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Abstract

In this report stress analysis in a multi - layers joint, a plate or a cylinder, shall be presented. In a metal - ceramic joint, creep behavior of the metal is studied at high temperature. Equations to calculate analytically the stresses far away from the ends of the joint are given in an explicit form. The stresses obtained by these equations and by the finite element method (FEM) are compared to show the agreement of them. The effects of the material creep behavior on the stresses in a plate and cylinder are compared. Finally, an example is given for a multi - layers joint with a functionally graded material (FGM).

Kriechverhalten in einem Mehrschichtverbund

Zusammenfassung

In diesem Bericht wird eine Spannungsanalyse in einem Mehrschichtverbund, einer Platte oder einem Zylinder, durchgeführt. In einem Metall - Keramik - Verbund wird das Kriechverhalten des Metalls bei Hochtemperatur untersucht. Die Gleichungen zur analytischen Berechnung der Spannungen weit weg vom Rand des Verbundes werden in einer expliziten Form gegeben. Die Spannungen, die durch diese Gleichungen und durch die Finite-Element-Methode (FEM) berechnet werden, werden verglichen. Die Effekte des materiellen Kriechverhaltens auf die Spannungen in einer Platte und in einem Zylinder werden auch verglichen. Schließlich wird ein Beispiel für einen Mehrschichtverbund mit Gradientenwerkstoff (FGM) gegeben.

Contents

Chapter 1

Introduction

In engineering applications, a number of ceramic - metal joints are used under high-temperature conditions, in which the metal is used as a mechanical support structure and the ceramic as a coating for resisting the high temperature. For most metals creep behavior exists when the temperature is higher than $800°C$. Therefore, material creep behavior must be considered in the stress analysis.

In a multi - layers joint of ceramic and metal, very high stresses develop after a homogeneous temperature change (e.g. the residual stress) due to the difference in the thermal expansion coefficients and the elastic constants. For linear elastic material behavior, stress singularities exist at the intersection of the interface and the free edges. To reduce these stresses and to avoid the stress singularity in a quarter planes joint, a functionally graded material (FGM) can be introduced as interlayer.

In this report a multi - layers joint, with and without a functionally graded material as interlayer, is considered. Stress analysis in a plate and cylinder will be performed taking into account the material creep behavior. In the range far away from the free edge (or from the ends) of a joint, the stresses can be calculated analytically. However, near the free edge the stresses can be obtained only from the finite elment method for materials following the multiaxial creep law. Stresses in a joint with and without FGM layer, with and without considering the creep behavior, in a plate and in a cylinder will be compared in order to determine the effect of introducing a FGM layer and the influence of the material creep behavior on the stress distribution. The results show that the values of some stress components are increased when considering the material creep behavior. In a joined cylinder, for example, this applies to the stress component in radial direction, which may lead to a delamination of the joint.

Chapter 2

Basic Equations for Multi - Layers Cylinders

In this chapter equations to calculate the stresses far away from the ends of a multi - layers cylinder will be given for elastic behavior (time equals zero) and for taking into account the material creep behavior (time dependent). Three cases will be treated, namely (a) the strain in the axial direction is assumed to be zero, (b) the strain in the axial direction is an arbitrary constant, (c) the strain in the axial direction is arbitrary.

2.1 Solutions for elastic behavior

2.1.1 Zero strain in axial direction

If the loading of a multi - layers cylinder is axial symmetric, the solution of the stresses is also symmetric. For the case of a cylinder being subjected to a temperature change (i.e. T=T(r)), the stress distribution is axial symmetric. If $\epsilon_z = 0$ is assumed (coordinate system see Fig.1), the elastic stress - strain relations in each homogeneous material read

$$\epsilon_r = \frac{1}{E'}\left(\sigma_r - \frac{\nu}{1-\nu}\sigma_\phi\right) + (1+\nu)\alpha T \tag{2.1}$$

$$\epsilon_\phi = \frac{1}{E'}\left(\sigma_\phi - \frac{\nu}{1-\nu}\sigma_r\right) + (1+\nu)\alpha T \tag{2.2}$$

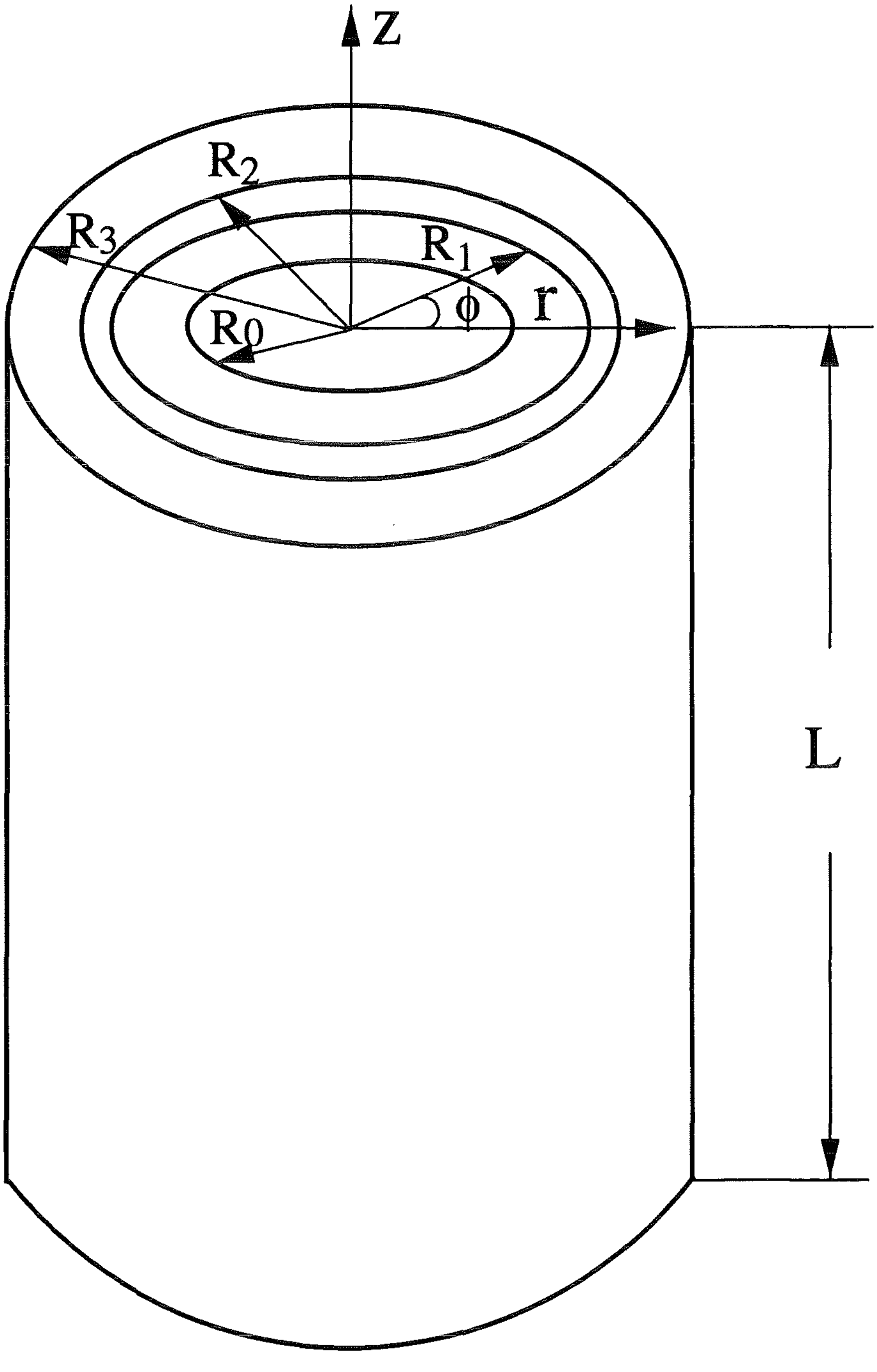


Fig.1 The investigated geometry and the coordinate system.

with $E' = E/(1-\nu^2)$, or

$$\sigma_r = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)}\left(\epsilon_r + \frac{\nu}{1-\nu}\epsilon_\phi - \frac{1+\nu}{1-\nu}\alpha T\right) \tag{2.3}$$

$$\sigma_\phi = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)}\left(\epsilon_\phi + \frac{\nu}{1-\nu}\epsilon_r - \frac{1+\nu}{1-\nu}\alpha T\right) \tag{2.4}$$

$$\sigma_z = \nu(\sigma_r + \sigma_\phi) - E\alpha T, \tag{2.5}$$

where E is the Young's modulus, ν the Poisson's ratio, α the thermal expansion coefficient and T is the temperature distribution in the cylinder T=T(r) or a homogeneous change of the temperature from a stress-free state. The strains are related to the displacement as

$$\epsilon_r = \frac{du}{dr} \tag{2.6}$$

$$\epsilon_\phi = \frac{u}{r} \tag{2.7}$$

where u is the displacement in r - direction. The equilibrium equation for this problem is

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\phi}{r} = 0. \tag{2.8}$$

Substituting Eqs.(2.6-2.7) into Eqs.(2.3-2.4) then in Eq.(2.8) yields

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = \frac{d}{dr}\left(\frac{1}{r}\frac{d(r\,u)}{dr}\right) = \frac{1+\nu}{1-\nu}\alpha\frac{dT(r)}{dr}. \tag{2.9}$$

The solution of Eq.(2.9) is

$$u = \frac{1+\nu}{1-\nu}\alpha\frac{1}{r}\int T(r)rdr + \frac{A}{2}r + \frac{B}{r}, \tag{2.10}$$

where A and B are unknown constants.

Corresponding to this solution of u, the strains and stresses can be calculated from

$$\epsilon_r = \frac{1+\nu}{1-\nu}\alpha\left\{-\frac{1}{r^2}\int T(r)rdr + T(r)\right\} + \frac{A}{2} - \frac{B}{r^2} \tag{2.11}$$

$$\epsilon_\phi = \frac{1+\nu}{1-\nu}\alpha\frac{1}{r^2}\int T(r)rdr + \frac{A}{2} + \frac{B}{r^2} \tag{2.12}$$

$$\sigma_r = -\frac{E}{1-\nu}\frac{\alpha}{r^2}\int T(r)rdr + \frac{E}{1+\nu}\left(\frac{A}{2}\frac{1}{1-2\nu} - \frac{B}{r^2}\right) \tag{2.13}$$

$$\sigma_\phi = \frac{E}{1-\nu}\left(\frac{\alpha}{r^2}\int T(r)rdr - \alpha T(r)\right) + \frac{E}{1+\nu}\left(\frac{A}{2}\frac{1}{1-2\nu} + \frac{B}{r^2}\right) \tag{2.14}$$

$$\sigma_z = \nu\frac{E}{1+\nu}\frac{A}{2}\frac{2}{1-2\nu} - \frac{E}{1-\nu}\alpha T(r). \tag{2.15}$$

If the temperature distribution in the cylinder is a constant, i.e. there is a homogeneous temperature change in the cylinder, the above equations can be simplified as

$$u = \frac{1+\nu}{1-\nu}\alpha\frac{T}{2}r + \frac{A}{2}r + \frac{B}{r}, \tag{2.16}$$

$$\epsilon_r = \frac{1+\nu}{1-\nu}\alpha\frac{T}{2} + \frac{A}{2} - \frac{B}{r^2} \tag{2.17}$$

$$\epsilon_\phi = \frac{1+\nu}{1-\nu}\alpha\frac{T}{2} + \frac{A}{2} + \frac{B}{r^2} \tag{2.18}$$

$$\sigma_r = -\frac{E}{1-\nu}\alpha\frac{T}{2} + \frac{E}{1+\nu}\left(\frac{A}{2}\frac{1}{1-2\nu} - \frac{B}{r^2}\right) \tag{2.19}$$

$$\sigma_\phi = -\frac{E}{1-\nu}\alpha\frac{T}{2} + \frac{E}{1+\nu}\left(\frac{A}{2}\frac{1}{1-2\nu} + \frac{B}{r^2}\right). \tag{2.20}$$

To determine the unknown constants A and B in each material, boundary and interface conditions have to be used. For a three-layers cylinder, the conditions are:

at r=R_0 (coordinates see Fig.1)

$$\sigma_r^{(1)} = 0, \tag{2.21}$$

at r=R_1

$$\begin{aligned} \sigma_r^{(1)} &= \sigma_r^{(2)} \\ u^{(1)} &= u^{(2)}, \end{aligned} \tag{2.22}$$

at r=R_2

$$\sigma_r^{(2)} = \sigma_r^{(3)}$$
$$u^{(2)} = u^{(3)}, \tag{2.23}$$

and at r=R_3

$$\sigma_r^{(3)} = 0, \tag{2.24}$$

where the superscript (i) denotes the quantity being in material i.
The boundary conditions lead to the following equations:

$$-\frac{E_1}{1-\nu_1}\alpha_1\frac{T}{2} + \frac{E_1}{1+\nu_1}\left(\frac{A_1}{2}\frac{1}{1-2\nu_1} - \frac{B_1}{R_0^2}\right) = 0 \tag{2.25}$$

$$\frac{E_1}{1-\nu_1}\alpha_1\frac{T}{2} - \frac{E_1}{1+\nu_1}\left(\frac{A_1}{2}\frac{1}{1-2\nu_1} - \frac{B_1}{R_1^2}\right) = \frac{E_2}{1-\nu_2}\alpha_2\frac{T}{2} - \frac{E_2}{1+\nu_2}\left(\frac{A_2}{2}\frac{1}{1-2\nu_2} - \frac{B_2}{R_1^2}\right) \tag{2.26}$$

$$\frac{1+\nu_1}{1-\nu_1}\alpha_1\frac{T}{2}R_1 + \frac{A_1}{2}R_1 + \frac{B_1}{R_1} = \frac{1+\nu_2}{1-\nu_2}\alpha_2\frac{T}{2}R_1 + \frac{A_2}{2}R_1 + \frac{B_2}{R_1} \tag{2.27}$$

$$\frac{E_2}{1-\nu_2}\alpha_2\frac{T}{2} - \frac{E_2}{1+\nu_2}\left(\frac{A_2}{2}\frac{1}{1-2\nu_2} - \frac{B_2}{R_2^2}\right) = \frac{E_3}{1-\nu_3}\alpha_3\frac{T}{2} - \frac{E_3}{1+\nu_3}\left(\frac{A_3}{2}\frac{1}{1-2\nu_3} - \frac{B_3}{R_2^2}\right) \tag{2.28}$$

$$\frac{1+\nu_2}{1-\nu_2}\alpha_2\frac{T}{2}R_2 + \frac{A_2}{2}R_2 + \frac{B_2}{R_2} = \frac{1+\nu_3}{1-\nu_3}\alpha_3\frac{T}{2}R_2 + \frac{A_3}{2}R_2 + \frac{B_3}{R_2} \tag{2.29}$$

$$-\frac{E_3}{1-\nu_3}\alpha_3\frac{T}{2} + \frac{E_3}{1+\nu_3}\left(\frac{A_3}{2}\frac{1}{1-2\nu_3} - \frac{B_3}{R_3^2}\right) = 0. \tag{2.30}$$

Using the definitions of

$$\lambda_1 = \frac{1+\nu_1}{1-\nu_1}$$
$$\lambda_2 = \frac{1+\nu_2}{1-\nu_2}$$
$$\lambda_3 = \frac{1+\nu_3}{1-\nu_3} \tag{2.31}$$

$$\eta_1 = \frac{1}{1-2\nu_1}$$
$$\eta_2 = \frac{1}{1-2\nu_2}$$
$$\eta_3 = \frac{1}{1-2\nu_3} \tag{2.32}$$

$$r_{01} = \left(\frac{R_1}{R_0}\right)^2$$
$$r_{21} = \left(\frac{R_1}{R_2}\right)^2$$
$$r_{31} = \left(\frac{R_1}{R_3}\right)^2 \tag{2.33}$$

$$B^* = \frac{B}{R_1^2}$$
$$A^* = \frac{A}{2} \tag{2.34}$$

$$E_{12} = \frac{E_1}{E_2}\frac{1+\nu_2}{1+\nu_1}$$
$$E_{32} = \frac{E_3}{E_2}\frac{1+\nu_2}{1+\nu_3}, \tag{2.35}$$

Eqs.(2.25-2.30) can be rewritten as

$$A_1^*\eta_1 - B_1^* r_{01} = \frac{T}{2}\lambda_1\alpha_1 \tag{2.36}$$

$$-E_{12}\eta_1 A_1^* + E_{12}B_1^* + \eta_2 A_2^* - B_2^* = \frac{T}{2}(\lambda_2\alpha_2 - E_{12}\lambda_1\alpha_1) \tag{2.37}$$

$$A_1^* + B_1^* - A_2^* - B_2^* = \frac{T}{2}(\lambda_2\alpha_2 - \lambda_1\alpha_1) \tag{2.38}$$

$$-\eta_2 A_2^* + r_{21}B_2^* + E_{32}\eta_3 A_3^* - E_{32}B_3^* r_{21} = \frac{T}{2}(E_{32}\lambda_3\alpha_3 - \lambda_2\alpha_2) \tag{2.39}$$

$$A_2^* + B_2^* r_{21} - A_3^* - B_3^* r_{21} = \frac{T}{2}(\lambda_3\alpha_3 - \lambda_2\alpha_2) \tag{2.40}$$

$$\eta_3 A_3^* - r_{31} B_3^* = \frac{T}{2}\lambda_3\alpha_3, \tag{2.41}$$

where the subscript i is for material i.

By solving Eqs.(2.36-2.41), the coefficients $A_1, B_1, A_2, B_2, A_3, B_3$ can be determined. After using the program MATHEMATICA and simplifying the solution, there is

$$\begin{aligned}
\Delta &= \eta_3 (E_{32}-1)\Big(\eta_1(\eta_2+E_{12})+(\eta_2-\eta_1 E_{12})r_{01}\Big)r_{21}^2 + \\
&+\eta_1(E_{12}-1)(\eta_3E_{32}-\eta_2)r_{31}+(1+\eta_1E_{12})(\eta_2-\eta_3E_{32})r_{01}r_{31}+ \\
&+r_{21}\Big[\eta_1\eta_3(1-E_{12})(\eta_2+E_{32})-\eta_1(\eta_2+E_{12})(1+\eta_3E_{32})r_{31}+ \\
&+r_{01}\Big(\eta_3(1+\eta_1E_{12})(\eta_2+E_{32})+(\eta_1E_{12}-\eta_2)(1+\eta_3E_{32})r_{31}\Big)\Big]
\end{aligned} \tag{2.42}$$

$$\begin{aligned}
A_1' &= \alpha_3(1+\eta_2)(1+\eta_3)E_{32}\lambda_3 r_{01}(r_{21}-r_{31})+\alpha_2(1+\eta_2)\lambda_2 r_{01}(1-r_{21})\times \\
&\times\Big(\eta_3(1-E_{32})r_{21}+(1+\eta_3E_{32})r_{31}\Big)+\alpha_1\lambda_1(r_{01}-1)\Big[\eta_3(\eta_2+E_{12})\times \\
&\times(1-E_{32})r_{21}^2+(E_{12}-1)(\eta_2-\eta_3E_{32})r_{31}+ \\
&+r_{21}\Big(\eta_3(E_{12}-1)(\eta_2+E_{32})+(\eta_2+E_{12})(1+\eta_3E_{32})r_{31}\Big)\Big]
\end{aligned} \tag{2.43}$$

$$\begin{aligned}
B_1' &= \alpha_3\eta_1(1+\eta_2)(1+\eta_3)E_{32}\lambda_3(r_{21}-r_{31})+ \\
&+\alpha_2\eta_1(1+\eta_2)\lambda_2(1-r_{21})\Big(\eta_3(1-E_{32})r_{21}+(1+\eta_3E_{32})r_{31}\Big)+ \\
&+\alpha_1(1+\eta_1)\lambda_1\Big[-\eta_3(\eta_2+E_{32})r_{21}+\eta_2\eta_3(1-E_{32})r_{21}^2+ \\
&+\Big(\eta_3E_{32}-\eta_2+\eta_2(1+\eta_3E_{32})r_{21}\Big)r_{31}\Big]
\end{aligned} \tag{2.44}$$

$$\begin{aligned}
A_2' &= \alpha_3(1+\eta_3)E_{32}\lambda_3\Big(\eta_1(1-E_{12})+(1+\eta_1E_{12})r_{01}\Big)(r_{21}-r_{31})+ \\
&+\alpha_2\lambda_2\Big(\eta_1(1-E_{12})+(1+\eta_1E_{12})r_{01}\Big)(1-r_{21})\times \\
&\times\Big(\eta_3(1-E_{32})r_{21}+(1+\eta_3E_{32})r_{31}\Big)+ \\
&+\alpha_1(1+\eta_1)E_{12}\lambda_1(r_{01}-1)r_{21}\Big(\eta_3(1-E_{32})r_{21}+(1+\eta_3E_{32})r_{31}\Big)
\end{aligned} \tag{2.45}$$

$$\begin{aligned}
B_2' &= \alpha_3(1+\eta_3)E_{32}\lambda_3\Big(\eta_1(\eta_2+E_{12})+(\eta_2-\eta_1E_{12})r_{01}\Big)(r_{21}-r_{31})- \\
&-\alpha_1(1+\eta_1)E_{12}\lambda_1(1-r_{01})\Big(\eta_3(\eta_2+E_{32})r_{21}+(\eta_2-\eta_3E_{32})r_{31}\Big)+ \\
&+\alpha_2(1+\eta_2)\lambda_2\Big[\eta_1\eta_3(E_{12}-E_{32})r_{21}+\eta_1(E_{12}+\eta_3E_{32})r_{31}- \\
&-r_{01}\Big(\eta_3(\eta_1E_{12}+E_{32})r_{21}+(\eta_1E_{12}-\eta_3E_{32})r_{31}\Big)\Big]
\end{aligned} \tag{2.46}$$

$$
\begin{aligned}
A'_3 &= \alpha_2 (1+\eta_2) \lambda_2 \Big(\eta_1 (1-E_{12}) + (1+\eta_1 E_{12}) r_{01}\Big) (1-r_{21}) r_{31} + \\
&+\alpha_1 (1+\eta_1)(1+\eta_2) E_{12}\lambda_1 (r_{01}-1) r_{21} r_{31} + \alpha_3 \lambda_3 (r_{31}-r_{21}) \times \\
&\times \Big\{\eta_1 (\eta_2 + E_{32})(E_{12}-1) + \eta_1 (\eta_2 + E_{12})(1-E_{32}) r_{21} - \\
&-r_{01}\Big[(\eta_2 + E_{32})(1+\eta_1 E_{12}) - (\eta_2 - \eta_1 E_{12})(1-E_{32}) r_{21}\Big]\Big\}
\end{aligned} \tag{2.47}
$$

$$
\begin{aligned}
B'_3 &= \alpha_2 (1+\eta_2) \eta_3 \lambda_2 \Big(\eta_1 (1-E_{12}) + (1+\eta_1 E_{12}) r_{01}\Big) (1-r_{21}) + \alpha_1 (1+\eta_1) \times \\
&\times (1+\eta_2) \eta_3 E_{12} \lambda_1 (r_{01}-1) r_{21} + \alpha_3 (1+\eta_3) \lambda_3 \Big[\eta_1 \eta_2 (E_{12}-1) + \\
&+\eta_1 (\eta_2 + E_{12}) r_{21} + r_{01}\Big((\eta_2 - \eta_1 E_{12}) r_{21} - \eta_2 (1+\eta_1 E_{12}) \Big)\Big]
\end{aligned} \tag{2.48}
$$

and

$$
A_1 = T\frac{A'_1}{\Delta}, \qquad A_2 = T\frac{A'_2}{\Delta}, \qquad A_3 = T\frac{A'_3}{\Delta} \tag{2.49}
$$

$$
B_1 = \frac{T}{2}R_1^2\frac{B'_1}{\Delta}, \qquad B_2 = \frac{T}{2}R_1^2\frac{B'_2}{\Delta}, \qquad B_3 = \frac{T}{2}R_1^2\frac{B'_3}{\Delta}. \tag{2.50}
$$

Now the coefficients $A_1, B_1, A_2, B_2, A_3, B_3$ are known. The stress distribution in a multi - layers cylinder can be calculated analytically from Eqs.(2.19,2.20,2.15) for the case of $\epsilon_z = 0$ and T = constant.

2.1.2 Constant strain in axial direction

In this section a multi - layers cylinder under mechanical loading at the ends and having a constant strain in axial direction in the range far away from the ends shall be dealt with. For the case of $\epsilon_z = d$, the solution in each material is

$$
u = A_c r + \frac{B_c}{r} \tag{2.51}
$$

$$
\epsilon_r = A_c - \frac{B_c}{r^2} \tag{2.52}
$$

$$
\epsilon_\phi = A_c + \frac{B_c}{r^2} \tag{2.53}
$$

$$\epsilon_z = d \tag{2.54}$$

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left\{A_c - \frac{(1-2\nu)B_c}{r^2} + \nu d\right\} \tag{2.55}$$

$$\sigma_\phi = \frac{E}{(1+\nu)(1-2\nu)}\left\{A_c + \frac{(1-2\nu)B_c}{r^2} + \nu d\right\} \tag{2.56}$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left\{2A_c\nu + (1-\nu)d\right\}, \tag{2.57}$$

where d is the constant strain in axial direction and A_c, B_c are unknown constants. To determine the unknown constants A_{1c}, B_{1c}, A_{2c}, B_{2c},A_{3c}, B_{3c} in each material, boundary conditions have to be used. From the boundary conditions, the following equations are obtained:

$$A_{1c} - \frac{(1-2\nu_1)B_{1c}}{R_0^2} + \nu_1 d = 0 \tag{2.58}$$

$$\frac{E_1}{(1+\nu_1)(1-2\nu_1)}\left\{A_{1c} - \frac{(1-2\nu_1)B_{1c}}{R_1^2} + \nu_1 d\right\} = \frac{E_2}{(1+\nu_2)(1-2\nu_2)}\left\{A_{2c} - \frac{(1-2\nu_2)B_{2c}}{R_1^2} + \nu_2 d\right\} \tag{2.59}$$

$$A_{1c}R_1 + \frac{B_{1c}}{R_1} = A_{2c}R_1 + \frac{B_{2c}}{R_1} \tag{2.60}$$

$$\frac{E_2}{(1+\nu_2)(1-2\nu_2)}\left\{A_{2c} - \frac{(1-2\nu_2)B_{2c}}{R_2^2} + \nu_2 d\right\} = \frac{E_3}{(1+\nu_3)(1-2\nu_3)}\left\{A_{3c} - \frac{(1-2\nu_3)B_{3c}}{R_2^2} + \nu_3 d\right\} \tag{2.61}$$

$$A_{2c}R_2 + \frac{B_{2c}}{R_2} = A_{3c}R_2 + \frac{B_{3c}}{R_2} \tag{2.62}$$

$$A_{3c} - \frac{(1-2\nu_3)B_{3c}}{R_3^2} + \nu_3 d = 0. \tag{2.63}$$

The solution of Eqs.(2.58-2.63) is

$$\begin{aligned}
\Delta_c &= (1-E^c_{32})\, r_{31}\Big(E^c_{12}\,(1-2\nu_1)-(1-2\nu_2)\Big)\,(1-2\nu_3)+\\
&+ r^2_{21}\Big(1+E^c_{12}\,(1-2\nu_1)\Big)\Big(1-2\nu_2-E^c_{32}\,(1-2\nu_3)\Big)-\\
&- r_{21}\Big[(1-2\nu_2)+E^c_{32}\,(1-2\nu_2)\,(1-2\nu_3)-r_{31}\Big(1+E^c_{12}\,(1-2\nu_1)\Big)\times\\
&\quad\times(1+E^c_{32}-2\nu_2)\,(1-2\nu_3)-E^c_{12}\,(1-2\nu_1)\,\Big(1+E^c_{32}\,(1-2\nu_3)\Big)\Big]+\\
&+ r_{01}\Big\{\,(E^c_{32}-1)\,r_{31}\,(1-2\nu_1)\,(1+E^c_{12}-2\nu_2)\,(1-2\nu_3)+\\
&\quad+(1-E^c_{12})\,r^2_{21}\,(1-2\nu_1)\,\Big(1-2\nu_2-E^c_{32}\,(1-2\nu_3)\Big)-\\
&\quad-r_{21}\Big[\,(1-2\nu_1)\,(1-2\nu_2)+E^c_{12}\,(1-2\nu_1)\,\Big(1+E^c_{32}\,(1-2\nu_3)\Big)+\\
&\quad+E^c_{32}\,(1-2\nu_1)\,(1-2\nu_2)\,(1-2\nu_3)-(1-E^c_{12})\,r_{31}\,(1-2\nu_1)\times\\
&\quad\times(1-2\nu_2+E^c_{32})\,(1-2\nu_3)\Big]\Big\}
\end{aligned} \tag{2.64}$$

$$\begin{aligned}
A^*_{1c} &= (1-E^c_{32})\,r_{31}\nu_1\Big(1-2\nu_2-E^c_{12}\,(1-2\nu_1)\Big)\,(1-2\nu_3)-\\
&-r^2_{21}\nu_1\Big(1+E^c_{12}\,(1-2\nu_1)\Big)\Big(1-2\nu_2-E^c_{32}\,(1-2\nu_3)\Big)+\\
&+r_{21}\nu_1\Big[\,(1-2\nu_2)-E^c_{12}\,(1-2\nu_1)\,\Big(1+E^c_{32}\,(1-2\nu_3)\Big)-r_{31}\Big(1+E^c_{12}\,(1-2\nu_1)\Big)\\
&\quad\times(1+E^c_{32}-2\nu_2)\,(1-2\nu_3)+E^c_{32}\,(1-2\nu_2)\,(1-2\nu_3)\Big]+\\
&+r_{01}\Big\{r^2_{21}\,(1-2\nu_1)\,(E^c_{12}\nu_1-\nu_2)\Big(1-2\nu_2-E^c_{32}\,(1-2\nu_3)\Big)+r_{31}\,(1-2\nu_1)\times\\
&\quad\times(1-2\nu_3)\,\Big(E^c_{12}\,(1-E^c_{32})\,\nu_1+(1-2\nu_2)\,\nu_2+E^c_{32}\,(\nu_2-2\nu_3+2\nu_2\nu_3)\Big)+\\
&+r_{21}\Big[\,(1-2\nu_1)\,\nu_2\,(1-2\nu_2)+E^c_{12}\nu_1\,(1-2\nu_1)\,\Big(1+E^c_{32}\,(1-2\nu_3)\Big)+\\
&+r_{31}\,(1-2\nu_1)\,(1+E^c_{32}-2\nu_2)\,(E^c_{12}\nu_1-\nu_2)\,(1-2\nu_3)-\\
&-E^c_{32}\,(1-2\nu_1)\,(1-2\nu_3)\,(\nu_2-2\nu_3+2\nu_2\nu_3)\Big]\Big\}
\end{aligned} \tag{2.65}$$

$$\begin{aligned}
B^*_{1c} &= r^2_{21}\,(\nu_1-\nu_2)\,\Big(1-2\nu_2-E^c_{32}\,(1-2\nu_3)\Big)-r_{31}\,(1-2\nu_3)\times\\
&\quad\times\Big[\,(\nu_1-\nu_2)\,(1-2\nu_2)-E^c_{32}\Big(\,(1-\nu_1)\,\nu_2+(1-\nu_2)\,(\nu_1-2\nu_3)\Big)\Big]-\\
&- r_{21}\Big[\,(\nu_1-\nu_2)\,(1-2\nu_2)-r_{31}\,(1+E^c_{32}-2\nu_2)\,(\nu_1-\nu_2)\,(1-2\nu_3)+\\
&\quad+E^c_{32}\,(1-2\nu_3)\,\Big(\,(1-\nu_1)\,\nu_2+(1-\nu_2)\,(\nu_1-2\nu_3)\Big)\Big]
\end{aligned} \tag{2.66}$$

$$\begin{aligned}
A^*_{2c} &= r_{31}\Big(1-2\nu_2-E^c_{12}\,(1-2\nu_1)\Big)\,(1-2\nu_3)\,(\nu_2-E^c_{32}\nu_3)-\\
&- r^2_{21}\Big(E^c_{12}\nu_1\,(1-2\nu_1)+\nu_2\Big)\Big(1-2\nu_2-E^c_{32}\,(1-2\nu_3)\Big)+\\
&+ r_{21}\Big[\,(1-2\nu_2)\,\nu_2-r_{31}\,(1+E^c_{32}-2\nu_2)\,\Big(E^c_{12}\nu_1\,(1-2\nu_1)+\nu_2\Big)\,(1-2\nu_3)+
\end{aligned}$$

$$
\begin{aligned}
&+E^c_{32}(1-2\nu_2)\,\nu_3\,(1-2\nu_3) - E^c_{12}(1-2\nu_1)\Big(\nu_2 + E^c_{32}(1-2\nu_3)\,\nu_3\Big)\Big] + \\
+\ & r_{01}\Big\{r_{31}(1-2\nu_1)(1-2\nu_2+E^c_{12})(1-2\nu_3)(\nu_2 - E^c_{32}\nu_3) + \\
+\ & r^2_{21}(1-2\nu_1)(E^c_{12}\nu_1-\nu_2)\Big(1-2\nu_2-E^c_{32}(1-2\nu_3)\Big) + \\
+\ & r_{21}\Big[(1-2\nu_1)\,\nu_2\,(1-2\nu_2) + r_{31}(1-2\nu_1)(1+E^c_{32}-2\nu_2)\times \\
&\times(E^c_{12}\nu_1-\nu_2)(1-2\nu_3) + E^c_{32}(1-2\nu_1)(1-2\nu_2)\,\nu_3\times \\
&\times(1-2\nu_3) + E^c_{12}(1-2\nu_1)\Big(\nu_2+E^c_{32}(1-2\nu_3)\,\nu_3\Big)\Big]\Big\}
\end{aligned}
\tag{2.67}
$$

$$
\begin{aligned}
B^*_{2c} =\ & r_{31}\Big[E^c_{12}(1-2\nu_1)\Big(\nu_2-\nu_1+E^c_{32}(\nu_1-\nu_3)\Big)(1-2\nu_3) + \\
+\ & E^c_{32}(\nu_2-\nu_3)(1-2\nu_3)\Big] - r_{21}\Big[E^c_{32}(\nu_2-\nu_3)(1-2\nu_3) + \\
+\ & E^c_{12}(1-2\nu_1)\Big(\nu_1-\nu_2+E^c_{32}(\nu_1-\nu_3)(1-2\nu_3)\Big)\Big] + r_{01}\times \\
\times\ & \Big\{r_{31}(1-2\nu_1)(1-2\nu_3)\Big[E^c_{12}(\nu_1-\nu_2)+E^c_{32}\Big(\nu_2-\nu_3-E^c_{12}(\nu_1-\nu_3)\Big)\Big] - \\
-\ & r_{21}\Big[E^c_{32}(1-2\nu_1)(\nu_2-\nu_3)(1-2\nu_3) - E^c_{12}(1-2\nu_1)\times \\
& \times\Big(\nu_1-\nu_2+E^c_{32}(\nu_1-\nu_3)(1-2\nu_3)\Big)\Big]\Big\}
\end{aligned}
\tag{2.68}
$$

$$
\begin{aligned}
A^*_{3c} =\ & r^2_{21}\Big(1+E^c_{12}(1-2\nu_1)\Big)\Big(E^c_{32}(1-2\nu_3)-(1-2\nu_2)\Big)\nu_3 + \\
+\ & r_{31}\Big(1-2\nu_2-E^c_{12}(1-2\nu_1)\Big)(1-2\nu_3)(\nu_2-E^c_{32}\nu_3) + \\
+\ & r_{21}\Big\{(1-2\nu_2)\,\nu_3 - E^c_{12}(1-2\nu_1)\Big(1+E^c_{32}(1-2\nu_3)\Big)\nu_3 + \\
& +E^c_{32}(1-2\nu_2)\,\nu_3\,(1-2\nu_3) - r_{31}(1-2\nu_3)\Big[(1-2\nu_2)\,\nu_2 + \\
& +E^c_{12}(1-2\nu_1)(2\nu_1-\nu_2-2\nu_1\nu_2) + E^c_{32}\Big(1+E^c_{12}(1-2\nu_1)\Big)\nu_3\Big]\Big\} + \\
+\ & r_{01}\Big\{(1-E^c_{12})\,r^2_{21}(1-2\nu_1)\Big(E^c_{32}(1-2\nu_3)-(1-2\nu_2)\Big)\nu_3 + \\
& +r_{31}(1-2\nu_1)(1-2\nu_2+E^c_{12})(1-2\nu_3)(\nu_2-E^c_{32}\nu_3) + \\
& +r_{21}\Big[(1-2\nu_1)(1-2\nu_2)\,\nu_3 + E^c_{12}(1-2\nu_1)\Big(1+E^c_{32}(1-2\nu_3)\Big)\nu_3 + \\
& +E^c_{32}(1-2\nu_1)(1-2\nu_2)\,\nu_3\,(1-2\nu_3) + r_{31}(1-2\nu_1)(1-2\nu_3)\times \\
& \times\Big(E^c_{12}(2\nu_1-\nu_2-2\nu_1\nu_2) - \nu_2(1-2\nu_2) - E^c_{32}\nu_3(1-E^c_{12})\Big)\Big]\Big\}
\end{aligned}
\tag{2.69}
$$

$$
\begin{aligned}
B^*_{3c} =\ & (1-2\nu_2)\,\nu_2 - E^c_{12}(1-2\nu_1)(\nu_2-\nu_3) - (1-2\nu_2)\,\nu_3 - r_{21}\times \\
& \times\Big[(1-2\nu_2)(\nu_2-\nu_3) + E^c_{12}(1-2\nu_1)\Big(2\nu_1-(1+2\nu_1)\,\nu_2-(1-2\nu_2)\,\nu_3\Big)\Big] \\
& +r_{01}(1-2\nu_1)\Big\{(1-2\nu_2)(\nu_2-\nu_3) - r_{21}(1-2\nu_2)(\nu_2-\nu_3) + \\
& +E^c_{12}\Big[\nu_2-\nu_3+r_{21}\Big(2\nu_1-(1+2\nu_1)\,\nu_2-(1-2\nu_2)\,\nu_3\Big)\Big]\Big\}
\end{aligned}
\tag{2.70}
$$

with

$$E_{12}^c = \frac{E_1 (1+\nu_2)(1-2\nu_2)}{E_2 (1+\nu_1)(1-2\nu_1)}$$
$$E_{32}^c = \frac{E_3 (1+\nu_2)(1-2\nu_2)}{E_2 (1+\nu_3)(1-2\nu_3)} \qquad (2.71)$$

and

$$A_{1c} = d\frac{A_{1c}^*}{\Delta_c}, \qquad A_{2c} = d\frac{A_{2c}^*}{\Delta_c}, \qquad A_{3c} = d\frac{A_{3c}^*}{\Delta_c} \qquad (2.72)$$

$$B_{1c} = dR_1^2\frac{B_{1c}^*}{\Delta_c}, \qquad B_{2c} = dR_1^2\frac{B_{2c}^*}{\Delta_c}, \qquad B_{3c} = dR_1^2\frac{B_{3c}^*}{\Delta_c}. \qquad (2.73)$$

Using the coefficients $A_{1c}, B_{1c}, A_{2c}, B_{2c}, A_{3c}, B_{3c}$, stresses in the cylinder can be determined analytically from Eqs.(2.55-2.57) for the case of a cylinder having ϵ_z=d.

2.1.3 Arbitrary strain in axial direction

For a multi - layers cylinder under thermal loading, in fact, the strain ϵ_z is not zero and the stress σ_z at the ends of the cylinder is zero. To find an exact analytical solution, satisfying $\sigma_z = 0$ at each point of the ends of a cylinder, is very difficult. Although $\sigma_z = 0$ at the ends of the cylinder cannot be satisfied at each point, the solution of the resulting force at the ends of the cylinder being zero

$$2\pi \int_{R_i}^{R_a} \sigma_z r dr = 0 \qquad (2.74)$$

is useful, where R_i is the inner radius and R_a is the outer radius of the cylinder. For the case of σ_z in each layer being a constant, Eq.(2.74) can be rewritten as

$$\pi \sum_{i=1}^{N} \sigma_z^{(i)} \left[(R_a^{(i)})^2 - (R_i^{(i)})^2\right] = 0 \qquad (2.75)$$

where N is the number of the layers, $R_a^{(i)}$ the outer radius and $R_i^{(i)}$ the inner radius of each layer.

Under the assumption of $\epsilon_z = 0$, the resulting force of σ_z at the ends is

$$F_0 = \sum_{i=1}^{N} \left\{ \nu_i \frac{E_i}{1+\nu_i} \frac{A_i}{1-2\nu_i} - \frac{E_i}{1-\nu_i} \alpha_i T \right\} \pi \left[(R_a^{(i)})^2 - (R_i^{(i)})^2 \right]. \tag{2.76}$$

The real situation is that the resulting force is zero at the ends of the cylinder. To describe the true stress distribution in the range far away from the ends, according to the Saint-Venant principle, the solution for loading with a resulting force -F_0 should be superposed to that of $\epsilon_z = 0$. Under loading with a resulting force -F_0, the cylinder has a constant ϵ_z in the range far away from the ends - denoted as d. From section 2.1.2 it is known that the resulting force of σ_z corresponding to ϵ_z=d is

$$F_d = \sum_{i=1}^{N} \frac{E_i}{(1+\nu_i)(1-2\nu_i)} \left\{ 2A_{ic}\nu_i + (1-\nu_i)d \right\} \pi \left[(R_a^{(i)})^2 - (R_i^{(i)})^2 \right] \tag{2.77}$$

where A_{ic} is a function of d (see Eq.(2.72)). Following

$$F_d = -F_0 \tag{2.78}$$

the quantity d can be determined. From Eqs.(2.76,2.77,2.72,2.78), there is

$$d = -\frac{\sum_{i=1}^{N} \left\{ \left[\nu_i \frac{E_i}{1+\nu_i} \frac{A_i}{1-2\nu_i} - \frac{E_i}{1-\nu_i} \alpha_i T \right] \left[(R_a^{(i)})^2 - (R_i^{(i)})^2 \right] \right\}}{\sum_{i=1}^{N} \left\{ \frac{E_i}{(1+\nu_i)(1-2\nu_i)} \left[2\frac{A_{ic}^*}{\Delta_c}\nu_i + (1-\nu_i) \right] \left[(R_a^{(i)})^2 - (R_i^{(i)})^2 \right] \right\}} \tag{2.79}$$

where A_i, A_{ic}^*, Δ_c see chapters 2.1.1 and 2.1.2. Finally, the true stress distribution in a multi - layers cylinder under thermal loading is

$$\sigma_{ij} = \sigma_{ij}^0 + \sigma_{ij}^c \tag{2.80}$$

where σ_{ij}^0 will be calculated from section 2.1.1 and σ_{ij}^c will be determined from section 2.1.2 using the value of d from Eq.(2.79). In chapter 3, some examples will be presented to see the agreement between the stresses calculated from the finite element method and those obtained by the analytical equations given in this chapter.

2.2 Solutions for creep behavior

2.2.1 The case of $\dot{\epsilon}_z = 0$

Following Norton's law for creep material, the relation between the rates of stress and strain in the multi - axial form is

$$\dot{\epsilon}_{ij} = \frac{1+\nu}{E}\dot{\sigma}_{ij} - \frac{\nu}{E}\dot{\sigma}_{kk}\delta_{ij} + \frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}S_{ij}, \tag{2.81}$$

where . denotes the rate, D and n are material creep constants, S_{ij} is the deviator of the stress tensor σ_{ij}, σ_{kk} is the sum of the three normal stresses, δ_{ij} is Kronecker's tensor and σ_{eff} is the effective stress of the stress tensor, which are defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if i=j} \\ 0 & \text{if i}\neq\text{j} \end{cases}$$

$$\sigma_{kk} = \sigma_1 + \sigma_2 + \sigma_3 = \sigma_r + \sigma_\phi + \sigma_z \tag{2.82}$$

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{2.83}$$

$$\sigma_{\text{eff}} = \sqrt{\frac{3}{2}S_{ij}S_{ij}} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_r - \sigma_\phi)^2 + (\sigma_r - \sigma_z)^2 + (\sigma_z - \sigma_\phi)^2}. \tag{2.84}$$

The rates of strains and displacement satisfy

$$\dot{\epsilon}_r = \frac{d\dot{u}}{dr} \tag{2.85}$$

$$\dot{\epsilon}_\phi = \frac{\dot{u}}{r}. \tag{2.86}$$

The equilibrium equation of the stress rate is

$$\frac{d\dot{\sigma}_r}{dr} + \frac{\dot{\sigma}_r - \dot{\sigma}_\phi}{r} = 0. \tag{2.87}$$

For the case of $\dot{\epsilon}_z = 0$, the following relations can be obtained from Eq.(2.81):

$$\dot{\epsilon}_r = \frac{1}{E'}\left(\dot{\sigma}_r - \frac{\nu}{1-\nu}\dot{\sigma}_\phi\right) + \frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}\left(S_r + \nu S_z\right) \tag{2.88}$$

$$\dot{\epsilon}_\phi = \frac{1}{E'}\left(\dot{\sigma}_\phi - \frac{\nu}{1-\nu}\dot{\sigma}_r\right) + \frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}\left(S_\phi + \nu S_z\right) \tag{2.89}$$

and

$$\dot{\sigma}_r = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)}\left\{\dot{\epsilon}_r + \frac{\nu}{1-\nu}\dot{\epsilon}_\phi - \frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}\left[(S_r+\nu S_z) + \frac{\nu}{1-\nu}(S_\phi + \nu S_z)\right]\right\} \tag{2.90}$$

$$\dot{\sigma}_\phi = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)}\left\{\dot{\epsilon}_\phi + \frac{\nu}{1-\nu}\dot{\epsilon}_r - \frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}\left[(S_\phi+\nu S_z) + \frac{\nu}{1-\nu}(S_r + \nu S_z)\right]\right\} \tag{2.91}$$

$$\dot{\sigma}_z = \nu(\dot{\sigma}_r + \dot{\sigma}_\phi) - \frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}ES_z. \tag{2.92}$$

It can be seen that S_r and S_ϕ always appear in the combination of $S_r + \nu S_z$ and $S_\phi + \nu S_z$. Therefore, the definitions

$$\begin{aligned} S_r' &= S_r + \nu S_z \\ S_\phi' &= S_\phi + \nu S_z \end{aligned} \tag{2.93}$$

will be used in the following. Insertion of Eqs.(2.85-2.86) into Eqs.(2.90-2.91) then into Eq.(2.87) yields

$$\begin{aligned} &\frac{d^2\dot{u}}{dr^2} + \frac{1}{r}\frac{d\dot{u}}{dr} - \frac{\dot{u}}{r^2} = \frac{d}{dr}\left(\frac{1}{r}\frac{d(r\,\dot{u})}{dr}\right) \\ = &\frac{3}{2}D\frac{d}{dr}\left[\sigma_{\text{eff}}^{(n-1)}(S_r' + \frac{\nu}{1-\nu}S_\phi')\right] + \frac{1-2\nu}{1-\nu}\frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}\frac{S_r'-S_\phi'}{r}. \end{aligned} \tag{2.94}$$

The solution of Eq.(2.94) is

$$\begin{aligned} \dot{u} \;=\; &\frac{1}{r}\Bigg\{\int\left[r\frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}(S_r' + \frac{\nu}{1-\nu}S_\phi')\right]dr + \\ &+\frac{1-2\nu}{1-\nu}\int r'\int\left[\frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}\frac{S_r'-S_\phi'}{r}\right]dr\,dr' + \mathcal{A}r^2 + \mathcal{B}\Bigg\}, \end{aligned} \tag{2.95}$$

where $\mathcal{A}, \mathcal{B}$ are unknown constants. In the general case, the quantities $\sigma_{\text{eff}}^{(n-1)}$, S_r' and S_ϕ' are a function of r. Therefore, Eq.(2.95) cannot be integrated.
If the thickness of the creep layer is very thin, however, $\sigma_{\text{eff}}^{(n-1)}$, S_r' and S_ϕ' can be assumed as being constant in the integration range. This is the case for a

coated structure with an interlayer exhibiting a creep behavior. For this case, the solution can be simplified as

$$\dot{u} = \mathcal{A}r + \frac{\mathcal{B}}{r} + \frac{1}{2}\times\frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}\left\{r(S_r' + \frac{\nu}{1-\nu}S_\phi') + \frac{1-2\nu}{1-\nu}(S_r' - S_\phi')\left[r\ln(r) - \frac{r}{2}\right]\right\}. \tag{2.96}$$

To determine the constants $\mathcal{A}, \mathcal{B}$, boundary conditions have to used. For simplifying the equations obtained from boundary conditions, the solution of $\dot{u}$ can be rewritten as

$$\begin{aligned}\dot{u} &= \frac{1}{r}\left\{\int_{R_i}^{r}\left[r'\frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}(S_r' + \frac{\nu}{1-\nu}S_\phi')\right]dr' + \right.\\ &\left. + \frac{1-2\nu}{1-\nu}\int_{R_i}^{r} r'\int_{R_i}^{r'}\left[\frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}\frac{S_r' - S_\phi'}{r''}\right]dr''\,dr' + \mathcal{A} + \mathcal{B}r^2\right\},\end{aligned} \tag{2.97}$$

where R_i is the inner radius of the creep layer. Integrating Eq.(2.97) results as

$$\begin{aligned}\dot{u} &= \frac{\mathcal{A}}{r} + \mathcal{B}r + \frac{1}{2}\times\frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}\left\{(S_r' + \frac{\nu}{1-\nu}S_\phi')(r - \frac{R_i^2}{r}) + \right.\\ &\left. + \frac{1-2\nu}{1-\nu}(S_r' - S_\phi')\left[r\ln(r) - \frac{r}{2} - \frac{R_i^2}{r}\ln(R_i) + \frac{R_i^2}{2r}\right]\right\}.\end{aligned} \tag{2.98}$$

It is noted that the values of $\mathcal{A}$ and $\mathcal{B}$ in Eq.(2.96) and Eq.(2.98) are different. The rates of strains and stresses according to $\dot{u}$ as given in Eq.(2.98) are

$$\begin{aligned}\dot{\epsilon}_r &= -\frac{\mathcal{A}}{r^2} + \mathcal{B} + \frac{1}{2}\times\frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}\left\{(S_r' + \frac{\nu}{1-\nu}S_\phi')(1 + \frac{R_i^2}{r^2}) + \right.\\ &\left. + \frac{1-2\nu}{1-\nu}(S_r' - S_\phi')\left[\ln(r) + \frac{1}{2} + \frac{R_i^2}{r^2}\ln(R_i) - \frac{R_i^2}{2r^2}\right]\right\}.\end{aligned} \tag{2.99}$$

$$\begin{aligned}\dot{\epsilon}_\phi &= \frac{\mathcal{A}}{r^2} + \mathcal{B} + \frac{1}{2}\times\frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}\left\{(S_r' + \frac{\nu}{1-\nu}S_\phi')(1 - \frac{R_i^2}{r^2}) + \right.\\ &\left. + \frac{1-2\nu}{1-\nu}(S_r' - S_\phi')\left[\ln(r) - \frac{1}{2} - \frac{R_i^2}{r^2}\ln(R_i) + \frac{R_i^2}{2r^2}\right]\right\}.\end{aligned} \tag{2.100}$$

$$\begin{aligned}\dot{\sigma}_r &= \frac{1}{2}\times\frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}\frac{E}{1-\nu^2}\left\{S_r'\left[\ln(r) + \frac{1}{2}\frac{R_i^2}{r^2} + (1-2\nu)\frac{R_i^2}{r^2}\ln(R_i) - \frac{1}{2}\right] + \right.\\ &\left. + S_\phi'\left[-\ln(r) + \frac{1}{2}\frac{R_i^2}{r^2} - (1-2\nu)\frac{R_i^2}{r^2}\ln(R_i) - \frac{1}{2}\right]\right\} + \\ &+ \frac{E}{(1-2\nu)(1+\nu)}\left[\mathcal{B} - \frac{\mathcal{A}}{r^2}(1-2\nu)\right]\end{aligned} \tag{2.101}$$

$$
\begin{aligned}
\dot{\sigma}_\phi &= \frac{1}{2} \times \frac{3}{2} D\sigma_{\text{eff}}^{(n-1)} \frac{E}{1-\nu^2} \left\{ S'_r \left[\ln(r) - \frac{1}{2}\frac{R_i^2}{r^2} - (1-2\nu)\frac{R_i^2}{r^2}\ln(R_i) + \frac{1}{2} \right] + \right. \\
&\left. + S'_\phi \left[-\ln(r) - \frac{1}{2}\frac{R_i^2}{r^2} + (1-2\nu)\frac{R_i^2}{r^2}\ln(R_i) - \frac{3}{2} \right] \right\} + \\
&+ \frac{E}{(1-2\nu)(1+\nu)} \left[\mathcal{B} + \frac{\mathcal{A}}{r^2}(1-2\nu) \right] \qquad (2.102)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\sigma}_z &= \nu \left\{ \frac{3}{2} D\sigma_{\text{eff}}^{(n-1)} \frac{E}{1-\nu^2} \left[(S'_r - S'_\phi)\ln(r) - S'_\phi \right] + \frac{2E}{(1-2\nu)(1+\nu)} \mathcal{B} \right\} - \\
&- \frac{3}{2} D\sigma_{\text{eff}}^{(n-1)} E S_z \qquad (2.103)
\end{aligned}
$$

Now a three-layers cylinder will be studied. It is assumed that the interlayer only exhibits creep behavior and is very thin. Therefore, Eqs.(2.98-2.103) can be used for the stress analysis. Although materials 1 and 3 do not show any creep behavior, the rates of displacement ($\dot{u}$), strains ($\dot{\epsilon}_r, \dot{\epsilon}_\phi$) and stresses ($\dot{\sigma}_r, \dot{\sigma}_\phi, \dot{\sigma}_z$) are not zero during material 2 creeping. The conditions for determining the unknown constants $\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_2, \mathcal{B}_2, \mathcal{A}_3, \mathcal{B}_3$ are:

at r=R_0

$$\dot{\sigma}_r^{(1)} = 0, \qquad (2.104)$$

at r=R_1

$$
\begin{aligned}
\dot{\sigma}_r^{(1)} &= \dot{\sigma}_r^{(2)} \\
\dot{u}^{(1)} &= \dot{u}^{(2)}, \qquad (2.105)
\end{aligned}
$$

at r=R_2

$$
\begin{aligned}
\dot{\sigma}_r^{(2)} &= \dot{\sigma}_r^{(3)} \\
\dot{u}^{(2)} &= \dot{u}^{(3)}, \qquad (2.106)
\end{aligned}
$$

and at r=R_3

$$\dot{\sigma}_r^{(3)} = 0. \qquad (2.107)$$

They yield the following equations:

$$\mathcal{B}_1 - \frac{\mathcal{A}_1}{R_0^2}(1-2\nu_1) = 0 \qquad (2.108)$$

$$\frac{E_1}{(1-2\nu_1)(1+\nu_1)}\left[\mathcal{B}_1-\frac{\mathcal{A}_1}{R_1^2}(1-2\nu_1)\right]-\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\left[\mathcal{B}_2-\frac{\mathcal{A}_2}{R_1^2}(1-2\nu_2)\right]$$
$$=\frac{3}{2}\,D_2\,\sigma_{2,\text{eff}}^{(n_2-1)}\,\frac{E_2}{1-\nu_2^2}(S_r'^{(2)}-S_\phi'^{(2)})(1-\nu_2)\ln(R_1) \qquad (2.109)$$

$$\mathcal{B}_1R_1+\frac{\mathcal{A}_1}{R_1}-\mathcal{B}_2R_1-\frac{\mathcal{A}_2}{R_1}=0 \qquad (2.110)$$

$$\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\left[\mathcal{B}_2-\frac{\mathcal{A}_2}{R_2^2}(1-2\nu_2)\right]-\frac{E_3}{(1-2\nu_3)(1+\nu_3)}\left[\mathcal{B}_3-\frac{\mathcal{A}_3}{R_2^2}(1-2\nu_3)\right]$$
$$=-\frac{1}{2}\times\frac{3}{2}\,D_2\,\sigma_{2,\text{eff}}^{(n_2-1)}\,\frac{E_2}{1-\nu_2^2}\left\{S_r'^{(2)}\left[\ln(R_2)+\frac{1}{2}\frac{R_1^2}{R_2^2}+(1-2\nu_2)\frac{R_1^2}{R_2^2}\ln(R_1)-\frac{1}{2}\right]\right.$$
$$\left.+S_\phi'^{(2)}\left[-\ln(R_2)+\frac{1}{2}\frac{R_1^2}{R_2^2}-(1-2\nu_2)\frac{R_1^2}{R_2^2}\ln(R_1)-\frac{1}{2}\right]\right\} \qquad (2.111)$$

$$\mathcal{B}_2R_2+\frac{\mathcal{A}_2}{R_2}-\mathcal{B}_3R_2-\frac{\mathcal{A}_3}{R_2}$$
$$=-\frac{1}{2}\times\frac{3}{2}\,D_2\,\sigma_{2,\text{eff}}^{(n_2-1)}\left\{\left(S_r'^{(2)}+\frac{\nu_2}{1-\nu_2}S_\phi'^{(2)}\right)\left(R_2-\frac{R_1^2}{R_2}\right)\right.$$
$$\left.+\frac{1-2\nu_2}{1-\nu_2}(S_r'^{(2)}-S_\phi'^{(2)})\left[R_2\ln(R_2)-\frac{R_2}{2}-\frac{R_1^2}{R_2}\ln(R_1)+\frac{R_1^2}{2R_2}\right]\right\} \qquad (2.112)$$

$$\mathcal{B}_3-\frac{\mathcal{A}_3}{R_3^2}(1-2\nu_3)=0. \qquad (2.113)$$

The solution of Eqs.(2.108-2.113) is

$$\Delta_t = \frac{E_2}{(1-2\nu_2)(1+\nu_2)}\frac{E_3}{(1-2\nu_3)(1+\nu_3)}(r_{31}-r_{21})\,(1+r_{01}(1-2\nu_1))\times$$
$$\times(r_{21}+(1-2\nu_2))\,(1-2\nu_3)+\left(\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\right)^2(r_{21}-1)\,(1+r_{01}(1-2\nu_1))$$
$$\times(1-2\nu_2)\,(r_{21}+r_{31}(1-2\nu_3))+\frac{E_1}{(1-2\nu_1)(1+\nu_1)}(1-r_{01})\,(1-2\nu_1)\times$$
$$\times\left[\frac{E_3}{(1-2\nu_3)(1+\nu_3)}(1-r_{21})\,(r_{21}-r_{31})\,(1-2\nu_3)+\right.$$
$$\left.+\frac{E_2}{(1-2\nu_2)(1+\nu_2)}(1+r_{21}(1-2\nu_2))\,(r_{21}+r_{31}(1-2\nu_3))\right] \qquad (2.114)$$

$$
\begin{aligned}
\mathcal{A}_1' &= \mathcal{R}_{t5}\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\frac{E_3}{(1-2\nu_3)(1+\nu_3)}\left(r_{31}-r_{21}\right)2(1-\nu_2)(1-2\nu_3) \\
&- \mathcal{R}_{t4}\frac{E_2}{(1-2\nu_2)(1+\nu_2)}2(1-\nu_2)\left(r_{21}+r_{31}(1-2\nu_3)\right)+ \\
&+ \mathcal{R}_{t2}\left[\frac{E_3}{(1-2\nu_3)(1+\nu_3)}\left(r_{21}-1\right)\left(r_{21}-r_{31}\right)(1-2\nu_3)-\right. \\
&\left.-\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\left(1+r_{21}(1-2\nu_2)\right)\left(r_{21}+r_{31}(1-2\nu_3)\right)\right] \qquad (2.115)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_1' &= \mathcal{R}_{t5}\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\frac{E_3}{(1-2\nu_3)(1+\nu_3)}r_{01}\left(r_{31}-r_{21}\right)(1-2\nu_1)\times \\
&\times 2(1-\nu_2)(1-2\nu_3)- \\
&- \mathcal{R}_{t4}\frac{E_2}{(1-2\nu_2)(1+\nu_2)}r_{01}(1-2\nu_1)2(1-\nu_2)\left(r_{21}+r_{31}(1-2\nu_3)\right)+ \\
&+ \mathcal{R}_{t2}r_{01}(1-2\nu_1)\left[\frac{E_3}{(1-2\nu_3)(1+\nu_3)}\left(r_{21}-1\right)\left(r_{21}-r_{31}\right)(1-2\nu_3)-\right. \\
&\left.-\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\left(1+r_{21}(1-2\nu_2)\right)\left(r_{21}+r_{31}(1-2\nu_3)\right)\right] \qquad (2.116)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_2' &= \mathcal{R}_{t5}\frac{E_3}{(1-2\nu_3)(1+\nu_3)}\left(r_{31}-r_{21}\right)\left[\frac{E_1}{(1-2\nu_1)(1+\nu_1)}\left(1-r_{01}\right)(1-2\nu_1)+\right. \\
&\left.+\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\left(1+r_{01}(1-2\nu_1)\right)\right](1-2\nu_3)+ \\
&+ \mathcal{R}_{t4}\left(\frac{E_1}{(1-2\nu_1)(1+\nu_1)}\left(r_{01}-1\right)(1-2\nu_1)-\right. \\
&\left.-\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\left(1+r_{01}(1-2\nu_1)\right)\right)\left(r_{21}+r_{31}(1-2\nu_3)\right)+ \\
&+ \mathcal{R}_{t2}\left(1+r_{01}(1-2\nu_1)\right)\left[\frac{E_3}{(1-2\nu_3)(1+\nu_3)}\left(r_{31}-r_{21}\right)(1-2\nu_3)-\right. \\
&\left.-\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\left(r_{21}+r_{31}(1-2\nu_3)\right)\right] \qquad (2.117)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_2' &= \mathcal{R}_{t5}\frac{E_3}{(1-2\nu_3)(1+\nu_3)}\left(r_{31}-r_{21}\right)\left[\frac{E_1}{(1-2\nu_1)(1+\nu_1)}\left(r_{01}-1\right)(1-2\nu_1)+\right. \\
&\left.+\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\left(1+r_{01}(1-2\nu_1)\right)(1-2\nu_2)\right](1-2\nu_3)+ \\
&+ \mathcal{R}_{t4}\left[\frac{E_1}{(1-2\nu_1)(1+\nu_1)}\left(1-r_{01}\right)(1-2\nu_1)-\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\left(1+r_{01}(1-2\nu_1)\right)(1-2\nu_2)\Big]\left(r_{21}+r_{31}(1-2\nu_3)\right)+\\
&+\ \mathcal{R}_{t2}r_{21}\left(1+r_{01}(1-2\nu_1)\right)\left[\frac{E_3}{(1-2\nu_3)(1+\nu_3)}\left(r_{21}-r_{31}\right)(1-2\nu_3)-\right.\\
&\left.-\frac{E_2}{(1-2\nu_2)(1+\nu_2)}(1-2\nu_2)\left(r_{21}+r_{31}(1-2\nu_3)\right)\right] \qquad (2.118)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_3' =& -\mathcal{R}_{t2}\frac{E_2}{(1-2\nu_2)(1+\nu_2)}r_{21}\left(1+r_{01}(1-2\nu_1)\right)2(1-\nu_2)+\\
&+\ \mathcal{R}_{t4}\left[\frac{E_1}{(1-2\nu_1)(1+\nu_1)}\left(r_{01}-1\right)\left(r_{21}-1\right)(1-2\nu_1)-\right.\\
&\left.-\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\left(1+r_{01}(1-2\nu_1)\right)\left(r_{21}+(1-2\nu_2)\right)\right]+\\
&+\ \mathcal{R}_{t5}\left[\left(\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\right)^2\left(1-r_{21}\right)\left(1+r_{01}(1-2\nu_1)\right)(1-2\nu_2)+\right.\\
&\left.+\frac{E_1}{(1-2\nu_1)(1+\nu_1)}\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\left(r_{01}-1\right)(1-2\nu_1)\left(1+r_{21}(1-2\nu_2)\right)\right]
\end{aligned}
$$
(2.119)

$$
\begin{aligned}
\mathcal{B}_3' =& -\mathcal{R}_{t2}\frac{E_2}{(1-2\nu_2)(1+\nu_2)}r_{21}r_{31}\left(1+r_{01}(1-2\nu_1)\right)2(1-\nu_2)(1-2\nu_3)+\\
&+\ \mathcal{R}_{t4}\left[\frac{E_1}{(1-2\nu_1)(1+\nu_1)}\left(r_{01}-1\right)\left(r_{21}-1\right)r_{31}(1-2\nu_1)(1-2\nu_3)-\right.\\
&\left.-\frac{E_2}{(1-2\nu_2)(1+\nu_2)}r_{31}\left(1+r_{01}(1-2\nu_1)\right)\left(r_{21}+(1-2\nu_2)\right)(1-2\nu_3)\right]+\\
&+\ \mathcal{R}_{t5}\left[\left(\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\right)^2\left(1-r_{21}\right)r_{31}\left(1+r_{01}(1-2\nu_1)\right)(1-2\nu_2)(1-2\nu_3)\right.\\
&+\frac{E_1}{(1-2\nu_1)(1+\nu_1)}\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\left(r_{01}-1\right)r_{31}(1-2\nu_1)\times\\
&\left.\times\left(1+r_{21}(1-2\nu_2)\right)(1-2\nu_3)\right] \qquad (2.120)
\end{aligned}
$$

with

$$
\mathcal{A}_1 = R_1^2\frac{\mathcal{A}_1'}{\Delta_t}, \qquad \mathcal{A}_2 = R_1^2\frac{\mathcal{A}_2'}{\Delta_t}, \qquad \mathcal{A}_3 = R_1^2\frac{\mathcal{A}_3'}{\Delta_t}, \qquad (2.121)
$$

$$
\mathcal{B}_1 = \frac{\mathcal{B}_1'}{\Delta_t}, \qquad \mathcal{B}_2 = \frac{\mathcal{B}_2'}{\Delta_t}, \qquad \mathcal{B}_3 = \frac{\mathcal{B}_3'}{\Delta_t}, \qquad (2.122)
$$

where $\mathcal{R}_{t2}, \mathcal{R}_{t4}, \mathcal{R}_{t5}$ make up the right side of Eqs.(2.109,2.111,2.112) and r_{01}, r_{21}, r_{31} see Eq.(2.33).

When the coefficients $\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_2, \mathcal{B}_2, \mathcal{A}_3, \mathcal{B}_3$ are known, the rate of stresses can be determined from Eqs.(2.101-2.103). To calculate the stresses at any time t_i, an iteration should be performed, i.e.

$$\sigma_{ij}^{(i)}(r, t_i) = \sigma_{ij}^{(i-1)}(r, t_{i-1}) + \dot{\sigma}_{ij}^{(i)}(r, t_i)dt^{(i)} \tag{2.123}$$

where

$$t_i = \sum_{k=0}^{i} dt^{(k)}. \tag{2.124}$$

The solution of $t_i = 0$ corresponds to the elastic behavior. To calculate $\dot{\sigma}_{ij}^{(i)}(r, t_i)$, the stresses at the time t_{i-1} are used. In chapter 3, some examples will be presented to illustrate the agreement between the stresses calculated from the finite element method and those obtained by the analytical equations given in this chapter.

2.2.2 The case of $\dot{\epsilon}_z$ being constant

It is assumed that the rate $\dot{\epsilon}_z$ is a constant - denoted as d. From Eq.(2.81), there is

$$\dot{\sigma}_z = Ed - \frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}ES_z + \nu(\dot{\sigma}_r + \dot{\sigma}_\phi) \tag{2.125}$$

and

$$\dot{\epsilon}_r = \frac{1}{E'}\left(\dot{\sigma}_r - \frac{\nu}{1-\nu}\dot{\sigma}_\phi\right) + \frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}(S_r + \nu S_z) - \nu\, d \tag{2.126}$$

$$\dot{\epsilon}_\phi = \frac{1}{E'}\left(\dot{\sigma}_\phi - \frac{\nu}{1-\nu}\dot{\sigma}_r\right) + \frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}(S_\phi + \nu S_z) - \nu\, d \tag{2.127}$$

$$\dot{\sigma}_r = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)}\left\{\dot{\epsilon}_r + \frac{\nu}{1-\nu}\dot{\epsilon}_\phi - \frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}\left[S_r' + \frac{\nu}{1-\nu}S_\phi'\right] + \frac{\nu}{1-\nu}\,d\right\} \tag{2.128}$$

$$\dot{\sigma}_\phi = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)}\left\{\dot{\epsilon}_\phi + \frac{\nu}{1-\nu}\dot{\epsilon}_r - \frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}\left[S'_\phi + \frac{\nu}{1-\nu}S'_r\right] + \frac{\nu}{1-\nu}\,d\right\}. \tag{2.129}$$

Substituting Eqs.(2.85-2.86) into Eqs.(2.128-2.129) and then in Eq.(2.87) yields a differential equation for $\dot{u}$, which is the same as that given in Eq.(2.94). Therefore, the solution for $\dot{u}$ is the same as that in Eq.(2.98), and the strains $\dot{\epsilon}_r$ and $\dot{\epsilon}_\phi$ are the same as those given in Eqs.(2.99) and (2.100) (it is noted that the values of the coefficients $\mathcal{A}, \mathcal{B}$ are different). The stresses, however, differ from those of Eqs.(2.101) and (2.102) as follows

$$\begin{aligned}\dot{\sigma}_r &= \frac{1}{2}\times\frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}\frac{E}{1-\nu^2}\left\{S'_r\left[\ln(r) + \frac{1}{2}\frac{R_i^2}{r^2} + (1-2\nu)\frac{R_i^2}{r^2}\ln(R_i) - \frac{1}{2}\right] + \right.\\ &\left. + S'_\phi\left[-\ln(r) + \frac{1}{2}\frac{R_i^2}{r^2} - (1-2\nu)\frac{R_i^2}{r^2}\ln(R_i) - \frac{1}{2}\right]\right\} + \\ &+ \frac{E}{(1-2\nu)(1+\nu)}\left[\mathcal{B}_d - \frac{\mathcal{A}_d}{r^2}(1-2\nu)\right] + \frac{E\nu\, d}{(1-2\nu)(1+\nu)}\end{aligned} \tag{2.130}$$

$$\begin{aligned}\dot{\sigma}_\phi &= \frac{1}{2}\times\frac{3}{2}D\sigma_{\text{eff}}^{(n-1)}\frac{E}{1-\nu^2}\left\{S'_r\left[\ln(r) - \frac{1}{2}\frac{R_i^2}{r^2} - (1-2\nu)\frac{R_i^2}{r^2}\ln(R_i) + \frac{1}{2}\right] + \right.\\ &\left. + S'_\phi\left[-\ln(r) - \frac{1}{2}\frac{R_i^2}{r^2} + (1-2\nu)\frac{R_i^2}{r^2}\ln(R_i) - \frac{3}{2}\right]\right\} + \\ &+ \frac{E}{(1-2\nu)(1+\nu)}\left[\mathcal{B}_d + \frac{\mathcal{A}_d}{r^2}(1-2\nu)\right] + \frac{E\nu\, d}{(1-2\nu)(1+\nu)}.\end{aligned} \tag{2.131}$$

Use of the boundary conditions results in

$$\mathcal{B}_{1d} - \frac{\mathcal{A}_{1d}}{R_0^2}(1-2\nu_1) = -\nu_1\, d \tag{2.132}$$

$$\begin{aligned}&\frac{E_1}{(1-2\nu_1)(1+\nu_1)}\left[\mathcal{B}_{1d} - \frac{\mathcal{A}_{1d}}{R_1^2}(1-2\nu_1)\right] - \frac{E_2}{(1-2\nu_2)(1+\nu_2)}\left[\mathcal{B}_{2d} - \frac{\mathcal{A}_{2d}}{R_1^2}(1-2\nu_2)\right]\\ &= \frac{3}{2}\,D_2\,\sigma_{2,\text{eff}}^{(n_2-1)}\,\frac{E_2}{1-\nu_2^2}(S_r'^{(2)} - S_\phi'^{(2)})(1-\nu_2)\ln(R_1)\\ &+ \frac{E_2\nu_2\, d}{(1-2\nu_2)(1+\nu_2)} - \frac{E_1\nu_1\, d}{(1-2\nu_1)(1+\nu_1)}\end{aligned} \tag{2.133}$$

$$\mathcal{B}_{1d}R_1 + \frac{\mathcal{A}_{1d}}{R_1} - \mathcal{B}_{2d}R_1 - \frac{\mathcal{A}_{2d}}{R_1} = 0 \tag{2.134}$$

$$\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\left[\mathcal{B}_{2d}-\frac{\mathcal{A}_{2d}}{R_2^2}(1-2\nu_2)\right]-\frac{E_3}{(1-2\nu_3)(1+\nu_3)}\left[\mathcal{B}_{3d}-\frac{\mathcal{A}_{3d}}{R_2^2}(1-2\nu_3)\right]$$
$$=-\frac{1}{2}\times\frac{3}{2}\,D_2\,\sigma_{2,\mathrm{eff}}^{(n_2-1)}\,\frac{E_2}{1-\nu_2^2}\left\{S_r'^{(2)}\left[\ln(R_2)+\frac{1}{2}\frac{R_1^2}{R_2^2}+(1-2\nu_2)\frac{R_1^2}{R_2^2}\ln(R_1)-\frac{1}{2}\right]\right.$$
$$\left.+S_\phi'^{(2)}\left[-\ln(R_2)+\frac{1}{2}\frac{R_1^2}{R_2^2}-(1-2\nu_2)\frac{R_1^2}{R_2^2}\ln(R_1)-\frac{1}{2}\right]\right\}$$
$$+\frac{E_3\nu_3\,d}{(1-2\nu_3)(1+\nu_3)}-\frac{E_2\nu_2\,d}{(1-2\nu_2)(1+\nu_2)} \tag{2.135}$$

$$\mathcal{B}_{2d}R_2+\frac{\mathcal{A}_{2d}}{R_2}-\mathcal{B}_{3d}R_2-\frac{\mathcal{A}_{3d}}{R_2}$$
$$=-\frac{1}{2}\times\frac{3}{2}\,D_2\,\sigma_{2,\mathrm{eff}}^{(n_2-1)}\left\{\left(S_r'^{(2)}+\frac{\nu_2}{1-\nu_2}S_\phi'^{(2)}\right)\left(R_2-\frac{R_1^2}{R_2}\right)\right.$$
$$\left.+\frac{1-2\nu_2}{1-\nu_2}(S_r'^{(2)}-S_\phi'^{(2)})\left[R_2\ln(R_2)-\frac{R_2}{2}-\frac{R_1^2}{R_2}\ln(R_1)+\frac{R_1^2}{2R_2}\right]\right\} \tag{2.136}$$

$$\mathcal{B}_{3d}-\frac{\mathcal{A}_{3d}}{R_3^2}(1-2\nu_3)=-\nu_3\,d, \tag{2.137}$$

from which the coefficients $\mathcal{A}_d, \mathcal{B}_d$ can be determined. Simplifying the solutions, the following relations are obtained:

$$\mathcal{A}'_{1d} = Eq.(2.115)+d\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\left\{\frac{E_2}{(1-2\nu_2)(1+\nu_2)}(r_{21}-1)(1-2\nu_2)\times\right.$$
$$\times(r_{21}+r_{31}(1-2\nu_3))(\nu_1-\nu_2)+\frac{E_3}{(1-2\nu_3)(1+\nu_3)}(r_{31}-r_{21})(1-2\nu_3)\times$$
$$\left.\times\Big[(r_{21}+(1-2\nu_2))\,\nu_1+(1-r_{21})\,\nu_2-2\,(1-\nu_2)\,\nu_3\Big]\right\} \tag{2.138}$$

$$\mathcal{B}'_{1d} = Eq.(2.116)+d\left\{\frac{E_1}{(1-2\nu_1)(1+\nu_1)}\left[\frac{E_3}{(1-2\nu_3)(1+\nu_3)}(r_{01}-1)(r_{21}-1)\times\right.\right.$$
$$\times(r_{31}-r_{21})(1-2\nu_1)(1-2\nu_3)\nu_1+$$
$$\left.+\frac{E_2}{(1-2\nu_2)(1+\nu_2)}(r_{01}-1)(1-2\nu_1)(1+r_{21}(1-2\nu_2))(r_{21}+r_{31}(1-2\nu_3))\,\nu_1\right]$$
$$+\left(\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\right)^2(1-r_{21})(1-2\nu_2)(r_{21}+r_{31}(1-2\nu_3))(\nu_1+r_{01}(1-2\nu_1)\nu_2)$$
$$+\frac{E_2}{(1-2\nu_2)(1+\nu_2)}\frac{E_3}{(1-2\nu_3)(1+\nu_3)}(r_{21}-r_{31})(1-2\nu_3)\Big[(r_{21}+(1-2\nu_2))\,\nu_1+$$
$$\left.+r_{01}\,(r_{21}-1)(1-2\nu_1)\nu_2+r_{01}(1-2\nu_1)2(1-\nu_2)\nu_3\Big]\right\} \tag{2.139}$$

$$
\begin{aligned}
\mathcal{A}'_{2d} \;=\;& Eq.(2.117) + d\Bigg\{ \frac{E_2}{(1-2\nu_2)(1+\nu_2)} \frac{E_3}{(1-2\nu_3)(1+\nu_3)} (r_{31} - r_{21}) \times \\
& \times (1 + r_{01}(1-2\nu_1)) (1-2\nu_3) (\nu_2 - \nu_3) + \frac{E_1}{(1-2\nu_1)(1+\nu_1)} \Big[\frac{E_2}{(1-2\nu_2)(1+\nu_2)} \times \\
& \times (r_{01} - 1) (1-2\nu_1) (r_{21} + r_{31}(1-2\nu_3)) (\nu_1 - \nu_2) + \\
+\;& \frac{E_3}{(1-2\nu_3)(1+\nu_3)} (r_{01} - 1) (r_{31} - r_{21}) (1-2\nu_1)(1-2\nu_3) (\nu_3 - \nu_1) \Big] \Bigg\}
\end{aligned} \tag{2.140}
$$

$$
\begin{aligned}
\mathcal{B}'_{2d} \;=\;& Eq.(2.118) + d\Bigg\{ \left(\frac{E_2}{(1-2\nu_2)(1+\nu_2)} \right)^2 (1 - r_{21}) (1 + r_{01}(1-2\nu_1)) \times \\
& \times (1-2\nu_2) (r_{21} + r_{31}(1-2\nu_3)) \nu_2 + \frac{E_1}{(1-2\nu_1)(1+\nu_1)} \Big[\frac{E_2}{(1-2\nu_2)(1+\nu_2)} \times \\
& \times (r_{01} - 1) (1-2\nu_1) (r_{21} + r_{31}(1-2\nu_3)) (r_{21}(1-2\nu_2)\nu_1 + \nu_2) + \\
+\;& \frac{E_3}{(1-2\nu_3)(1+\nu_3)} (r_{01} - 1) (r_{31} - r_{21}) (1-2\nu_1)(1-2\nu_3) (r_{21}\nu_1 - \nu_3) \Big] + \\
+\;& \frac{E_2}{(1-2\nu_2)(1+\nu_2)} \frac{E_3}{(1-2\nu_3)(1+\nu_3)} (r_{21} - r_{31}) (1 + r_{01}(1-2\nu_1)) (1-2\nu_3) \\
& \times (r_{21}\nu_2 + (1-2\nu_2)\nu_3) \Bigg\}
\end{aligned} \tag{2.141}
$$

$$
\begin{aligned}
\mathcal{A}'_{3d} \;=\;& Eq.(2.119) + d\frac{E_2}{(1-2\nu_2)(1+\nu_2)} \Bigg\{ \frac{E_2}{(1-2\nu_2)(1+\nu_2)} (r_{21} - 1) \times \\
& \times (1 + r_{01}(1-2\nu_1)) (1-2\nu_2) (\nu_3 - \nu_2) + \frac{E_1}{(1-2\nu_1)(1+\nu_1)} (r_{01} - 1) \times \\
& \times (1-2\nu_1) \Big[2r_{21}\nu_1 (1-\nu_2) + (1 - r_{21}) \nu_2 - (1 + r_{21}(1-2\nu_2)) \nu_3 \Big] \Bigg\}
\end{aligned} \tag{2.142}
$$

$$
\begin{aligned}
\mathcal{B}'_{3d} \;=\;& Eq.(2.120) + d\Bigg\{ \frac{E_2}{(1-2\nu_2)(1+\nu_2)} \frac{E_3}{(1-2\nu_3)(1+\nu_3)} (r_{21} - r_{31}) \times \\
\times\;& (1 + r_{01}(1-2\nu_1)) (r_{21} + (1-2\nu_2)) (1-2\nu_3)\nu_3 + \left(\frac{E_2}{(1-2\nu_2)(1+\nu_2)} \right)^2 \times \\
\times\;& (1 - r_{21}) (1 + r_{01}(1-2\nu_1)) (1-2\nu_2) (r_{31}(1-2\nu_3)\nu_2 + r_{21}\nu_3) + \\
+\;& \frac{E_1}{(1-2\nu_1)(1+\nu_1)} \Big[\frac{E_3}{(1-2\nu_3)(1+\nu_3)} (r_{01} - 1) (r_{21} - 1) (r_{31} - r_{21}) \times \\
& \times (1-2\nu_1)(1-2\nu_3)\nu_3 + \frac{E_2}{(1-2\nu_2)(1+\nu_2)} (r_{01} - 1) (1-2\nu_1) \Big(2r_{21}r_{31} \times
\end{aligned}
$$

$$\times(1-2\nu_3)\nu_1\,(1-\nu_2)+(1-r_{21})\,r_{31}(1-2\nu_3)\nu_2+r_{21}\,(1+r_{21}(1-2\nu_2))\,\nu_3\Big)\Big]\Big\} \tag{2.143}$$

and

$$\mathcal{A}_{1d}=R_1^2\frac{\mathcal{A}'_{1d}}{\Delta_t},\qquad \mathcal{A}_{2d}=R_1^2\frac{\mathcal{A}'_{2d}}{\Delta_t},\qquad \mathcal{A}_{3d}=R_1^2\frac{\mathcal{A}'_{3d}}{\Delta_t}. \tag{2.144}$$

$$\mathcal{B}_{1d}=\frac{\mathcal{B}'_{1d}}{\Delta_t},\qquad \mathcal{B}_{2d}=\frac{\mathcal{B}'_{2d}}{\Delta_t},\qquad \mathcal{B}_{3d}=\frac{\mathcal{B}'_{3d}}{\Delta_t}, \tag{2.145}$$

where Δ_t see Eq.(2.114).
When the coefficients $\mathcal{A}_{1d}, \mathcal{B}_{1d}, \mathcal{A}_{2d}, \mathcal{B}_{2d}, \mathcal{A}_{3d}, \mathcal{B}_{3d}$ are known, the rate of stresses can be determined from Eqs.(2.125,2.130-2.131). The stresses can be calculated by means of the procedure given in chapter 2.2.1.

Chapter 3

Examples for Multi - Layers Cylinders

In this chapter, some examples will be presented to show the agreement of the stresses calculated from the analytical equations given in chapter 2 and from the finite element method (FEM) in a cylinder with a finite length. The length (2L) of the cylinder is varied to find out in which case (i.e. which ratio of the length (L) and thickness (H)) they have a good agreement.
The radii of the cylinder are

$$\begin{aligned} R_0 &= 2\,mm \\ R_1 &= 4\,mm \\ R_2 &= 4.1\,mm \\ R_3 &= 4.355\,mm. \end{aligned}$$

The material data of the three layers are

$$\begin{aligned} E_1 = 215\text{ GPa}, \quad \nu_1 &= 0.3, \quad \alpha_1 = 16.28 \times 10^{-6}/K \\ E_2 = 180\text{ GPa}, \quad \nu_2 &= 0.3, \quad \alpha_2 = 16.6 \times 10^{-6}/K \\ E_3 = 125\text{ GPa}, \quad \nu_3 &= 0.225, \quad \alpha_3 = 10.8 \times 10^{-6}/K \end{aligned}$$

and

$$D_2 = 1.4 \times 10^{-8} \quad (\sigma \text{ in MPa and t in hours}), \qquad n_2 = 2.25.$$

For the stress analysis from FEM, the program ABAQUS with 8 nodes element is applied. The used mesh is shown in Fig.2, where only one half of the cylinder in z direction is presented due to the symmetry. In the following, the given FEM results are located along this symmetrical line.

3.1 Results for elastic behavior

For the case of a cylinder under thermal loading, the temperature change is 980K.

Stress distributions along the symmetrical line of the cylinder are plotted in Fig.3a for σ_r, in Fig.3b for σ_ϕ and in Fig.3c for σ_z. The analytical solution (lines) is for the case of $\epsilon_z = 0$ (given in chapter 2.1.1). The FEM results (symbols) are for the case of $\epsilon_z = 0$ and for ϵ_z being arbitrary with a variable ratio of L/H. It can be seen that for all three stress components only the FEM results for the case of $\epsilon_z = 0$ exhibit a very good agreement with those from the analytical solution, irrespective of the ratio of L/H.

For the case of a cylinder having a constant ϵ_z, the stress distributions along the symmetrical line of the cylinder are plotted in Fig.4a for σ_r, in Fig.4b for σ_ϕ and in Fig.4c for σ_z. The analytical solution (lines) is for the case of $\epsilon_z = d$ (given in chapter 2.1.2). The FEM results (symbols) are for the case of the cylinder being loaded with a uniform σ_z (=1 MPa) at the ends and the cylinder with a variable ratio of L/H. It can be seen that for all three stress components FEM results are in very good agreement with those from the analytical solution when the ratio $L/H \geq 6$.

For the case of a cylinder under a temperature change of 980K and with an arbitrary ϵ_z, the stress distributions along the symmetrical line of the cylinder are plotted in Fig.5a for σ_r, in Fig.5b for σ_ϕ and in Fig.5c for σ_z. The analytical solution (lines) is taken from chapter 2.1.3. The FEM results (symbols) are for the case of a cylinder with a free σ_z at the ends and the cylinder with a variable ratio of L/H. It can be seen that for all three stress components FEM results are in very good agreement with those from the analytical solution when the ratio

$L/H \geq 5$.

The comparisons have shown that the analytical solutions for the elastic behavior are accurate for a multi - layers cylinder with $L/H \geq 5$. In fact, for cylinder with $L/H \geq 3$ and stress free at the ends, the analytical solution can be used to describe the stress distribution in the center of a joint well. The equations for the case of $\epsilon_z = 0$ cannot be used to calculate the stresses in the center of a cylinder, irrespective of the ratio of L/H.

3.2 Results for creep behavior

To test the equations given in chapter 2.2, FEM calculations are performed for the case of a joint having an initial temperature of $1000°C$, an end temperature of $20°C$ and subjected to t hours of creeping. It is noted that in this case stress relaxation and not creep occurs. It is the true creep process if the initial temperature is $20°C$, the end temperature is $1000°C$ and then t hours of creeping. However, the equations to calculate the stresses are the same.

For the case of $\epsilon_z = 0$, stress distributions along the symmetrical line of the cylinder are plotted in Fig.6 for σ_r, σ_ϕ and σ_z with t=0.1 hours, in Fig.7 for σ_r, σ_ϕ and σ_z with t=1.4472 hours, and in Fig.8 for σ_r, σ_ϕ and σ_z with t=10 hours. The analytical solution (lines) is given for the case of $\epsilon_z = 0$ and $\dot{\epsilon}_z = 0$ (see chapter 2.1.1 and chapter 2.2.1). The FEM results (symbols) are for the case of $\epsilon_z = 0$ and $\dot{\epsilon}_z = 0$. It can be seen that in elastic materials they always are in very good agreement, irrespective of time. In creep material, the results obtained for the stress components σ_r and σ_z also are always in very good agreement, irrespective of time. However, for the stress components σ_ϕ they have a slight difference.

For the case of a cylinder having a free σ_z at the ends, the stress distributions along the symmetrical line of the cylinder are plotted in Fig.9 for σ_r, σ_ϕ and σ_z with t=0.1 hours, and in Fig.10 for σ_r, σ_ϕ and σ_z with t=1.66 hours. The analytical solution (lines) is given for the case of ϵ_z being arbitrary and $\dot{\epsilon}_z = 0$

(see chapter 2.1.3 and chapter 2.2.1). The FEM results (symbols) are for the cylinder with a free σ_z at the ends, which corresponds to ϵ_z being arbitrary and $\dot{\epsilon}_z = 0$. It can be seen that for all stress components they are always in very good agreement, irrespective of time (here, $L/H = 5.1$).

The results have shown that for a multi - layers cylinder with a thin creep interlayer, the analytical solution given in chapter 2.2 can describe the stresses very well, especially for the case of a cylinder with a free σ_z at the ends, which is the relevant case in practice.

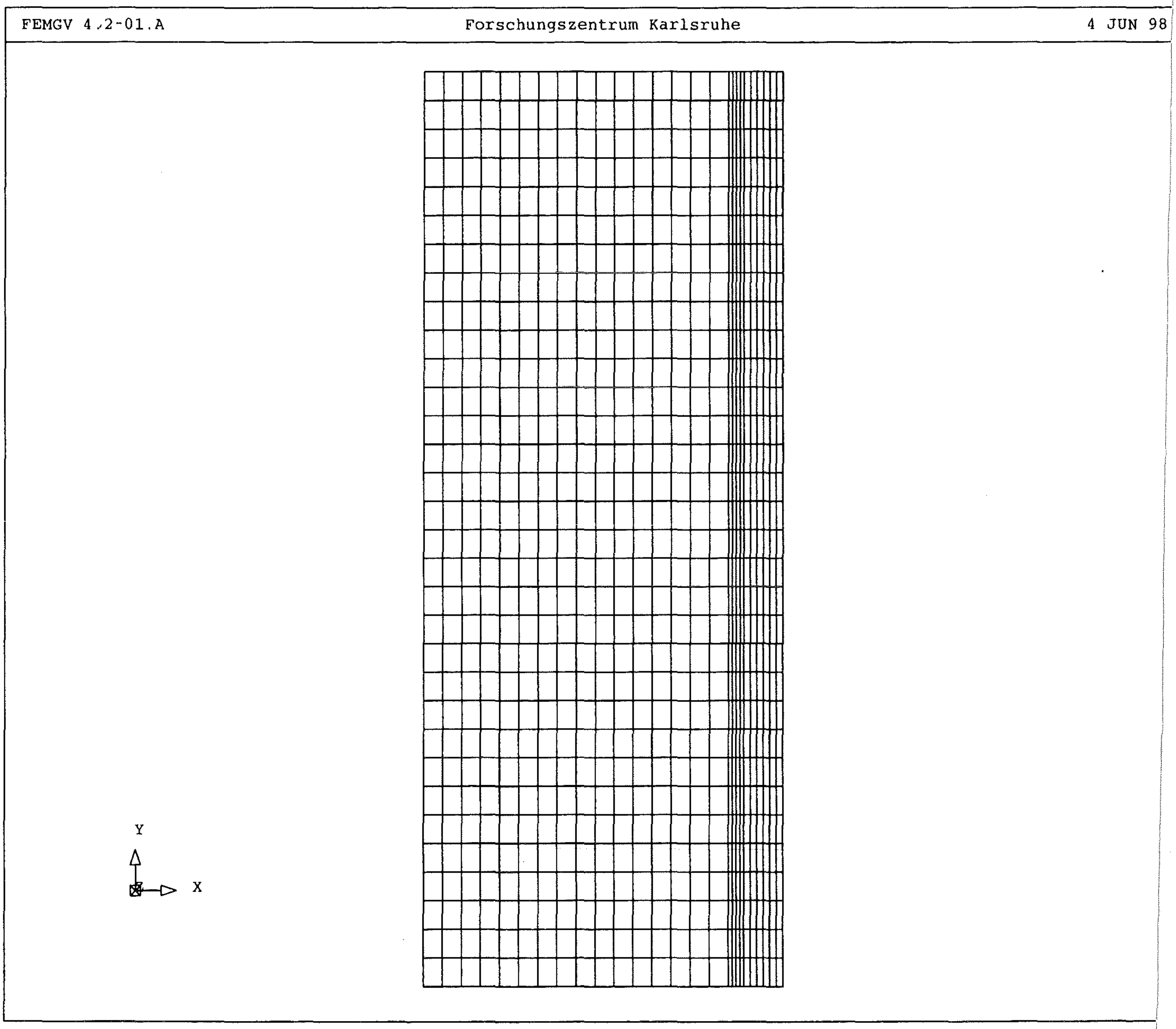


Fig.2 FEM mesh used.

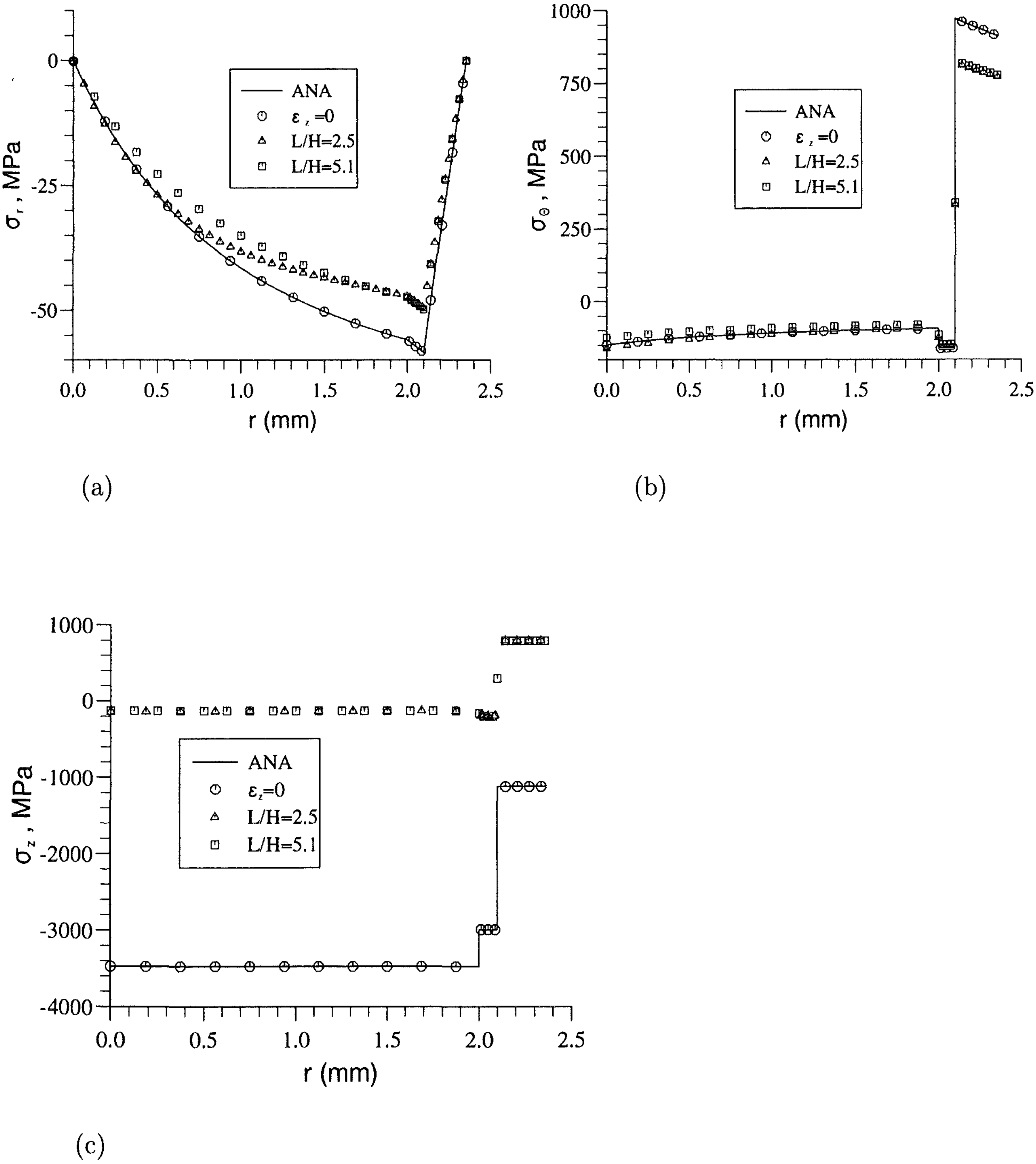
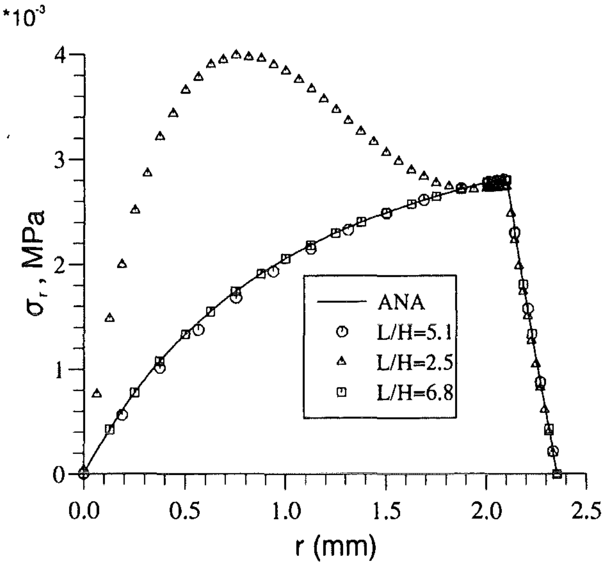
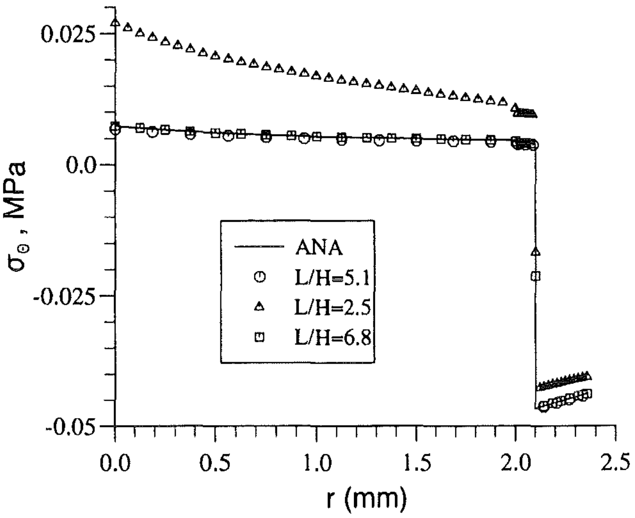


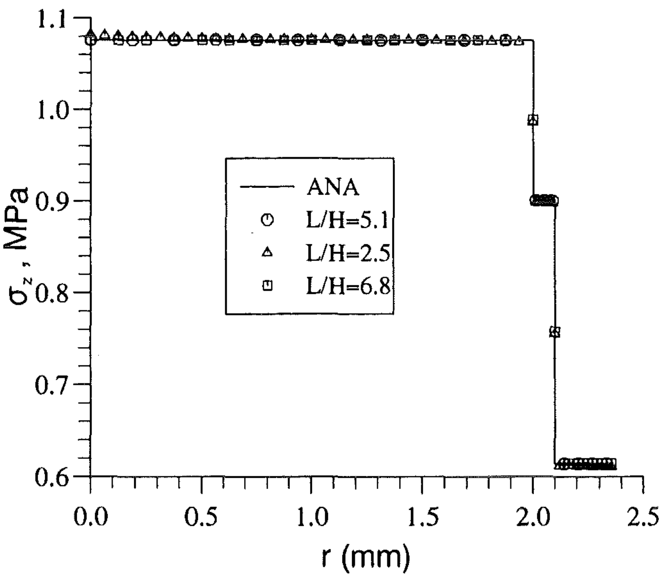
Fig.3 Stresses for cylinders with $\epsilon_z = 0$ and arbitrary ϵ_z.



(a)



(b)



(c)

Fig.4 Stresses for cylinders with $\epsilon_z = d$.

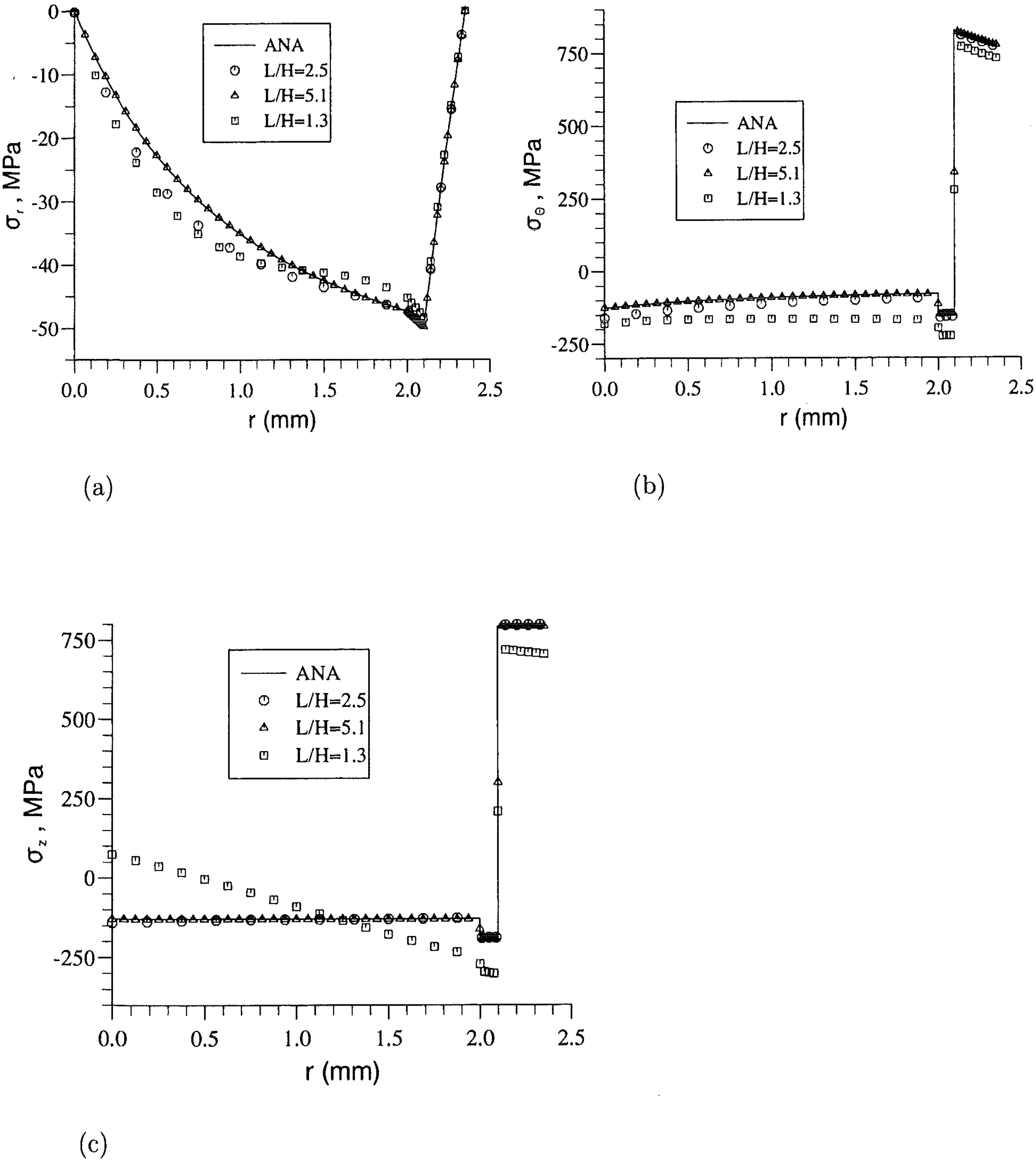


Fig.5 Stresses for cylinders with an arbitrary ϵ_z.

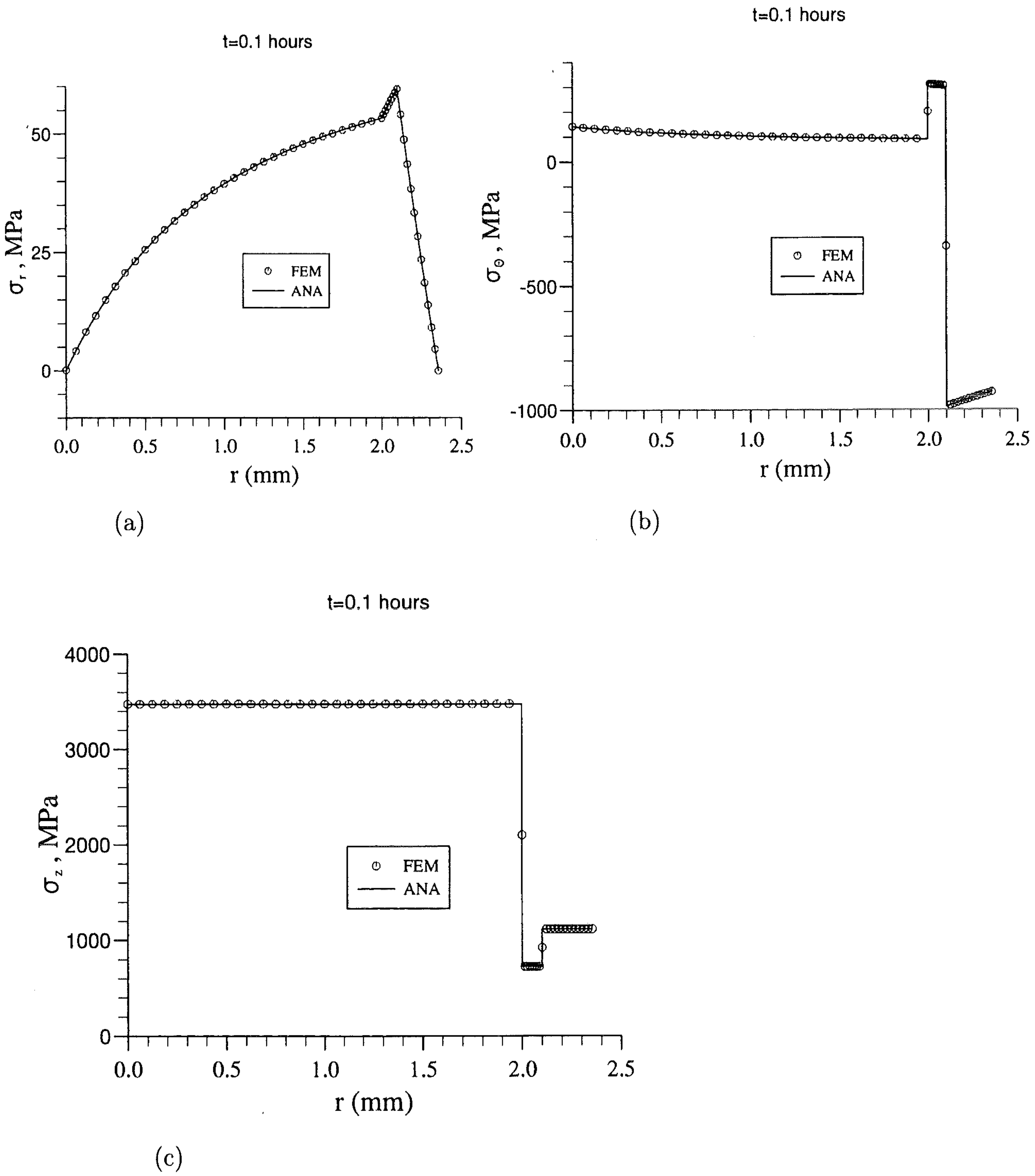
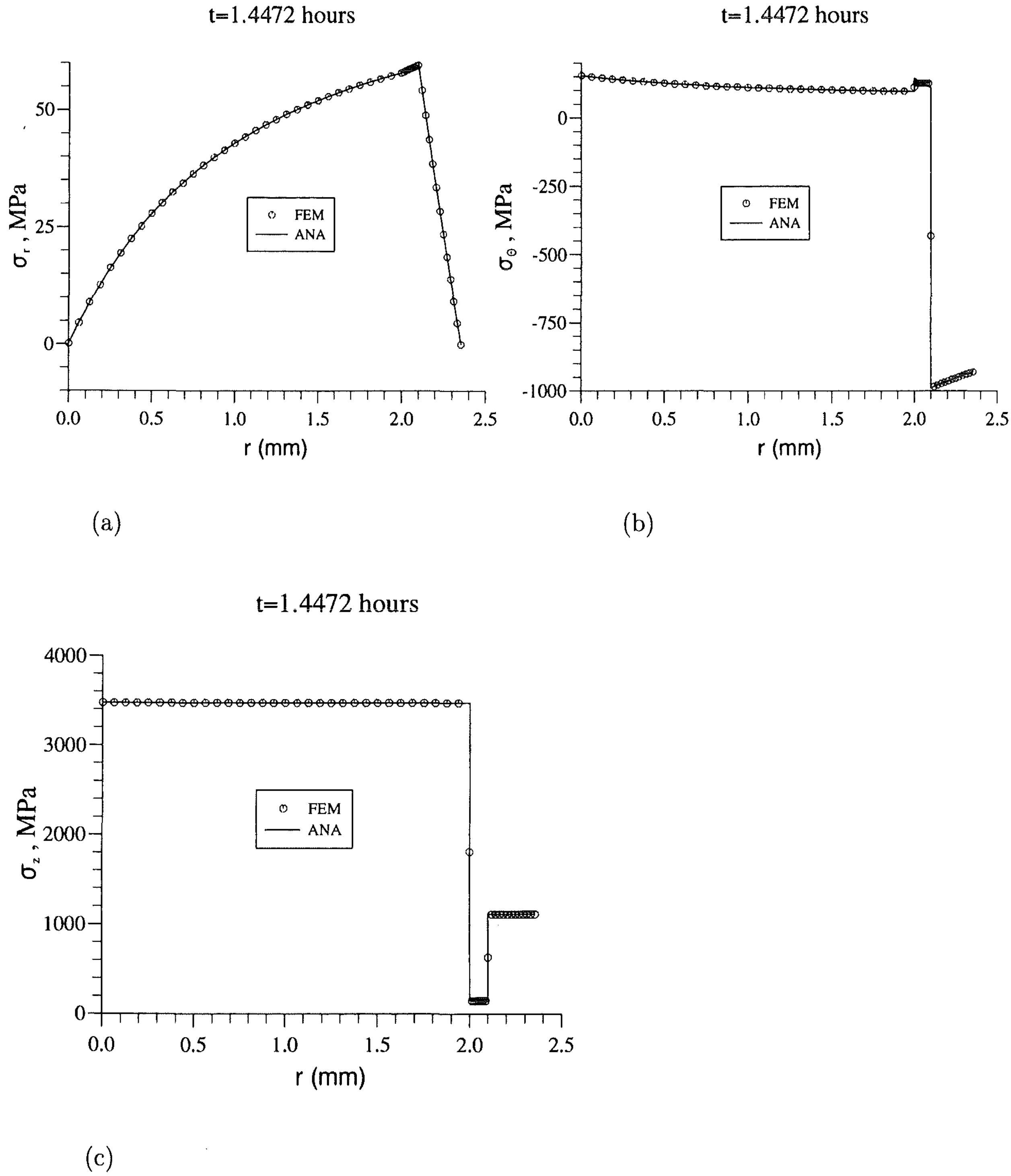
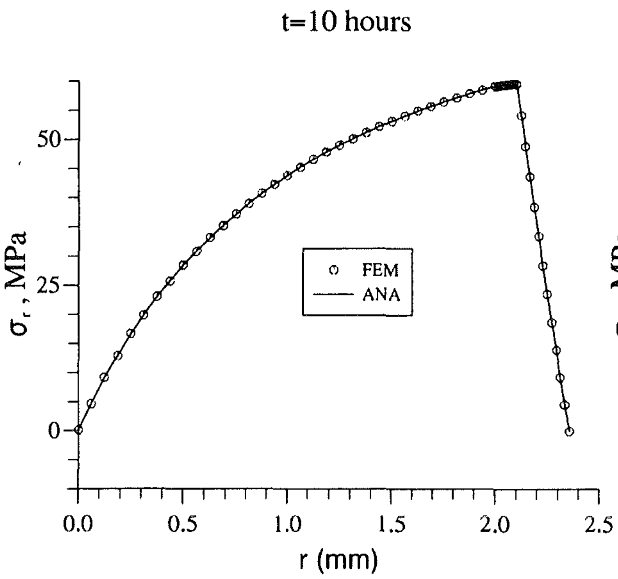


Fig.6 Stresses for cylinders with $\epsilon_z = 0$ and $\dot{\epsilon}_z = 0$ (t=0.1 hours).

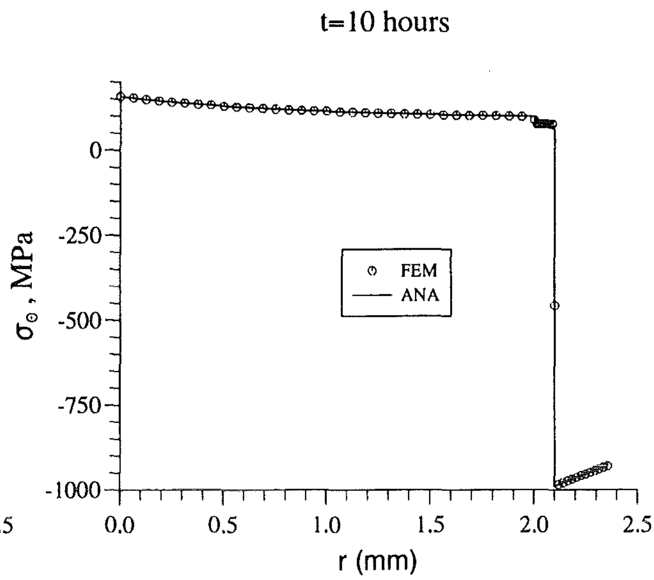


(a) (b) (c)

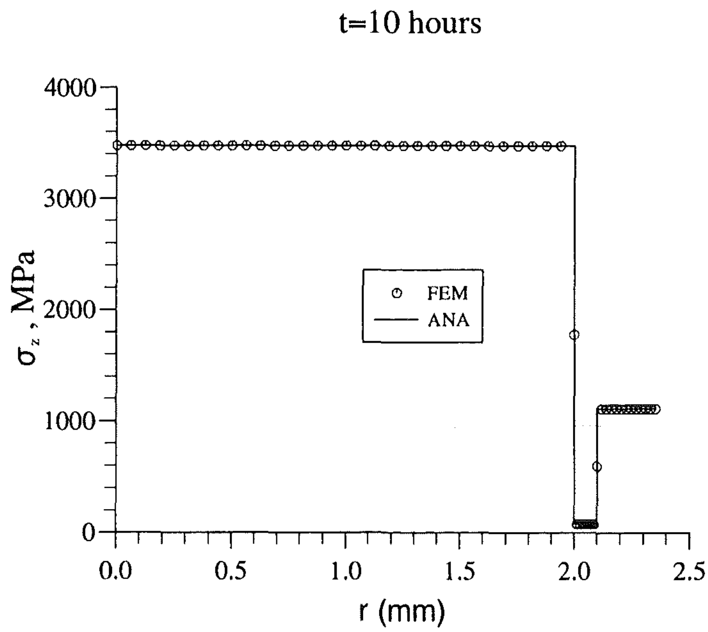
Fig.7 Stresses for cylinders with $\epsilon_z = 0$ and $\dot{\epsilon}_z = 0$ (t=1.4472 hours).



(a)

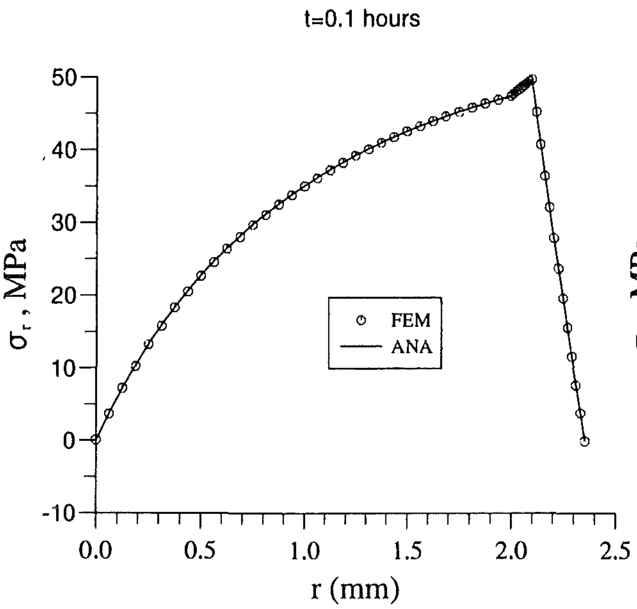


(b)

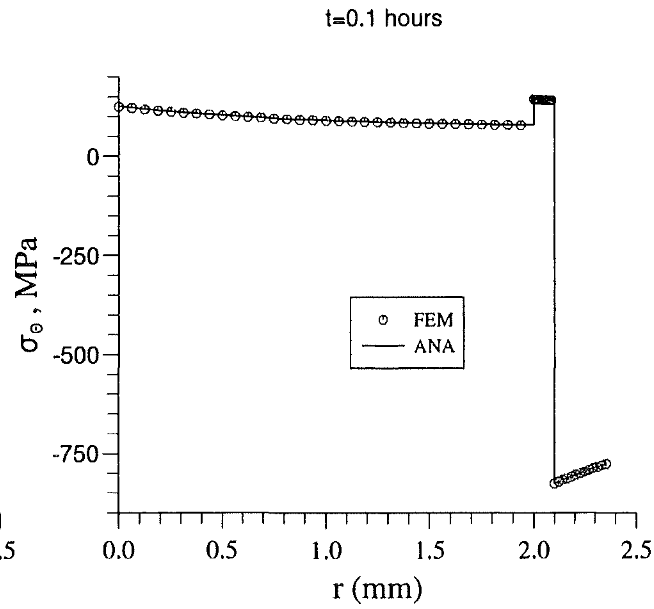


(c)

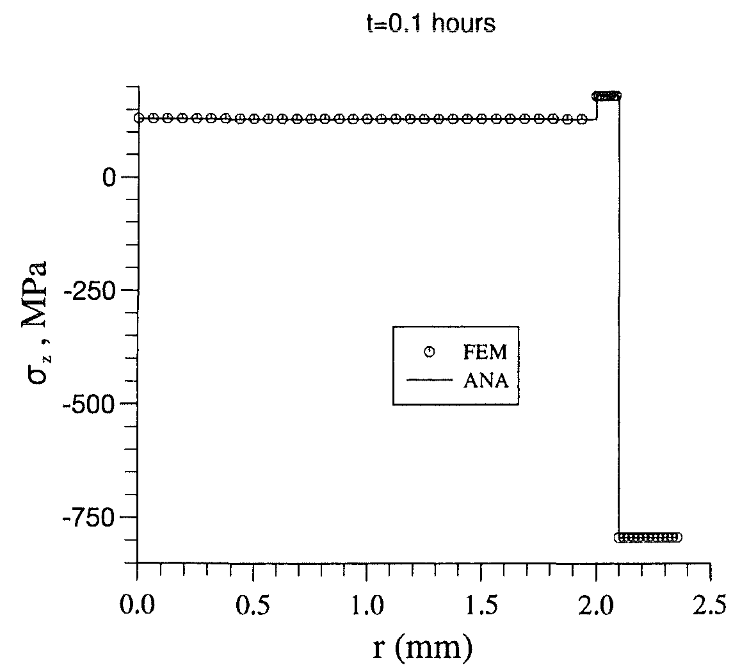
Fig.8 Stresses for cylinders with $\epsilon_z = 0$ and $\dot{\epsilon}_z = 0$ (t=10 hours).



(a)



(b)



(c)

Fig.9 Stresses for cylinders with an arbitrary ϵ_z and $\dot{\epsilon}_z = 0$ (t=0.1 hours).

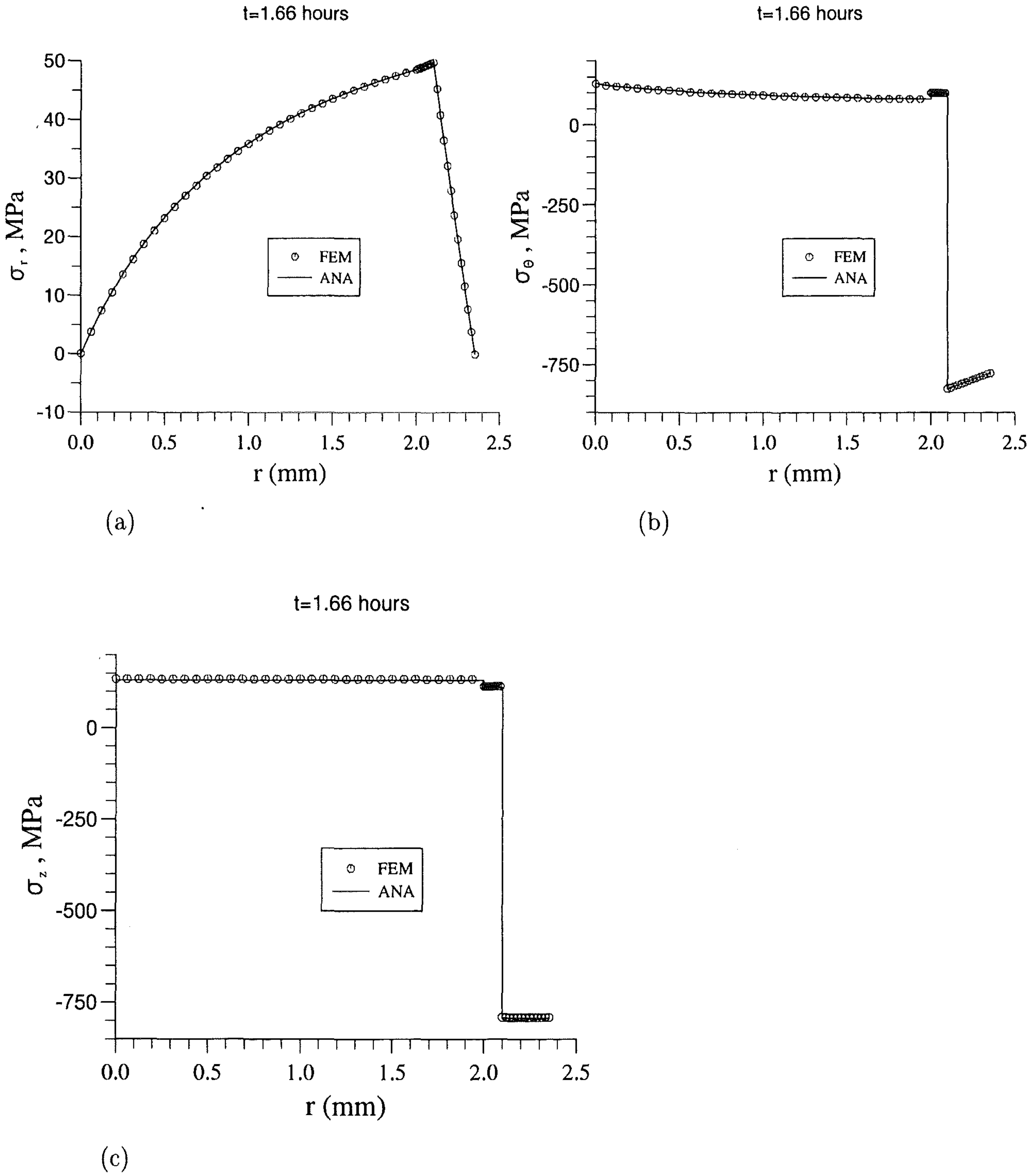


Fig. 10 Stresses for cylinders with an arbitrary ϵ_z and $\dot{\epsilon}_z = 0$ (t=1.66 hours) .

Chapter 4

Stress Analysis in a Plate

In this chapter, stress analysis in a multi - layers plate will be presented briefly for elastic behavior and for a creep process, with only plane strain being considered. Then, the stress distributions in the range far away from the ends of a multi - layers cylinder and of a multi - layers plate are compared to see the effect of the material creep behavior on stresses.

4.1 Solutions for elastic behavior

In this section, it is assumed that the plate is in plane strain. This means that $\epsilon_z = 0$. In the plane x-y (for coordinates see Fig.11), following Bernoulli's law the strain can be always characterized, irrespective of materials property, by

$$\epsilon_x = A + By \tag{4.1}$$

where A and B are unknown constants. The relation between the stress and the strain is

$$\sigma_x = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} \left(\epsilon_x + \frac{\nu}{1-\nu}\epsilon_y - \frac{1+\nu}{1-\nu}\alpha T \right) \tag{4.2}$$

$$\sigma_y = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} \left(\epsilon_y + \frac{\nu}{1-\nu}\epsilon_x - \frac{1+\nu}{1-\nu}\alpha T \right) \tag{4.3}$$

$$\sigma_z = \nu(\sigma_x + \sigma_y) - E\alpha T. \tag{4.4}$$

Because the thickness of the plate is much smaller than the size of the plate in x- and z- direction, the stress σ_y can be assumed to be zero (FEM calculations have shown that this is reasonable). Then, the stress and strain relation can be simplified as

$$\epsilon_x = \frac{1}{E'}\sigma_x + \alpha' T \tag{4.5}$$

$$\epsilon_y = -\frac{\nu'}{E'}\sigma_x + \alpha' T \tag{4.6}$$

$$\epsilon_z = 0 \tag{4.7}$$

and

$$\sigma_x = E'(\epsilon_x - \alpha' T) \tag{4.8}$$

$$\sigma_y = 0 \tag{4.9}$$

$$\sigma_z = \nu\sigma_x - \alpha E T, \tag{4.10}$$

with $E' = E/(1-\nu^2), \nu' = \nu/(1-\nu), \alpha' = (1+\nu)\alpha$. In each material,

$$\sigma_x = E_i'(A + By - \alpha_i' T) \qquad \text{(i=1,2,...,N)} \tag{4.11}$$

is valid. When the parameters A and B are known, the stress and strain in the multi - layers plate can be calculated. To determine the constants A and B, the equilibrium conditions of the plate have to be used. As no force and moment is applied in the joint, the equilibrium conditions are:

$$F_x = \int_{y_0}^{y_3} \sigma_x dy = 0 \tag{4.12}$$

$$M_x = \int_{y_0}^{y_3} \sigma_x y dy = 0. \tag{4.13}$$

Substituting Eq.(4.11) in Eqs.(4.12) and (4.13) yields the general equations to determine A and B:

$$A\sum_{i=1}^{N} E_i' \left[y_i - y_{i-1}\right] + \frac{B}{2}\sum_{i=1}^{N} E_i' \left[(y_i)^2 - (y_{i-1})^2\right] = \sum_{i=1}^{N} E_i'\alpha_i' T \left[y_i - y_{i-1}\right] \tag{4.14}$$

$$\frac{A}{2}\sum_{i=1}^{N} E_i' \left[(y_i)^2 - (y_{i-1})^2\right] + \frac{B}{3}\sum_{i=1}^{N} E_i' \left[(y_i)^3 - (y_{i-1})^3\right]$$
$$= \sum_{i=1}^{N} E_i'\alpha_i' T/2 \left[(y_i)^2 - (y_{i-1})^2\right], \qquad (4.15)$$

where y_i see Fig.11 and N is the number of the layers in the joint. Using the definitions of

$$a_{11} = \sum_{i=1}^{N} E_i' \left[y_i - y_{i-1}\right] \qquad (4.16)$$

$$a_{12} = \frac{1}{2}\sum_{i=1}^{N} E_i' \left[(y_i)^2 - (y_{i-1})^2\right] \qquad (4.17)$$

$$a_{21} = a_{12} \qquad (4.18)$$

$$a_{22} = \frac{1}{3}\sum_{i=1}^{N} E_i' \left[(y_i)^3 - (y_{i-1})^3\right] \qquad (4.19)$$

$$R_1 = \sum_{i=1}^{N} E_i'\alpha_i' T \left[y_i - y_{i-1}\right] \qquad (4.20)$$

$$R_2 = \sum_{i=1}^{N} E_i'\alpha_i' T/2 \left[(y_i)^2 - (y_{i-1})^2\right], \qquad (4.21)$$

gives

$$A = \frac{R_1 a_{22} - R_2 a_{12}}{\Delta}$$
$$B = \frac{R_2 a_{11} - R_1 a_{21}}{\Delta}, \qquad (4.22)$$

with $\Delta = a_{11}a_{22} - a_{12}a_{21}$. The stress at any point in a N - layers joint can be calculated analytically from Eq.(4.11) with the coefficients given in Eq.(4.22). To check the analytical solution, one FEM calculation has been carried out. The comparison is shown in Fig.12 (joint geometry is the same as in chapter 4.3 and the material data are the same as in chapter 3). It can be seen that they are in good agreement and that the assumption of $\sigma_y = 0$ is true.

4.2 Solutions for creep behavior

For creep material, Norton's law given in chapter 2.2 will be used. For a thin plate and in plane strain

$$\begin{aligned} \dot{\epsilon}_z &= 0 \\ \dot{\sigma}_y &= 0 \end{aligned} \tag{4.23}$$

are true. Equations (2.81) and (4.23) lead to

$$\dot{\epsilon}_x = \frac{\dot{\sigma}_x}{E'} + \frac{3}{2} D \sigma_{\text{eff}}^{(n-1)} (S_x + \nu S_z) \tag{4.24}$$

$$\dot{\epsilon}_y = -\frac{\nu'}{E'} \dot{\sigma}_x + \frac{3}{2} D \sigma_{\text{eff}}^{(n-1)} (S_y + \nu S_z) \tag{4.25}$$

and

$$\dot{\sigma}_x = E' \left\{ \dot{\epsilon}_x - \frac{3}{2} D \sigma_{\text{eff}}^{(n-1)} (S_x + \nu S_z) \right\} \tag{4.26}$$

$$\dot{\sigma}_z = \nu \dot{\sigma}_x - \frac{3}{2} D \sigma_{\text{eff}}^{(n-1)} E S_z. \tag{4.27}$$

Following Eq.(4.1)

$$\dot{\epsilon}_x = \mathcal{A} + \mathcal{B} y \tag{4.28}$$

and therefore

$$\dot{\sigma}_x = E' \left\{ \mathcal{A} + \mathcal{B} y - \frac{3}{2} D \sigma_{\text{eff}}^{(n-1)} (S_x + \nu S_z) \right\}. \tag{4.29}$$

The thickness of the layer with creep material is very thin. Therefore, the quantities S_x, S_y, S_z and $\sigma_{\text{eff}}^{(n-1)}$ can be assumed to be constant. The equilibrium condition of the plate yields

$$\dot{F}_x = \int_{y0}^{y3} \dot{\sigma}_x dy = 0 \tag{4.30}$$

$$\dot{M}_x = \int_{y0}^{y3} \dot{\sigma}_x y dy = 0. \tag{4.31}$$

From Eqs.(4.29-4.31), the following equations can be obtained:

$$\mathcal{A}\sum_{i=1}^{N} E_i' [y_i - y_{i-1}] + \frac{\mathcal{B}}{2}\sum_{i=1}^{N} E_i' \left[(y_i)^2 - (y_{i-1})^2\right] = \sum_{i=1}^{N} \frac{3}{2} D_i \sigma_{\mathrm{i,eff}}^{(n_i-1)} E_i' [S_x^{(i)} + \nu_i S_z^{(i)}] [y_i - y_{i-1}] \quad (4.32)$$

$$\frac{\mathcal{A}}{2}\sum_{i=1}^{N} E_i' \left[(y_i)^2 - (y_{i-1})^2\right] + \frac{\mathcal{B}}{3}\sum_{i=1}^{N} E_i' \left[(y_i)^3 - (y_{i-1})^3\right] = \sum_{i=1}^{N} \frac{3}{2} D_i \sigma_{\mathrm{i,eff}}^{(n_i-1)} E_i'/2 [S_x^{(i)} + \nu_i S_z^{(i)}] \left[(y_i)^2 - (y_{i-1})^2\right]. \quad (4.33)$$

Let

$$\mathcal{R}_1 = \sum_{i=1}^{N} \frac{3}{2} D_i \sigma_{\mathrm{i,eff}}^{(n_i-1)} E_i' [S_x^{(i)} + \nu_i S_z^{(i)}] [y_i - y_{i-1}] \quad (4.34)$$

$$\mathcal{R}_2 = \sum_{i=1}^{N} \frac{3}{2} D_i \sigma_{\mathrm{i,eff}}^{(n_i-1)} E_i'/2 [S_x^{(i)} + \nu_i S_z^{(i)}] \left[(y_i)^2 - (y_{i-1})^2\right]. \quad (4.35)$$

This results in

$$\mathcal{A} = \frac{\mathcal{R}_1 a_{22} - \mathcal{R}_2 a_{12}}{\Delta}$$
$$\mathcal{B} = \frac{\mathcal{R}_2 a_{11} - \mathcal{R}_1 a_{21}}{\Delta}. \quad (4.36)$$

Now, the rate of the stress can be determined from Eq.(4.29) with the coefficients given in Eq.(4.36). The stress can be calculated by using the process shown in chapter 2.2.1. A comparison of the stresses obtained from Eqs.(4.36) and (4.29) and from FEM is made in Figs.13, 14 and 15 for different times (t=0.05943 hours (Fig.13), t=0.1617 hours (Fig.14) and t=5.3496 hours (Fig.15)) (joint geometry is the same as in chapter 4.3 and the material data are the same as in chapter 3). It can be seen that they are in good agreement, irrespective of time.

4.3 Effect of the material creep behavior on stresses in a plate and in a cylinder

To compare the creep effect on stresses in a plate and in a cylinder, the stress distributions are plotted together for various times and for plane strain.

The geometry of the plate is

$$\begin{aligned} y_0 &= 0\ mm \\ y_1 &= 2\ mm \\ y_2 &= 2.1\ mm \\ y_3 &= 2.355\ mm \\ L/H &= 5.1. \end{aligned}$$

For the cylinder, the geometry is the same as that given in chapter 3.
As a first example, the material data of the three layers are

$$\begin{aligned} E_1 = 142\ \text{GPa}, \quad \nu_1 &= 0.3, \quad \alpha_1 = 15.8 \times 10^{-6}/K \\ E_2 = 160\ \text{GPa}, \quad \nu_2 &= 0.3, \quad \alpha_2 = 16.6 \times 10^{-6}/K \\ E_3 = 150\ \text{GPa}, \quad \nu_3 &= 0.3, \quad \alpha_3 = 10.4 \times 10^{-6}/K \end{aligned}$$

and

$$D_2 = 9.7 \times 10^{-9} \quad (\sigma \text{ in MPa and t in hours}), \qquad n_2 = 3.4.$$

The loading results from the joint having an initial temperature of $20°C$, an end temperature of $900°C$ and being subjected to t hours of creeping.
Fig.16 shows the stress distribution in a plate and in a cylinder for the elastic behavior and for a short time. It can be seen that (a) with an increasing t, the absolute value of σ_r increases, even in creep material, (b) in creep material, the absolute value of the stresses, which are parallel to the interface, decreases with increasing t; however, in elastic materials they may decrease or increase, (c) stress components parallel to the interface $(\sigma_\theta, \sigma_x, \sigma_z)$ in a cylinder are nearly constant in each material, whereas in the plate they vary strongly.
The time dependence of the stresses at different points (inside radius R_i, outside radius R_a and the center of each layer) in the joint is given in Figs.17 - 19 for materials 1 (Fig.17), 2 (Fig.18) and 3 (Fig.19). The stress component σ_z is not true due to the assumption of $\epsilon_z = 0$. Therefore, in the following σ_z is not plotted (σ_z is similar to σ_θ or σ_x). It is shown that (a) in elastic materials (1 and 3) with increasing t stresses tend to a constant, which is not zero, although they decrease to zero in the creep material, (b) in creep material, with increasing t the stresses

parallel to the interface decrease, but the stress perpendicular to the interface σ_r does not, (c) in creep material the rate of stress $\dot{\sigma}$ in the plate is larger than that of the cylinder (Fig.18b).

As a second example, the material data of the three layers are

$$E_1 = 158.5 \text{ GPa}, \quad \nu_1 = 0.3, \quad \alpha_1 = 15.66 \times 10^{-6}/K$$
$$E_3 = 15 \text{ GPa}, \quad \nu_3 = 0.3, \quad \alpha_3 = 10.75 \times 10^{-6}/K$$

and material 2 is the same as that in the first example. For this example, E_3 is much smaller than E_1 and E_2.

Fig.20 shows the stress distribution in a plate and in a cylinder for the elastic behavior and for a short time. It can be seen that for a very small time, $|\sigma|$ is much larger than that at t=0 and for very long time (even stress changes the sign, this is not the case for example 1), which may indicate fracture or failure of the joint.

The time dependence of the stresses at different points in the joint is given in Figs.21 - 23 for materials 1, 2 and 3. It is shown that (a) in the elastic materials stresses may change the sign with increasing t, (b) in creep material, stresses parallel to the interface decrease to zero with increasing t, the stress perpendicular to the interface tends to a constant, (c) for the plate, in elastic material the rate of stress $\dot{\sigma}$ is much larger than for the cylinder.

To sum up, it may be concluded that for a joint with the same thickness, same materials combination and with three Young's moduli being similar (as in example 1) in a plate the stress situation is more beneficial than in a cylinder under considering material creep behavior; if E_3 is much smaller than E_1 and E_2 (as in example 2), σ_r is very small and a cylinder is more beneficial than a plate.

So far only the stresses in the center of a joint are studied. Along the free edges of a multi - layers plate or along the free ends of a multi - layers cylinder, there is a stress singularity due to the difference of the elastic constants in the joined components. Therefore, the values of the stresses obtained from FEM have no real meaning. However, the tendency of the stress distribution obtained by the

FEM calculation is true. FEM results have shown that along the free edge of a plate and along the free ends of a cylinder stress distributions are similar, but the rate of stresses in a plate is larger that in a cylinder.

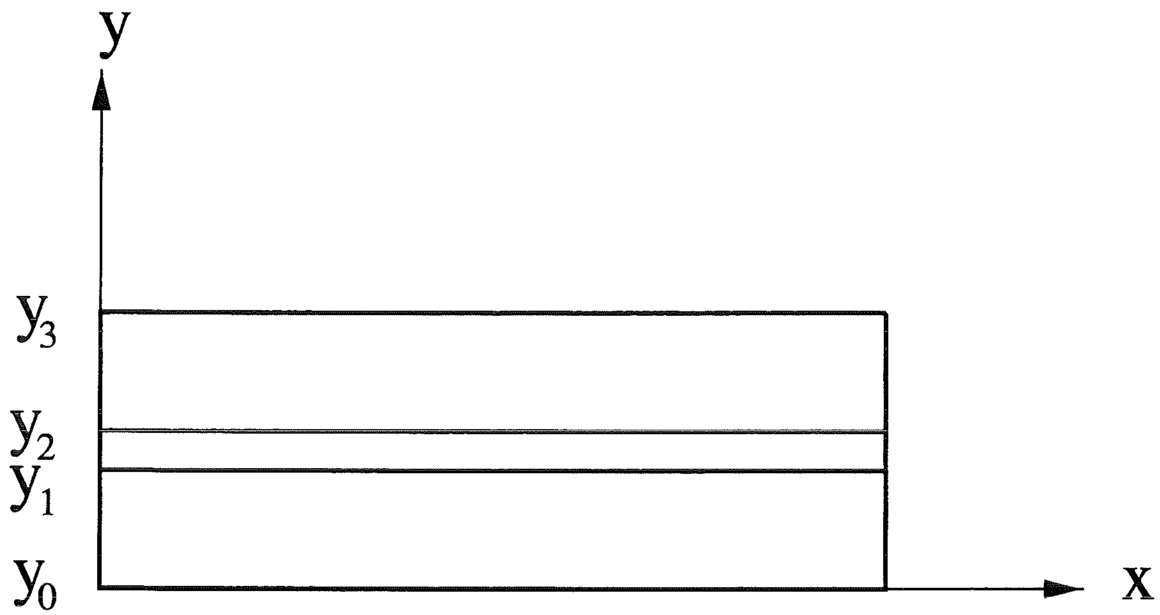


Fig.11 The geometry and coordinates of a plate.

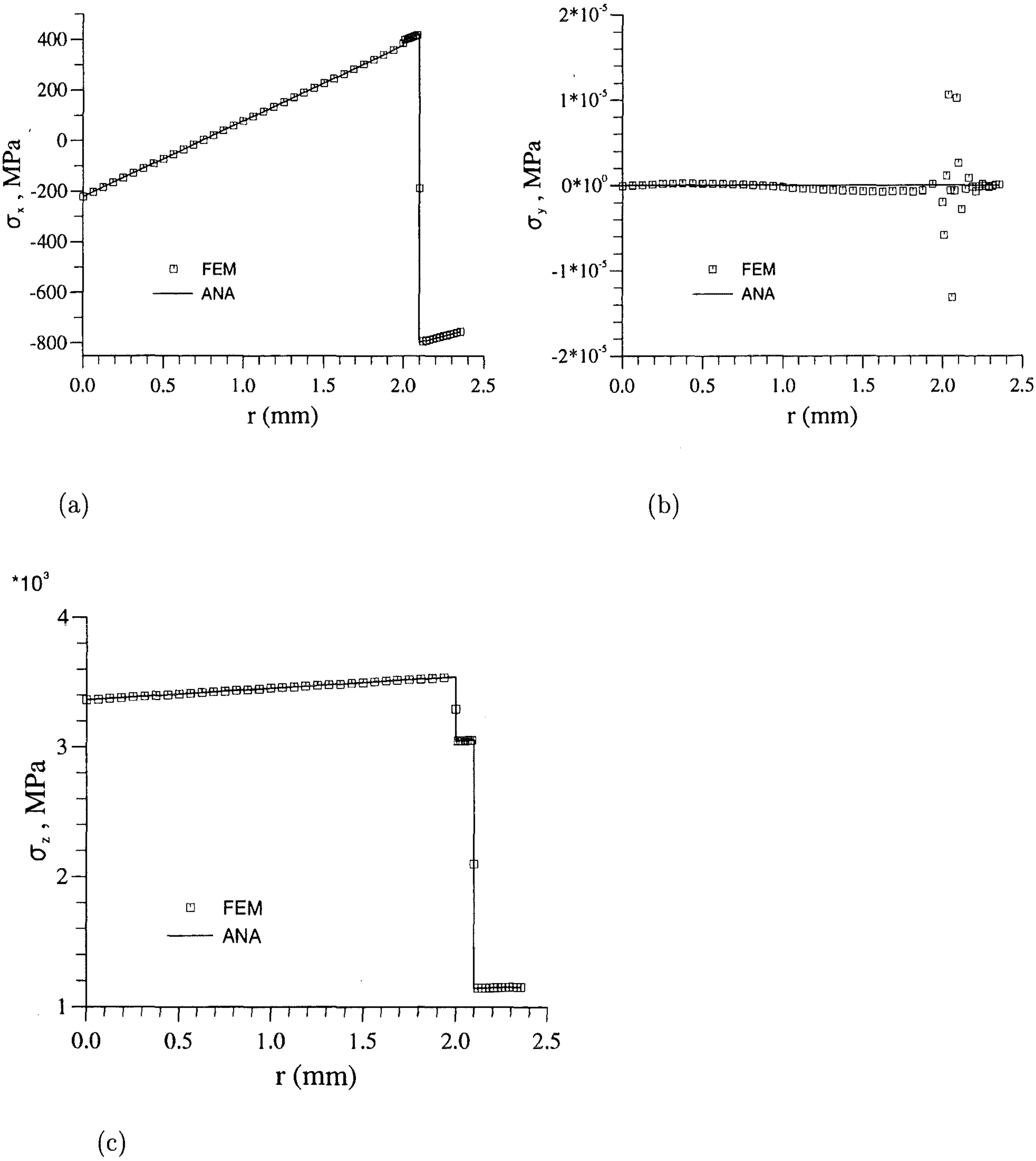


Fig.12 Comparison of stresses for a plate with $\epsilon_z = 0$.

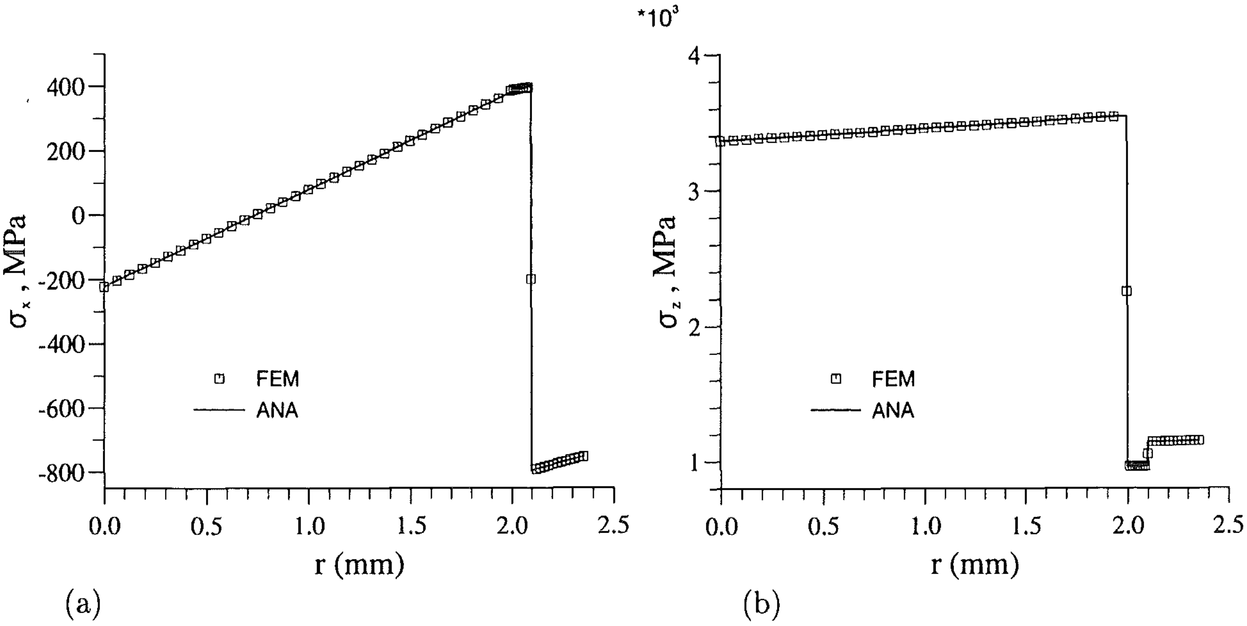


Fig.13 Comparison of stresses for a plate with $\epsilon_z = 0$ and $\dot{\epsilon}_z = 0$ (t=0.05943 hours).

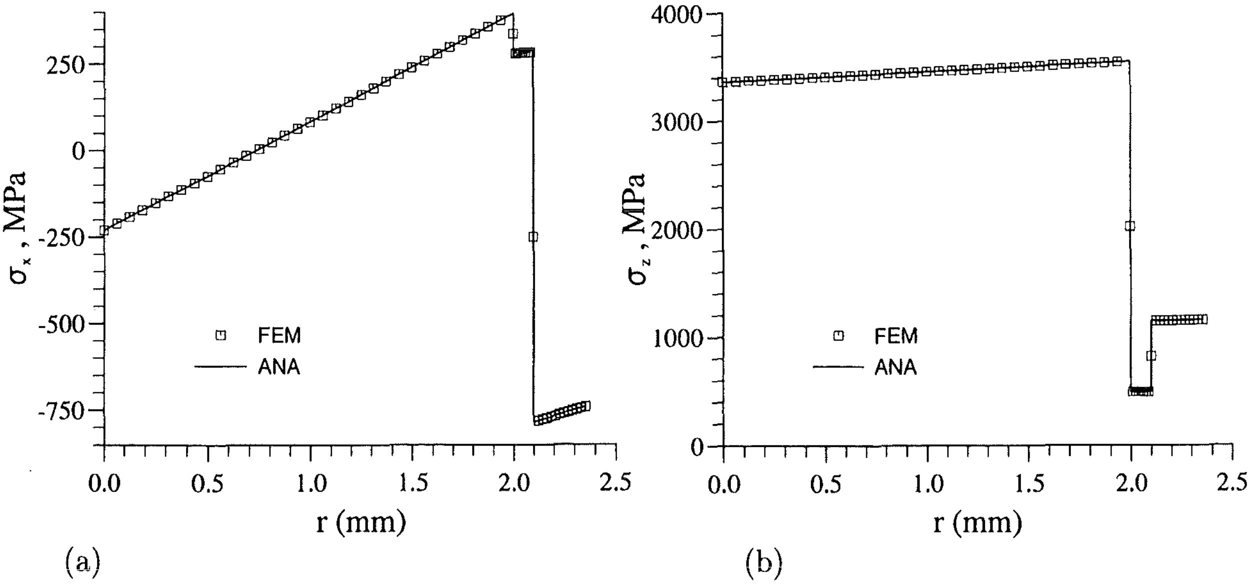


Fig.14 Comparison of stresses for a plate with $\epsilon_z = 0$ and $\dot{\epsilon}_z = 0$ (t=0.1617 hours).

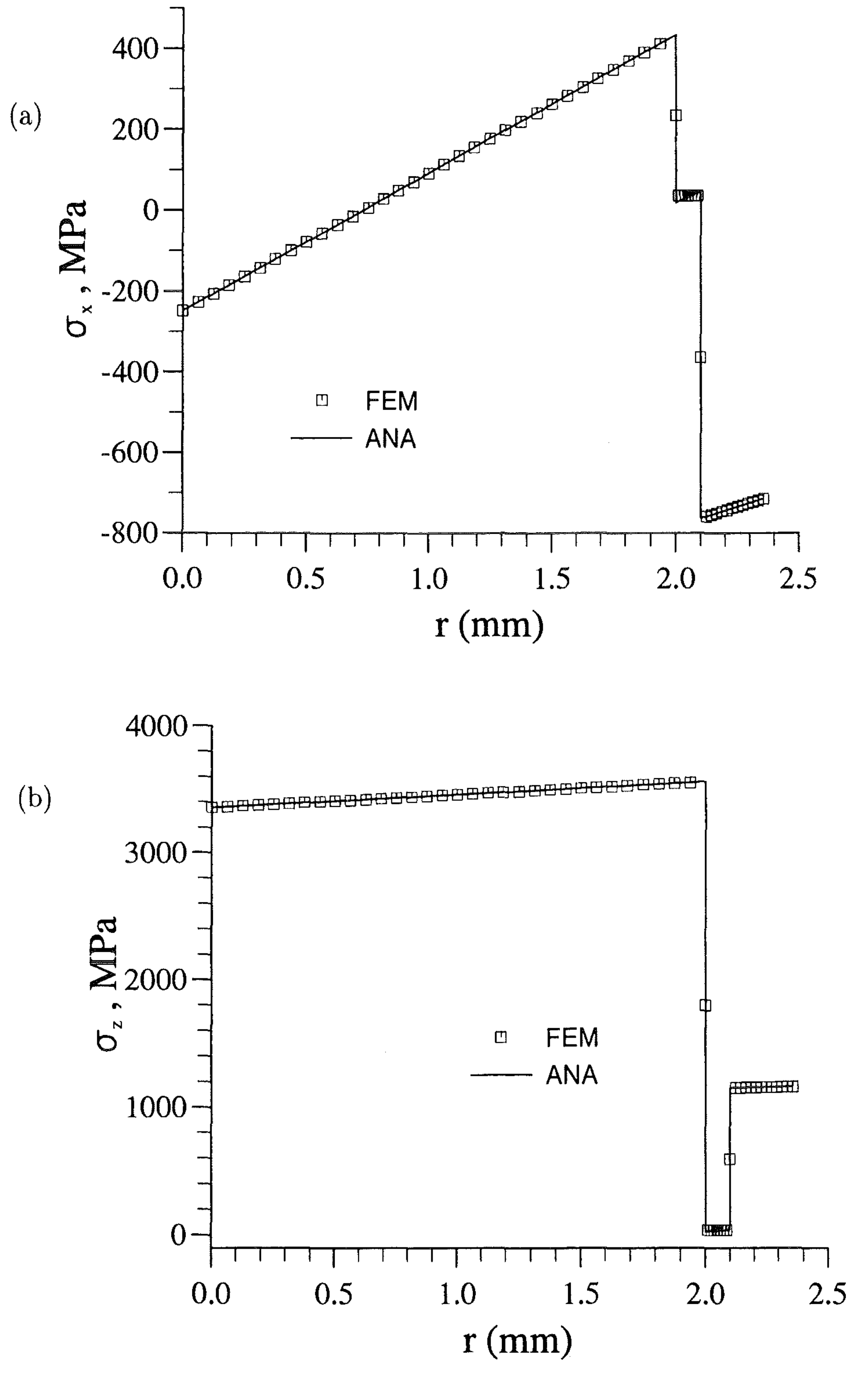


Fig.15 Comparison of stresses for a plate with $\epsilon_z = 0$ and $\dot{\epsilon}_z = 0$ (t=5.3496 hours).

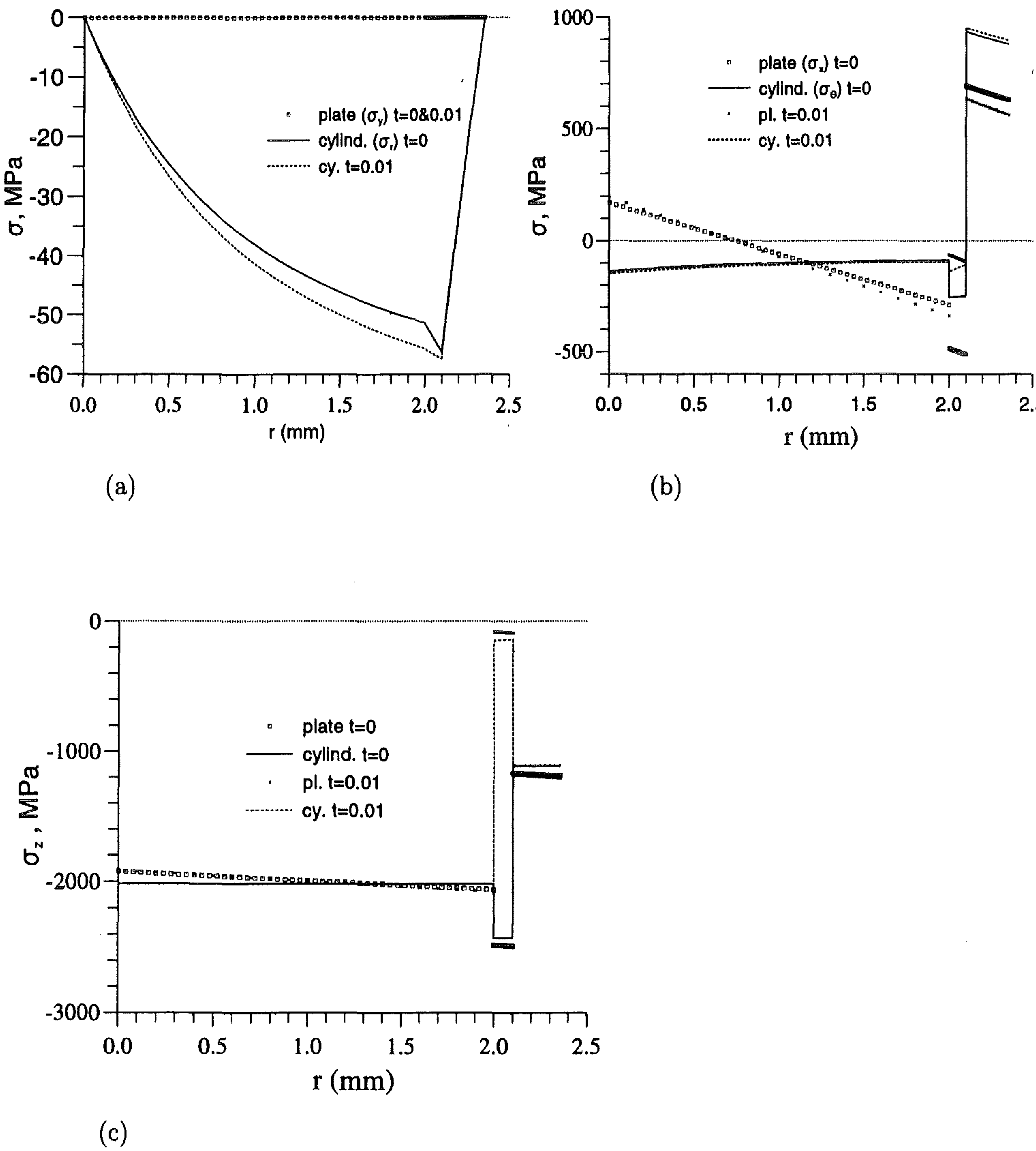


Fig.16 Stresses for a plate and a cylinder with $\epsilon_z = 0$ and $\dot{\epsilon}_z = 0$ (t in hours).

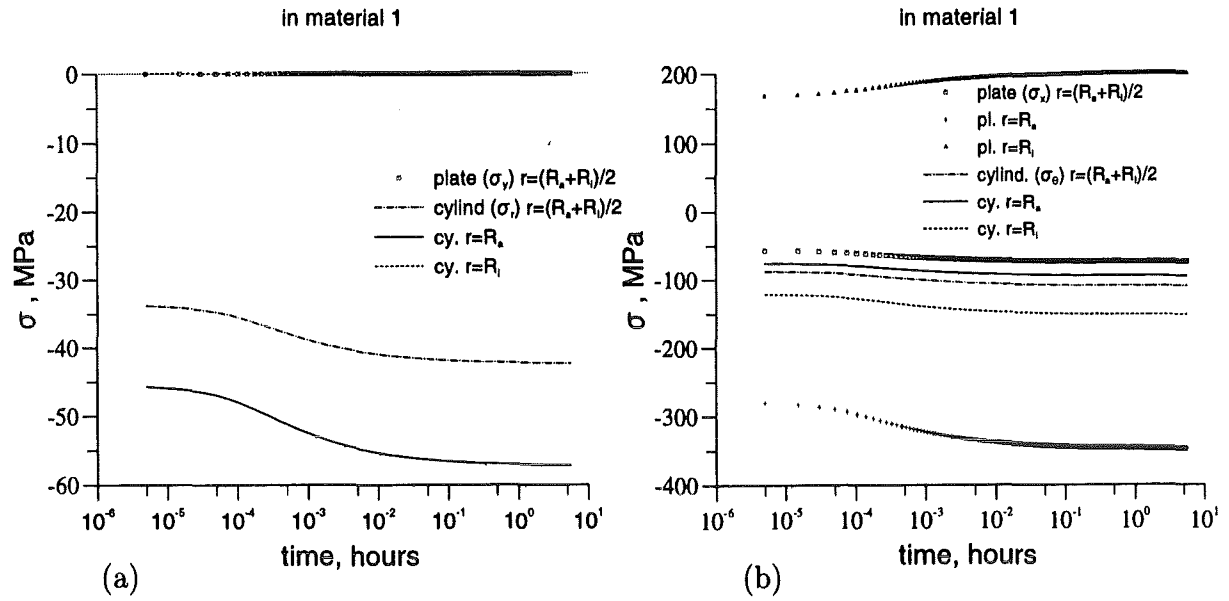


Fig.17 Stresses at different points of a plate and a cylinder in material 1.

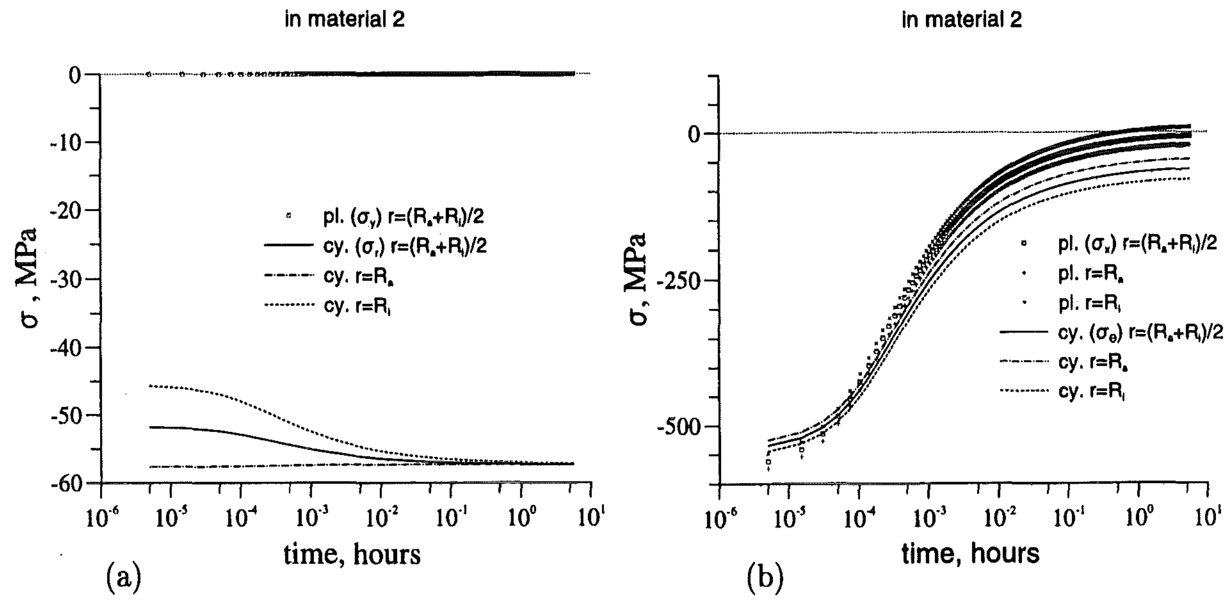


Fig.18 Stresses at different points of a plate and a cylinder in material 2.

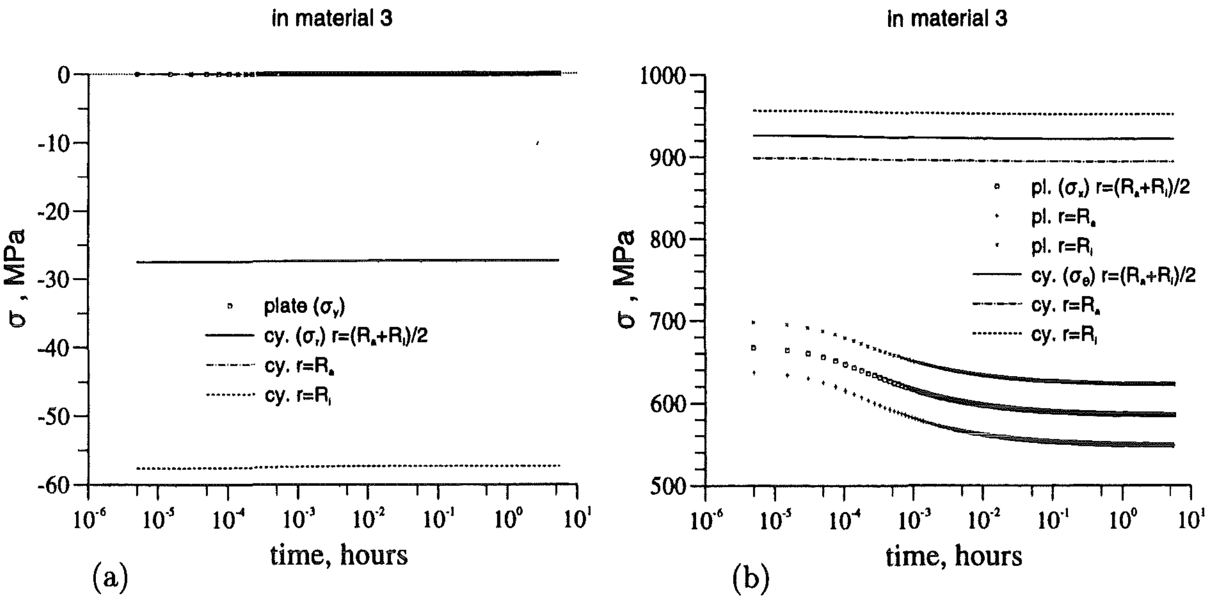


Fig.19 Stresses at different points of a plate and a cylinder in material 3.

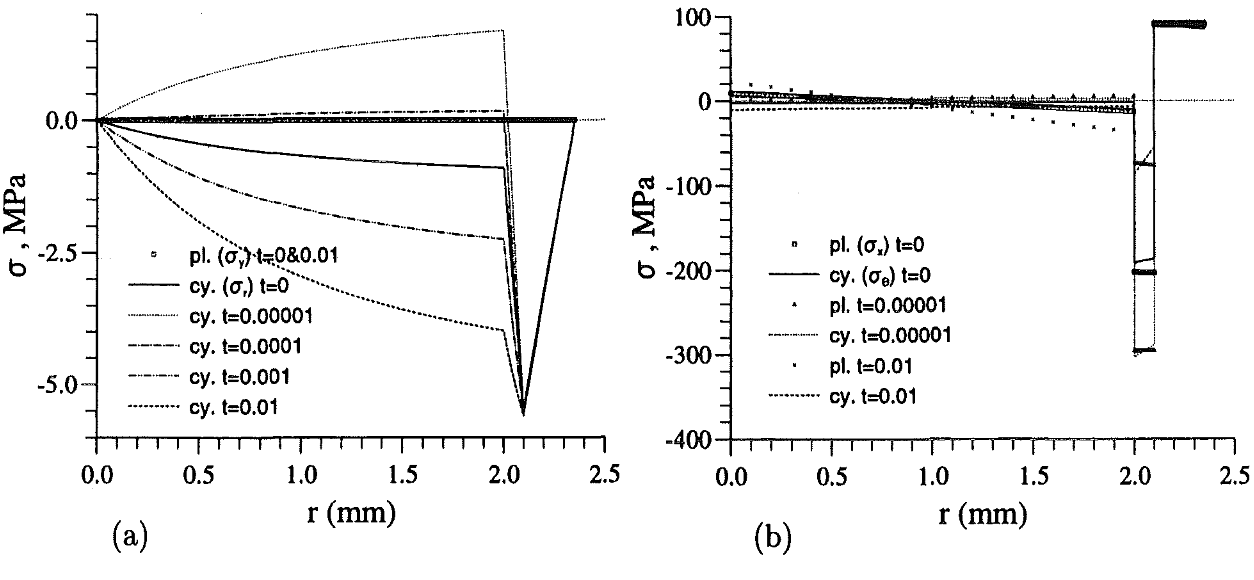


Fig.20 Stresses for a plate and a cylinder with $\epsilon_z = 0$ and $\dot{\epsilon}_z = 0$ (example 2).

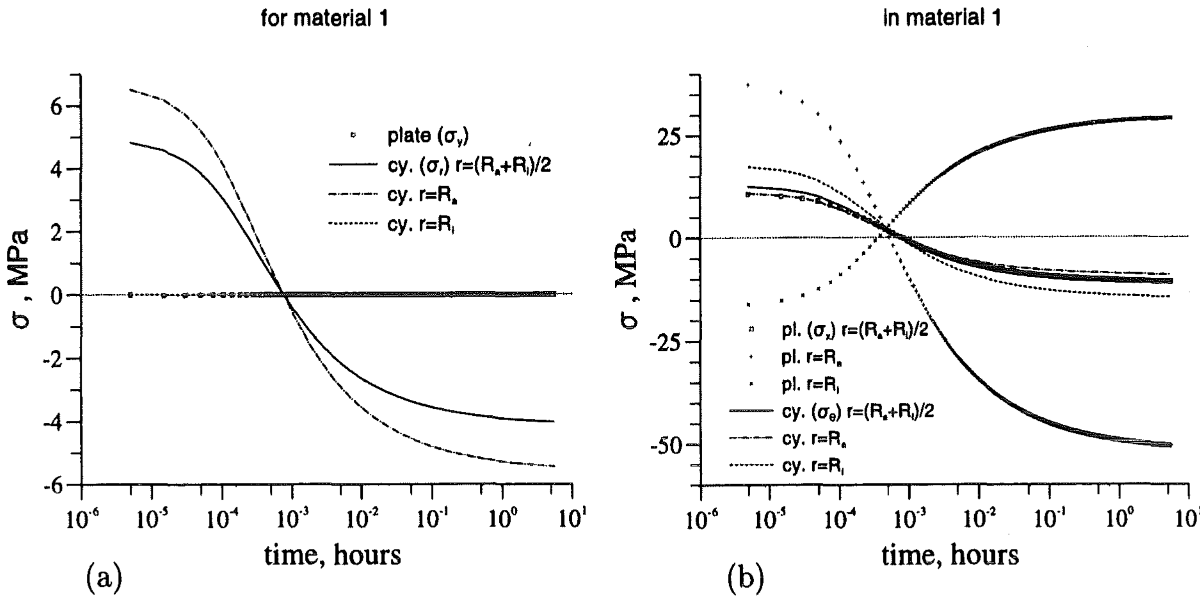


Fig.21 Stresses at different points of a plate and a cylinder in material 1 (Ex.2).

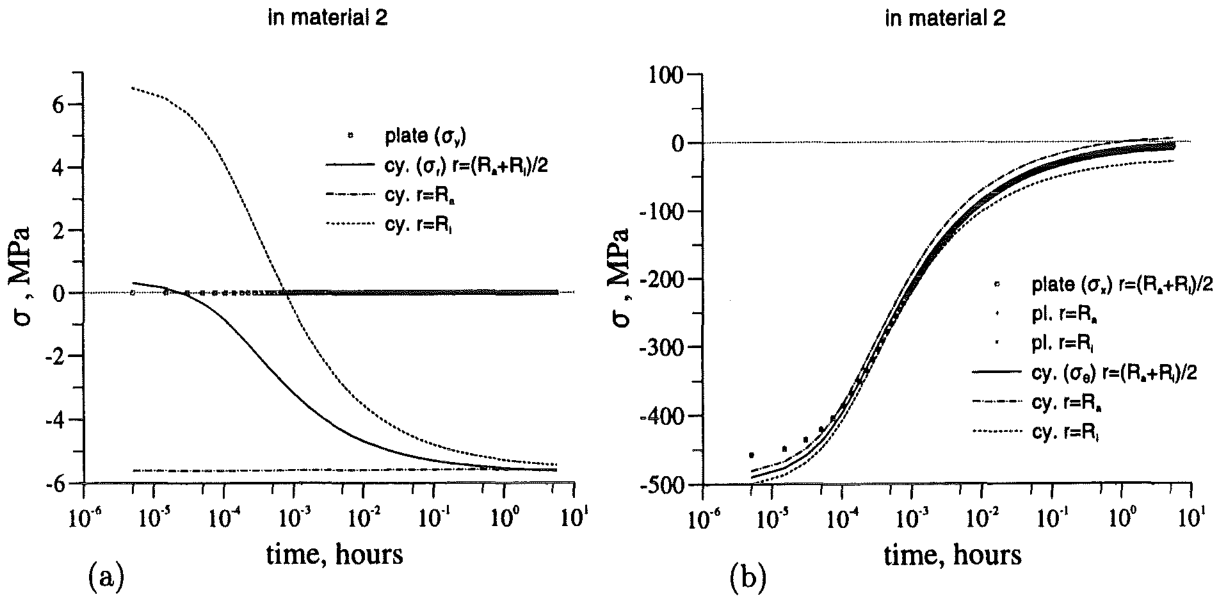


Fig.22 Stresses at different points of a plate and a cylinder in material 2 (Ex.2).

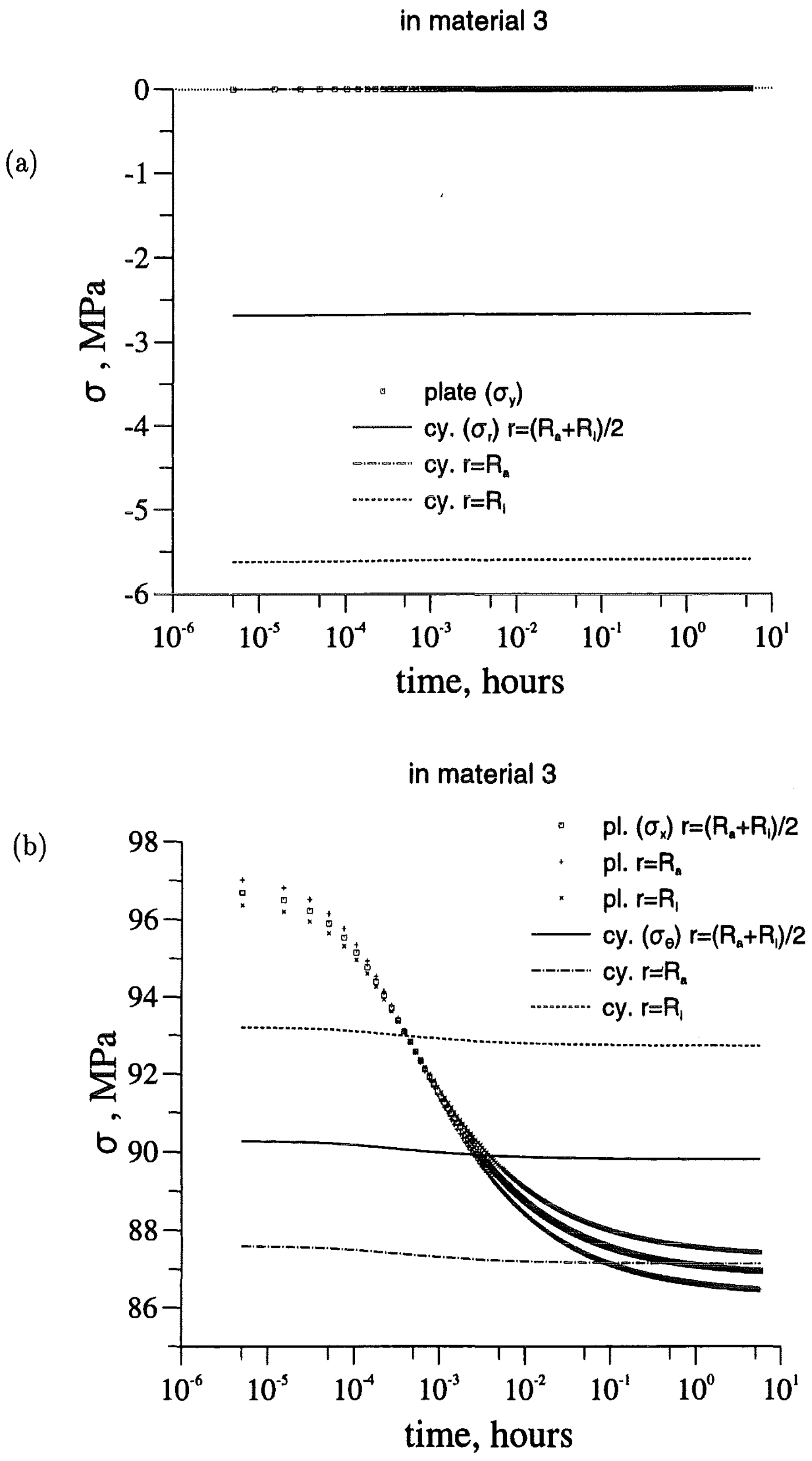


Fig.23 Stresses at different points of a plate and a cylinder in material 3 (Ex.2).

Chapter 5

Creep Behavior in a Joint with Functionally Graded Material

In this chapter, an example will be presented to see whether a joint with a Functionally Graded Material (FGM) is beneficial or not, under material creep behavior. The joint (multi - layers cylinder) is made of a substrate (Ni - Superalloy) and a bond coat layer with a NiCoCrAlY layer, a Al_2O_3 layer, a FGM layer and a ZrO_2 layer. The inner radius of the cylinder is 2mm and the outer radius is 4.355mm. The thicknesses of the layers are: Substrate - 2mm; NiCoCrAlY layer - 100 μm; Al_2O_3 layer - 5μm; FGM layer - 50 μm; ZrO_2 layer - 200 μm (with FGM) or 250 μm (without FGM). The ratio of the length and thickness is L/H=5.1.

The used material data are:

$$\begin{aligned}
E_{sub.} &= 148 \text{ GPa}, & \nu_{sub.} &= 0.3, & \alpha_{sub.} &= 16.28 \times 10^{-6}/K \\
E_{NiCo.} &= 70 \text{ GPa}, & \nu_{NiCo.} &= 0.3, & \alpha_{NiCo.} &= 16.6 \times 10^{-6}/K \\
E_{Al_2O_3} &= 319 \text{ GPa}, & \nu_{Al_2O_3} &= 0.24, & \alpha_{Al_2O_3} &= 8 \times 10^{-6}/K \\
E_{ZrO_2} &= 16 \text{ GPa}, & \nu_{ZrO_2} &= 0.286, & \alpha_{ZrO_2} &= 10.8 \times 10^{-6}/K
\end{aligned}$$

and only the NiCoCrAlY layer has a creep behavior with the data of

$$D = 7.3 \times 10^{-9} \quad (\sigma \text{ in MPa and t in hours}), \qquad n = 2.7.$$

Graded material is introduced between the Al_2O_3 layer and the ZrO_2 layer with a linear transition function for Young's modulus E and the thermal expansion coefficient α. Loading is obtained from the initial temperature being 1200°C, the end temperature being 800°C, t hours of creeping and cooling down to 20 °C.

The stress distribution in the range far away from the ends of the cylinder is plotted in Fig.24 for the elastic behavior. The stress components σ_θ and σ_z are always similar, therefore, σ_z is not presented. Stresses are calculated from FEM. It can be seen that for elastic behavior, the joint with a thick FGM layer and without the ZrO_2 layer (but the surface has the material data of ZrO_2) is the worst one, while the joint without FGM is the best one.

In Figs.25 -26, the stress distributions for t=0.1 hours and t=1.5 hours are given, where no cooling down to room temperature has taken place. Fig.27 shows the stresses after t=1.5 hours of creeping and subsequent cooling down to room temperature. It can be seen that the joint without FGM is better suited than the joint with FGM. In the case of a joint with FGM, the joint without a ZrO_2 layer is better.

The time dependences of the stresses in a joint without FGM (Fig.28) and with FGM (Fig.29) are presented in Figs.28 - 29. It can be seen that (a) after creeping, the stress σ_r near the interface substrate / NiCoCrAlY layer is increased. Especially in the joint without FGM, the stress changes the sign (see Fig.28), which may introduce a delamination of the joint. (b) with t exceeding 1.5 hours, the creep process is almost completed.

If the stress σ_r at the interface is larger than the strength of the interface a delamination may occur. When the stress σ_θ at the surface is higher than strength of the material a crack may initiate. To analyze the effect of material creep behavior on the failure, e.g. interface delamination or surface crack initiates, time dependence of stress σ_r at the interface and σ_θ at the surface for different joint will be presented.

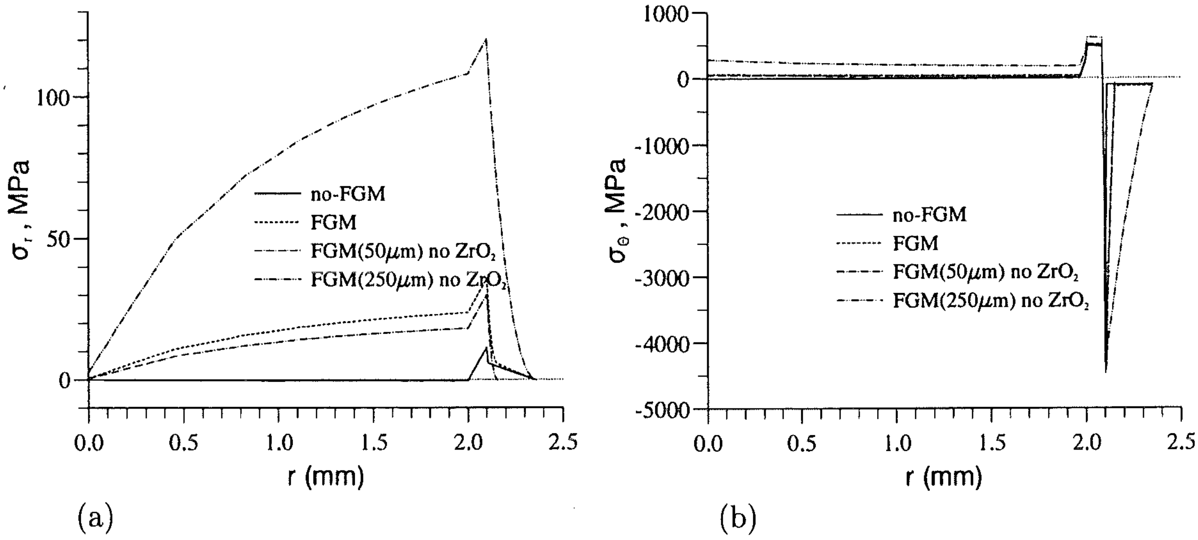


Fig.24 Comparison of stresses in a joint with and without FGM (t=0).

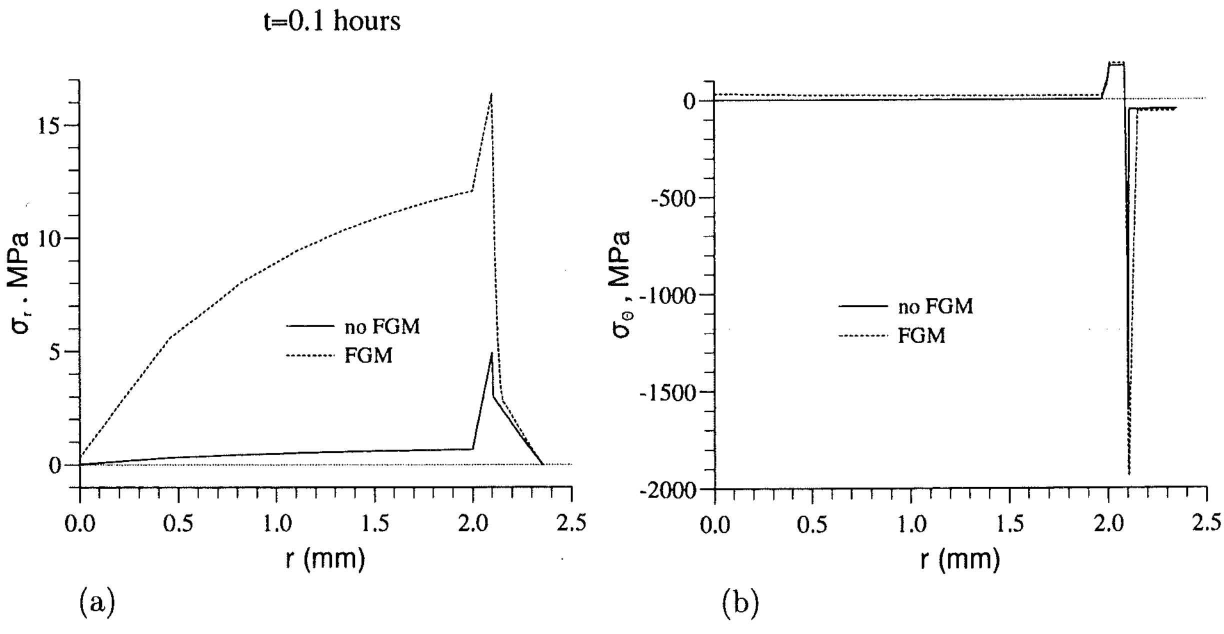
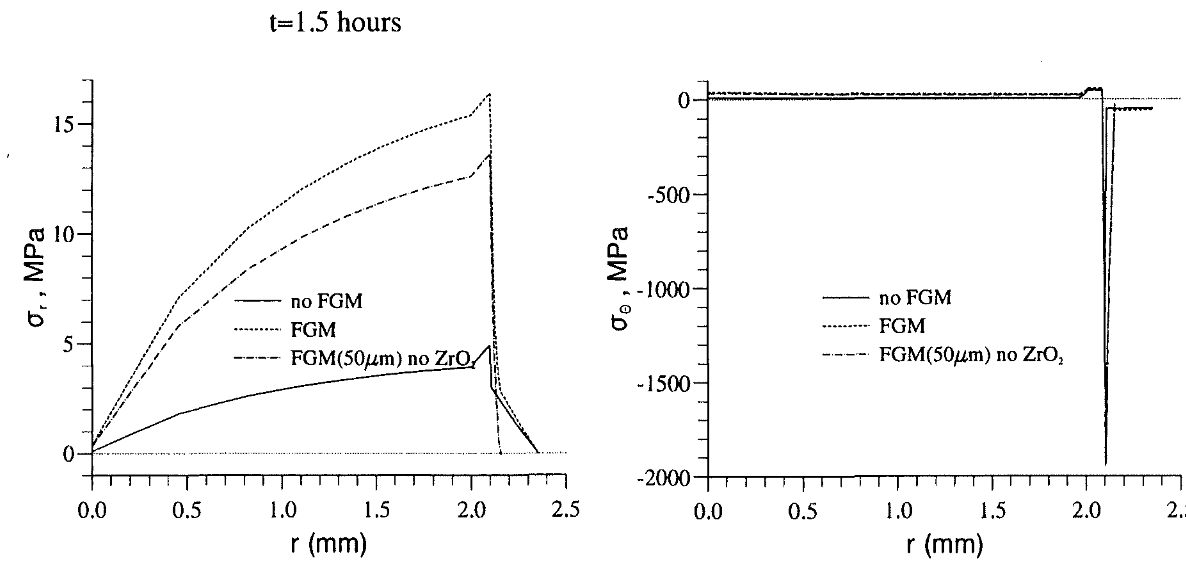
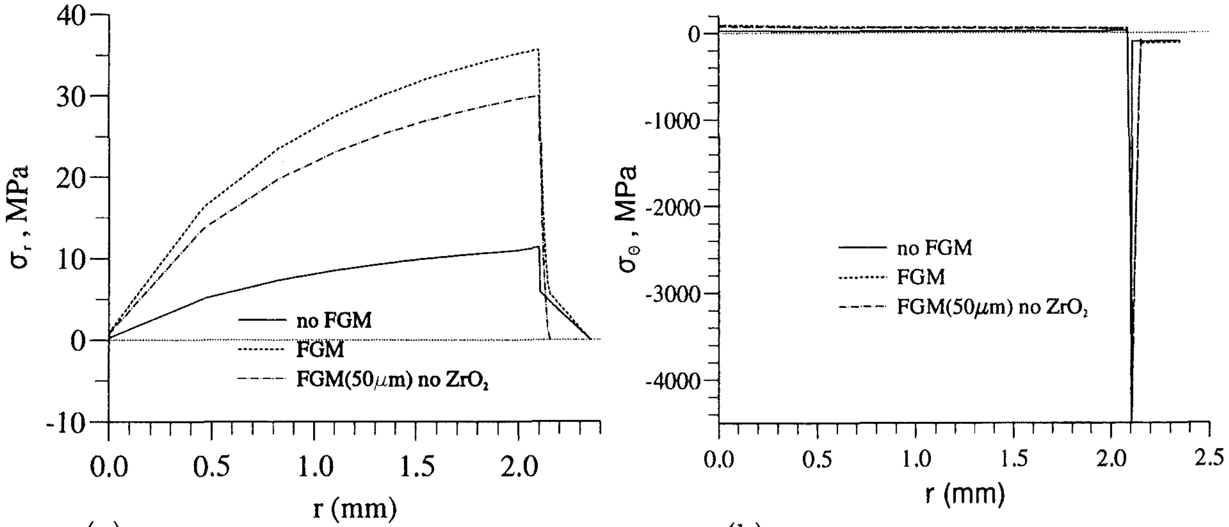


Fig.25 Comparison of stresses in a joint with and without FGM (t=0.1 hours).



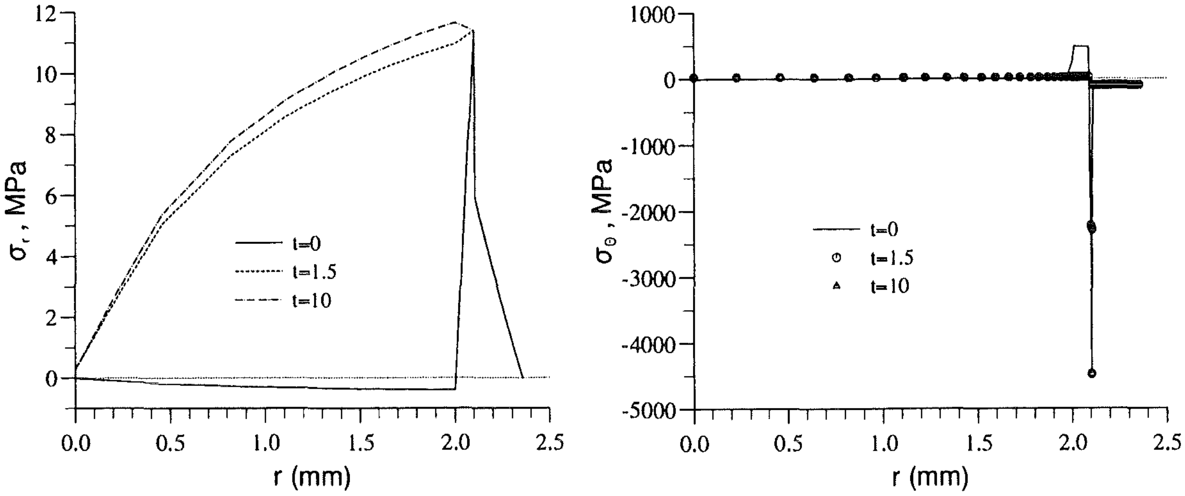
(a) (b)

Fig.26 Comparison of stresses in a joint with and without FGM (t=1.5 hours).



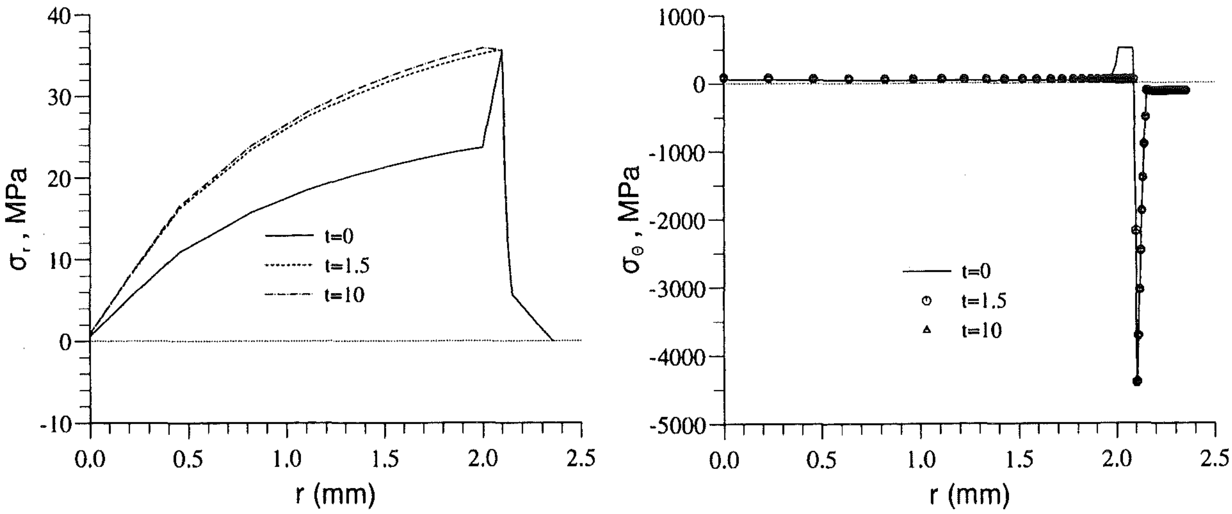
(a) (b)

Fig.27 Comparison of stresses in a joint with and without FGM (t=1.5 hours & cooling down).



(a) (b)

Fig.28 Time dependence of stresses in a joint without FGM.



(a) (b)

Fig.29 Time dependence of stresses in a joint with FGM.

For a joint <u>without</u> FGM, stresses σ_r at the interfaces substrate / NiCoCrAlY layer (denoted as A), NiCoCrAlY layer / Al_2O_3 layer (denoted as B) and Al_2O_3 layer / ZrO_2 layer (denoted as C) are plotted in Fig.30. In Fig.31 stresses σ_θ at the inside surface (substrate side) (denoted as D) and the outside surface (ZrO_2 side) (denoted as E) are given. In Figs.30 - 37 the time is in hours.

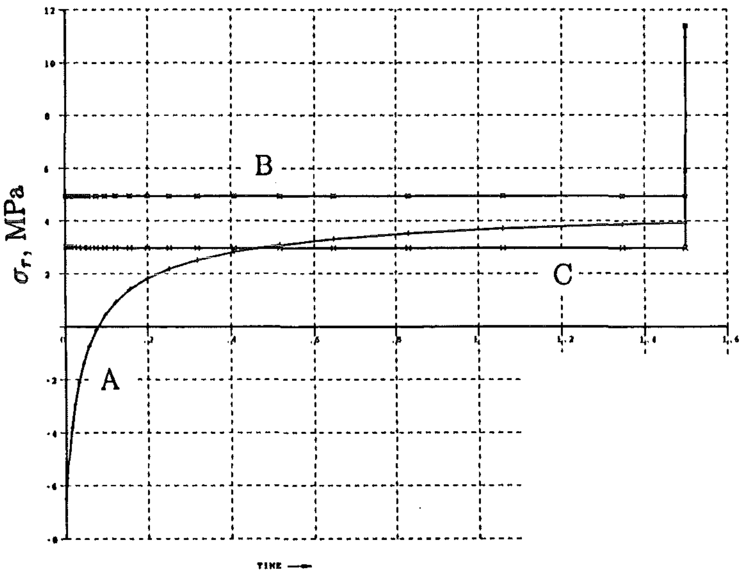


Fig.30 Stresses σ_r at interfaces A, B and C for cylinder <u>without</u> FGM.

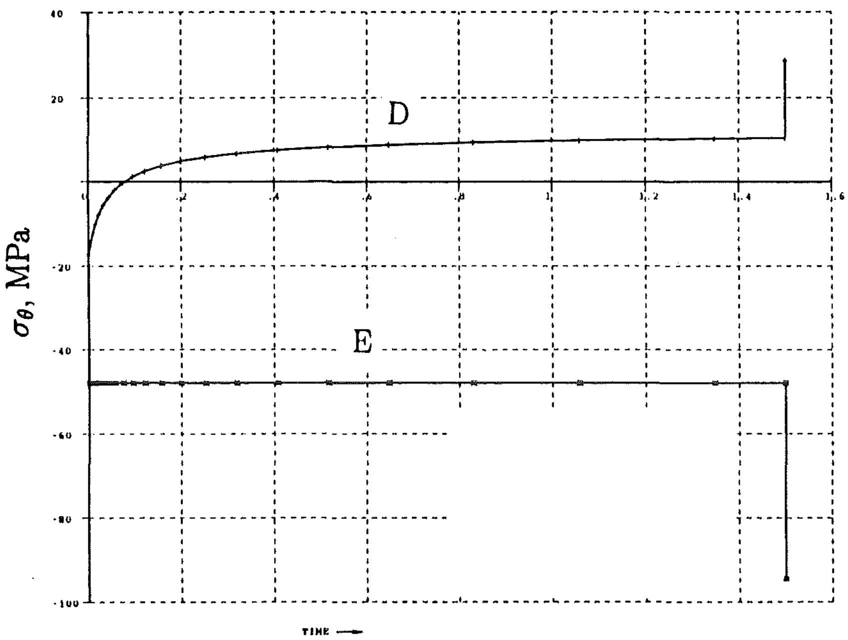


Fig.31 Stresses σ_θ at surfaces D and E for cylinder <u>without</u> FGM.

For a joint with FGM, stresses σ_r at the interfaces substrate / NiCoCrAlY layer (denoted as A), NiCoCrAlY layer / Al_2O_3 layer (denoted as B) and Al_2O_3 layer / FGM (denoted as C) are plotted in Fig.32. In Fig.33 stresses σ_θ at the inside surface (substrate side) (denoted as D) and the outside surface (ZrO_2 side) (denoted as E) are shown.

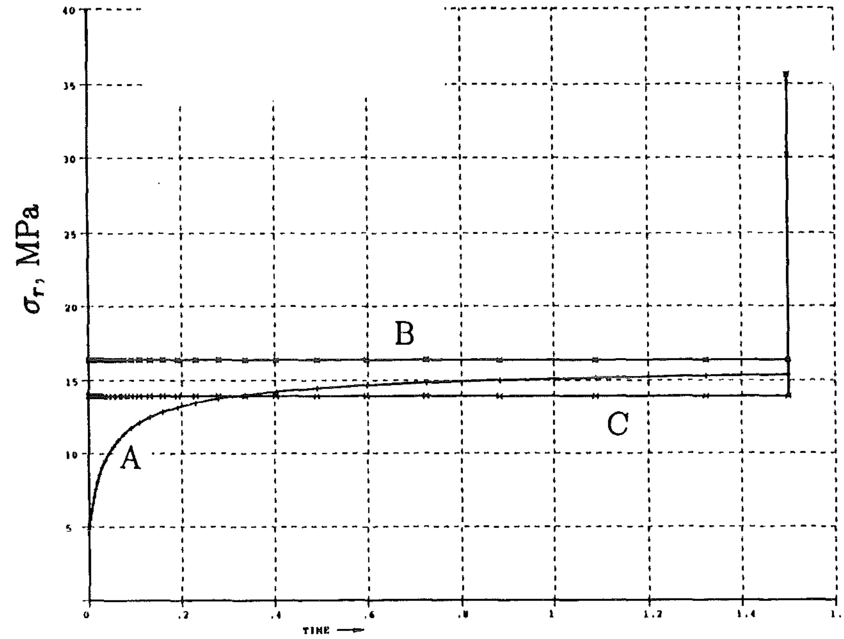


Fig.32 Stresses σ_r at interfaces A, B and C for cylinder with FGM.

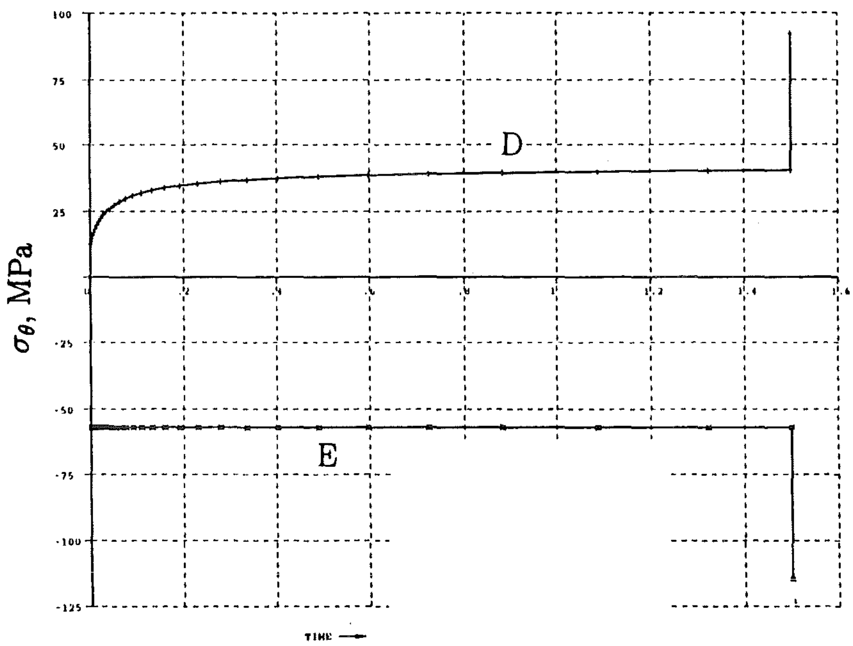


Fig.33 Stresses σ_θ at surfaces D and E for cylinder with FGM.

For a joint with FGM but without the ZrO_2 layer (surface has the material data as ZrO_2), stresses σ_r at the interfaces A, B and C are plotted in Fig.34. In Fig.35 stresses σ_θ at the inside surface (substrate side) (denoted as D) and the outside surface (ZrO_2 side) (denoted as E) are presented.

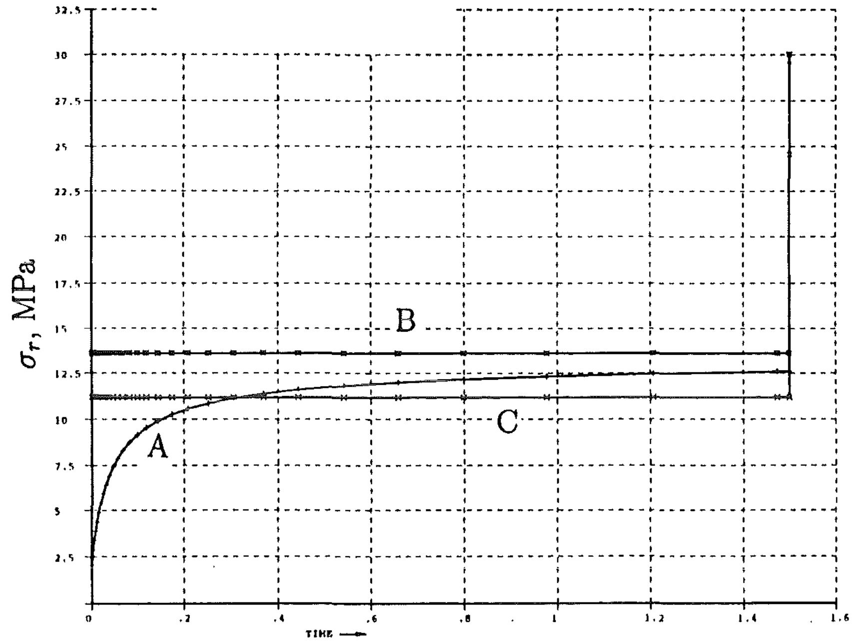


Fig.34 Stresses σ_r at A, B and C for cylinder with FGM (no ZrO_2 layer).

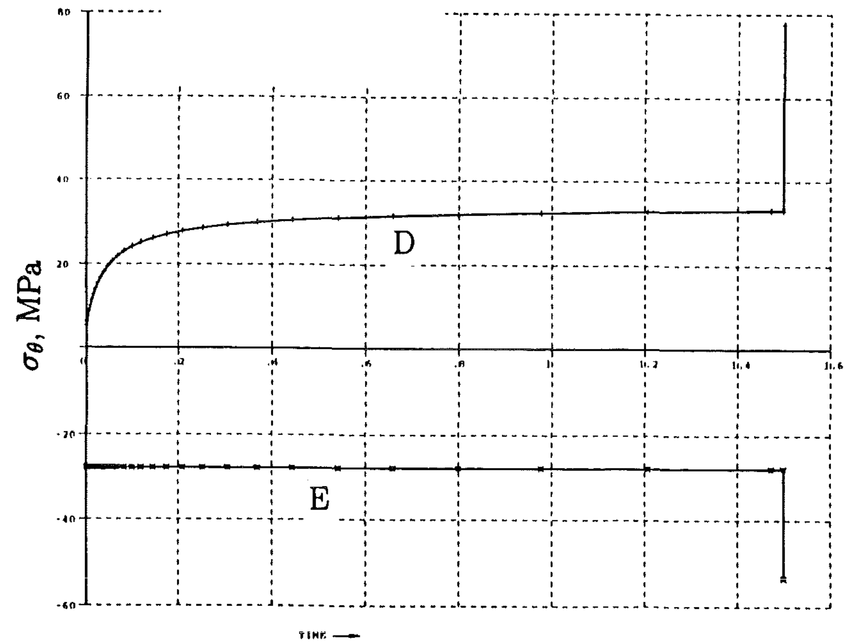


Fig.35 Stresses σ_θ at D and E for cylinder with FGM (no ZrO_2 layer).

For a multi - layers plate the stress situation is, however, different. For a plate in the range far away from the edge, only stress σ_x at the surface is dangerous. In Fig.36 stresses σ_x at the surface substrate (denoted as D) and the surface ZrO_2 (denoted as E) are presented for a plate <u>without</u> FGM. Stresses σ_x at the surfaces D and E are shown in Fig.37 for a plate <u>with</u> FGM.

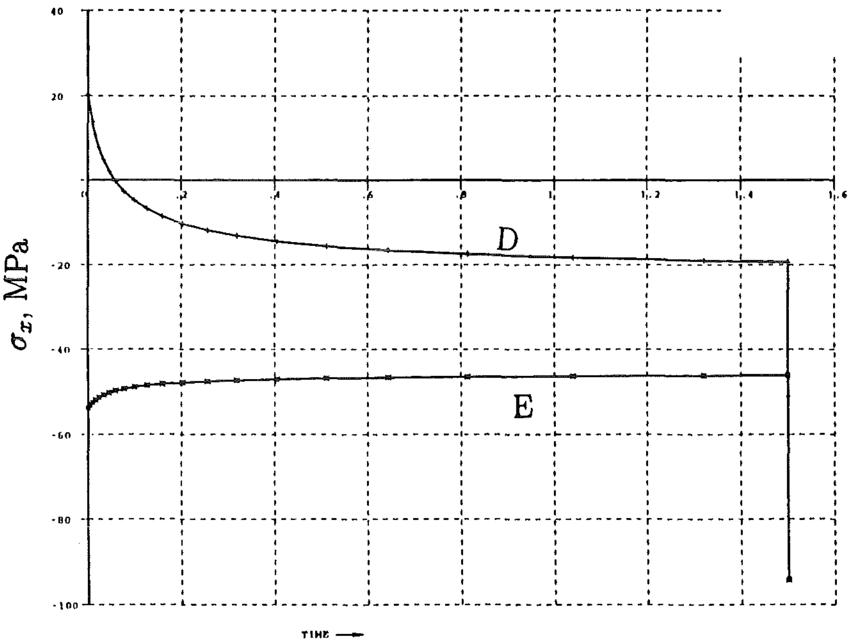


Fig.36 Stresses σ_x at the surfaces D and E for a plate <u>without</u> FGM.

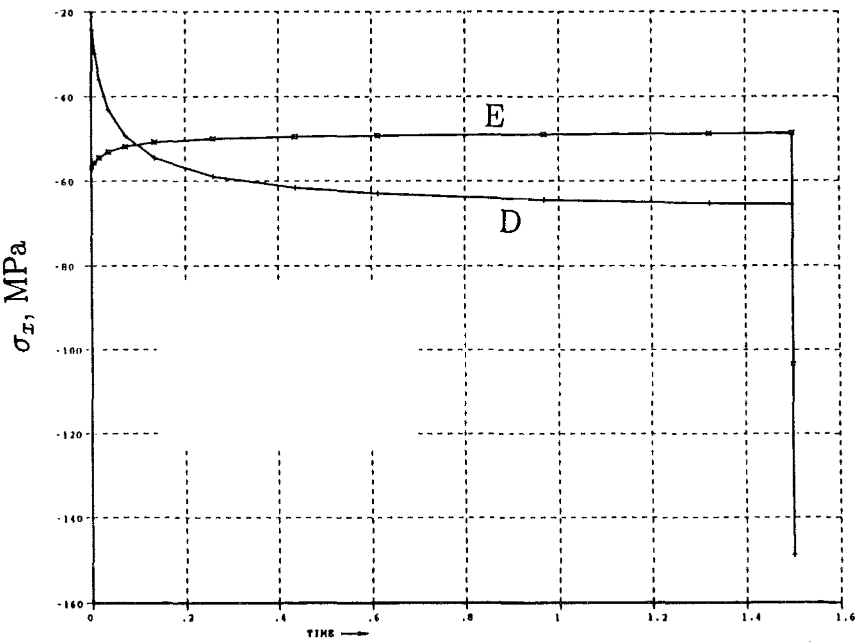


Fig.37 Stresses σ_x at the surfaces D and E for a plate <u>with</u> FGM.

From Figs. 30, 32 and 34 it can be seen that stress σ_r at interfaces B and C is almost constant during the creep process, however, at interface A it varys strongly for the short time and it may change the sign (in the joint without FGM). The joint with FGM and ZrO_2 layer is the worst one and the joint without FGM is the best one from the sense of delamination.

It is shown from Figs. 31, 33 and 35 that stress σ_θ at the surface ZrO_2 is almost constant during the creep process, however, at the surface substrate it varys strongly for the short time and it may change the sign (in the joint without FGM). The joint with FGM and a ZrO_2 layer has the largest tensile stress at the surface substrate and the joint with FGM but without ZrO_2 has the lowest compressive stress at the surface ZrO_2.

For a multi - layers plate the situation is another. At the surface ZrO_2 stress σ_x is always compression, but at the surface substrate σ_x is tensile for the joint without FGM and for a short time.

Comparing plate and cylinder it can be seen that at the surface ZrO_2 the stress component parallel to the surface (σ_x and σ_θ) is always compression, however, at the surface substrate the stress component parallel to the surface is almost tensile for cylinder and compression for plate.

For the given joint geometry and materials combination, the introduction of a FGM layer is not beneficial and after creeping the stress situation is even worse, especially, for the cylinder geometry.

Chapter 6

Conclusions

Stress distribution in a multi - layers joint, a plate or a cylinder, has been analyzed taking material creep behavior of one interlayer into consideration.

Explicit equations to calculate the stresses far away from the ends of a multi - layers cylinder under thermal and mechanical loading have been given for the cases of $\epsilon_z = 0$, $\epsilon_z =$ constant and ϵ_z being arbitrary, and for the cases of $\dot{\epsilon}_z = 0$ and $\dot{\epsilon}_z$ being a constant. FEM results have shown that for cylinder with $L/H \geq 3$ and stress free at the ends, the analytical solution can be used well to describe the stress distribution in the center of a joint. The equations for the case of $\epsilon_z = 0$ cannot be used to calculate the stresses in the center of a cylinder, irrespective of the ratio of L/H.

Equations to calculate the stresses far away from the free edge of a multi - layers plate under thermal loading have been presented explicitly for the cases of $\epsilon_z = 0$ and $\dot{\epsilon}_z = 0$.

In elastic materials (1 and 3) with increasing t stresses tend to a constant, which may be larger or smaller than the value at t=0. In creep material, with increasing t the stresses parallel to the interface decrease to zero, but the stress component perpendicular to the interface (e.g. σ_r) tends to a constant, which is not zero.

Stress distribution in a multi - layers plate and in a multi - layers cylinder has been compared taking into account the material creep behavior. It is shown that for a joint with the same thickness and same materials combination, in a plate the stress situation is more beneficial than in a cylinder if the three Young's moduli are similar (as in example 1); if E_3 is much smaller than E_1 and E_2 (as in example 2), σ_r is very small and a cylinder is more beneficial than a plate. In creep material, the rate of stress $\dot{\sigma}$ in the plate is larger than that of the cylinder.

An example has been presented for a multi - layers cylinder with a functionally graded material (FGM) interlayer. For this given joint geometry and materials combination, the introduction of a FGM layer is not beneficial and after creeping, the stress situation is even worse. It should be noted that the effect of introducing a FGM layer and the influence of the material creep behavior on the stresses are strongly dependent on the ratios of thicknesses and Young's moduli between the layers. This means that for another ratios of thicknesses and Young's moduli between the layers, introducing a FGM layer may be beneficial.

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