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NON-UNIQUENESS OF THE MOTIONS OF A FRICTION OSCILLATOR WITH SELF-EXCITATION

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Abstract—Self-exciting non-unique motions of a system with a finite number of degrees of freedom are investigated. By means of an example, analytically calculated limit cycles can be compared with numerically determined attractors. This procedure allows the discussion of the initial conditions, the initial state, the step size and the influence of switch time with regard to the valuation of numerical results. © 1997 Elsevier Science Ltd.

Keywords: non-linear vibrations, multibody-systems, dry friction, self-excitation, non-uniqueness

1. INTRODUCTION

A special class of contact problems with friction is considered. An elastic body is moving on a rigid plain contact surface such that the contact surface and the law of the applied forces are known at any time. The classical example is a rigid body which is pulled across a plane by means of an elastic spring with constant velocity. As known the stationary motion is not unique. Besides the trivial solution (sliding at any time) a self-excited stick–slip oscillation can exist where the two states sticking and sliding are succeeding one another alternating periodically. Continua can be mapped on systems with a finite number of degrees of freedom by means of the finite element method. For that purpose the influence of the friction in all contact nodes has to be taken into account [1]. In the following the algorithm for forming the equations of motion is derived for the simplest continuum, that is the rod, by a discrete representation with n elements. In the case $n = 2$ non-unique solutions can be calculated analytically. A comparison with numerical results allows to discuss the problematic nature of the determination of self-excited limit cycles from transient phenomena by means of numerical integration.

2. MECHANICAL MODEL AND EQUATIONS OF MOTION

A heavy elastic rod (density ρ, cross section A, length L_s, modulus of elasticity E) on a rough (coefficient of sliding friction μ) horizontal and rigid plane is considered in the gravitational field (gravitational acceleration g). The motion of the rod's right end is constrained by a constant velocity v (Fig. 1).

If we discretize the rod by n finite elements with the element length $L = L_s/n$, then the absolute coordinates of the $n + 1$ nodes are $x_0, \ldots, x_n$. The system of equations reads

$$\frac{\rho AL}{6}\mathbf{M}\ddot{\mathbf{x}} + \frac{EA}{L}\mathbf{C}\mathbf{x} = \mathbf{F} \tag{1}$$

with the matrices

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & . & . & 0 \\ 1 & 4 & 1 & 0 & 0 & . & . & 0 \\ 0 & 1 & 4 & 1 & 0 & . & . & 0 \\ & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & 0 & . & 1 & 2 \end{pmatrix}, \tag{2}$$

Contributed by P. Hagedorn.

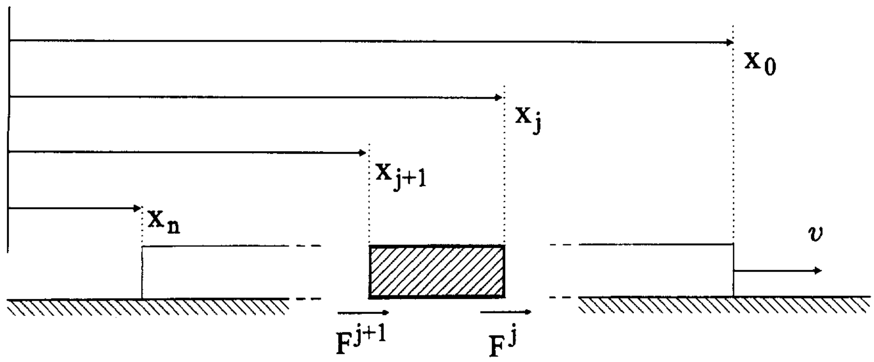

Fig. 1. Discrete representation of the rod.

$$\mathrm{C} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & . & . & 0 \\ -1 & 2 & -1 & 0 & 0 & . & . & 0 \\ 0 & -1 & 2 & -1 & 0 & . & . & 0 \\ & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & 0 & . & -1 & 1 \end{pmatrix} \tag{3}$$

and the vector of coordinates

$$\mathbf{x} = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}. \tag{4}$$

The vector **F** of nodal forces can be split up into the sum

$$\mathbf{F} = \mathbf{A} + \mathbf{P}. \tag{5}$$

The vector **A** contains the applied (active) nodal forces, which are determined considering Coulomb's friction law, whereas the vector **P** contains the constraints (passive forces). The friction law of Coulomb is only valid in the case of sliding. For the arbitrary node j with a velocity $\dot{x}_j \neq 0$ we get the nodal force

$$A^j = \begin{cases} -\mu\rho g \dfrac{A}{2} L \operatorname{sgn}(\dot{x}_j), & j = 0 \vee j = n \\ -\mu\rho g A L \operatorname{sgn}(\dot{x}_j), & j = 1, \ldots, n-1. \end{cases} \tag{6}$$

$A^j = 0$ for the sticking node j and the contact force is a time-varying passive force P^j which can be calculated by the actual state of the system. The element at the right end with the coordinate x_0 can be treated separately. Here the kinematic condition

$$\dot{x}_0 = v > 0. \tag{7}$$

exists at any time. This yields

$$A^0 = -\mu\rho g \frac{A}{2} L \operatorname{sgn}(v),$$
$$P^0 = P^0(t). \tag{8}$$

For the calculation of the motion the passive force P^0 is of no interest. Therefore the first row in equation (1) can be canceled and

$$x_0 = vt,$$
$$\ddot{x}_0 \equiv 0 \tag{9}$$

is known in all remaining equations. Now we consider an arbitrary state of the system at time t, when l nodes are sliding and $n-l$ nodes are sticking. Then the system (1) is split into l coupled equations of motion and $n-l$ algebraic equations. Consequently the total number of all equations is constantly n. For their description the quantity

$$\delta_j = \begin{cases} 0, & \text{if } j\text{th node is sticking,} \\ 1, & \text{if } j\text{th node is sliding,} \end{cases} \tag{10}$$

is introduced for $j = 1, \dots, n$, where

$$l = \sum_{j=1}^{n} \delta_j \tag{11}$$

The equations of motion now read

$$\frac{\rho AL}{6} M^{(l)}\ddot{\mathbf{x}}^{(l)} + \frac{EA}{L} \mathrm{C}^{(l)}\mathbf{x}^{(l)} + \frac{EA}{L} \mathrm{d}^{(l)} = \mathbf{A}^{(l)}. \tag{12}$$

The matrix

$$M^{(l)} = [m_{ik}^{j}]_{i,k=1,l}^{j=1,n} \tag{13}$$

is the actual mass matrix at time t. Here and in the following the index j marks the considered node. The number of the equation in system (12) which describes the motion of node j is named i and i is determined by

$$i = \sum_{p=1}^{j} \delta_p. \tag{14}$$

In particular the elements of the actual mass matrix are

$$m_{ik}^{j} = \begin{cases} 4, & k = i \\ \delta_{j+1}, & k = i+1 \quad i \neq 1, j \neq n, i \neq l \\ \delta_{j-1}, & k = i-1 \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

Specifically for the first and lth row we have

$$m_{11}^{j} = 4, m_{12}^{j} = \delta_{j+1}, m_{l,l-1}^{n} = \delta_{n-1}, m_{ll}^{n} = 2, m_{l,l-1}^{j} = \delta_{j-1}, m_{ll}^{j} = 4. \tag{16}$$

Correspondingly the actual stiffness matrix

$$C^{(l)} = [c_{ik}^{j}]_{i,k=1,l}^{j=1,n} \tag{17}$$

consists of the elements

$$c_{ik}^{j} = \begin{cases} 2, & k = i \\ -\delta_{j+1}, & k = i+1 \quad i \neq 1, i \neq l, j \neq n \\ -\delta_{j-1}, & k = i-1 \\ 0, & \text{otherwise} \end{cases} \tag{18}$$

Specifically for the first and lth row we have

$$c_{11}^{j} = 2, c_{12}^{j} = -\delta_{j+1}, c_{l,l-1}^{n} = -\delta_{n-1}, c_{ll}^{n} = 1, c_{l,l-1}^{j} = -\delta_{j-1}, c_{ll}^{j} = 2. \tag{19}$$

All other elements of the first and lth row of the matrices $\mathrm{M}^{(l)}$ and $\mathrm{C}^{(l)}$ are zero. The equations (12) describe a defined state of the mechanical system at time t. Its starting time t_0 results from the history of motion and is assumed to be known. All $n-l$ nodes which are sticking in this state have a vanishing velocity and acceleration, but a displacement which is also known from the history of motion. Thus we have

$$\begin{aligned} \mathbf{x}^{(n-l)} &= \mathbf{x}^{(n-l)}(t_0 - 0), \\ \dot{\mathbf{x}}^{(n-l)} &\equiv 0, \\ \ddot{\mathbf{x}}^{(n-l)} &\equiv 0. \end{aligned} \tag{20}$$

This leads to the matrix of the inner active forces

$$\mathrm{d}^{(l)} = [d_i^j]_{i=1,l}^{j=1,n} \tag{21}$$

with the elements

$$d_i^j = -(1-\delta_{j-1})x_{j-1}(t_0) - (1-\delta_{j+1})x_{j+1}(t_0), j \neq 1, j \neq n,$$

$$d_l^n = -(1-\delta_{n-1})x_{n-1}(t_0), d_1^1 = -(1-\delta_2)x_2(t_0) - vt. \tag{22}$$

At all l sliding nodes Coulomb's friction is pressent. Thus the vector of the external active forces is

$$\mathbf{A}^{(l)} = [A_i^j]_{i=1,l}^{j=1,n} \tag{23}$$

with the elements

$$A_i^j = -\mu\rho A\boldsymbol{g}L\,\mathrm{sgn}(\dot{x}_j), j \neq n,$$

$$A_l^n = -\mu\rho A\boldsymbol{g}\frac{L}{2}\mathrm{sgn}(\dot{x}_n). \tag{24}$$

The time-varying passive forces

$$P_i^j = P_i^j(t),$$

$$i = 1, n-l; j = 1, n; i = \sum_{p=1}^{j}(1-\delta_p). \tag{25}$$

are acting on the $n-l$ sticking nodes. Knowing an actual state of the system these forces are calculated from

$$P_i^j(t) = -\frac{\rho AL}{6}(\delta_{j-1}\ddot{x}_{j-1}(t) + \delta_{j+1}\ddot{x}_{j+1}(t)) + \frac{EA}{L}(\delta_{j-1}x_{j-1}(t) + \delta_{j+1}x_{j+1}(t)$$

$$-(\delta_{j-1}-1)x_{j-1}(t_0) - 2x_j(t_0) - (\delta_{j+1}-1)x_{j+1}(t_0)), j \neq 1, j \neq n,$$

$$P_{n-l}^n(t) = -\frac{\rho AL}{6}\delta_{n-1}\ddot{x}_{n-1}(t) + \frac{EA}{L}(\delta_{n-1}x_{n-1}(t) - (\delta_{n-1}-1)x_{n-1}(t_0) - x_n(t_0)), \tag{26}$$

$$P_1^1(t) = -\frac{\rho AL}{6}\delta_2\ddot{x}_2 + \frac{EA}{L}(\delta_2 x_2(t) + \vartheta t - (\delta_2-1)x_2(t_0) - 2x_1(t_0)).$$

The equations (12), (20) and (26) describe a state of the system with l sliding nodes and $n-l$ sticking nodes. The value of l ranges from 0 to n. The number l is determined by the history of motion. We asume, that at time $t = t_0$ the number l and all state variables of the system are known. Then the actual state at any time $t > t_0$ is completely calculable. This state is valid only during a certain time interval. The next switch time t_1 for a transition to the following new state is indicated by switch conditions. Generally they determine the transition from sticking to sliding and vice versa, and also specify the node where the transition takes place. In the case $\delta_j = 1$ (sliding) for node j, a changing sign of the velocity $\dot{x}_j$ during the open time interval $[t_0, t[$ with $t > t_0$ defines the switch time t_j. In the case $\delta_j = 0$ (sticking) for node j, the amount of the sticking force according to equation (25) has to be less or equal than a given maximum value $P_{\max}$ during the open time interval $[t_0, t[$ with $t > t_0$. With the condition

$$|P_i^j(t)| = P_{\max} \geqslant |A_i^j| \tag{27}$$

we get the switch time t_j. In the following the difference between this maximum value and the active Coulomb's force is described by the dimensionless parameter $\varepsilon > 0$, according to

$$|P_{\max}| = (1+\varepsilon)|A_i^j| \tag{28}$$

In addition to the integration of equation (12) during an actual state, n equality conditions (control equations) of l velocities and $n-l$ passive forces have to be checked for eventual switch events at time t_j. Therefore the switch time t_1 in the open time interval $[t_0, t[$ is

$$t_1 = \min_{j=1,n} \{t_j\} \tag{29}$$

The entire motion is constructed by joining the solutions for all successive states, which begin at time t_0 and end at time t_1. All state variables are known at the end of each state. Displacements and velocities are continuous. The values at the end of the last state yield the initial values of the new state. The time t_1 at the end of the last state is the initial time t_0 of the new state.

3. STATES OF MOTION FOR THE CASE $n = 2$

The integration of the l coupled linear ordinary differential equations existing during the time interval $t_0 < t < t_1$ causes no difficulties. Therefore we focus our main attention to the series of switch times t_0 during the entire course of motion. Here we are confronted with the question, how sensitively the solution responds to the inevitably inexact calculation of the switch times. On this occasion it is appropriate to take a small number n of finite elements, so that we are able to compare exact analytical solutions with numerical results. This situation is given in the case $n = 2$. The advantage of this selection is the easily clear procedure for the construction of solutions with no restriction of the universal validity for $n > 2$.

The mechanical system can acquire four different states for two finite elements. This results from the velocity $v = \text{const}$ at the node 0. Correspondingly the states of sliding at the nodes 1 and 2 simplify to $\operatorname{sgn}(\dot{x}_j) = +1$. In the following G marks the sliding case and H the sticking case, while the added index corresponds to the node. So the combinations G_1G_2, H_1G_2, G_1H_2 and H_1H_2 are possible. Appropriately we are changing to a dimensionless description. With the introduction of the only parameter

$$\alpha = \frac{\varepsilon}{v}\sqrt{\frac{\rho}{E}}\,\mu g L, \quad 0 < \alpha < \infty \tag{30}$$

all mechanical properties of the system are captured. The non-dimensional time is

$$\tau = \sqrt{\frac{E}{\rho}}\frac{t}{L}, \tag{31}$$

where in the following a dash marks the differentiation of a quantity with respect to non-dimensional time τ. The motion itself will be described by the relative coordinates

$$\xi_1 = \sqrt{\frac{E}{\rho}}\frac{x_1}{Lv} - \tau + \frac{3}{2}\sqrt{\frac{\rho}{E}}\frac{\mu g L}{v},$$
$$\xi_2 = \sqrt{\frac{E}{\rho}}\frac{x_2}{Lv} - \tau + 2\sqrt{\frac{\rho}{E}}\frac{\mu g L}{v}. \tag{32}$$

Thus the trivial solution of the problem has the form

$$\xi_1(\tau) \equiv 0,$$
$$\xi_2(\tau) \equiv 0 \tag{33}$$

for all values of the parameter α. All nontrivial solutions can clearly be compared with this solution.

For the calculation of non-trivial solutions four states have to be considered.

State $G_1G_2: \tau \in [\tau_0, \min\{\tau_{11}, \tau_{12}\}], l = 2$

$$\begin{gathered}
4\xi_1'' + \xi_2'' + 12\xi_1 - 6\xi_2 = 0, \\
\xi_1'' + 2\xi_2'' - 6\xi_1 + 6\xi_2 = 0, \\
\xi_1(\tau_0) = \xi_1(\tau_0 - 0), \xi_1'(\tau_0) = \xi_1'(\tau_0 - 0), \\
\xi_2(\tau_0) = \xi_2(\tau_0 - 0), \xi_2'(\tau_0) = \xi_2'(\tau_0 - 0), \\
\xi_1' + 1 > 0, \\
\xi_2' + 1 > 0.
\end{gathered} \tag{34}$$

State $G_1H_2: \tau \in [\tau_0, \min\{\tau_{11}, \tau_{12}\}], l = 1$

$$\begin{gathered}
\xi_2(\tau) = \xi_2(\tau_0 - 0) - (\tau - \tau_0), \xi_2'(\tau) = -1, \\
\xi_1'' + 3\xi_1 - \tfrac{3}{2}\xi_2 = 0, \\
\xi_1(\tau_0) = \xi_1(\tau_0 - 0), \xi_1'(\tau_0) = \xi_1'(\tau_0 - 0), \\
\xi_1' + 1 > 0, \\
-\frac{1}{6}\xi_1'' + \xi_1 - \xi_2 - \frac{\alpha}{2} < 0.
\end{gathered} \tag{35}$$

State $H_1G_2: \tau \in [\tau_0, \min\{\tau_{11} \Lambda \tau_{12}\}], l = 1$

$$\begin{gathered}
\xi_1(\tau) = \xi_1(\tau_0 - 0) - (\tau - \tau_0), \xi_1'(\tau) = -1, \\
\xi_2'' + 3\xi_2 - 3\xi_1 = 0, \\
\xi_2(\tau_0) = \xi_2(\tau_0 - 0), \xi_2'(\tau_0) = \xi_2'(\tau_0 - 0), \\
\xi_2' + 1 > 0, \\
-\tfrac{1}{6}\xi_2'' + \xi_2 - 2\xi_1 - \alpha < 0.
\end{gathered} \tag{36}$$

State $H_1H_2: \tau \in [\tau_0, \min\{\tau_{11}, \tau_{12}\}], l = 0$

$$\begin{gathered}
\xi_1(\tau) = \xi_1(\tau_0 - 0) - (\tau - \tau_0), \xi_1'(\tau) = -1, \\
\xi_2(\tau) = \xi_2(\tau_0 - 0) - (\tau - \tau_0), \xi_2'(\tau) = -1, \\
\xi_2 - 2\xi_1 - \alpha < 0, \\
\xi_1 - \xi_2 - \frac{\alpha}{2} < 0.
\end{gathered} \tag{37}$$

All initial conditions are known from the past history at the time $\tau = \tau_0 - 0$. The validity of each state is checked by two inequalities each for velocities or sticking forces, respectively. A transition to a new state takes place, when in one of these inequalities the sign of equality appears. In this connexion the switch time is the smaller of the two values τ_{11} and τ_{12}.

4. ANALYTICAL CALCULATION OF LIMIT CYCLES

For all four states we can present explicit solutions of the equations of motion. Because of the extensive formulas the solutions are not written down. The whole problem can be completely reduced to an algebraic form. For this purpose the explicit solutions of the equations of motion are inserted into the corresponding control equations where the inequality signs are replaced by equality signs. This leads to transcendental algebraic equations which contain the initial conditions and switch times τ_0 as the only unknowns. Periodical solutions, that are limit cycles, are of particular interest. Unfortunately the four states given above take place in a sequence, which is unknown. Setting up the conditions of periodicity demands to choose a certain sequence.

We have investigated several sequences—without demand of completeness—and found the following periodical solutions for different parameters α:

$$\begin{gathered}
\alpha = 10.: G_1G_2, H_1G_2, H_1H_2, G_1H_2, \\
\alpha = 3.3: G_1G_2, H_1G_2, H_1H_2, G_1H_2, G_1G_2, G_1H_2, \\
\alpha = 2.5: G_1G_2, G_1H_2, G_1G_2, H_1G_2, H_1H_2, G_1H_2, G_1G_2, H_1G_2, \\
\alpha = 2.0: G_1G_2, H_1G_2, G_1G_2, G_1H_2, \\
\alpha = 1.5: G_1H_2, G_1G_2, H_1G_2, G_1G_2, G_1H_2, G_1G_2, \\
\alpha = 1.0: G_1H_2, G_1G_2, \\
\alpha = 0.5: G_1H_2, G_1G_2, G_1H_2, G_1G_2.
\end{gathered} \tag{38}$$

In contrast to the classical stick–slip motion in the case $n = 1$, the type of the solution depends on the value of the parameter α.

The order $G_1H_2, \tau \in [\tau_0, \tau_2]$; $G_1G_2, \tau \in [\tau_2, \tau_3]$; $G_1H_2, \tau \in [\tau_3, \tau_4]$; $G_1G_2, \tau \in [\tau_4, \tau_5]$ is exemplarily considered for the general procedure. The initial time is arbitrarily set to $\tau_0 = 0$. Then the time τ_5 is the standardized period. Because of computational reasons we use the transformations

$$y_1 = \frac{1}{2}\left(\xi_1 + \frac{1}{\sqrt{2}}\xi_2\right),$$

$$y_2 = \frac{1}{2}\left(\xi_1 - \frac{1}{\sqrt{2}}\xi_2\right). \tag{39}$$

They correspond to main coordinates of the system valid in the state G_1G_2. Thus the conditions of periodicity can be expressed with the algebraic system of equations

$$\begin{gathered}
y_1(\tau_0) = y_1(\tau_5),\\
y_2(\tau_0) = y_2(\tau_5),\\
y_1'(\tau_0) = y_1'(\tau_5),\\
\left(\frac{3}{2} - \frac{5\sqrt{2}}{4}\right)y_1(\tau_2) + \left(\frac{3}{2} + \frac{5\sqrt{2}}{4}\right)y_2(\tau_2) = \frac{\alpha}{2},\\
\left(\frac{3}{2} - \frac{5\sqrt{2}}{4}\right)y_1(\tau_4) + \left(\frac{3}{2} + \frac{5\sqrt{2}}{4}\right)y_2(\tau_4) = \frac{\alpha}{2},\\
y_1'(\tau_3) = y_2'(\tau_3) - \frac{1}{\sqrt{2}},\\
y_1'(\tau_5) = y_2'(\tau_5) - \frac{1}{\sqrt{2}}.
\end{gathered} \tag{40}$$

The initial condition $y_2'(\tau_0)$ is known with the condition of sticking in equation (35) when using equation (39). So the unknown quantities are three initial conditions $y_1(\tau_0)$, $y_2(\tau_0)$, $y_1'(\tau_0)$ and four switch times $\tau_2, \tau_3, \tau_4, \tau_5$. One should notice, that the formulas for the functions y_1, y_1', y_2 and y_2', which are containing the initial conditions, are known. Here these functions will not be explicitly given. We are searching for the roots of the algebraic system of equations (40), because they allow the explicit formulation of the solution in the form of piecewise defined functions.

Taking the value $\alpha = 0.5$ we find three solutions of the system of equations (40). The simplest periodical process is the sequence G_1H_2, G_1G_2. The limit cycle has a frequency ω_1 which is nearly equal to the higher natural frequency of the unconstrained system without frictional contact. For the amplitudes in main coordinates $|\max y_1| \ll |\max y_2|$ holds. The solution is similar to the system with one degree of freedom ($n = 1$). The second limit cycle has the same sequence. But the corresponding frequency ω_2 is close to the lower natural frequency of the unconstrained system. The amplitudes are $|\max y_2| \ll |\max y_1|$. The third limit cycle describes a higher-periodical sequence G_1H_2, G_1G_2, G_1H_2, G_1G_2. For its frequency ω_3 we get the conditions $7\omega_3 \approx \omega_1$ and $2\omega_3 \approx \omega_2$. All results are shown in Fig. 2 in the form of trajectories in two phase planes. The assigned period τ_5 is designated as T. The letters a, b, c as indices mark the type of the solution.

5. NUMERICAL CALCULATION OF ATTRACTORS FROM TRANSIENT MOTIONS

The preceding considerations lead us to the conclusion that for more that two finite elements ($n > 2$) we are no longer able to realize the analytical procedure described above, because of the large number of possible states. Thus we have to depend on numerical integration. Here three principal problems arise:

I. The choice of the initial conditions and the state of the system at time $\tau = 0$.
II. The selection of the method and the time-step for a numerical integration of the equations of motion.
III. The required accuracy for the determination of the switch times.

I. Each periodical solution corresponds to a limit cycle in the phase space. If we choose for example the initial conditions resulting from equations (40) and the initial state $G_1 H_2$ for the numerical integration, then we will get a limit cycle immediately. A limit cycle can be a stable attractor or an unstable repeller. A transient motion only leads to an attractor. Every attractor has an attracting basin of unknown size. If a trajectory starts in the interior of this basin, then the trajectory will asymptotically move towards the attractor because of the dissipative property of friction. The choice of the initial conditions and the initial state turns out to be absolutely heuristic. In general, for unique solutions this choice will cause no problems but only influence the duration of the transient phenomenon. For non-unique solutions we are short of a safe procedure.

II. All numerical integration methods are based on the idea, that starting from a known solution vector at time τ, the next following solution vector at time $\tau + \Delta\tau$ is calculated. For our problem with discontinuous second derivatives (a system with intermittent constraints) the choice of the step size $\Delta\tau$ has to be considered with great attention [2]. At time $\tau + \Delta\tau$ we have to calculate the sticking forces and velocities at the sticking or sliding nodes, respectively. If a velocity changes the sign or a sticking force exceeds its maximum value, then at least one switch time will occur within the interval $\Delta\tau$. This condition is not at all sufficient for the capture of all switch times. If the step size is too large, we may possibly not register a switch time at all or more than one switch time may lie within the interval. Thus a maximum step size is assigned to each solution. Then only one switch time will exist within this maximum $\max \Delta\tau$. This unknown maximum value depends on the unknown type of the solution. Altogether the time step and the specifically selected integration algorithm determine the numerical stability. In the following we will only investigate the influence of the time step $\Delta\tau$. Integration faults depending on the used algorithm are avoidable, because explicit solutions of the equations of motion (34), (35), (36) and (37) are known for all occurring states. So the numerical integration is simulated by using these analytical solution for successive states during the discrete time steps $\Delta\tau$. But the required initial conditions at the beginning of each state and the switch times themselves result from numerical calculations.

III. From the practical point of view, the time step $\Delta\tau$ should be chosen as large as possible, so that even time consuming transient motions can be calculated with acceptable computational expenditure. Proceeding from the maximum step size for the registration of all switch times, the remaining problem is the occurring inaccuracy of the determination of each switch time within the interval $\Delta\tau$. During the transient phenomenon each of these faults is propagating. Different shapes of motion show different sensitivities against such systematic faults. In the following examples we will use the method of bisection. The interval $\Delta\tau$, in which a switch time is found, will be gradually bisected. The number of bisections will be called m. Exemplarily the parameter $\alpha = 0.5$ is taken, for which three analytically calculated attractors are shown in Fig. 2. Furthermore all switch times and also the maximum permissible step sizes for the numerical integration are known. These quantities are $\max \Delta\tau_a = \tau_2 - \tau_0 \approx 0.21458$, $\max \Delta\tau_b = \tau_2 - \tau_0 \approx 1.7134$ and $\max \Delta\tau_c = \tau_2 - \tau_0 \approx 0.23573$. Consequently numerical results with different accuracies can be compared with the exact analytical results.

A first example for the initial conditions is choosen as

$$\xi_1(0) = 7.5, \xi_2(0) = 10.0,$$

$$\xi_1'(0) = \xi_2'(0) = -1. \tag{41}$$

The motion starts with the initial state $H_1 H_2$, at which the system is undeformed and in rest. All following results are represented in Poincaré maps, because the transient

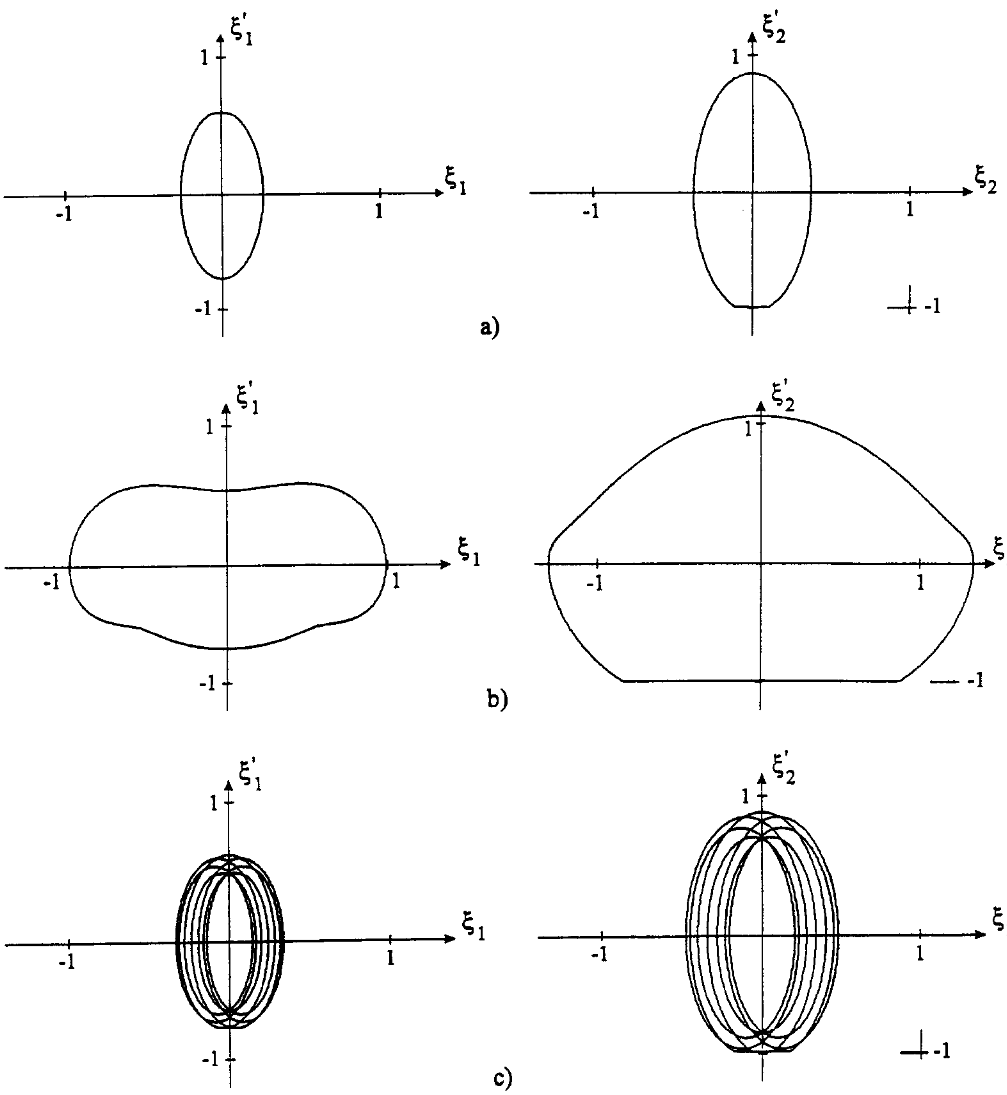

Fig. 2. Periodical solutions for $\alpha = 0.5$; (a) $T_a = 2.2365$; (b) $T_b = 7.9012$; (c) $T_c = 15.6374$.

phenomenon may last for a long time. The selected scanning time interval $\Delta T = 2.2323$ is the period of the first natural mode of the unconstrained system. The motion is described by 22,400 supporting points for the time interval $0 \leqslant \tau \leqslant 50{,}000$ of calculation. Periodical processes are indicated by a collapse of several supporting points or by the formation of structures, respectively. The result for the step size $\Delta\tau = 1.0 < \max \tau_b$ without accurate determination of the switch times ($m = 0$) is shown in Fig. 3.

After a short transient phenomenon we get an attractor with a long period and many switch times. The switch times for the velocities have been registered with great inaccuracy. (The switching of the contact force cannot be detected in the Poincaré maps.) An improvement can be expected by means of a refinement of the step size. The result for the step size $\Delta\tau = 0.1 < \max \tau_a < \max \tau_c$ and $m = 0$ is shown in Fig. 4.

The transient phenomenon lasts longer. The stationary state still has a long period and many switch times. But an improved determination of the switch times for the velocities is evident. On an average the result is very similar to the analytical solution of type a. This trend is confirmed by a further refinement of the step size to $\Delta\tau = 0.01$ and $m = 0$. In Fig. 5 we see the solution of type a after a very long transient phenomenon.

In reality the accuracy of the switch times is still not sufficient. This is evident, when the switch times are additionally determined more accurately with 10 bisections ($m = 10$). After a further expansion of the response time the solution of type c appears in Fig. 6.

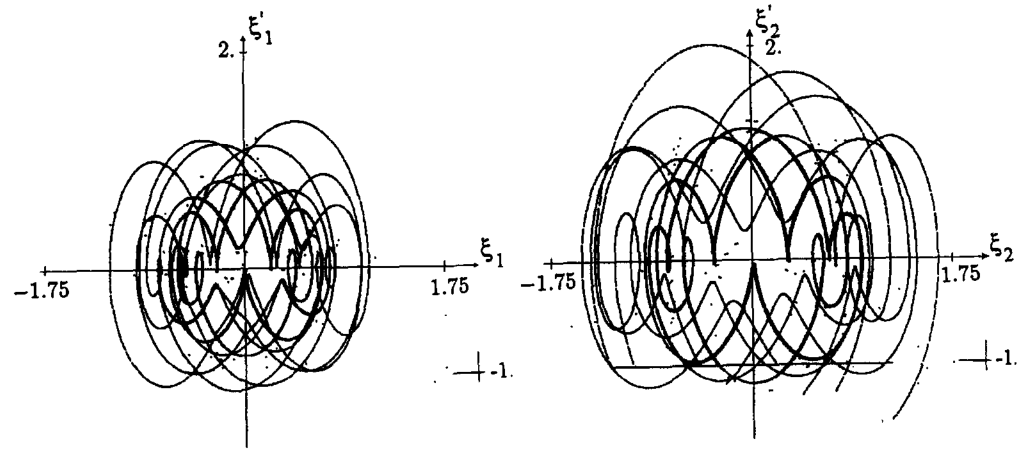

Fig. 3. Poincaré map for $\Delta\tau = 1.0$, $m = 0$.

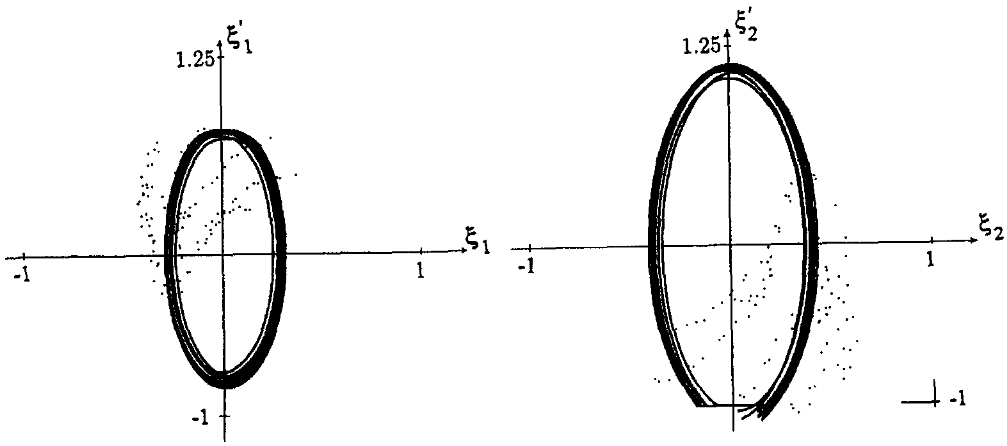

Fig. 4. Poincaré map for $\Delta\tau = 0.1$, $m = 0$.

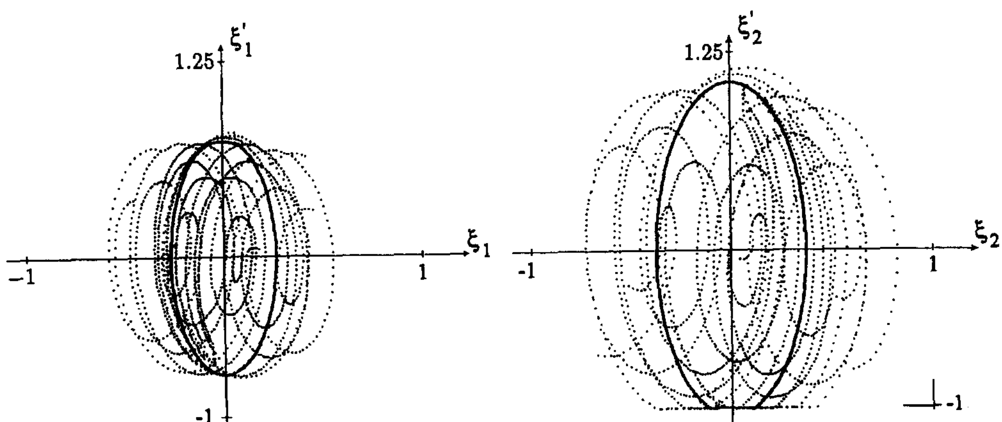

Fig. 5. Poincaré map for $\Delta\tau = 0.01$, $m = 0$.

In a second example we use the arbitrarily chosen initial conditions

$$\xi_1(0) = \xi_2(0) = 0,$$

$$\xi'_1(0) = 0.5, \xi_2(0) = -1. \tag{42}$$

The corresponding initial state is $G_1 H_2$. For the point map, the same assumptions are valid as before. We take the large value $\Delta\tau = 0.1$ as step size. The determination of switch times with $m = 10$ is very accurate, however. The result is shown in Fig. 7.

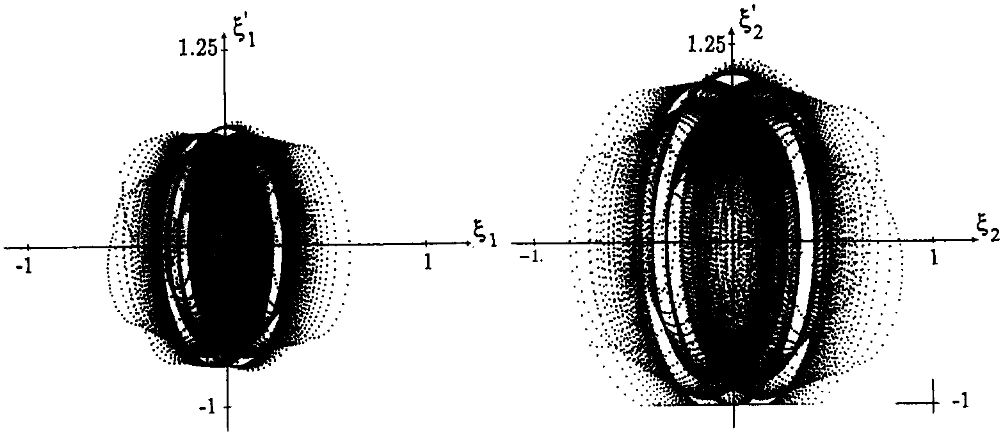

Fig. 6. Poincarè map for $\Delta\tau = 0.01$, $m = 10$.

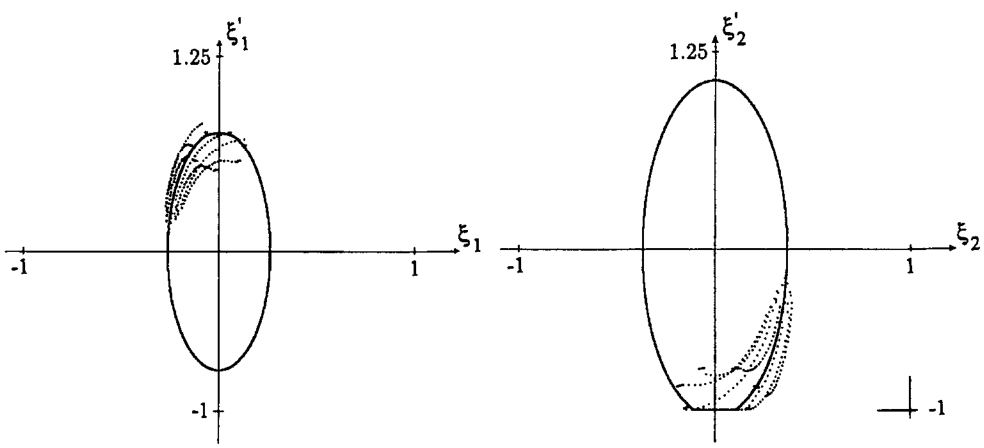

Fig. 7. Poincare map for $\Delta\tau = 0.1$, $m = 10$.

After a very short transient phenomenon we obtain the solution of type a. This solution does not change with increasing accuracy.

The described procedure requires single zeros of the control equations for velocities and sticking forces. Double zeros are not detected. This case always occurs at the transition of one solution type to another in list (38), when the parameter α changes. Even in the neighborhood of such a transition, two switch times are succeeding one another so closely, that an accurate calculation of the attractor is practically no longer possible. These circumstances have already been discussed in more detail for a similar problem [3].

6. CONCLUSIONS

The discretization of continua with frictional contact leads to mechanical systems with finite numbers of degrees of freedom and intermittent constraints. This results in a sequence of linear problems with different numbers of DOF. The reasons for the non-linearity are the transitions between successive states of sticking or sliding in all contact points. Depending on the value of the system parameter, non-unique periodic solutions can exist. Both an analytical and numerical analysis, are exemplarily carried out for a simple mechanical model.

The analytical procedure leads to the question of the ambiguity of zeros of algebraic systems of equations. These systems of equations proceed from the assumption that certain successive chronological states of sticking and sliding exist in all discrete contact points, i.e. the stipulation of certain periodicity conditions. Initial values are needed for the numerical treatment of the transcendent systems of equations. With ambiguous zeros, no significant method is known which yields all solutions. An increase of the maximum number of degrees of freedom beyond the number in the considered example would aggravate the nature of the problem: due to the larger number of possible stick–slip transitions, the number of periodicity conditions to be investigated would increase. Simultaneously, the dimension of the assigned transcendent systems of equations would expand. From the practical point of view, we would have to confine ourselves to the investigation of special motions.

For the qualitative investigation of the periodical solutions of the piecewise linear systems of differential equations, other methods like point mapping [4] and cell mapping [5] could also be used. In [6] the same example has been investigated by a point mapping method. Similar problems would be caused by an extrapolation to many degrees of freedom. As early as $n = 2$, the sequence function gets rather complicated and therefore hard to analyse. To construct this function, definite assumptions concerning the character of the investigated limit cycle would have to be made. The cell mapping method [5], on the other hand, requires initial information about the possible states of motion and assigned domains of attraction.

Using the numerical procedure, stationary states are obtained from the transient phenomena. It requires the knowledge of appropriate initial values. As already indicated the duration of a transient phenomenon depends on the initial values. Their choice is purely heuristic. Moreover, the numerical calculation is extremely sensitive to errors in the precision of the calculated switch points. Increase of the number of DOF and hence the corresponding occurrence of higher frequencies could result in a more rapid succession of the switch points. Thus, the demands on precision are increasing. Here it turns out that a strict coupling between the spatial and time discretization of contact problems with friction exists.

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