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Spreading with Variable Viscosity

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Abstract

The isothermal spreading of a volume of liquid which increases with time as qt^{α}, $\alpha \geq 0$ was described in terms of similarity solutions by Huppert. Sakimoto and Zuber derived a similarity solution for the axisymmetric release of a fixed volume of fluid $(\alpha = 0)$ with time dependent viscosity of the form $\nu = \gamma t^{\beta}$, $0 \leq \beta < 1$. This result is extended to flows with $\alpha \geq 0$ and $0 \leq \beta < 3\alpha + 1$ in one-dimensional and axisymmetric geometries.

Ausbreitung mit variabler Viskosität

Zusammenfassung

Die isotherme Ausbreitung eines Volumenstromes, qt^{α}, $\alpha \geq 0$, in Form von dünnen Schichten kann mit Hilfe selbstähnlicher Lösungen beschrieben werden (Huppert). Für die axialsymmetrische Ausbreitung eines konstanten Volumens $(\alpha = 0)$ mit einer variablen Viskosität von der Form $\nu = \gamma t^{\beta}$, $0 \leq \beta < 1$ existieren ebenfalls selbstähnliche Lösungen (Sakimoto, Zuber). Dieses Ergebnis wird auf den Fall einer eindimensionalen und einer axialsymmetrischen Ausbreitung für alle $\alpha \geq 0$, $0 \leq \beta < 3\alpha + 1$ erweitert.

Contents

1. Introduction

The lubrication approximation for the isothermal spreading under gravity leads to a nonlinear degenerate parabolic equation [1]. A number of solutions of this equation has been found [6], [7], [11]. One of the most important differences between the solution of this nonlinear parabolic equation and the solution of the linear diffusion equation is the existence of solutions which are zero outside a closed and bounded set. Furthermore, the interfaces propagate with finite speed.

The isothermal spreading of a volume of liquid which increases with time as $q\,t^{\alpha}$, $q > 0$, $\alpha \geq 0$ was described in terms of similarity solutions in [5]. In many applications the cooling process will lead to variable flow properties such as viscosity and, consequently, to a coupled system of the temperature and flow equations. It is impossible to satisfactorily treat this problem analytically.

Bercovici [15] developed a model for an axisymmetric gravity current which accounts for thermo-viscous effects, i. e., the spatial variation of the viscosity. The numerical results showed significant deviations from the similarity profiles of the constant viscosity case. Sakimoto and Zuber [14] derived a similarity solution for the axisymmetric release of a fixed volume of fluid $(\alpha=0)$ with time dependent viscosity of the form $\nu = \gamma\, t^{\beta}$, $0 \leq \beta < 1$. In this paper this result is transferred to flows with $\alpha \geq 0$, $0 \leq \beta < 3\alpha + 1$ in one-dimensional and axisymmetric geometries.

2. Theory

A number of important physical processes are governed by a nonlinear diffusion equation [1]

$$\frac{\partial h}{\partial t}=\frac{\partial}{\partial x}\left(h^n\frac{\partial h}{\partial x}\right)\ in\ \Re^1\text{x}\ (0,\ T), \tag{1}$$

$$h(x,0)=h_0(x)\ in\ \Re^1, \tag{2}$$

where $h_0(x)$ is a bounded continuous non-negative function and $h_0\in L^1(\Re^1)\cap L^2(\Re^1)$. This equation arises in the study of the following physical problems:

1. Thin saturated regions in porous media, n = 1 [2].
2. Flow of homogeneous fluids through porous media, n ≥ 1 [3].
3. Flow of thin liquid films spreading under gravity, n = 3 [4], [5].

Recently, the spreading under gravity has received much attention in the field of nuclear safety research.

One of the most important differences between the solutions of Eq. (1) and the solutions of the linear diffusion equation (Eq. (1) with n = 0) is the existence of solutions which have compact support (i. e. , h ≡ 0 outside a closed and bounded set in $\Re^1$). Furthermore, the interface propagates with finite speed except possibly at t=0. This can be seen from the similarity solution of Eq. (1) obtained by Zel'dovich and Kompameets [6].

$$h(x,t)=\begin{cases}(t+1)^{-\frac{1}{n+2}}\left[C-\frac{n}{2(n+2)}x^2(t+1)^{-\frac{2}{n+2}}\right]^{1/n} & \text{for }\ x^2(t+1)^{-\frac{2}{n+2}}\leq\frac{2(n+2)}{n}C\\ 0 & \text{for }\ x^2(t+1)^{-\frac{2}{n+2}}>\frac{2(n+2)}{n}C\end{cases} \tag{3}$$

with the initial data of

$$h_0(x)=\begin{cases}\left[C-\frac{n}{2(n+2)}x^2\right]^{1/n} & \text{for } x^2\leq\frac{2(n+2)}{n}C\\ 0 & \text{for } x^2>\frac{2(n+2)}{n}C,\end{cases}$$

where C is a constant related to the initial integral of h. The support of the solution (3) is bounded by two monotonic cont inuous curves

$$x_f^{\pm}(t) = \pm\left[\frac{2(n+2)}{n}C\right]^{1/2}(t+1)^{\frac{1}{n+2}}. \tag{4}$$

These curves are called interface curves. From the general results obtained in [7] for the speed of the interfaces $\dot{x}_f^{\pm}(t)$ the following relation holds

$$\dot{x}_f^{\pm}(t) = -\frac{\partial}{\partial x}\Phi(x,t)\Big|_{x=x_f^{\pm}} \tag{5}$$

where

$$\Phi = \frac{1}{n}h^n. \tag{6}$$

A solution of (1) with initial data (in the limes of distributions with compact support $\boldsymbol{D}'(\Re^1)$)

$$h(x,t) \xrightarrow[t\to 0]{\boldsymbol{D}'(\Re^1)} q\delta(x),\ q \in \Re^1,\ q > 0, \tag{7}$$

where $\delta(x)$ denotes the Dirac distribution, is given by

$$h(x,t) = \begin{cases} t^{-\frac{1}{n+2}}\left[C - \frac{n}{2(n+2)}\,x^2 t^{-\frac{2}{n+2}}\right]^{1/n} & \text{for } |x| \le \sqrt{\frac{2(n+2)}{n}C}\ t^{\frac{1}{n+2}} \\ 0 & \text{for } |x| > \sqrt{\frac{2(n+2)}{n}C}\ t^{\frac{1}{n+2}} \end{cases} \tag{8}$$

with

$$C = q\sqrt{\frac{n}{2(n+2)}}\left(\int_{-1}^{1}dy(1-y^2)^{\frac{1}{n}}\right)^{-1}. \tag{9}$$

The right hand side of Eq. (1), $\frac{\partial}{\partial x}(h^n\frac{\partial h}{\partial x})$, is not defined at the free boundary where h=0 so that Eq. (3) is not a solution of (1) and (2) in the classical sense. The solution given by Eq. (3) may only be regarded as a weak solution of equations (1) and (2) [7], [8]. For this reason, caution must be exercised in interpreting any results near the leading edge of a spreading viscous liquid. The existence and uniqueness of weak solutions of (1), (2) was proven in [9]. It was also shown that if the initial distribution h_0 has a compact support, then the solution has a compact support for all times for which the solution exists.

In [10] it was proven that the solution of Eq. (1) with the initial data given by Eq. (2) behaves asymptotically as $t \to \infty$ like the similarity solution (8) with initial data (7), i. e.

$$t^{\frac{1}{n+2}} \sup_{|x| \le At^{\frac{1}{n+2}}} \left| h_{h_0}(x,t) - h(x,t) \right| \underset{t\to\infty}{\rightarrow} 0, \; A > 0.$$

In [7], [11], [12] a proof of the existence of a different type of so-called waiting-time solutions of Eq. (1) was given. These solutions are characterized by having a fixed support over a non-zero time interval. A solution of Eq. (1) with this property is

$$h(x,t) = \begin{cases} \left[\dfrac{n}{2(n+2)} \dfrac{x^2}{t_0 - t} \right]^{1/n} & \text{for } x \ge 0, \; t < t_0 \\ 0 & \text{for } x < 0, \; t < t_0. \end{cases} \tag{10}$$

Although the interface does not move for $t < t_0$, motion takes place behind the interface. The solution (10) is a classical solution, i. e. the right hand side of Eq. (1) exists even at the free boundary $x = 0$. This solution becomes unbounded at the waiting time t_0. No information is provided by the solution (10) about the solution after the front starts to move, i. e. for $t \ge t_0$. In [8] similarity solutions were found with a waiting-time property which are valid when the front starts to move. Using the singular-perturbation theory for $0 < n \ll 1$ [13], an approximate solution was constructed which contains information such as an appropriate waiting-time and a determination of the ways in which the interface can begin to move.

3. Spreading with variable viscosity

3.1 One-dimensional spreading

Using the lubrication theory for low Reynolds-number-flow for which the volume of the fluid increases with time as

$$\int_{-x_f}^{x_f} dx\, h(x,t) = qt^{\alpha}, \; \alpha \ge 0, q > 0, \tag{11}$$

the following equation holds [5]:

$$\frac{\partial h}{\partial t} - \frac{1}{3}\frac{g}{\nu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right) = 0, \tag{12}$$

$$t^{-\alpha}h(x,t) \xrightarrow[t\to 0]{\mathbf{D}'(\Re^1)} h_0(x) = q\delta(x),\ \alpha\geq 0,\ q>0, \tag{13}$$

where ν is the kinematic viscosity and g the gravity constant. Equations (12) and (13) have been solved for time dependent viscosity for $\alpha = 0$ in an axisymmetric geometry in [14]. Assuming the viscosity to be a function of time of the form

$$\nu = \gamma t^{\beta},\ \gamma = \frac{\nu_0}{t_0^{\beta}},\ \beta \geq 0 \tag{14}$$

a solution of (12) with initial data given by (13) for every $\alpha \geq 0$ will be constructed.

Consider similarity solutions expressed in terms of the similarity variable η:

$$h(x,t) = \eta_f^{2/3}\, C(t) \begin{cases} \Psi\left(\frac{|\eta|}{\eta_f}\right) & \text{for } x_f^- \leq x \leq x_f^+ \\ 0 & \text{for } x_f^+ < x < x_f^-, \end{cases} \tag{15}$$

$$\eta = \left(\frac{3\gamma}{gq^3}\right)^{1/5} x t^{-\frac{1}{5}(3\alpha-\beta+1)}, \tag{16}$$

$$\eta_f = \eta\big|_{x=x_f^+}. \tag{17}$$

From the global continuity equation (11) follows

$$C(t) = \left(\frac{3q^2\gamma}{g}\right)^{1/5} t^{\frac{1}{5}(2\alpha+\beta-1)} \tag{18}$$

and

$$\eta_f = \left(2\int_0^1 dy\,\Psi(y)\right)^{-\frac{3}{5}}. \tag{19}$$

Insertion of the form (15) into the basic equation (12) gives

$$\left(\Psi^3\Psi'\right)' + \frac{3\alpha-\beta+1}{5}\, y\Psi' - \frac{2\alpha+\beta-1}{5}\,\Psi = 0 \tag{20}$$

with

$$y = \frac{|\eta|}{\eta_f} \in [0,\ 1],\ \ \Psi(1) = 0. \tag{21}$$

The following relation holds for $\alpha = 0$, i. e. for a spreading of a volume q of a liquid with variable viscosity

$$\left(\Psi^3\Psi'\right)' + \frac{1-\beta}{5}\left(y\Psi\right)' = 0,\ y \in [0,\ 1],\ \beta<1. \tag{22}$$

The solution of equation (22) with the initial data given in eq. (21) is of the form

$$\Psi(y) = \left(\frac{3}{10}\right)^{1/3} (1-\beta)^{1/3} \left(1-y^2\right)^{1/3}. \tag{23}$$

The constant η_f can be easily evaluated from (19)

$$\eta_f = 2^{-\frac{3}{5}}\left[\frac{3}{10}(1-\beta)\right]^{-1/5}\left[\frac{1}{5}\pi^{1/2}\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{6}\right)}\right]^{-3/5}. \tag{24}$$

The solution of the original equation (11), (12), (13) can be written as

$$h(x,t) = \begin{cases} \left(\frac{3}{10}\right)^{1/3}\left(\frac{3q^2\gamma}{g}\right)^{1/5} \eta_f^{2/3}(1-\beta)^{1/3}\, t^{\frac{\beta-1}{5}}\left[1 - \left(\frac{3\gamma}{gq^3}\right)^{2/5} \eta_f^{-2} x^2 t^{\frac{2}{5}(\beta-1)}\right]^{1/3} & \text{for } x_f^- \le x \le x_f^+ \\ 0 & \text{for } x_f^+ < x < x_f^-, \end{cases} \tag{25}$$

where

$$x_f^{\pm} = \pm\eta_f\left(\frac{gq^3}{3\gamma}\right)^{1/5} t^{\frac{1}{5}(1-\beta)} \tag{26}$$

describes the positions of the interfaces as a function of time. The speed of the interfaces can be calculated from (5) or directly from (26):

$$\dot{x}_f^{\pm} = \pm\eta_f \frac{1-\beta}{5}\left(\frac{gq^3}{3\gamma}\right)^{1/5} t^{-\frac{1}{5}(4+\beta)}. \qquad (27)$$

For every $\alpha > 0$ the solution of equation (22) has to fulfil the condition $\left.\frac{\partial h(x,t)}{\partial x}\right|_{x=0} \neq 0$. This reflects the fact that the fluid is continuously introduced at $x = 0$. In accordance with other properties of solutions of Eq. (1), the approximate solution of (20) may be obtained in terms of the following expansion:

$$\Psi(y) = c_1(1-y)^{1/3}\left[1 + c_2(1-y) + c_3(1-y)^2 + \ldots\right]. \qquad (28)$$

Insertion of (28) into equation (20) gives

$$c_1 = \left[\frac{3}{5}(3\alpha-\beta+1)\right]^{1/3},\; c_2 = \frac{3a+4\beta-4}{24(3\alpha-\beta+1)},\; 0 \le \beta < 3a+1. \qquad (29)$$

From the relation (19) the following expression is obtained for the constant η_f:

$$\eta_f = \left[\frac{3}{2}c_1\left(1 + \frac{4}{7}c_2 + \ldots\right)\right]^{-3/5}. \qquad (30)$$

Then, $h(x,t)$ is determined by the following equation with $\alpha > 0,\; 0 \le \beta < 3\alpha + 1$:

$$h(x,t) = \begin{cases} \eta_f^{2/3}\left(\frac{3q^2\gamma}{g}\right)^{1/5} c_1 t^{\frac{1}{5}(2\alpha+\beta-1)}\left(1-\eta_f^{-1}|\eta|\right)^{1/3}\left\{1+c_2\left(1-\eta_f^{-1}|\eta|\right)+\ldots\right\} & \text{for } x_f^- \le x \le x_f^+ \\ 0 & \text{for } x_f^+ < x < x_f^-, \end{cases} \qquad (31)$$

where the position of the interfaces is given by

$$x_f^{\pm} = \pm\eta_f\left(\frac{gq^3}{3\gamma}\right)^{1/5} t^{\frac{1}{5}(3\alpha-\beta+1)}. \qquad (32)$$

In this case the speed of the interfaces can be calculated from

$$\dot{x}_f^{\pm} = \pm\eta_f \left(\frac{gq^3}{3\gamma}\right)^{1/5} \frac{3a-\beta+1}{5}\, t^{\frac{1}{5}(3a-\beta-4)}. \tag{33}$$

3.2 Axisymmetric spreading

The radial spreading of an axisymmetric flow is governed by the following equation:

$$\frac{\partial h}{\partial t} - \frac{1}{3}\frac{g}{\nu}\frac{1}{r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial h}{\partial r}\right) = 0,\ \nu = \gamma\, t^{\beta},\ \beta \geq 0 \tag{34}$$

with the initial data of

$$t^{-a}h(r,t) \xrightarrow[t\to 0]{\mathbf{D}'(\Re^1)} Q\delta(r),\ a\geq 0,\ Q>0. \tag{35}$$

The volume of the fluid is assumed to increase with time as

$$2\pi \int_0^{r_f} rdrh(r,t) = Qt^{\alpha}. \tag{36}$$

The solution procedure is similar to that described in section 3.1. The similarity solution is of the form

$$h(r,t) = \xi_f^{2/3} C(t) \begin{cases} \Psi\left(\dfrac{\xi}{\xi_f}\right) & \text{for } 0 \leq r \leq r_f \\ 0 & \text{for } r > r_f, \end{cases} \tag{37}$$

$$\xi = \left(\frac{3\gamma}{gQ^3}\right)^{1/8} r\, t^{-\frac{1}{8}(3a-\beta+1)}, \tag{38}$$

$$\xi_f = \xi\big|_{r=r_f} = \left[2\pi \int_0^1 dy\, y\Psi(y)\right]^{-3/8} \tag{39}$$

and

$$C(t) = \left(\frac{3\gamma Q}{g}\right)^{1/4} t^{\frac{1}{4}(a+\beta-1)}. \tag{40}$$

Ψ has to fulfil the equation

$$\left(y\Psi^3\Psi'\right)' + \frac{3\alpha+1-\beta}{8}\, y^2\, \Psi' - \frac{\alpha+\beta-1}{4}\, y\, \Psi = 0 \tag{41}$$

with

$$y = \frac{\xi}{\xi_f}.$$

For a spreading of a constant fluid volume, i. e. for $\alpha = 0$, the following analytical solution was derived in [14]:

$$\Psi(y) = \left(\frac{3}{16}\right)^{1/3} (1-\beta)^{1/3} \left(1-y^2\right)^{1/3},\ 0 \le \beta < 1, \tag{42}$$

$$\xi_f = \left[\frac{3^4}{2^{10}}\, \pi^3 (1-\beta)\right]^{-1/8}. \tag{43}$$

Using the expansion given by equation (28), an approximate solution of (41) is obtained for $\alpha > 0$ of the form

$$\Psi(y) = b_1 (1-y)^{1/3} \left[1 + b_2 (1-y) + \ldots\right] \tag{44}$$

with

$$b_1 = \left[\frac{3}{8}(3\alpha-\beta+1)\right]^{1/3},\ b_2 = \frac{1}{6}\,\frac{3\ \alpha+\beta-1}{3\ \alpha-\beta+1},\ 0 \le \beta < 3\alpha+1. \tag{45}$$

The evolution of the interface r_f is given by

$$r_f = \xi_f \left(\frac{g\,Q^3}{3\,\gamma}\right)^{1/8} t^{\frac{1}{8}(3\alpha-\beta+1)}, \tag{46}$$

where

$$\xi_f = \left\{\frac{9}{7}\pi b_1 \left[\frac{1}{2} + \frac{1}{5} b_2 + \ldots\right]\right\}^{-3/8}. \tag{47}$$

4. Discussion

The theory presented here is anticipated to be valid only if the viscous forces are much larger than the inertial forces, i. e. the ratio of viscous to inertial forces must be $\frac{F_v}{F_i} >> 1$. Using the arguments established in [5], the following expression holds:

$$\frac{F_v}{F_i} \approx \eta_f^2 \left(\frac{g^2 v_0^3}{q^4} \right)^{1/5} t_0^{-\frac{3}{5}\beta} t^{-\frac{1}{5}(4\alpha-3\beta-7)} \quad \text{for one-dimensional spreading} \quad (48)$$

and

$$\frac{F_v}{F_i} \approx \xi_f^4 \left(\frac{g v_0}{Q} \right)^{1/2} t_0^{-\frac{1}{2}\beta} t^{-\frac{1}{2}(\alpha-\beta-3)} \quad \text{for axisymmetric spreading.} \quad (49)$$

The transition time at which both forces are comparable is given by

$$t_{tr}(\alpha,\beta) \approx \left(\frac{1}{\eta_f^{10}} \frac{q^4}{g^2 v_0^3} \right)^{-\frac{1}{4\alpha-3\beta-7}} t_0^{\frac{-3\beta}{4\alpha-3\beta-7}}, \; 4\alpha-3\beta-7 \neq 0 \quad \text{for one-dimensional spreading} \quad (50)$$

and

$$t_{tr}(\alpha,\beta) \approx \left(\frac{1}{\xi_f^8} \frac{Q}{g v_0} \right)^{-\frac{1}{\alpha-\beta-3}} t_0^{-\frac{\beta}{\alpha-\beta-3}}, \; \alpha-\beta-3 \neq 0 \quad \text{for radial spreading.} \quad (51)$$

$t_{tr}(\alpha,\beta)$ is a monotonically decreasing function of β. Depending on the sign of $4\alpha - 3\beta - 7$ in case of one-dimensional spreading and the sign of $\alpha-\beta-3$ for axisymmetric spreading, the ratio $\frac{F_v}{F_i}$ is either a monotonically increasing or a monotonically decreasing function of time. Let $4\alpha - 3\beta - 7 < 0$ and $\alpha - \beta - 3 < 0$, then the spreading will be dominated by viscous forces for all $t >> t_{tr}$. If the opposite relations hold, i. e. $4\alpha - 3\beta - 7 > 0$ and $\alpha - \beta - 3 > 0$, then $\frac{F_v}{F_i} >> 1$ for all $t << t_{tr}$. For $4\alpha - 3\beta - 7 = 0$ and $\alpha - \beta - 3 = 0$, conditions under which the spreading is dominated by viscous forces for all times are obtained from (48) and (49)

$$t_0 << \begin{cases} \left(\dfrac{\eta_{0f}^{10}}{\eta_f^{10}} \right)^{-\frac{1}{3\beta}} \bar{\tau}_{tr} & \text{for one-dimensional spreading} \\ \left(\dfrac{\xi_{0f}^{8}}{\xi_f^{8}} \right)^{-\frac{1}{\beta}} \bar{\tau}_{tr} & \text{for axisymmetric spreading,} \end{cases} \tag{52}$$

where

$$\bar{\tau}_{tr} \approx \begin{cases} \left(\dfrac{1}{\eta_{0f}^{10}} \dfrac{q^4}{g^2 \nu_0^3} \right)^{-\frac{1}{3\beta}} & \text{for one-dimensional spreading} \\ \left(\dfrac{1}{\xi_{0f}^{8}} \dfrac{Q}{g \nu_0} \right)^{-\frac{1}{\beta}} & \text{for axisymmetric spreading} \end{cases} \tag{53}$$

is the transition time for the spreading with constant fluid viscosity ν_0 and the flow rates given by $\alpha = \frac{1}{4}(3\beta + 7)$ and $\alpha = \beta + 3,\ \beta \neq 0$ for one-dimensional spreading and axisymmetric spreading [5], respectively. It can easily be seen from (50) and (51) that if the parameter t_0 in Eq. (14) is chosen to be

$$t_0 = \tau_{tr} = \begin{cases} \left(\dfrac{1}{\eta_{0f}^{10}} \dfrac{q^4}{g^2 \nu_0^3} \right)^{-\frac{1}{4\alpha-7}} & \text{for one-dimensional spreading} \\ \left(\dfrac{1}{\xi_{0f}^{8}} \dfrac{Q}{g \nu_0} \right)^{-\frac{1}{\alpha-3}} & \text{for axisymmetric spreading} \end{cases} \tag{54}$$

which is the transition time for spreading with constant viscosity and the flow rates determined by α, then the transition time is

$$t_{tr}(\alpha,\beta) \approx \begin{cases} \left(\dfrac{\eta_{0f}^{10}}{\eta_f^{10}} \right)^{-\frac{1}{4\alpha-7}} \tau_{tr} & \text{for one-dimensional spreading} \\ \left(\dfrac{\xi_{0f}^{8}}{\xi_f^{8}} \right)^{-\frac{1}{\alpha-3}} \tau_{tr} & \text{for axisymmetric spreading,} \end{cases} \tag{55}$$

provided that $4\alpha - 7 \neq 0$ and $\alpha - 3 \neq 0$. τ_{tr} is given by the formula (54). In case of $4\alpha - 7 = 0$ and $\alpha - 3 = 0$, it results

$$t_{tr}(\alpha,\beta) \approx \begin{cases} \left(\dfrac{\eta_{0f}^{10}}{\eta_f^{10}}\right)^{\frac{1}{3\beta}} \tilde{\tau}_{tr}^{-1}\, t_0 & \text{for one-dimensional spreading} \\ \left(\dfrac{\xi_{0f}^{8}}{\xi_f^{8}}\right)^{\frac{1}{\beta}} \tilde{\tau}_{tr}^{-1}\, t_0 & \text{for axisymmetric spreading.} \end{cases} \quad (56)$$

$\tilde{\tau}_{tr}$ is obtained from eq. (54) by setting $\alpha = \dfrac{3\beta+7}{4}$ for one-dimensional spreading and $\alpha = \beta + 3$ for axisymmetric spreading.

Consider the one-dimensional spreading of a constant flux, i. e. for $\alpha = 1$. Setting $t_0 = \tau_{tr}$ in Eq. (14), where τ_{tr} is given by Eq. (54) and shifting the time $t \to t + t^*, t^* = \max_{\beta \in [0,3\alpha+1)} t_{tr}(\alpha,\beta) = t_{tr}(\alpha,0) = \tau_{tr}$, one ends up with the Cauchy problem (12), (13) with initial data $h_0(x) = h(x,t+t^*)\big|_{t=0} = h(x,t^*)$, where h is given by (31). The shape of $h(x^*,t+t^*)$, $x^* = x - x_f^+(t^*) \geq 0$ for various values of β and $t = 2t^*$ as well as $t = 10t^*$ with $t^* = 1s$ is depicted in the following figures (Fig. 1 and Fig. 2). The evolution of the interface $x_f^+(t+t^*) - x_f^+(t^*)$, $t \in [0,100t^*]$ is shown in Fig. 3.

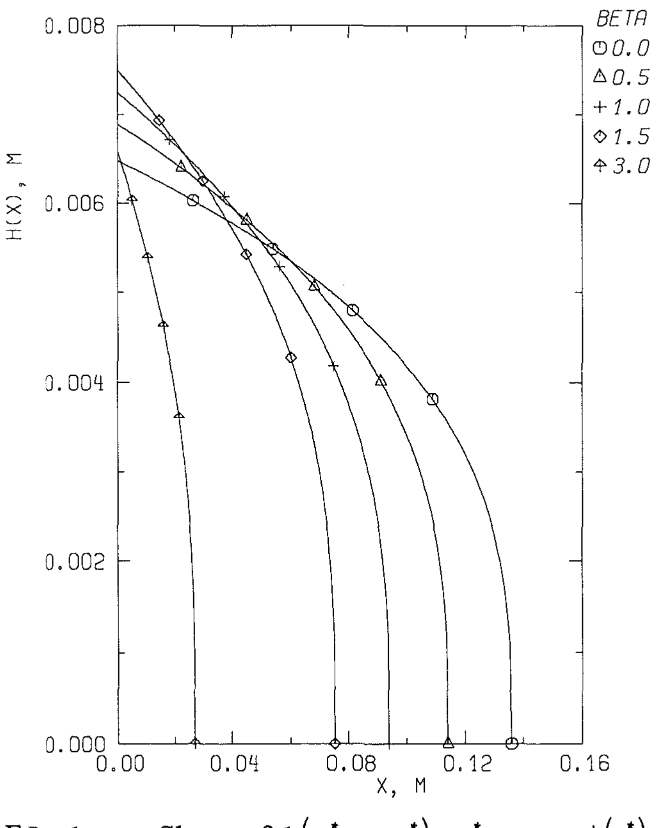

Fig. 1: Shape of $h(x^*,t+t^*)$, $x^* = x - x_f^+(t^*) \geq 0$ for $t = 2t^*$ $(t^* = \tau_{tr} = 1s)$.

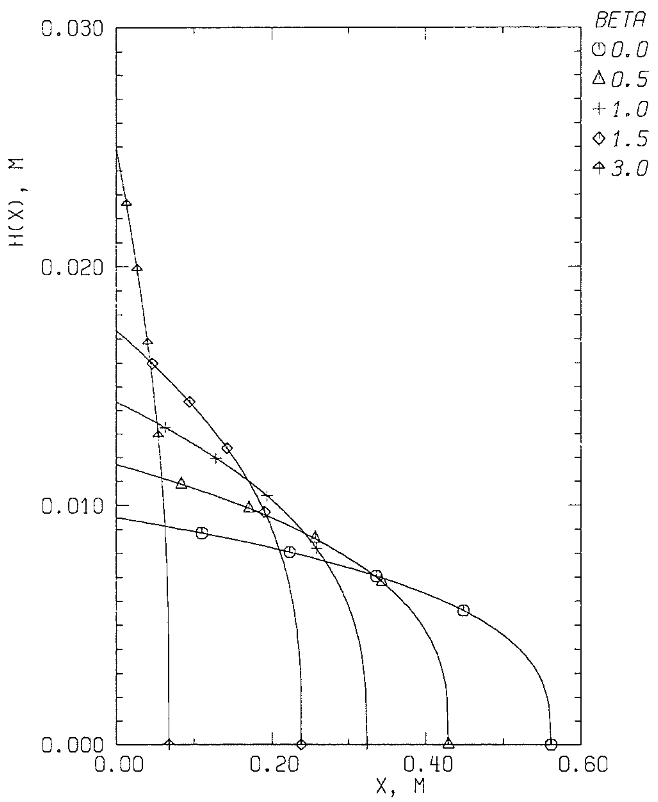

Fig. 2: Shape of $h\left(x^{*}, t+t^{*}\right)$, $x^{*}=x-x_{f}^{+}\left(t^{*}\right) \geq 0$ for $t=10 t^{*}$ $\left(t^{*}=1 s\right)$.

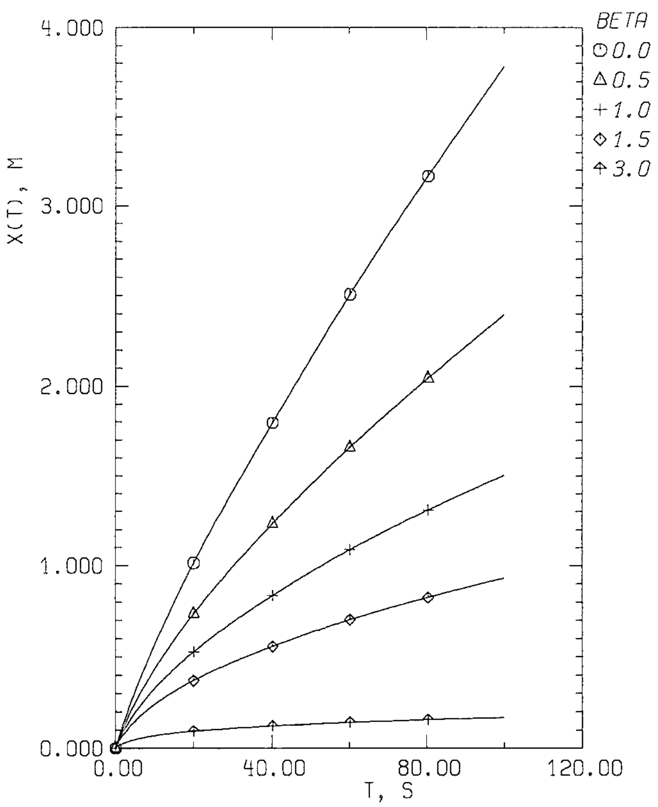

Fig. 3: Evolution of the interface $x_{f}^{+}\left(t+t^{*}\right)-x_{f}^{+}\left(t^{*}\right)$, $t^{*}=1 s$.

5. Conclusions

The lubrication approximation for the spreading under gravity leads to a nonlinear degenerate parabolic equaion. This equation admits solutions which have important properties, namely, they are zero outside a closed and bounded set and the interfaces propagate with finite speed. In this paper a similarity solution for the spreading of a volume of liquid which increases with time as qt^{α}, $q > 0$, $\alpha \geq 0$ and a time dependent viscosity of the form $\nu = \gamma t^{\beta}$, $0 \leq \beta < 3\alpha + 1$ was derived. The solutions are expected to be valid only if the viscous forces are much larger than the inertial forces. For flows for which $4\alpha - 3\beta - 7 < 0$ (in case of one-dimensional spreading) and $\alpha - \beta - 3 < 0$ (for axisymmetric spreading) the viscous forces are dominant for all $t >> t_{tr}$. In the opposite case, i. e. for $4\alpha - 3\beta - 7 > 0$ and for $\alpha - \beta - 3 > 0$ the viscous forces are dominant for all $t << t_{tr}$. t_{tr} is the transition time at which the inertial and viscous forces are equal. For $4\alpha - 3\beta - 7 = 0$ and $\alpha - \beta - 3 = 0$, conditions under which the spreading is dominated by viscous forces for all times are also given. Huppert's solutions for the constant viscosity are the formal limes of the solutions with time dependent viscosity as β tends to zero.

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