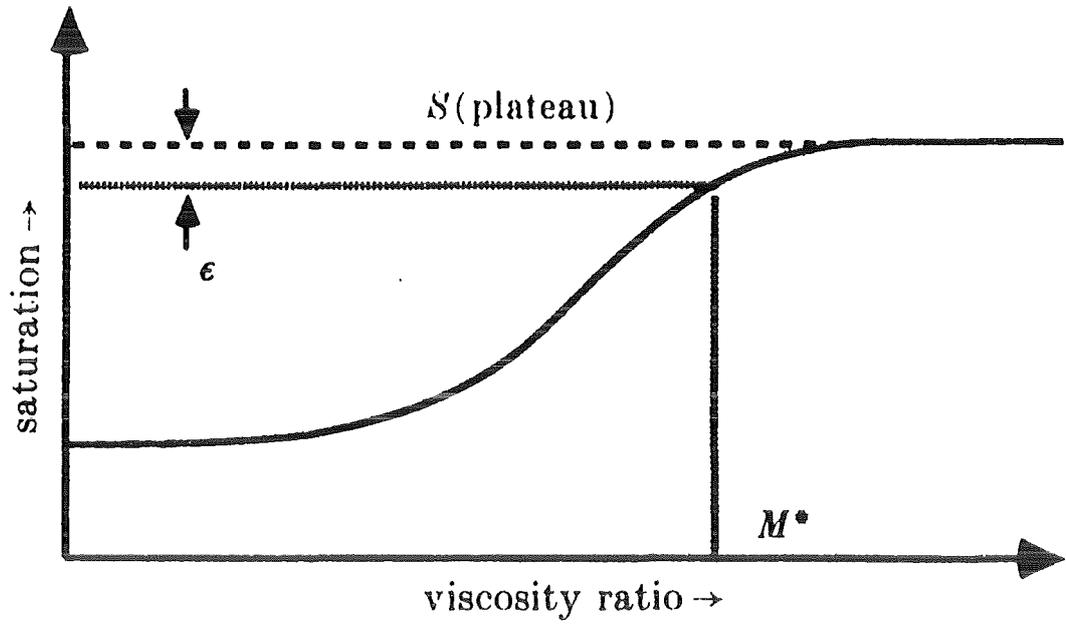
Fig A3-4   Function characterizing invading fluid
(saturation), with variable M
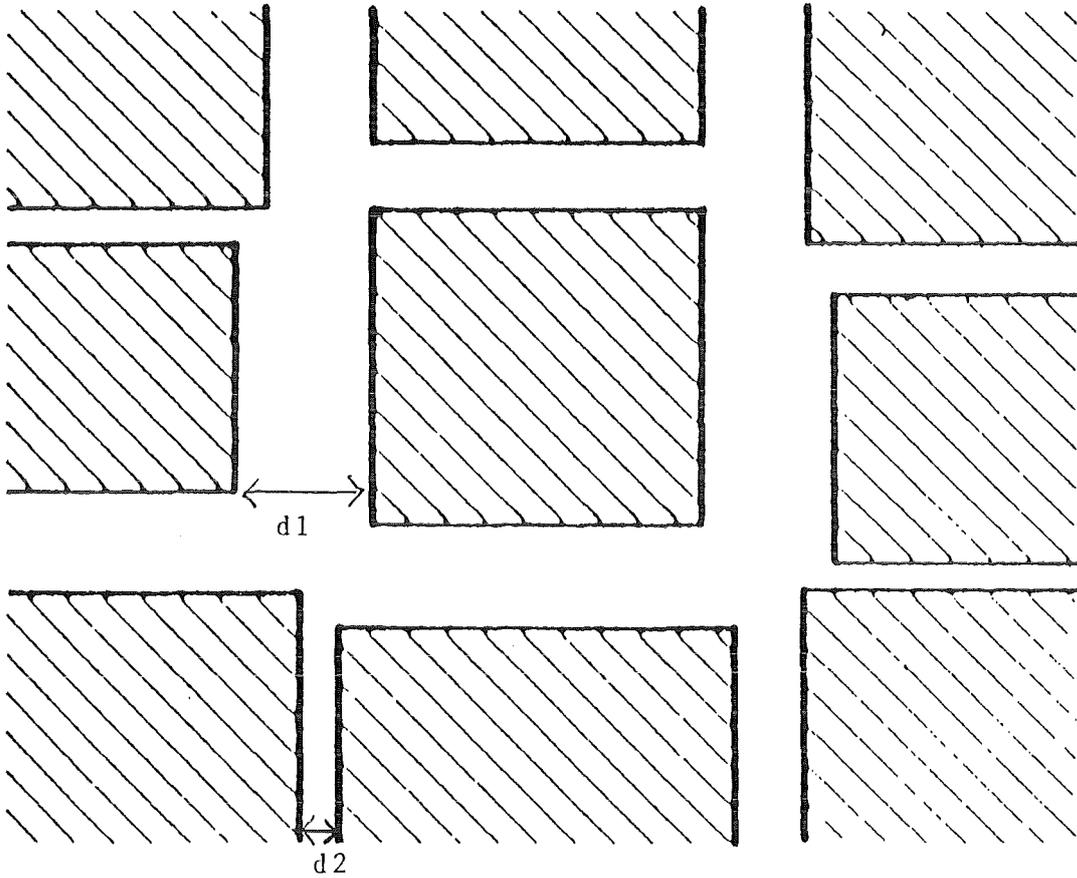
Fig. A3-5 Porous medium, with capillaries of
variable diameters:

d1 = 2*(1 + δ)*Ro   (maximum)

d2 = 2*(1 - δ)*Ro   (minimum)

This can be used to formulate the limits for the mechanisms.

A3.8 Boundaries for mechanisms

With a few considerations, relating a number of considerations due to percolation theory and due to some formulas of A3.7 we can write the following formulas.

Boundaries for invasion percolation domain

(a1) Injecting the more viscous fluid

As can be seen, the this boundary is a horizontal line, which is given as follows:

$$Ca* = A* \varepsilon * \delta (L/a)^{\tau} *(Ro/a)^3 \qquad (A3-16)$$

where

A   a constant (empirical factor due to simulations),

ε difference of S(M*) and S(plateau, (see (A3-10)) which is also due to simulations,

δ is a variation parameter due to pore geometry, see (A3-11),

a   mesh size of percolation network

L   size of percolation network

τ = $(t + 1 + \nu)/ \nu$

         this exponent is related to the following exponents:
         ν exponent for correlation length (A3-7),
         t   exponent for conductivity.

Since t = 1.3 and   = 4/3 holds, we can approximately write

$$t \cong \nu = 4/3$$

and also

$$\tau = -2.75 = - 11/4$$

Here also seems to be a certain relation to the "hull exponent"
(A3-9). Of course, such a statement has to be taken with care.
It seems to us, that for the exponent   assumptions, similar
to A3.6 apply (we have also a finite system with a network where
L >> a applies. We have L = 100, a = 1.

(a2) injecting the less viscous fluid

As can be seen, this is anyway a monotonously increasing function
of M, in a certain range, it is a linear function of M.

We can write for this limit:

$$Ca* = B* \mathcal{E} * \sigma * (L/a)^{ah} *(Ro/a)*M \qquad (A3-17)$$

Here most variables used remain the same as for (A3-16) unless
explicitly stated:

B   is an empirical numerical factor, obtained by simulation,
     but of a different value.

ah = $(\gamma + 1)/\gamma$ = 7/4 is the exponent clearly used as hull
exponent (see A3.6). This means, that (L/a) scales with the
ah exponent (defining the hull).
It has to be noted, that the exponent ah, used in (A2-17) applies
for all networks where L >> a, (see [Isichenko]), we have L = 100,
a = 1.

Boundary for DLA domain

(b1) viscous limit M*

$$M* = \mathcal{E} *(L/a) \qquad (A3-18)$$

We note that in (b1) as well as in (b2), and in (c1), (c2)
we have no critical exponent for this evaluation.

(b2) Capillary limit Ca*

$$Ca* = C* \mathcal{E} * \sigma *(a/L)*(Ro/a)^{3} *M \qquad (A3-19)$$

C   is also a constant, but of a different value. Here we also
have a linear fuction of M, as has been discussed for (a2).
This is parallel to the transition show in (A3-17).

Boundaries of anti-DLA domain

(c1) capillary limit Ca*

$$Ca* = D* \varepsilon * \delta *(Ro/a)^3 \qquad \text{(A3-20)}$$

We have here D = 80, a constant due to simulation.
Here we also have a horizontal line.

(c2) viscous limit M*

$$M* = E* \varepsilon \qquad \text{(A3-21)}$$

We have here E = 1000, a constant due to simulation.
Here we have again a vertical line.

A3.9 Evaluation

We give a graphic representation, showing all the boundaries
of these mechanisms (Fig. A3-5):

- invasion percolation (IP), (a1), (a2)
- diffusion limited aggregation (DLA), (b1), (b2)
- anti-DLA, (c1), (c2).

In Table A3-1 all parameters used in this model are listed.
Now Fig. (A3-1) is quantified, giving Fig. A3-5. Here all
the coefficients and parameters, mentioned in the model of
A3 and shown in Table A3-1 have been used.
For a generalization of this model, and some of its limits.
see sect. A3.10.

A3.9 Generalization and some limits of the model

We assume, that only L (size of the network) is changed,
all other parameters are kept fixed (Table A3-1). Then
obtain for increasing (decreasing) values of L the following
changes for the boundaries.

A few examples of these changes are given in Tables A3-1a,b:
A3-1a : increasing L
A3-1b : decreasing L.
We assume that always L >> a (network dimension >> mesh size)
holds.

We obtain (compare Fig. A3-1 and Fig. A3-5) the following
changes for boundaries (in (log M, log Ca)-plot):

IF L IS INCREASING (Table A3-2a),

(a1), (c1)
- both boundaries remain horizontal
- distance between boundaries increases
- c1 stays fixed, a1 moves downward

(b1), (c2)
- both boundaries remain vertical
- distance between boundaries increases
- c2 stays fixed, b1 moves to the left

(a2), (b2)
- both boundaries keep the direction (45° to ln M-axis)
- distance between boundaries increases
- a2 moves downwards, b2 moves downwards

We also note, that for DECREASING L (Table A3-2b),
distances are decreasing, the movents indicated in
the case of increasing L have to be reversed.

Limitations of this model

Having no results from other simulated grids, the model
used here for L > 100 (L < 100) is only of a limited
precision, but stays at least qualitatively the same.

In addition to the mechanisms which relate to capillary
number Ca and to viscosity ratio M (as discussed here in
detail), there are also some considerations referring
to a parameter called "crossover length", which means this:
Sometimes the types of mechanisms also strongly depend on
the scale of observation, i.e. whether mechanisms are
observed in a scale significantly larger than a certain
length scale (crossover length) or below this length
scale [Fernandez].
This way, mechanisms sometimes may depend (in addition
to Ca and M) also on a third physical quantity. It is
out of the scope of this report to include also these
considerations in detail.

log Ca∗ = f(log M)
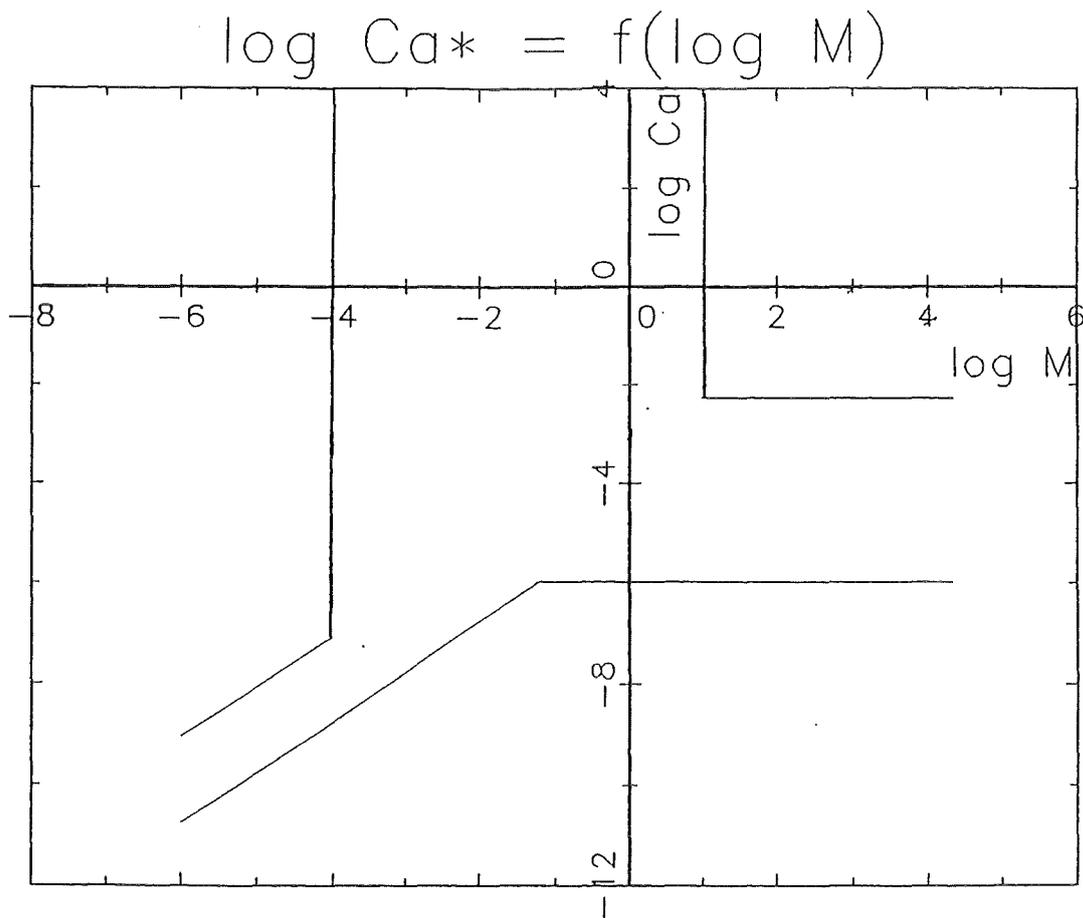
Fig. A3-6 Quantitative representation

of the phase diagram introduced in Fig. A3-1
using parameters given in Table A3-1

```
┌─────────────────────────────────────────────────────────┐
│                                                         │
│    Parameters for the model                             │
│  ───────────────────────────────────────────────────    │
│                                                         │
│    (1) General parameters        (2) Exponents          │
│                                                         │
│    $\varepsilon$   = 0.01              ah = 7/4               │
│                                                         │
│    $\sigma$   = 0.56              $\tau$ = 2.75              │
│                                                         │
│    L   = 100                                            │
│                                                         │
│    a   = 1.0                                            │
│                                                         │
│    Ro  = 0.23                                           │
│                                                         │
│                                                         │
│  (3) Special coefficients for boundaries                │
│                                                         │
│    A   = 5000   (a1)                                    │
│                                                         │
│    B   =  800   (a2)                                    │
│                                                         │
│    --           (b1)     (no special coefficient)       │
│                                                         │
│    C   = 1300   (b2)                                    │
│                                                         │
│    D   =   80   (c1)                                    │
│                                                         │
│    E   = 1000   (c2)                                    │
│                                                         │
└─────────────────────────────────────────────────────────┘
```

TABLE A3-1 Listing of all coefficients for model
──────────

RESULTS

varying L and coordinates for boundary
(1)log Ca*-values,(2) log M*-values,(3) typical log Ca*-values
----

(1)	L:	dist.(a1),(c1):	coo. (c1):	coo.(a1):
	100.00000	3.7041200	-2.2635385	-5.9676585
	200.00000	4.5319525	-2.2635385	-6.7954909
	300.00000	5.0162034	-2.2635385	-7.2797419
	400.00000	5.3597850	-2.2635385	-7.6233234
	500.00000	5.6262875	-2.2635385	-7.8898260
(2)	L:	dist.(b1),(c2):	coo. (c2):	coo.(b1):
	100.00000	5.0000000	1.0000000	-4.0000000
	200.00000	5.3010300	1.0000000	-4.3010300
	300.00000	5.4771213	1.0000000	-4.4771213
	400.00000	5.6020600	1.0000000	-4.6020600
	500.00000	5.6989700	1.0000000	-4.6989700
(3)	L:	dist.(a2),(b2):	coo. (b2):	coo.(a2):
	100.00000	1.7108534	-8.0526851	-9.7635385
	200.00000	2.2376559	-8.3537151	-10.290341
	300.00000	2.5458156	-8.5298064	-10.598501
	400.00000	2.7644584	-8.6547451	-10.817143
	500.00000	2.9340509	-8.7516551	-10.986736

RESULTS

varying L and coordinates for boundary
(1)log Ca*-values,(2) log M*-values,(3) typical log Ca*-values
----

(1)	L:	dist.(a1),(c1):	coo. (c1):	coo.(a1):
	100.00000	3.7041200	-2.2635385	-5.9676585
	90.000000	3.5782869	-2.2635385	-5.8418254
	80.000000	3.4376174	-2.2635385	-5.7011559
	70.000000	3.2781396	-2.2635385	-5.5416781
	60.000000	3.0940359	-2.2635385	-5.3575744
(2)	L:	dist.(b1),(c2):	coo. (c2):	coo.(b1):
	100.00000	5.0000000	1.0000000	-4.0000000
	90.000000	4.9542425	1.0000000	-3.9542425
	80.000000	4.9030900	1.0000000	-3.9030900
	70.000000	4.8450980	1.0000000	-3.8450980
	60.000000	4.7781513	1.0000000	-3.7781513
(3)	L:	dist.(a2),(b2):	coo. (b2):	coo.(a2):
	100.00000	1.7108534	-8.0526851	-9.7635385
	90.000000	1.6307778	-8.3537151	-9.6834629
	80.000000	1.5412608	-8.5298064	-9.5939460
	70.000000	1.4397749	-8.6547451	-9.4924600
	60.000000	1.3226181	-8.7516551	-9.3753032

Table A3-2   Dependence of diferent characteristics

(capillary number, ratio of viscosities)
on the total length of the 2D-network

A4 Relations of permeability and saturation

A4.1 Introduction

Now in relation to section 4 (Evaluation of phase flow in a
porous medium, 4.1 - 4.3) relations of permeability to
saturation will be discussed. For this reason, first we
recall some of the concepts used in A3 and introduce some
more concepts.

A4.2 A few concepts from percolation

In a percolation network, some of the relevant quantities are
percolation probability p and critical probability pc (also
called critical concentration) as well as correlation length
(see A3.4).
Also other quantities are needed for our considerations:

(1) Accessible fraction XA.
That is the fraction of occupied bonds, belonging to an
(infinite) cluster.
(2) Backbone fraction XB.
That is the fraction of occupied bonds, belonging to an
(infinite) cluster, which actually carry flow (or current),
excluding the dead ends, which do not effectively contribute
to flow.
It is to be noted, that the fractions (1), (2) are very
relevant for various transport properties, giving measures
for all transport paths. Here it will be interesting to ask if
there exist some relations between permeability K and the
quantities (1) and (2).
It has been shown (by theoretical considerations, simulations
and by experiments) that quantities like ξ ,XA ,XB follow
polynomial laws (see [Sahimi2], [Nakayama]). For illustration,
see Fig. A3-1. It has been noted in section A3.4, that for the
correlation length, a polynomial law exists.

$$\xi(p) \sim |p - pc|^{-\gamma} \tag{A3-7}$$

We also state, that XA, XB follow polynomial laws.

$$XA(p) \sim (p - pc)^{\beta_p} \tag{A4-1}$$

$$XB(p) \sim (p - pc)^{\beta_B} \tag{A4-2}$$

Here we recall the concept of critical point (see also (A3-1)).
At some well defined value of p, there must be a transition for
the structure of the network, where the network obtains at least
one connected cluster which spans the whole network. This
transition probability is called "bond percolation threshold pc".
Below pc there exists no sample spanning cluster. Now, β_p is a
critical exponent related to the percolation probability P as
follows:

$$P \sim (p - pc)^{\beta_p} \qquad\qquad (A4-3)$$

Similarly, β_B is related to the backbone fractal dimension.
Thus we have for 2 dimensions:

$$\beta_p = 5/26 \qquad\qquad (A4-4a)$$

$$\beta_B = 0.48 \qquad\qquad (A4-4b)$$

A4.3 Mass and fractal dimension Df

From (A3-6) we note, that

$$Rs \sim s^{\omega} \qquad\qquad (A4-5)$$

holds, which is useful if we are sufficiently close to pc, the
critical point where percolation starts. Tis can be written,
since s is the cluster size as

$$s \sim Rs^{1/\omega} \qquad\qquad (A4-6a)$$

Using for $1/\omega$ the notation Df, fractal dimension (or Hausdorff
dimension) we can write also

$$s(Rs) \sim Rs^{Df} \qquad\qquad (A4-6b)$$

As is well known, for Df the following relation holds

$$Df = d - \beta/\nu \qquad\qquad (A4-7)$$

Df is usuall smaller than the Euclidean dimension d. It can be
seen that M, (number of all bonds of the sample spanning cluster)
can be given as a power law of Rs (rel. to correlation length),
where the exponent is Df. The quantity Df is important for
percolation and also inversion percolation (see e.g. [Pietronero],
[Weber]). It has to be noted that the so-called "mass" M(R) of the
cluster scales as

$$M(R) \sim Rs^{Df} \qquad\qquad (A4-8)$$

Also note, that (p), the correlation length (sect. A3.4) is a
threshold, up to which the cluster can be regarded as fractal.
For p > pc and for length scales L > ξ the percolation network
can be regarded as homogeneous.
We thus can write as follows (for a quantity analogous to M(R)):

$$M(L) \sim \begin{cases} L^{Df} & \text{for } L \ll \xi \\ \\ L^{d} & \text{for } L \gg \xi \end{cases} \tag{A4-9}$$

where d = 2.

This change in polynomial behavior depending on the question, if L is small or large compared to a quantity, such as correlation length ξ , is called "cross over".
We also can relate to M(L) - especially for two-phase fluid flow - quantities like saturation S. This will be discussed in detail.

A4.4 Saturation and polynomial laws

We consider a porous medium with volume $Vv = \phi > \phi c$ (ϕ porosity, ϕc critical porosity, below ϕc the original fluid is separated in different parts. Assume that the entire pore space of this medium is at the begining filled with a non-wetting (nw) fluid (see [Larson]). Now the nw-fluid is removed beginning with some pores and is now replaced by a second phase, the wetting fluid. We usually assume, that the wetting fluid (w-fluid) remains still randomly distributed, but anyway will prefer smaller pores due to capillary forces. (see 3.2 criteria for different mechanisms). Let Vnw be the volume of the nw-fluid, being distributed over the porous medium. For the nw-fluid, still randomly distributed, the fraction of accessible pores occupied by the nw-fluid, is called saturation Snw. Until now, we have pores which are separated, i.e. do not belong to a backbone, (a cluster of pores which span the medium).
Note that here for Snw the equivalence to XA, accessible fraction, is evident. We also have

$$Vnw = \phi *Snw \tag{A4-10}$$

If random replacement of the nw-fluid causes Vnw = *Snw to drop below c, then the nw-fluid is no longer continuous across the medium. Then the residual nw-saturation is called Srnw, defined by

$$\phi \, Srnw = \phi c \tag{A4-11}$$

Now since Snw is closely related to XA (A4-1) we can write for

(a) drainage (a procedure, analogous to the procedure discussed above, where the non wetting fluid replaces the wetting fluid).
(b) imbibition (exactly the procedure discussed above)

the following relations (similarly to (A4-1), (A4-2)):

(a) drainage

$$Snw \sim (p - pc)^{\beta p} \qquad (A4-12)$$

(b) imbibition

$$Snw - Srnw \sim (p - pc)^{1 + \beta p} \qquad (A4-13)$$

A4.5 Relation to permeability

It is also known from experiments, simulation and theoretical considerations ([Larson], [Sahimi]) that for permeability K the following power law holds (similar to conductivity):

$$K \sim (p - po)^{\mu} \qquad (A4-14)$$

We take into account, that for some quantities a cross over is taking place (as stated in relation to (A4-9)). This means, that we can use equations referring to (A4-4)-(A4-6) for a suitable scaling. Thus we obtain, for the relative permeability kr the following equations (assuming $\xi \gg L$):

(a) during drainage

$$krnw \sim (Snw)^{\mu / \beta p} \qquad (A4-15)$$

(b) during imbibition

$$krnw \sim (Snw - Srnw)^{\mu / (1 + \beta p)} \qquad (A4-16)$$

We see, that the two formulas are based on

- two variables, Snw and (Snw - Srnw) and
- two exponents, $\mu / \beta p$ and $\mu / (1 + \beta p)$.

It can be seen that for (b) we have an increasing function, and for (a) a decreasing function. The respective angles of the two curves in relation to the abszissa are considerably different (as can be seen from (A4-15) and (A4-16)). This leads us to the statements of sects. 4.2 and 4.3.

A5. Remarks on Programs for analytic solution

A5.1 Introduction

In relation to the discussions in sections 4 to 8 a number
of programs were required. Some of these programs are here
shown in detail. They belong to the GAUSS-program system
(see [GAUSS] for more details. We prefer two types for
an illustration of the programs used in our research.

Similarity Solution

It is interesting to note, that for an analytic solution,
i.e. for a similarity solution of the Riccati equation
was used. We recall this differential equation, discussed
in sect. 4.5:

$$f' = Nc*Nc(f^n - (1/2)*f*\xi)$$ (4-36)

For n = 2, we have a Riccati-equation. As solution we have

$$f(\xi) = \frac{\exp(-0.25*Nc*\xi^2)}{(\pi*Nc)^{0.5} * (1/\tanh((0.5*Nce) - erf(0.5*Nc^{0.5}*\xi))}$$ (4-41)

This is shown in sect A5.2.

Interpretation of a flux-curve

It may be noted, that an interesting criterions, do see if there
are some mechanisms available (such as drain, imbibition, for
different directions of the two liquids) (see Fig. 6-2) it is
required, to evaluate the flux-curve as is given in sect. 6:

$$J = t^{\wedge -1/2} * S - (1/Nc) * \frac{\partial S}{\partial \hat{x}}$$ (6-4)

This is shown in sect. A5.2.

It may be also noted, that it was possible, to use most programs
in a version which was either suited to

- a quick representation or to
- a representation for publication quality.

A5.2 Similarity Solution and Flux Curve

With the equ. (4-41) and (6-4) a program for the GAUSS
solution is given. For more information on this program
see [GAUSS].


```
/*FLUXA---------------------------------------------*/
/*Analytical solution for global fuel migration     */
/*FLUX of a phase, function of Saturation, Nc, t     */
/* Js(Se;Nc,t)                                       */
/*February 2, 1995                                   */
/*-------------------------------------------------*/


"fs(Se,Nc), as dimensionless function";

anf2:

/*qgraph-commands*/

     library qgraph;
     graphset;
     setgmode(16);
         _qmajor = 3;
         _qminor = 3;

     beggraph;

         r = 1;
         c = 1;

     setgmode(16);
         _qnum = 1;
         _qbox = 1;
         _qdfmode = 0;

     window(r,c);

"INPUT :";print;

     "no. of curves m1    = ";;m1 = con(1,1);
     "Nc            = ";;Nc = con(1,1);
     "times t            = ";;t = con(m1,1);
     "END of Input DATA";

/*"fixed parameters";*/

     let inc = 0.05;
     let n1  = 80;
```

```
/*Evaluation*/

declare x   ? = 0;
declare yma ? = 1;


    ni = seqa(0,1,n1);
    x  = ni*inc + 0.001;


    /*f(xi)-*/

    n1  = exp(-0.25*x^2.*nc);
    d11 = 1/tanh(0.5*nc) - erf(0.5*(nc^0.5)'*x);
    d12 = (pi*nc)^0.5;
     d1 = d12.*d11;

    f   = n1./d1;                   /*similarity solution f(xi)*/

/*diff(f(xi), x) = f'*/           /*see appendix A6.2*/

    n2  = -0.5*(nc^0.5)*x.*exp(-0.25*nc*x^2);
    d21 = 1/tanh(0.5*nc) - erf(0.5*(nc^0.5)*x);
    d2  = pi^0.5*d21;

diffa = n2./d2;

    n3  = (exp(-0.25*nc*x^2)).^2;
    d31 = 1/tanh(0.5*nc)  - erf(0.5*nc^0.5*x);
    d3  = pi*d31^2;

diffb = n3./d3;



diff  = (diffa + diffb);

    g  =  f^2 - (1/Nc).*diff;
    f2 = f^2;

    Is = g./t';                     /*flux      Is(Se)*/
    Se = f.*(t^(-1/2))';            /*saturation Se(xi)*/

    Is1 = maxc(Is);       ·    Se1 = maxc(se);
    xm1 = maxindc(is);         f3  = (Se1)^2;
    xm  = xm1*inc;

wait;cls;
    print;

anf1:

    "xmin, xmax = ";;v1 = con(1,2 );
    "ymin, ymax = ";;v2 = con(1,2);

        scale(v1,v2);
```

```
/*|================|*/
/*|  GRAPHIC   OUTPUT  |*/;
/*|================|*/

      format 3,3;

      title("f(x)");
      xy(x,f);

      title("diffa and diffb");
      xy(x,diffa~diffb);

      title("diff");
      xy(x,diff);

      title("Is(Se)");
      xlabel("Se");
      ylabel("Is");
      xy(Se,Is);

wait;cls;

      OUTPUT FILE = typea ON;

      print;

      "EXTREMA OF Is-CURVE ";

      "Nc   = ";;Nc;
      "--------------------------------";
      "time t       max(Se)         Is(Se) ";
       print t~se1~f3;
      "================================";
      OUTPUT OFF;

"use file typea, reseta";

"continue : ";;w2 = con(1,1);
      if w2 == 1;
          goto anf2;
      endif;

endgraph;

"reset <typea> with <run reseta>";

"END";
```

A6 Check of the behavior of the similarity solution

We discuss here some of the applications of the symbolic computation with MAPLE (Version 3) for obtaining partial derivatives of $f(xi, Nc)$. Note that most of the expressions shown here are automatically reproduced during a MAPLE-session.

A6.1 Analytical form of the similarity solution

Here the formula $f(x, Nc)$ is written to obtain (using numerator n1 and denominator d1). Note that this can also be used for a 3D-representation.

```
> n1 := exp(-0.25 nc x );
```

```
▶ d11(x,nc) := 1/tanh(0.5*nc) -erf(0.5*nc^0.5*x);
```

$$d11(x, nc) := \frac{1}{\tanh(.5\ nc)} - erf(.5\ nc^{.5}\ x)$$

```
▶ d12(x,nc) := (Pi*nc)^0.5;
```

$$d12(x, nc) := Pi^{.5}\ nc^{.5}$$


```
▶ d1(x,nc) := d12(x,nc)*d11(x,nc);
```

$$d1(x, nc) := Pi^{.5}\ nc^{.5} \left(\frac{1}{\tanh(.5\ nc)} - erf(.5\ nc^{.5}\ x) \right)$$

The similarity solution $f(x, nc)$ is represented as follows (A6-1):

```
▶ f(x,nc) := n1(x,nc)/d1(x,nc);
```

$$f(x, nc) := \frac{\exp(- .25\ nc\ x^2)}{Pi^{.5}\ nc^{.5} \left(\frac{1}{\tanh(.5\ nc)} - erf(.5\ nc^{.5}\ x) \right)} \quad (A6-1)$$

A6.2 Partial derivative with regard to x:

We now make a partial derivative for f(x,Nc) with regard to x, showing the slope of f(x,Nc) with regard to x.

▶ f (x,nc) := diff(f(x,nc), x);
   x

$$f_x(x, nc) = -.50 \frac{nc^{.5}\, x \exp(-.25\, nc\, x^2)}{Pi^{.5}\left(\dfrac{1}{\tanh(.5\, nc)} - erf(.5\, nc^{.5}\, x)\right)}$$

$$+ \frac{\exp(-.25\, nc\, x^2)\, \exp(-.25\, nc\, x^2)}{Pi\left(\dfrac{1}{\tanh(.5\, nc)} - erf(.5\, nc^{.5}\, x)\right)^2} \tag{A6-2}$$

A6.3 Partial derivative of f(x,nc) with regard to nc :

We also make a partial derivative of f(x, Nc) with regard to Nc. It shows in more detail the behavior of the reduced saturation f(x,Nc) with regard to possible mechanisms.

▶ f (x,nc) := diff(f(x,nc), nc);
   nc

$$f_{nc}(x,nc) := -.25 \frac{x^2 \exp(-.25\, nc\, x^2)}{Pi^{.5}\, nc^{.5}\, t1} - .5 \frac{\exp(-.25\, nc\, x^2)}{Pi^{.5}\, nc^{1.5}\, t1} \tag{A6-3}$$ (a)

(continued)

$$
- \cfrac{\exp(-.25\ nc\ x^2) - \cfrac{.5\ -.5\ \tanh(.5\ nc)^2}{\tanh(.5\ nc)}}{Pi^{.5}\ nc\ t1}
$$

(A6-3)
(b)

$$
+ .5\ \cfrac{\exp(-.25\ nc\ x)\ x^2}{Pi\ nc^{.5}\ t1}
$$

where t1 means

$$
t1 := \cfrac{1}{\tanh(.5\ nc)} - erf(.5\ nc^{.5}\ x)
$$

This answers the questions of increasing and decreasing of the similarity solutions with regard to Nc.

R E F E R E N C E S

[Abramowitz]   Pocket Book of Mathematical Functions
                  Abridged edition of Handbook of Mathematical
                  Functions, M. Abramowitz, I.A.Stegun (Eds.)
                  Verlag Harri Deutsch, Thun-Frankfurt/Main, 1984

[Barenblatt]   G.I.Barenblatt
                  Similarity, Self-Similarity and Intermediate
                  Asymptotics
                  Consultant Bureau, New York and London, 1979

[Bensimon]   D.Bensimon, L.Kadanoff, S.Liang,
                  B.I.Shraiman, Ch.Teng
                  Viscous flow in two dimensions
                  Rev.Mod.Phys.,Vol 58, pp.977-999, 1986

[Bluman]   G.W.Bluman, J.D.Cole
                  Similarity Methods for Differential Equations
                  (Applied Mathematical Sciences 13)
                  Springer Verlag, Berlin, 1974

[Davis]   H.T.Davis
                  Introduction to nonlinear differential and
                  integral equations
                  Dover Publications, Inc. New York, 1962

[Dresner]   L.Dresner
                  Similarity Solutions of Nonlinear Partial
                  Differential Equations
                  Pitman Advanced Publishing Program
                  Boston, London, Melbourne, 1983

[Fernandez]   J.F.Fernandez, R.Rangel, J.Rivero
                  Crossover Length from Invasion Percolation
                  to Diffusion Limited Aggregation in Porous
                  Media
                  Phys.Rev.Lett. 67, pp. 2958-2961, 1991

[GAUSS]   GAUSS
                  The GAUSS System Version 2.0
                  For IBM PC-XT-AT-PS2 & Compatibles
                  Aptech Systems Inc.
                  26250 196th Place S.E.
                  Kent, Washington 98042

[Isichenko]   M.B.Isichenko
                  Percolation, statistical topography, and transport
                  in random media
                  Rev.Mod.Phys. Vol.64, No.4, October 1992,
                  pp.961-1043

[Kessler]   D.E. Kessler
                  Pattern selection in fingered growth phenomena
                  Adv. in Phys. Vol. 37, 1988

[Koza]   Z.Koza
                  The Equivalence of the DLA and a Hydrodynamic
                  Model
                  J.Phys. A: Math. Gen., 24 (1991), pp. 4895-4905

[Lamb]   H.Lamb
                  Hydrodynamics
                  Cambridge University Press, Cambridge, 1932

[Larson]   R.G.Larson, H.T.Davis, L.e.Scriven
                  Displacement of residual nonwetting fluid
                  from porous media
                  Chem.Eng.Sci. Vol.36, pp.75-85, 1981

[Lenormand]   R.Lenormand
                  Flow through porous media: limits of fractal
                  patterns
                  Proc.R.Soc. London A 423, pp.159-168 (1989)

[Meakin]   P.Meakin, F.Family
                  Diverging length scales in diffusion-limited
                  aggregation
                  Phys.Rev. A 34, p.2558, 1986

[Nakayama]   T.Nakayama, K.Yakubo, R.L.Orbach
                  Dynamical properties of fractal networks: Scaling,
                  numerical simulations, and physical realizations
                  Rev.Mod.Phys. Vol.68, No.2, April 1994, pp. 381-443

[Pietronero]   L.Pietronero, W.R.Schneider
                  Invasion Percolation as a Fractal Growth Problem
                  Physica A, 170 (1990), pp.81-104

[Pistiner1]   A.Pistiner, M.Shapiro, H.R.Rubin
                 Analysis of Fuel Pollutant Migration in Water
                 Flow through Porous Media
                 Int.J.of Multiphase Flow, Vol.15, pp.135-154, 1989

[Pistiner2]   A.Pistiner, H.R.Rubin, M.Shapiro
                 Migration of an Immiscible Contaminant in a
                 Water Flow Through a Double Porosity Porous Medium
                 in [Rubin]

[Pistiner3]   A.Pistiner, M.Shapiro, H.R.Rubin
                 Self-Similar Analysis of Fuel Pollutant Migration
                 in Porous Media Including Gravity Segregation
                 in [Rubin]

[Po-zen Wong]   Po-zen Wong
                 The statistical physics of sedimentary rocks
                 Physics Today, December 1989

[Rogers]   C.Rogers, W.F.Shadwick
                 Bäcklund Transformations and their Applications
                 (Math.in Science and Engineering, Vol 161)
                 Academic Press, New York, 1982

[Rubin]   H.Rubin (Principal Investigator), E.Mechrez
                 Development of Methodologies for Restoring
                 Aquifers Contaminated by Oil Spills
                 Technion-Israel Institute of Technology
                 JOINT GERMAN ISRAELI RESEARCH PROJECTS
                 (Bundesministerium für Forschung und Technologie
                 (BMFT), Bonn,
                 Ministry of Science and Development, National
                 Council for Research and Development (NCRD))
                 Haifa and Kernforschungszentrum Karlsruhe
                 1987-1990

[Sahimi1]   M.Sahimi
                 Statistical and continuum models of fluid-solid
                 reactions in porous media
                 Chem.Eng.Sci. Vol.45, pp.1443-1502, 1990

[Sahimi2]   M.Sahimi
                 Flow phenomena in Rocks:
                 from continuum models to fractals, percolation,
                 cellular automata, and simulated annealing
                 Rev.Mod.Phys. Vol.65, pp.1395-1534, 1993

[Sahimi3]   M.Sahimi
                Applications of Percolation Theory
                Taylor & Francis, London 1994

[Saleur]   H.Saleur, B.Duplantier
                Exact determination of the percolation hull
                exponent in 2 dimension
                Phys.Rev. Lett. 58, p. 2325

[Sapoval]   B.Sapoval, M.Rosso, J.F.Gouyet
                The fractal nature of a diffusion front and
                the relation to percolation
                J.Phys.Lett., 46, L149

[Sonin]   A.A.Sonin
                Viscous Fingers: from Simple Amorphous Forms
                to Anisotropic Fractals
                Riv. del Nuovo Cimento, Vol. 14, No. 10,
                pp. 1-21, 1991

[Weber]   G.Weber
                Unveröffentlichte Resultate
                Kernforschungszentrum Karlsruhe, Juni 1992

[Wilkinson]   D.Wilkinson, J.F.Willemsen
                Invasion Percolation: a new form of percolation
                theory
                J.Phys. A: Math.Gen., 16, pp. 3365-3376, 1983

[Zhang]   Yu Zhang
                The Fractal Volume of the Two-Dimensional
                Invasion Percolation Cluster
                Commun. Mathematical Physics, Vol. 167,
                pp. 237-258 (1995)