# Martingale Restrictions and Implied Distributions for German Stock Index Option Prices

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# Martingale Restrictions and Implied Distributions for German Stock Index Option Prices

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**Abstract**

The paper investigates the pricing of stock index options on the Deutscher Aktienindex (DAX) traded on the Deutsche Terminbörse (DTB) as well as the distributions of terminal index values implied by the market prices of these options. As one main result we find that the martingale restriction is violated, meaning that the index level implied by option prices is significantly greater than the observed index price. The pricing differences can partly be explained by variables like the number of options available for the estimation, their average relative moneyness and their average relative bid-ask spread. Put options are consistently underpriced by the Black and Scholes model whereas calls are overpriced when both the volatility and the index price are estimated from option price data. Variables like moneyness and squared moneyness can explain part of the variation in pricing errors. Finally, implied distributions are systematically different from the lognormal or binomial distributions of the Black and Scholes [2] or Cox, Ross, and Rubinstein [4] model. For puts we observe an extra premium for Arrow-Debreu securities paying off in states with a very low index value.

## 1 Introduction

The idea to infer parameters of the underlying stock price distribution from market prices of options was first introduced by Latané and Rendleman [8]. As an estimate for the volatility of the stock they used the value $\hat{\sigma}$ which set the market price of the option equal to its theoretical value given by the Black and Scholes [2] formula. Several empirical studies, e.g. by Chiras and Manaster [3], showed that this implied volatility is not constant across all the options for a given underlying stock. As a consequence weighting schemes were developed to aggregate the different estimates into one number which then represented the volatility estimate for the stock. For example, Chiras and Manaster [3] suggested to use as a weight the elasticity of the option price with respect to its volatility, whereas Beckers [1] found that the best estimate was the implied volatlity of the option which was closest at the money. Other possibilities are to give equal weights to the individual implied volatilities or to use as weights the partial derivatives of the option prices with respect to volatility.

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A different approach to this weighting problem was devoloped by Whaley [16] who used a non-linear regression to obtain a common volatility estimate for all the options on a given underlying.

While all these methods just infer one parameter (the volatility) from the market prices of options Manaster and Rendleman [11] used pairs of prices to estimate volatilities and implied underlying prices simultaneously. Taking a pair of prices both of which do not violate basic no-arbitrage conditions there is always a solution for the implied parameters which will set the observed prices equal to the theoretical values. When there are more than two options there will in general be no perfect fit of theoretical to observed prices, so that again a non-linear regression procedure can be used to estimate the implied underlying value and volatility at the same time.

In a recent study Longstaff [9] used this approach to obtain estimates of the implied S&P 100 value from options traded on this index. He then compared these implied values to the observed index value to check what he calls *the martingale restriction*. If all the options were priced according to the same no-arbitrage model implied and observed index levels would have to be equal. In the absence of arbitrage any discrepancies between the two index prices must be due to market frictions like transaction costs or short selling restrictions. Longstaff's main result is that the implied value for the S&P 100 index is almost always larger than the observed index price so that the martingale restriction in its pure theoretical form is violated. By means of a regression analysis Longstaff shows that the differences are related to frictions in the options market (like the bid-ask spread) as well as option characteristics like moneyness (percentage difference between underlying price and strike price) and time to maturity. Furthermore, differences between theoretical and observed option prices are at least partly explained by market frictions like the bid-ask spread.

Finally, Rubinstein [12] proposed a technique that goes one step further and avoids an inherent weakness of Longstaff's approach, since a researcher cannot be sure that the model used by market participants for valuing options is the one that he uses to infer implied parameters and to test the martingale restricition. Instead of only estimating certain moments of the underlying distribution from market data Rubinstein infers the complete distribution for the underlying price from the prices of traded options.[2] To do so he first specifies an a priori distribution (in his case the binomial discretization of a lognormal distribution), and then an optimization is performed to obtain an implied distribution which is as close as possible to the lognormal in the sense of squared distances between state probabilities. The resulting distribution is constrained to exhibit positive probabilities for all terminal states and to yield theoretical option prices which fall between the bid and the ask quote of the options used in the estimation.[3] The basic idea behind Rubinstein's approach is the well-known fact that in a complete and frictionless market without arbitrage opportunities there exists a unique probability measure $Q$ called the *forward*

[2] Strictly speaking also Longstaff [9] infers the complete distribution from market prices by choosing a class of distributions that is completely specified by just the mean (the forward price of the implied index level) and the standard deviation (the implied volatility). The important difference between the two approaches is that Rubinstein does not prespecify the type of implied distribution.

[3] The resulting terminal stock price distribution can also be used as an input to compute an implied binomial tree which could then be used to compute theoretical prices for American options. See also Rubinstein [13] for a simplified exposition of this approach.

*risk-adjusted measure* which can be used to price all assets in the economy according to the formula $P_t = d_{t,T}\hat{E}_t^Q[P_T]$. Here $P_t$ ($P_T$) denotes the price of an asset at time $t$ ($T$), $d_{t,T}$ is the discount factor from $t$ to $T$, and $\hat{E}_t^Q$ denotes expectation under $Q$ conditional on all available information at time $t$.[4] Given a sufficient number of option price observations the pricing equation can be inverted to obtain an estimate of the market's assessment of state probabilities. In subsequent papers, e.g. by Jackwerth and Rubinstein [6], this approach was refined with respect to estimation techniques.

The purpose of this paper is to provide an empirical analysis of distributions implied by the prices of German stock index options as well as to test the validity of the martingale restriction for the pricing of these derivative contracts. European options on the most important German stock index, the DAX (Deutscher Aktienindex), are actively traded on the DTB (Deutsche Terminbörse) since August 19, 1991. The DAX is an index of thirty German blue chip stocks that is adjusted for dividends and capital changes of the component stocks, so that an option on the DAX can be valued easily as one on an underlying that does not pay dividends. Together with the fact that the options are European there are basically ideal conditions to apply simple valuation models, e.g. the Black and Scholes model or the binomial option pricing model developed by Cox, Ross, and Rubinstein [4].

Up to this point there are only a few empirical studies on the pricing of stock index options traded on the DTB. The paper thus fills a gap in empirical capital market research in Germany, since the DAX options traded on the DTB are one of the most liquid derivative contracts traded in Germany. Currently the typical daily volume for DAX options is well above 100,000 contracts, making it the most liquid contract on the DTB as a whole. The paper further integrates the approaches by Longstaff [9] and Rubinstein [12] by simultaneously testing the martingale restriction and calculating implied distributions based on the same samples. The results are therefore very useful to determine the sources of violations of the martingale restriction.

The main results of the analysis are as follows. The implied DAX value is almost always larger than the observed index price which is the same phenomenon as the one observed by Longstaff [9] for the U.S. market. A possible interpretation is that market participants consider the implied price the true price they would have to pay for the DAX to be able to duplicate a position in options, so that the difference to the observed index level could be due to transaction costs.

The direction of the DAX pricing differences as well as their regularity is the same for puts and calls with significantly larger differences for puts. Regression analyses show that for calls the average time to maturity of the options used in the estimation as well as

[4] Longstaff [9] calls $Q$ the *risk-neutral measure* instead of forward risk-adjusted measure. Under interest rate certainty the two measures are identical. However, in the literature on valuation of interest rate derivatives $Q$ is usually termed the forward risk-adjusted measure (see, e.g., Jamshidian [7]), since under $Q$ the forward prices of all assets are martingales. Note that $Q$ is horizon-specific, i.e. it dependes on $T$. The risk-neutral measure $\tilde{Q}$ has the property that under $\tilde{Q}$ the futures prices of all assets are martingales. The general pricing equation also shows the reason why Longstaff [9] calls his empirical analysis a test of the martingale restriction. If we consider the underlying asset $S$ of an option as an option with a zero strike price then the general pricing equation also has to hold for $S$, i.e. $S_t = d_{t,T}\hat{E}^Q[S_T]$ or, alternatively, after dividing through by $d_{t,T}$, $S_t d_{t,T}^{-1} = \hat{E}_t^Q[S_T]$. This, however, is equivalent to writing $S_t d_{t,T}^{-1} = \hat{E}_t^Q\left[S_T d_{T,T}^{-1}\right]$, since $d_{T,T} = 1$. The last equation shows that $S_t d_{t,T}^{-1}$ is a martingale under $Q$.

their average moneyness is an important determinant for the amount of the pricing error. When index pricing errors from put estimations are analyzed we find that the DAX pricing error decreases with the number of options used in the estimation process as well as with the average relative spread of the options whereas it increases with time to maturity and average moneyness.

Comparing volatilities implied from option prices when the observed index price is used in the estimation to the case when also the DAX level is inferred from market prices for options we again find systematic differences. For calls we obtain larger volatilities when the observed index price is used in the estimation whereas just the opposite result is obtained for puts.

Theoretical call prices are smaller than market prices when the observed index price is used in the course of volatility estimation. The differences change sign when also the index is implicitly estimated, whereas theoretical put prices are smaller than market prices in both cases. Call pricing errors (theoretical minus observed prices) are negatively related to time to maturity, relative spread, and moneyness, they increase with increasing option prices. For puts the most important determinants of pricing errors are again the moneyness of an option (with a negative impact) and its relative spread and time to maturity.

For implied distributions calculated according to Rubinstein's [12] method we find some interesting differences between puts and calls. For calls the market assigns a larger probabilitiy mass to states around the forward price of the index whereas in the market for puts state prices are higher than in the lognormal case for states with either very high or very low index prices. The average differences between implied and a priori distributions are not driven by outliers, their behavior across the different option series is remarkably stable.

The rest of the paper is organized as follows. The next section will describe the data that were used in the course of the study as well as some methodological items. In Section 3 the empirical results of the analysis are presented, and Section 4 contains a summary and some concluding remarks.

## 2 Data and Methodology

The basic sample for this study consists of all best bid and best ask quotes for DAX options traded on the DTB for the first six months of 1994. The quotes are time-stamped to fractions of a second, and they were considered good until changed. This yields a time-series of simultaneous best bid and best ask prices. The DTB is a fully computerized exchange, and there are no trades inside the spread, so that all transactions occur at either the bid or the ask. Therefore, we use the midpoint between bid and ask as an estimate of the true value of the option. Besides the option data we also use DAX prices from KISS (Kurs-Informations-Service-System) which are time-stamped to the nearest minute.

To be in the final sample for this study an option had to have a remaining time to maturity of at least five days. After deleting all option observations with a shorter maturity we selected the minute (or the minutes) with the highest aggregate quotation activity (in terms of the number of bid and ask quotes) for puts and calls to make sure that the data we use did not suffer from insufficient liquidity or stale quotes. In case there were several minutes on a given day with equal market activity we kept all of them in the sample to retain as many observations as possible.[5] Since the DAX option is a very liquid contract there may be several bid and ask quotes for a given option, i.e. a put or a call with a certain strike price and a certain maturity date. Only the first of these observations was kept in the sample. The option observations were then matched with the DAX prices for the corresponding minute. The descriptive statistics for our final sample are given in Table 1.

There is a total of 7,263 observations for calls in 485 series[6] and 6,955 observations for puts in 499 series, yielding an average of 14.97 options per series for calls and 13.94 for puts. There were always at least six individual options available for all the estimations with a maximum of 30 for calls and 29 for puts. Time to maturity ranged from seven days to about nine months for both option types. As expected the average moneyness of the options defined as the difference of the observed index level and the strike price divided by the strike price is close to zero for both puts (0.0176) and calls (0.0197). An interesting result are the numbers for the relative spreads of the options defined as the difference between ask and bid divided by the midpoint. With 7.7% and 10.8%, respectively, mean spreads as a measure of transaction costs in the options market are relatively high for both calls and puts, and relative spreads are signficantly larger for puts. The test for the null hypothesis of equal mean spreads yields a highly significant $t$-value. A similar result in terms of average spreads was also obtained by Lüdecke [10] for options on individual stocks traded on the DTB.

To estimate implied parameters from an option pricing model a discount factor for the maturity of the given option series is needed. Since there are no actively traded default risk-free discount bonds in Germany the discount factor was estimated implicitly as suggested by Shimko [14] together with an implied index value from the pair of the two closest at-the-money puts and calls using the standard put-call parity relationship. With $(C_i, P_i)$ $(i = 1, 2)$ denoting the two pairs of option prices with common maturity $\tau$ and strike prices $X_i$ $(i = 1, 2)$ the implied discount factor $\hat{d}_\tau$ and the implied index level $\widehat{DAX}$ are the solution to the following system of two equations:

$$\begin{aligned} C_1 - P_1 &= \widehat{DAX} - X_1 \hat{d}_\tau \\ C_2 - P_2 &= \widehat{DAX} - X_2 \hat{d}_\tau . \end{aligned}$$

The solution is given by

$$\begin{aligned} \hat{d}_\tau &= \frac{(C_1 - P_1) - (C_2 - P_2)}{X_2 - X_1} \\ \widehat{DAX} &= \frac{(C_1 - P_1) X_2 - (C_2 - P_2) X_1}{X_2 - X_1} . \end{aligned}$$

This implied discount factor $\hat{d}_\tau$ is used for all further computations. The calculation of the implied discount factor is the only case in this study when put and call prices are used

[5] This means that there may be more than one estimation on a given day.

[6] An option series is a collection of options of one type (calls or puts) with identical maturity dates and different strike prices. We do not use all the options observed in a given minute as one sample for the estimation of implied volatilities and implied index levels, since a deterministically varying volatility for different maturity dates does not contradict the assumptions of the Black and Scholes [2] model.

simultaneously. All the following estimations are performed separately for puts and calls to be able to detect systematic differences between the two option types.

Individual implied volatilities are estimated in the standard fashion, i.e. given all the other inputs the volatility is changed until the theoretical price of the option, given by the Black and Scholes formula, is equal to its market price.

To obtain a single implied volatility number for a given option series we used the procedure suggested by Whaley [16], i.e. a non-linear regression was run of observed option prices on the theoretical values given by the Black and Scholes formula [2]. More formally, given a set of $N$ observed option prices in one series we search for an estimate $\hat{\sigma}_I$ minimizing the expression

$$\sum_{i=1}^{N} \left[ P_i - \hat{P}_{I,i}(\hat{\sigma}_I) \right]^2$$

with $P_i$ and $\hat{P}_{I,i}(\hat{\sigma}_I)$ as the observed and theoretical price for option $i$ ($i = 1, \ldots, N$). We will refer to this method as the Longstaff I method.

To test the martingale restriction we use the Longstaff II (first suggested by Manaster and Rendleman [11]) method to estimate both the implied volatility and the implied DAX price, again via a non-linear regression minimizing the sum of squared distances between observed and theoretical option prices. The volatility and price estimates are denoted by $\hat{\sigma}_{II}$ and $\hat{P}_{II}$, and $\widehat{DAX}_{II}$ represents an estimate of the implied index level. Formally, $(\hat{\sigma}_{II}, \widehat{DAX}_{II})$ is the two-dimensional vector which minimizes the sum of squares

$$\sum_{i=1}^{N} \left[ P_i - \hat{P}_{II,i}(\hat{\sigma}_{II}, \widehat{DAX}_{II}) \right]^2 .$$

over all possible pairs of implied volatilities and index levels. The formal test of the martingale restriction will then be performed by comparing the observed and implied index prices $DAX$ and $\widehat{DAX}_{II}$ as well as the two volatility estimates $\hat{\sigma}_I$ and $\hat{\sigma}_{II}$.

To compute implied distributions the implied volatility of the option closest at the money was used to construct a binomial tree with fifty ending nodes, irrespective of the time to maturity of the options.[7] The tree was set up in the standard fashion according to Cox, Ross, and Rubinstein [4], i.e. in each step the risk-neutral probability $q$ for an up move was given by

$$q = \frac{e^{r\Delta t} - e^{-\hat{\sigma}\sqrt{\Delta t}}}{e^{\hat{\sigma}\sqrt{\Delta t}} - e^{-\hat{\sigma}\sqrt{\Delta t}}}$$

with $r$ as the estimated risk-free rate p.a. of interest derived from the implied discount factor $\hat{d}$ for the corresponding maturity, $\Delta t$ as the length of a time step, and $\hat{\sigma}$ as the estimate of implied volatility. With $n$ steps the a priori probability for state $i$ ($i = 1, \ldots, 50$) is denoted by $q_i$ and given by

$$q_i = \binom{n}{i-1} (1-q)^{i-1} q^{n-i-1},$$

[7] Alternatively, trees with 100 ending nodes were tried, but the main difference between the two approaches was the larger number of nodes with a zero implied probability when 100 nodes were used.

i.e. the nodes are numbered such that node 1 represents the highest terminal index price, followed by node 2, and so on. The associated terminal index values $DAX_i$ were later used to compute option payoffs in the different states.

Implied state probabilities $\hat{q}_i$ were then estimated by minimizing the sum of squared deviations between implied and a priori probabilities:

$$\sum_{i=1}^{50} (q_i - \hat{q}_i)^2 . \tag{1}$$

The constraints under which (1) was minimized were the same as those used by Rubinstein [12]:

1. All implied state probabilities had to be non-negative, i.e. $\hat{q}_i \geq 0$ $(i = 1, \ldots, 50)$.[8]
2. All the resulting theoretical option prices, i.e. the discounted expected payoffs, had to fall between the observed bid and ask prices of the options. Formally this means $P_i^a \geq \hat{P}_i \geq P_i^b$ with $P_i^a$ $(P_i^b)$ as the observed ask (bid) price and $\hat{P}_i$ as the theoretical price of the option. The theoretical price in turn is given by the standard binomial option pricing formula (Cox, Ross, and Rubinstein [4]). The terminal payoff in state $i$ is given as $\max\{DAX_i - X, 0\}$ for a call and $\max\{X - DAX_i, 0\}$ for a put.
3. The theoretical current index level when computed as the discounted expectation of terminal index values[9] had to fall within a band with a width of 1 percent of the observed index level, i.e. we assumed an 0.5% half spread for the index.

# 3 Empirical results

## 3.1 Testing the martingale restriction: implied index prices, implied volatilities, and theoretical option prices

Tables 2 and 3 show summary statistics for the estimation results for the call and put samples. The variables of interest are observed and implied index values as well as implied volatilities for different estimation methods and option pricing errors for the two Longstaff methods described in section 2.

The martingale restriction for option pricing does not hold if implied and observed underlying price are significantly different and if these differences are not due to market frictions like transaction costs or short selling restrictions. Looking at the first panel in Table 2 we find that the implied index value from the Longstaff II method $\widehat{DAX}_{II}$ is on average larger than the observed index price denoted by $DAX$. The mean difference is around twelve index points which corresponds to just 0.57 percent given an average index level of more

[8] Earlier computations using a simpler approach without the non-negativity constraint showed that negative probabilities occurred rather frequently. In a frictionless market this would already indicate an arbitrage opportunity, since the forward risk-adjusted measure does not exist in this case. A qualitatively similar result with some negative state probabilities was obtained by Rubinstein [12].

[9] This just means that the index is also priced according to the binomial model. It can be interpreted as an option with a strike price of zero.

than 2,100 points. Even the maximum mispricing is only around 2.5 percent of the average index value for negative mispricing and 3.5 percent for positive mispricing. However, the difference between implied and observed index level is statistically signficant, a $t$-test for the null hypothesis of a zero mean difference yields a test statistic of 24.7, which is far beyond conventional critical values. Furthermore, for 479 out of a total of 485 option series (98.76 percent) the implied index level is greater than the observed DAX price (see also Figure 2). This shows that the average mispricing is not caused by outliers in the data. The systematic pattern seems to support the hypothesis that market participants implicitly add transaction costs to the observed index value since this number would represent the true cost of setting up a duplicating portfolio for a DAX call option.[10] Of course, in frictionless markets the two index prices would have to be equal in the absence of arbitrage.

Table 3 shows similar tendencies for estimations done with put options. Here the implied index price is also on average higher than the observed DAX price. With an average of 30 index points the differences are larger than for call options, and the difference in DAX pricing quality between calls and puts is also statistically significant. A $t$-test for equal mean DAX pricing errors for puts and calls yielded a statistic with an absolute value of 24.6 which is highly significant. Again the signs of the differences are not random. 489 out of 499 observations (98 percent) exhibit a positive sign. This also becomes clear from Figure 3. As for calls the data points are heavily concentrated below the 45 degree line, i.e. in the area where the implied index level is greater than the observed price. For puts we also observe that the horizontal distance of the data points gets larger for larger index values, i.e. for larger observed DAX prices we also observe larger pricing errors.

Taken together these results suggests that similar to the findings by Longstaff [9] the martingale restriction seems to be violated. Table 6 shows the results of multiple regressions with (relative) DAX pricing errors as dependent variables. Explanatory variables are the number of options in a given series that were used for the estimation procedure ($N$), the average relative spread of the options in the series ($SPREAD$), the time to maturity ($T$) of the options (identical for all options of a given series and measured in years), and the average relative moneyness of the options ($MONEY$) in the given series.

The results are different for calls and puts. Whereas for calls pricing errors and relative pricing errors tend to become larger with an increasing number of observations, we observe the opposite tendency for puts. The coefficient of $N$ for pricing errors is, however, insignificant for calls, so we can not conclude that there is indeed a relationship between these two variables. Since the vast number of pricing errors is positive this means that the quality of implied DAX pricing improves for puts with an increasing number of options that are available for the estimation.

For the average spread we obtain coefficients that are similar for calls and puts. Again, the estimates in the regressions for calls are not statistically significant, and the impact is also numerically larger for puts. The general result is that for a series with a larger average relative spread we obtain a smaller pricing error.

The variable $MONEY$ has a postive sign for puts and a negative sign for calls, and it is statistically significant in all four regressions. For an interpretation of this result we have to

[10] Since the DAX contains thirty stocks a natural assumption is that a recorded DAX price contains a roughly equal number of most recent transactions at the bid and at the ask so that the price rather reflects a midpoint and not the bid or the ask.

keep in mind that the variable $MONEY$ is defined identically for puts and calls. Whereas a call is far in the money for a large positive value of $MONEY$ the opposite is true for a put. Thus the implied DAX price is in general closer to the observed index level if we use options that are 'far in the money' (calls with a large positive value for $MONEY$, puts with a large negative value for this variable) to estimate the implied index price. Option pricing theory provides a reasoning which may help to explain the better index pricing quality of in the money options. Since DAX options are European the value of an option that is far in the money approaches a boundary which is independent of the pricing model and allows for a static replication of the option. This static duplication is possible with much lower transaction costs than those accumulated in a dynamic strategy.[11] Finally, the average time to maturity also has a significant impact on the amount of the pricing error. For both types of options we find that with an increasing time to maturity of the options we also obtain larger DAX pricing errors.

Figure 1 shows a typical smile pattern for DAX options. We find a negative slope of the implied volatility curve with respect to the strike price of the options. This pattern is pretty stable for calls across the option series in our sample, for puts there are also cases when the curve is upward sloping. The general tendency is, however, that individual implied volatilities within a given series are not constant across strike prices which they should be for the Black and Scholes model to be valid. The second panel in Table 2 and Table 3 contains the average of the mean individual implied volatility per series denoted by $\bar{\sigma}$. This makes it possible to compare the single value estimated by the Longstaff methods to an estimate giving equal weight to each of the options in a given series.[12]

Comparing first the two implied volatilities from the Longstaff I and II methods we again find a systematic difference between the two estimates. Figures 4 and 5 show plots of $\hat{\sigma}_I$ versus $\hat{\sigma}_{II}$ for calls and puts. Whereas for puts $\hat{\sigma}_{II}$ is consistently greater than $\hat{\sigma}_I$ (489 out of 499 cases) the differences have the opposite sign for calls (479 out of 485 cases). This may seem surprising at a first glance, but the result is perfectly consistent with the findings for the implied estimation of the index level. For both calls and puts we observe that the implied index level is systematically larger than the observed DAX price. For the squared difference between theoretical and market prices to be as small as possible this means that in the Longstaff II procedure the implied volatility $\hat{\sigma}_{II}$ has to be lower for calls, since this reduces the theoretical price which increases due to a higher implied index level. For puts the argument works exactly in the opposite direction: the theoretical option value decreases with an increasing implied index price so that there is a tendency for $\hat{\sigma}_{II}$ to increase to compensate the decrease in the put price. The results for implied index levels and implied volatilities are therefore not independent of each other, they are a more or less direct consequence of the estimation procedure.

Table 7 shows the results of some regressions with the differences between $\hat{\sigma}_{II}$ and $\hat{\sigma}_I$ as the dependent variable. The set of regressors was the same as for the analysis of DAX pricing errors above. For calls the average relative moneyness as well as time to maturity are significant sources of variation in (absolute) volatility differences. With an increasing average moneyness the difference $\hat{\sigma}_{II} - \hat{\sigma}_I$ tends to become numerically larger which means

[11] In the case of a call option with maturity $\tau$ this boundary is given by $DAX - Xd_\tau$, and for a put the limiting value is $Xd_\tau - DAX$.

[12] In a recent paper Dumas, Fleming and Whaley [5] estimate a linear function relating the implied volatility of S&P 500 index options to the strike price and time to maturity of these options.

that due to the systematic sign it will be closer to zero. The oppostie effect is observed for time to maturity. The negative coefficient in the call regression means that the two volatility estimates will be further apart for longer term options.

For the implied volatilities from put estimations we obtain a more systematic result. The coefficients for all the four variables are significant in both regressions. The difference $\hat{\sigma}_{II} - \hat{\sigma}_I$ increases with time to maturity and average relative moneyness and decreases with an increasing number of options available for estimation and a larger average relative spread. Note that the interpretation of the coefficients here is opposite to the case of call options, since the differences $\hat{\sigma}_{II} - \hat{\sigma}_I$ are positive in most cases.

The grand mean of individual implied volatilities denoted by $\bar{\sigma}$ is the highest of all implied volatility estimates for calls whereas for puts its value is located between the two Longstaff estimators. Overall the common implied volatilities are pretty similar to the average of individual implied volatilities, so that the use of the arithmetic average of individual implied volatilities seems justified as a first approximation to a joint volatility estimate for all options of a given series.

Finally, one can look at the theoretical option prices generated by the estimation procedures. A number of descriptive statistics on the differences between these theoretical prices and the observed market prices (midpoints) are given in the third panel of Table 2 and Table 3. It is interesting to note that put options are underpriced by both the Longstaff I and II method (negative average difference between theoretical price and market price) whereas calls are underpriced by Longstaff I and overpriced by Longstaff II. The pricing errors for calls are statistically different from zero for both methods, standard $t$-tests for a zero mean error yield statistics of $-10.127$ for Longstaff I and 4.300 for Longstaff II. For puts we obtain $t$-statistics of $-9.662$ and $-9.245$, respectively. As expected the average pricing error is larger for the Longstaff I method than for Longstaff II, since with the implied index price there is additional free parameter so that a better fit should be obtained. The pricing quality of the Black and Scholes model is furthermore significantly better for calls than for puts. Except for the raw pricing error of the Longstaff I estimation raw and relative pricing errors are always signficantly larger for puts than for calls.[13]

The amount of the pricing difference can be considerable: the largest negative relative mispricing was around $-83$ percent of the observed price for calls and right around $-100$ percent for puts. These numbers may be at least partly caused by some market frictions like the tick size for option prices which do not exist in the theoretical model.[14] On the other hand these extreme value statistics are furthermore heavily influenced by a few outliers, since the 5 percent and 95 percent quantiles of call pricing errors are $-6.357$ and 5.597 for Longstaff I and $-2.666$ and 2.967 for Longstaff II, respectively. For relative call pricing errors these quantiles are also far away from the minimum and the maximum given in Table 2.[15]

Similar results are found for puts where only five percent of the Longstaff I pricing errors

[13] The $t$-statistics of the tests are signficant at levels much smaller than 1 percent.

[14] The effect of price discreteness may be expected to be especially severe for options far out of the money with a very low price. To take this into account the observed option price is included in the regression equations for option pricing differences.

[15] The exact values for the 5 percent and 95 percent quantiles are $-3.64$ percent and 21.75 percent for Longstaff I, and $-2.30$ percent and 9.55 percent for Longstaff II, respectively.

are smaller than $-7.110$ and only five percent greater than 6.631. Again the findings are similar for relative errors and for the Longstaff II method.[16]

Tables 4 and 5 show the results of an analysis concerned with the location of theoretical option prices generated by the Longstaff methods relative to bid and ask. Even a large difference between theoretical and observable prices does not necessarily imply that the model misprices options. Especially in the case when quote data are explicitly available we can check if the theoretical prices are still within the band created by the currently best bid ans ask prices. The observations have been classified according to moneyness and time to maturity. The options with the lowest moneyness are in the group identified by $M = 1$, and $MAT$ equals one for the options with the shortest maturity. $N$ denotes the total number of options in a given cell, and the following four lines indicate how many theoretical prices are larger than the ask ($\hat{P}_j > P_a$, $j = I, II$) or less than the bid ($\hat{P}_j < P_b$, $j = I, II$).

Looking at calls first we find that in total the number of mispricings outside the spread are lower for Longstaff II than for Longstaff I (1058 vs. 2949), which is expected due to the additional degree of freedom in the fitting procedure. Note, however, that for a test of the Black and Scholes model it is the Longstaff I method which is relevant, since pricing has to be done relative to the current observable index level. For options with low moneyness we mainly find that theoretical prices tend to be too large, since the number of observations above the ask clearly execeeds the number of observations with theoretical prices lower than the bid. For example, for $M = 1$ and Longstaff I there are 545 cases with overpricing as opposed to only 6 observations for underpricing. An even stronger tendency is found for $M = 2$ with (again for Longstaff I) 1,077 overpriced and only 17 underpriced options. For $M = 3$ overpricing and underpricing occurs with roughly similar frequencies, although overpricing is still found more often.

The picture is completely different for options with high moneyness. Here we find no overpricing at all by Longstaff I, but many cases of underpricing. So we can conclude that moneyness is a very important variable for predicting substantial deviations between theoretical Black and Scholes prices and market prices. The mispricing behavior also shows some variation with respect to time to maturity in that mispricing occurs more frequently for options with longer maturity, but the effect is not as pronounced as for moneyness.

Looking at the results of this analysis for puts we find exactly the same tendencies as for calls. This is somehow surprising, since a low value for $M$ means that the current index level is much lower than the strike price which means that puts are 'deep in the money'. Nevertheless, there is not a single case of a theoretical price being less than the bid for Longstaff I in groups $M = 1$ and $M = 2$. Again, for $M = 3$ the probability of underpricing is about the same as that for overpricing, but for $M = 4$ and $M = 5$ options are almost always underpriced by the model. Again, also time to maturity is an explanatory factor with more frequent mispricing for options with longer maturity, although again with less predictive power than moneyness.

Figures 6 to 9 show plots of relative pricing errors against relative moneyness. For both calls and puts we observe that pricing errors decline the further the option is in the money,

[16] The exact values for the 5 percent and 95 percent quantiles of relative pricing errors are $-55.71$ percent and 5.02 percent for Longstaff I, and $-34.07$ percent and 2.91 percent for Longstaff II, respectively. This provides some more evidence for the hypothesis that the pricing quality of the Black and Scholes model is worse for puts than for calls.

i.e. for a negative value of $MONEY$ for puts and a positive one for calls. The reason for this may be that the further an option is in the money the closer its value gets to a model-independent boundary, so that the pricing error is likely to decrease even if there are discrepancies between the assumed model and the true pricing mechanism used by market participants. It is interesting to note that options which are slightly out of the money tend to be overpriced by the model in the case of calls whereas we observe just the opposite for puts. Furthermore, calls that are not too far in the money seem to be underpriced, so that there seems to be a quadratic relationship between relative moneyness and relative pricing error. For this reason the regression model for pricing errors includes both the variables $MONEY$ and $MONSQR$, the squared value of $MONEY$.

The regression results are presented in Table 8. Except for four cases all the coefficients are significant at the 5 percent level. The explained portion of the pricing error variance is in general larger for the Longstaff I method as can be seen from the adjusted $R^2$ values for the individual regressions.

Pricing errors usually decrease with the variable $MONEY$, except for (absolute) pricing errors for calls although the coefficients are not significant at the 5% level. For squared moneyness $MONSQR$ the coefficient usually has a positive sign and is highly signficant with the only exception relative put pricing differences generated by Longstaff II. This supports the hypothesis presented above that it is rather the distance from zero than the actual location of moneyness that has an impact on pricing errors. A positive impact of $MONSQR$ on pricing errors is, however, offset by a negative coefficient for $MONEY$ for small values of this variable, since in this case $|MONEY| > |MONSQR|$ so that the impact of moneyness in linear form is dominant. Pricing errors exhhibit the tendency to increase with time to maturity for puts in general, whereas we obtain a negative coefficient for $T$ for relative call pricing error regressions. We obtain negative coefficients (except for one put regression) for the relative spread of an option which means that options with a lower spread are better priced by the model. This result has an intuitive explanation: our 'observed' price is taken to be the midpoint between the bid and the ask which is merely an assumption. If the true price is not exactly half way between bid and ask chances are that the *perceived* mispricing will be larger the wider the band between the best quotes even if the model hits the (unknown) true price exactly.

Finally, the observed option price $P$ has a negative coefficient in six of the eight regressions, the two exceptions being the relative pricing errors for calls. This means that for more expensive options model prices tend to be closer to market prices which is consistent with the impact of moneyness on pricing quality.

## 3.2 Implied distributions for DAX options

The algorithm to estimate implied state probabilities always converged rapidly without any numerical problems. The results are presented graphically in Figures 10 to 13. The graphs in Figures 10 and 12 show the average difference for each of the 50 nodes between the implied probability and the a priori specified lognormal (or binomial) distribution. Of course, the sum of these differences across all nodes has to be equal to zero. Figures 11 and 13 show the share of positive differences $\hat{q}_i - q_i$ for the 50 nodes. As described above the numbering of the terminal states along the horizontal axis is from the lowest terminal

DAX price (node 1) to the highest (node 50).

Taking a look at implied distributions for calls first we find that the market obviously puts more probability mass on events where the terminal index price is close to the forward price (which is the mean of the distribution under the forward risk-adjusted measure). Arrow-Debreu securities which have a payoff in one of these states of the world are therefore more expensive than predicted by the standard binomial (or the Black and Scholes) option pricing model. Note that this does not necessarily mean that certain types of calls are overpriced. Since the call price is the sum of the state dependent payoffs multiplied by the price of the associated Arrow-Debreu security, the positive pricing error induced by state prices which are too high may be (more than) offset by other state prices which are too low compared to standard option pricing models. Looking at the graph in Figure 10 we find that the implied state prices for relatively high terminal index values are lower than indicated by the binomial model. For example, a call with a strike price that generates positive payoffs in states 26 through 50 may well have a market price that is very similar to its theoretical Black and Scholes price, since there are some overpriced and some underpriced states in this range. However, we can deduce that if we are able to construct a portfolio of DAX call options yielding positive payoffs in states 26 to 31 then this portfolio will have a higher market price than predicted by the binomial model. Accordingly, portfolios of call options with payoffs only in states 32 through 50 or 1 through 25 are likely to be cheaper than predicted by the binomial model.[17] Around the forward price of the index the probability differences show a very systematic behavior which becomes obvious from the graph in Figure 11. In the center of the distribution the implied probability is greater than the a priori probability for more than 90 percent of the 485 series whereas for other areas this share is well below 50 percent, i.e. the majority of differences is negative. It is finally interesting to note that the average probability difference in states of high terminal index values is close to zero, although the share of positive differences is only around ten percent. This may seem strange, but the reason is that only the share of strictly positive differences is plotted. For calls more than 90 percent of the differences for nodes 48 through 50 are equal to zero and thus not strictly positive. The same phenomenon is observable for nodes 1 through 3, where the share of zero differences is around 82 percent. At the lower and upper tails of the distribution the binomial probabilities are so small that they are not different from zero with eight significant digits. If the optimization procedure also assigns a zero probability for these states we will obtain an observation with a zero difference. Except for the tails of the distribution there are no other cases when the two distributions assign numerically identical probabilities to a given state. In general we may conclude from the results obtained for calls that investors in the market for DAX calls seem to consider medium range changes in the index less likely than predicted by the binomial option pricing model. On the other hand there is stronger belief in the market that the terminal index value will be somewhere around the forward price. There is only slight evidence for a phenomenon which Rubinstein [12] calls 'crash-o-phobia', i.e. a larger implied probability for states with a very low index value, since for nodes 1 to 11 the differences between implied and a priori probabilities are positive but very small in absolute value.

[17] Shimko [15] uses this fact to design trading strategies based on an investor's subjective probability assessment. The investor should sell Arrow-Debreu securities for states for which his subjective probability is lower than the market estimate and vice versa. In the situation described above the investor should sell short a butterfly spread if he believes that the binomial model is the correct description for stock price movements.

The result for puts is different. Looking at the differences $\hat{q}_i - q_i$ for small $i$, i.e. for states with a low DAX price we find that investors are willing to pay more for state contingent payoffs in this range of index prices. This may be evidence for an extra premium for the insurance function of a put with a rather low strike price so that it has a positive payoff only in these states. Whereas we do not find such a pronounced tendency in the market for calls this finding represents a 'crash-o-phobia'. Figure 13 also supports this hypothesis, since for the vast number of option series we obtain indeed implied probabilities which are larger than their binomial counterparts. Another difference between implied distributions for calls and puts is obvious for nodes 32 through 50, i.e. for states with high index values. Whereas in the market for calls there is a tendency for these Arrow-Debreu securities to be cheaper than in the binomial model the opposite is observable for puts. All the states have a higher implied than a priori probability, i.e. a portfolio of puts with positive payoffs in the states and zero payoff in others is more expensive in the market than we would expect from the binomial model. If an investor has the binomial as his subjective probability distribution he would be willing to sell this portfolio short, since he considers it overpriced by the market. Note, however, that this is not an arbitrage transaction, it is based solely on speculative arguments (differences in expectations). As for calls we also observe negative average differences for nodes 14 to 25 which represent states in which the index has declined but they certainly do not represent a crash. The absolute values of the differences are much larger than for calls, and the shares of positive differences are close to zero in this range of states. Portfolios of put options with postive payoff in only these states are thus cheaper than predicted by our a priori distribution.

# 4 Summary and Conclusions

The two major purposes of this study were to conduct an empirical investigation into the pricing of DAX options on the DTB, the German financial futures and options exchange and to take a look at index price distributions implied by option prices. DAX options are of the European type and the underlying index DAX is a performance index, i.e. it is basically a non-dividend paying asset. These features would allow to use simple valuation models based on a lognormal or binomial distribution of terminal values of the underlying asset. The basic idea behind a test of the martingale restriction is to infer the implied current underlying price from option prices and to compare it to the contemporaneously observable market price. If there are significant differences then the conclusion is that there are either market frictions preventing strict no-arbitrage relationships from holding or that investors are using a valuation model that is different from the one assumed by the researcher. The results of a recent study by Longstaff [9] indicate that for S&P 100 options the implied index level is systematically higher than the observed price, meaning that the martingale restriction in its pure form is violated. Performing a similar test for DAX options on the DTB we find that also in the German market the implied cost of the index is systematically higher than the observed market price. In contrast to Longstaff we are also able to use put options to infer implied parameters, and the results for the implied DAX price are fully consistent with those for calls. Regression analyses show that the number of options used in the estimation process as well as the time to maturity and the average relative moneyness are significant determinants for the amount of the pricing error. Volatility differences between the two Longstaff approaches are perfectly consistent with the behavior of index pricing errors.

The second variable of interest is the pricing error of the options in the sample when the implied paramters are estimated. Using the observed DAX price to estimate implied volatilities yields theoretical call prices which are lower than market prices. Whereas the sign of the average difference changes when also the index price is estimated implicitly puts are consistently underpriced. Important determinants for the amount of the pricing error are the relative moneyness and its squared value, the relative bid-ask spread and the observed option price.

Given a sufficient number of option prices one can infer the distribution of the terminal underlying price that is implied by the market prices of these contracts. For our sample of DAX options we obtain different results for puts and calls. The prices of DAX calls imply that on average the market assigns a higher probability than predicted by the lognormal model to events close to the mean of the distribution of terminal index prices. On the other hand states a little further away from the mean are considered less likely by market participants. The findings for puts suggest that in this market there is something similar to a 'crash-o-phobia' effect, since the implied probability mass for states with a very low terminal index level is much higher than under the lognormal distribution.

It is interesting to compare the approaches by Longstaff [9] and by Rubinstein [12] with respect to their estimation approaches. Whereas Longstaff prespecifies the type of distribution for the terminal underlying price to infer only certain moments of it from market prices Rubinstein does not put any restricitions on the shape of the distribution. The cost for this extra degree of freedom is that Rubinstein uses the observed index price as one additional call option with a zero strike price in the estimation process whereas Longstaff focuses exactly on the relationship between observed and implied distribution. Given the empirical results of this paper we find that there are certainly pricing errors associated with the use of the Black and Scholes model (and thus the lognormal distribution). The shape of the implied distribution is signficantly different. Besides the implementation of alternative objective functions for the optimization procedure (as suggested already by Rubinstein [12]) one could design tests of trading strategies based on the differences between implied and assumed distributions. Furthermore using the algorithm described by Rubinstein [12] implied binomial trees can be constructed from the implied binomial distributions. These implied trees may then be used to price American options, e.g. the stock options traded on the DTB. All these tasks will be left for further research.

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Table 1:

Descriptive Statistics

| Calls (485 series) | | | | |
|---|---|---|---|---|
| Variable | Mean | Std Dev | Minimum | Maximum |
| N[a] | 14.97 | 5.39 | 6.00 | 30.00 |
| T[b] | 93.26 | 66.12 | 7.00 | 260.00 |
| MONEY[c] | 0.0197 | 0.0819 | -0.1649 | 0.3904 |
| SPREAD[d] | 0.0772 | 0.0898 | 0.0036 | 1.7113 |
| Puts (499 series) | | | | |
| Variable | Mean | Std Dev | Minimum | Maximum |
| N[a] | 13.94 | 4.75 | 6.00 | 29.00 |
| T[b] | 97.03 | 69.32 | 7.00 | 262.00 |
| MONEY[c] | 0.0176 | 0.0690 | -0.1716 | 0.3736 |
| SPREAD[d] | 0.1083 | 0.1616 | 0.0367 | 1.9683 |

[a] Number of observations per series. The total number of call (put) price observations is 7263 (6955).

[b] Time to maturity in days.

[c] Relative moneyness calculated for each individual option as $\frac{(DAX-X)}{X}$ with $X$ as the exercise price and $DAX$ as the observed DAX price.

[d] Mean relative spread calculated for each individual option as 2 (BAP-BBP)/(BAP+BBP) with $BAP$ ($BBP$) as the best ask (bid) price .

Table 2:
Estimation Results for Calls

| Variable | Mean | Std Dev | Minimum | Maximum |
|---|---|---|---|---|
| Index values (485 observations) | | | | |
| $DAX^a$ | 2152.56 | 67.5592 | 1967.36 | 2274.00 |
| $\widehat{DAX}_{II}{}^b$ | 2165.33 | 65.8756 | 1988.69 | 2299.86 |
| $\widehat{DAX}_{II} - DAX$ | 12.7695 | 11.3916 | -52.5922 | 71.6258 |
| $(\widehat{DAX}_{II} - DAX)/DAX$ | 0.0060 | 0.0054 | -0.0245 | 0.0336 |
| Implied volatilities (485 observations) | | | | |
| $\hat{\sigma}_I{}^c$ | 0.1983 | 0.0182 | 0.1532 | 0.2478 |
| $\hat{\sigma}_{II}{}^d$ | 0.1807 | 0.0223 | 0.1281 | 0.2393 |
| $\bar{\sigma}^e$ | 0.2065 | 0.0259 | 0.1649 | 0.3773 |
| $\hat{\sigma}_I - \bar{\sigma}$ | -0.0082 | 0.0190 | -0.1618 | 0.0057 |
| $(\hat{\sigma}_I - \bar{\sigma})/\bar{\sigma}$ | -0.0340 | 0.0690 | -0.4821 | 0.0271 |
| $\hat{\sigma}_{II} - \bar{\sigma}$ | -0.0258 | 0.0195 | -0.1736 | 0.0472 |
| $(\hat{\sigma}_{II} - \bar{\sigma})/\bar{\sigma}$ | -0.1219 | 0.0740 | -0.5146 | 0.2601 |
| $\hat{\sigma}_{II} - \hat{\sigma}_I$ | -0.0176 | 0.0109 | -0.0620 | 0.0451 |
| $(\hat{\sigma}_{II} - \hat{\sigma}_I)/\hat{\sigma}_I$ | -0.0900 | 0.0571 | -0.3126 | 0.2454 |
| Option prices (7263 observations) | | | | |
| $\hat{P}_I - P^f$ | -0.4702 | 3.9566 | -77.6279 | 37.2392 |
| $(\hat{P}_I - P)/P$ | 0.0317 | 0.1061 | -0.8326 | 1.2786 |
| $\hat{P}_{II} - P^g$ | 0.1112 | 2.2038 | -61.2940 | 34.5313 |
| $(\hat{P}_{II} - P)/P$ | 0.0092 | 0.0613 | -0.9472 | 0.6716 |

[a] Observed DAX price.
[b] Implied DAX price from Longstaff II method using the Black and Scholes formula.
[c] Implied volatility from Longstaff I method using the Black and Scholes formula.
[d] Implied volatility from Longstaff II method using the Black and Scholes formula.
[e] Mean of average individual implied volatilities per series.
[f] Difference between implied price from Longstaff I method and observed option price (midpoint between bid and ask).
[g] Difference between implied price from Longstaff II method and observed option price (midpoint between bid and ask).

Table 3:
Estimation Results for Puts

| Variable | Mean | Std Dev | Minimum | Maximum |
|---|---|---|---|---|
| Index values (499 observations) | | | | |
| $DAX^{a}$ | 2140.63 | 45.5523 | 2030.06 | 2259.36 |
| $\widehat{DAX}_{II}{}^{b}$ | 2170.60 | 57.4345 | 2034.84 | 2418.04 |
| $\widehat{DAX}_{II} - DAX$ | 29.9752 | 27.6484 | -29.7046 | 217.1555 |
| $(\widehat{DAX}_{II} - DAX)/DAX$ | 0.0140 | 0.0128 | -0.0140 | 0.1008 |
| Implied volatilities (499 observations) | | | | |
| $\hat{\sigma}_{I}{}^{c}$ | 0.1988 | 0.0182 | 0.1558 | 0.2486 |
| $\hat{\sigma}_{II}{}^{d}$ | 0.2340 | 0.0223 | 0.1580 | 0.3038 |
| $\bar{\sigma}^{e}$ | 0.2035 | 0.0192 | 0.1623 | 0.2598 |
| $\hat{\sigma}_{I} - \bar{\sigma}$ | -0.0047 | 0.0080 | -0.0688 | 0.0163 |
| $(\hat{\sigma}_{I} - \bar{\sigma})/\bar{\sigma}$ | -0.0220 | 0.0374 | -0.2921 | 0.0912 |
| $\hat{\sigma}_{II} - \bar{\sigma}$ | 0.0305 | 0.0211 | -0.0775 | 0.1145 |
| $(\hat{\sigma}_{II} - \bar{\sigma})/\bar{\sigma}$ | 0.1551 | 0.1086 | -0.3289 | 0.6046 |
| $\hat{\sigma}_{II} - \hat{\sigma}_{I}$ | 0.0352 | 0.0188 | -0.0400 | 0.1173 |
| $(\hat{\sigma}_{II} - \hat{\sigma}_{I})/\hat{\sigma}_{I}$ | 0.1810 | 0.1017 | -0.1786 | 0.6291 |
| Option prices (6955 observations) | | | | |
| $\hat{P}_{I} - P^{f}$ | -0.5162 | 4.4560 | -67.2018 | 42.5642 |
| $(\hat{P}_{I} - P)/P$ | -0.1008 | 0.2060 | -0.9999 | 1.2462 |
| $\hat{P}_{II} - P^{g}$ | -0.2294 | 2.0699 | -49.6477 | 50.0770 |
| $(\hat{P}_{II} - P)/P$ | -0.0486 | 0.1549 | -0.9999 | 0.8183 |

[a] Observed DAX price.

[b] Implied DAX price from Longstaff II method using the Black and Scholes formula.

[c] Implied volatility from Longstaff I method using the Black and Scholes formula.

[d] Implied volatility from Longstaff II method using the Black and Scholes formula.

[e] Mean of average individual implied volatilities per series.

[f] Difference between implied price from Longstaff I method and observed option price (midpoint between bid and ask).

[g] Difference between implied price from Longstaff II method and observed option price (midpoint between bid and ask).

Table 4:
Location of Theoretical Call Prices
Relative to Observed Bid and Ask Prices
by Moneyness[a] and Maturity[b]

| | | $MAT = 1$ | $MAT = 2$ | $MAT = 3$ | $MAT = 4$ | Total |
|---|---|---|---|---|---|---|
| | $N^c$ | 39 | 97 | 86 | 457 | 679 |
| | $\hat{P}_I > P_a{}^d$ | 12 | 83 | 84 | 366 | 545 |
| $M = 1$ | $\hat{P}_I < P_b{}^e$ | 6 | 0 | 0 | 0 | 6 |
| | $\hat{P}_{II} > P_a{}^f$ | 0 | 30 | 41 | 147 | 218 |
| | $\hat{P}_{II} < P_b{}^g$ | 15 | 7 | 1 | 10 | 33 |
| | $N$ | 211 | 398 | 272 | 809 | 1690 |
| | $\hat{P}_I > P_a$ | 163 | 317 | 208 | 389 | 1077 |
| $M = 2$ | $\hat{P}_I < P_b$ | 5 | 0 | 0 | 12 | 17 |
| | $\hat{P}_{II} > P_a$ | 91 | 151 | 88 | 134 | 464 |
| | $\hat{P}_{II} < P_b$ | 17 | 9 | 1 | 18 | 45 |
| | $N$ | 327 | 504 | 392 | 712 | 1935 |
| | $\hat{P}_I > P_a$ | 91 | 109 | 61 | 66 | 327 |
| $M = 3$ | $\hat{P}_I < P_b$ | 19 | 42 | 30 | 108 | 199 |
| | $\hat{P}_{II} > P_a$ | 40 | 48 | 9 | 12 | 109 |
| | $\hat{P}_{II} < P_b$ | 23 | 29 | 10 | 24 | 86 |
| | $N$ | 281 | 374 | 245 | 454 | 1354 |
| | $\hat{P}_I > P_a$ | 0 | 0 | 0 | 0 | 0 |
| $M = 4$ | $\hat{P}_I < P_b$ | 42 | 58 | 64 | 211 | 375 |
| | $\hat{P}_{II} > P_a$ | 5 | 3 | 0 | 3 | 11 |
| | $\hat{P}_{II} < P_b$ | 8 | 16 | 8 | 32 | 64 |
| | $N$ | 316 | 533 | 293 | 463 | 1605 |
| | $\hat{P}_I > P_a$ | 0 | 0 | 0 | 0 | 0 |
| $M = 5$ | $\hat{P}_I < P_b$ | 33 | 80 | 52 | 238 | 403 |
| | $\hat{P}_{II} > P_a$ | 2 | 8 | 5 | 8 | 23 |
| | $\hat{P}_{II} < P_b$ | 1 | 2 | 1 | 1 | 5 |
| | $N$ | 1174 | 1906 | 1288 | 2895 | 7263 |
| | $\hat{P}_I > P_a$ | 266 | 509 | 353 | 821 | 1949 |
| Total | $\hat{P}_I < P_b$ | 105 | 180 | 146 | 569 | 1000 |
| | $\hat{P}_{II} > P_a$ | 138 | 240 | 143 | 304 | 825 |
| | $\hat{P}_{II} < P_b$ | 64 | 63 | 21 | 85 | 233 |

[a] Moneyness classification: $M = 1$: $-0.075 \leq MONEY$, $M = 2$: $0.075 < MONEY \leq -0.025$, $M = 3$: $-0.025 < MONEY \leq 0.025$, $M = 4$: $0.025 < MONEY \leq 0.075$, $M = 4$: $0.075 < MONEY$.

[b] Maturity classification: $MAT = 1$: time to maturity at most 30 days, $MAT = 2$: time to maturity between 31 and 60 days, $MAT = 3$: time to maturity between 61 and 90 days, $MAT = 4$: time to maturity more than 90 days.

[c] Number of observations.

[d] Theoretical price from Longstaff I method greater than observed ask price; absolute frequency.

[e] Theoretical price from Longstaff I method less than observed bid price; absolute frequency.

[f] Theoretical price from Longstaff II method greater than observed ask price; absolute frequency.

[g] Theoretical price from Longstaff II method less than observed bid price; absolute frequency.

Table 5:
Location of Theoretical Put Prices
Relative to Observed Bid and Ask Prices
by Moneyness[a] and Maturity[b]

| | | $MAT = 1$ | $MAT = 2$ | $MAT = 3$ | $MAT = 4$ | Total |
|---|---|---|---|---|---|---|
| | $N^c$ | 79 | 92 | 87 | 144 | 402 |
| | $\hat{P}_I > P_a{}^d$ | 1 | 14 | 64 | 115 | 194 |
| $M = 1$ | $\hat{P}_I < P_b{}^e$ | 0 | 0 | 0 | 0 | 0 |
| | $\hat{P}_{II} > P_a{}^f$ | 0 | 0 | 0 | 2 | 2 |
| | $\hat{P}_{II} < P_b{}^g$ | 0 | 0 | 2 | 3 | 5 |
| | $N$ | 267 | 403 | 270 | 683 | 1623 |
| | $\hat{P}_I > P_a$ | 37 | 145 | 117 | 371 | 670 |
| $M = 2$ | $\hat{P}_I < P_b$ | 0 | 0 | 0 | 0 | 0 |
| | $\hat{P}_{II} > P_a$ | 10 | 6 | 13 | 12 | 41 |
| | $\hat{P}_{II} < P_b$ | 1 | 6 | 1 | 14 | 22 |
| | $N$ | 337 | 511 | 387 | 861 | 2096 |
| | $\hat{P}_I > P_a$ | 43 | 85 | 67 | 147 | 342 |
| $M = 3$ | $\hat{P}_I < P_b$ | 72 | 103 | 69 | 134 | 378 |
| | $\hat{P}_{II} > P_a$ | 48 | 53 | 50 | 98 | 249 |
| | $\hat{P}_{II} < P_b$ | 24 | 8 | 12 | 16 | 60 |
| | $N$ | 238 | 394 | 263 | 647 | 1542 |
| | $\hat{P}_I > P_a$ | 1 | 0 | 0 | 2 | 3 |
| $M = 4$ | $\hat{P}_I < P_b$ | 222 | 379 | 226 | 548 | 1375 |
| | $\hat{P}_{II} > P_a$ | 3 | 16 | 10 | 43 | 72 |
| | $\hat{P}_{II} < P_b$ | 167 | 131 | 43 | 47 | 388 |
| | $N$ | 96 | 357 | 240 | 599 | 1292 |
| | $\hat{P}_I > P_a$ | 0 | 0 | 0 | 0 | 0 |
| $M = 5$ | $\hat{P}_I < P_b$ | 95 | 356 | 240 | 595 | 1286 |
| | $\hat{P}_{II} > P_a$ | 0 | 0 | 1 | 10 | 11 |
| | $\hat{P}_{II} < P_b$ | 89 | 315 | 194 | 337 | 1292 |
| | $N$ | 1017 | 1757 | 1247 | 2934 | 6955 |
| | $\hat{P}_I > P_a$ | 82 | 244 | 248 | 635 | 1209 |
| Total | $\hat{P}_I < P_b$ | 389 | 838 | 535 | 1277 | 3039 |
| | $\hat{P}_{II} > P_a$ | 61 | 75 | 74 | 165 | 375 |
| | $\hat{P}_{II} < P_b$ | 281 | 460 | 252 | 417 | 1410 |

[a] Moneyness classification: $M = 1$: $-0.075 \leq MONEY$, $M = 2$: $0.075 < MONEY \leq -0.025$, $M = 3$: $-0.025 < MONEY \leq 0.025$, $M = 4$: $0.025 < MONEY \leq 0.075$, $M = 4$: $0.075 < MONEY$.

[b] Maturity classification: $MAT = 1$: time to maturity at most 30 days, $MAT = 2$: time to maturity between 31 and 60 days, $MAT = 3$: time to maturity between 61 and 90 days, $MAT = 4$: time to maturity more than 90 days.

[c] Number of observations.

[d] Theoretical price from Longstaff I method greater than observed ask price; absolute frequency.

[e] Theoretical price from Longstaff I method less than observed bid price; absolute frequency.

[f] Theoretical price from Longstaff II method greater than observed ask price; absolute frequency.

[g] Theoretical price from Longstaff II method less than observed bid price; absolute frequency.

Table 6:
Regression Analysis for DAX Pricing Errors[a]

| Dep. Variable | Constant | $N^b$ | $ARS^c$ | $T^d$ | $MONEY^e$ | Adj. $R^2$ |
|---|---|---|---|---|---|---|
| Calls (485 observations) | | | | | | |
| $\widehat{DAX}_{II} - DAX$ | 8.8658 | 0.2082[f] | -8.1552[f] | 30.1289 | -62.1085 | 0.4886 |
| $(\widehat{DAX}_{II} - DAX)/DAX$ | 0.0040 | 0.0001 | -0.0032[f] | 0.0136 | -0.0327 | 0.4849 |
| Puts (499 observations) | | | | | | |
| $\widehat{DAX}_{II} - DAX$ | 26.3637 | -3.4741 | -23.1028 | 120.3212 | 330.0830 | 0.6103 |
| $(\widehat{DAX}_{II} - DAX)/DAX$ | 0.0122 | -0.0016 | -0.0104 | 0.0553 | 0.1466 | 0.6189 |

[a] Coeffcients are signficant at the 5% level unless otherwise indicated.
[b] Number of options used for estimation.
[c] Average relative spread.
[d] Time to maturity in years.
[e] Average relative moneyness.
[f] Coefficient not significant at the 5% level.

Table 7:
Regression Analysis for Implied Volatility Differences[a]

| Dep. Variable | Constant | $N^b$ | $ARS^c$ | $T^d$ | $MONEY^e$ | Adj. $R^2$ |
|---|---|---|---|---|---|---|
| Calls (485 observations) | | | | | | |
| $\hat{\sigma}_{II} - \hat{\sigma}_I$ | -0.0214 | $< 0.0001$[f] | 0.0001[f] | -0.0069 | 0.0529 | 0.2003 |
| $(\hat{\sigma}_{II} - \hat{\sigma}_I)/\hat{\sigma}_I$ | -0.1115 | 0.0009[f] | 0.0233[f] | -0.0604 | 0.2164 | 0.2182 |
| Puts (499 observations) | | | | | | |
| $\hat{\sigma}_{II} - \hat{\sigma}_I$ | 0.0416 | -0.0026 | -0.0218 | 0.0358 | 0.2523 | 0.3859 |
| $(\hat{\sigma}_{II} - \hat{\sigma}_I)/\hat{\sigma}_I$ | 0.2085 | -0.0137 | -0.1000 | 0.2138 | 1.2691 | 0.3769 |

[a] Coefficients signifcant at the 5% level unless otherwise indicated.
[b] Number of options used for estimation.
[c] Average relative spread.
[d] Time to maturity.
[e] Average realtive moneyness.
[f] Coefficient not significant at the 5% level.

Table 8:
Regression Analysis of Option Pricing Errors[a]

| Dep. Variable | Constant | $T^b$ | $RS^c$ | $MONEY^d$ | $MONSQR^e$ | $P^f$ | Adj. $R^2$ |
|---|---|---|---|---|---|---|---|
| | | | Calls (7263 observations) | | | | |
| $\hat{P}_I - P$ | 3.5252 | 0.0095 | -6.8560 | 2.2561[g] | 219.6087 | -0.0454 | 0.5096 |
| $(\hat{P}_I - P)/P$ | 0.0127 | -0.0008 | -0.0405 | -2.2382 | 2.8711 | 0.0009 | 0.3897 |
| $\hat{P}_{II} - P$ | 0.7274 | 0.0024 | -2.2152 | 2.1353[g] | 72.1417 | -0.0093 | 0.0764 |
| $(\hat{P}_{II} - P)/P$ | 0.0327 | -0.0002 | -0.1767 | -0.4401 | 1.1988 | $< 0.0001$[g] | 0.1100 |
| | | | Puts (6955 observations) | | | | |
| $\hat{P}_I - P$ | 3.5234 | 0.0192 | 3.8091 | -138.8813 | 404.3095 | -0.0783 | 0.6108 |
| $(\hat{P}_I - P)/P$ | 0.0893 | 0.0010 | -0.3409 | -4.1603 | 3.8914 | -0.0025 | 0.8412 |
| $\hat{P}_{II} - P$ | 2.9836 | 0.0141 | -0.1650[g] | -58.4012 | 142.9362 | -0.0564 | 0.1774 |
| $(\hat{P}_{II} - P)/P$ | 0.0585 | 0.0005 | -0.4872 | -1.2929 | -1.3629 | -0.0009 | 0.7307 |

[a] Coefficients are significant at the 5% level using standard $t$-statistics unless otherwise indicated.
[b] Time to maturity.
[c] Relative spread.
[d] Relative moneyness.
[e] Relative moneyness squared.
[f] Observed price.
[g] Coefficient not signficant at the 5% level.

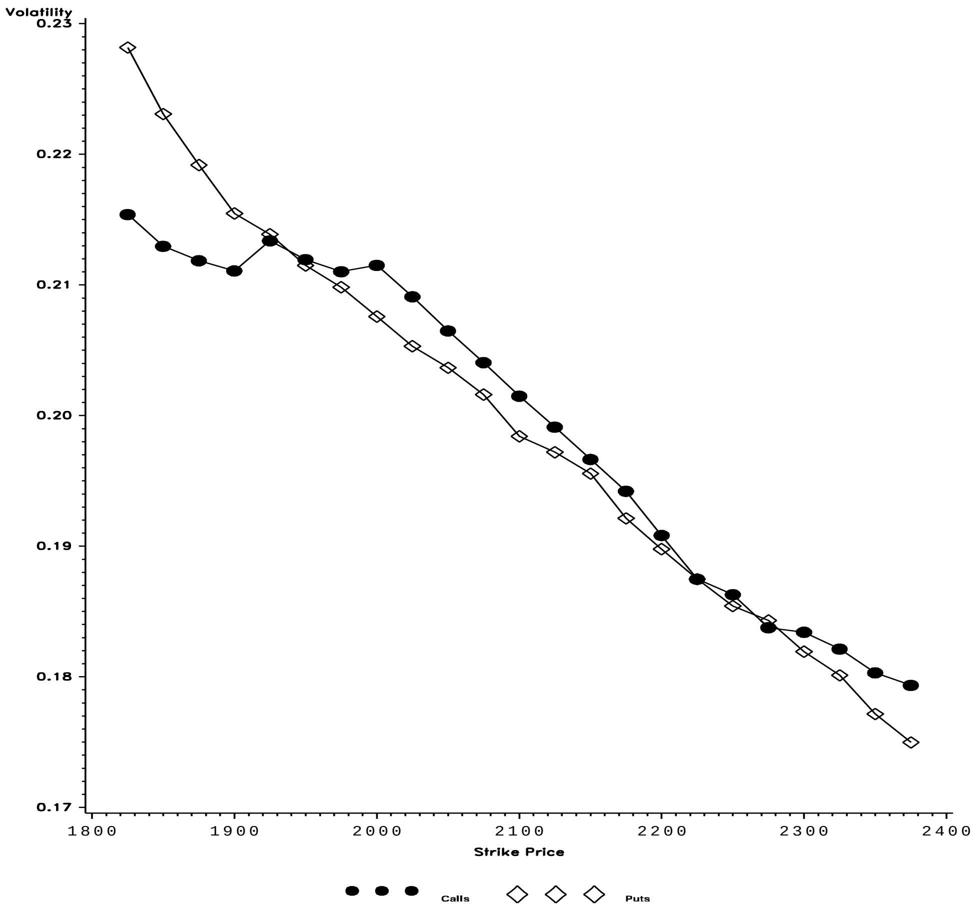


Figure 1:
Typical smile pattern for DAX options

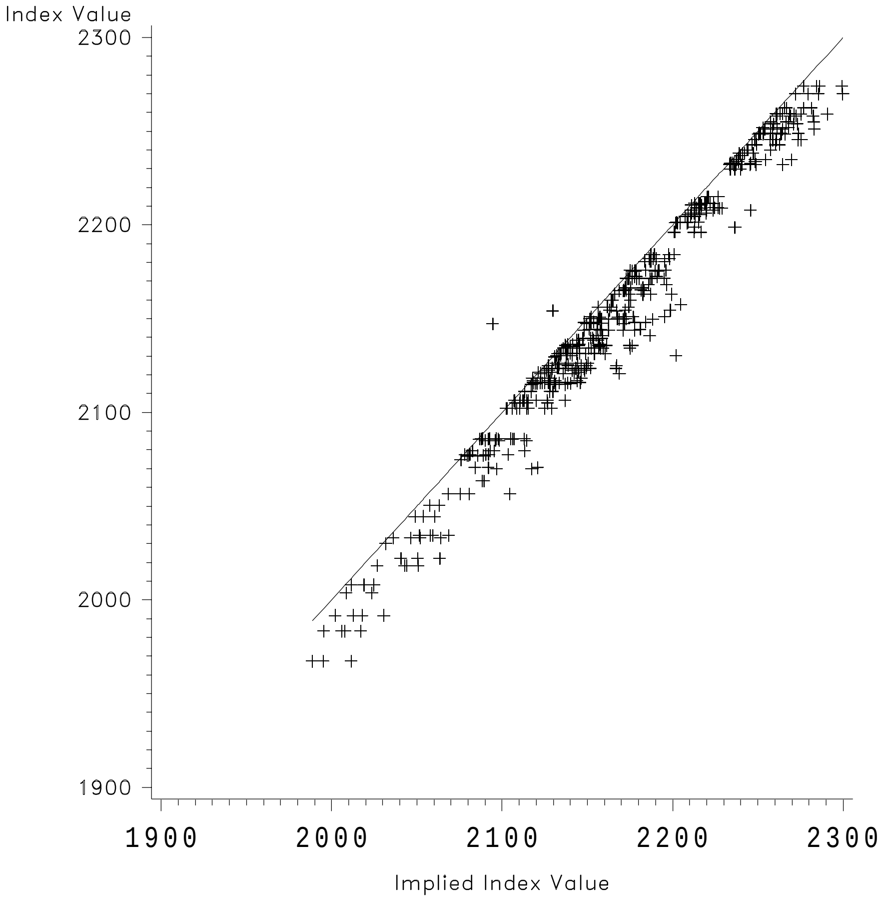


Figure 2:
Observed vs. implied DAX prices for calls

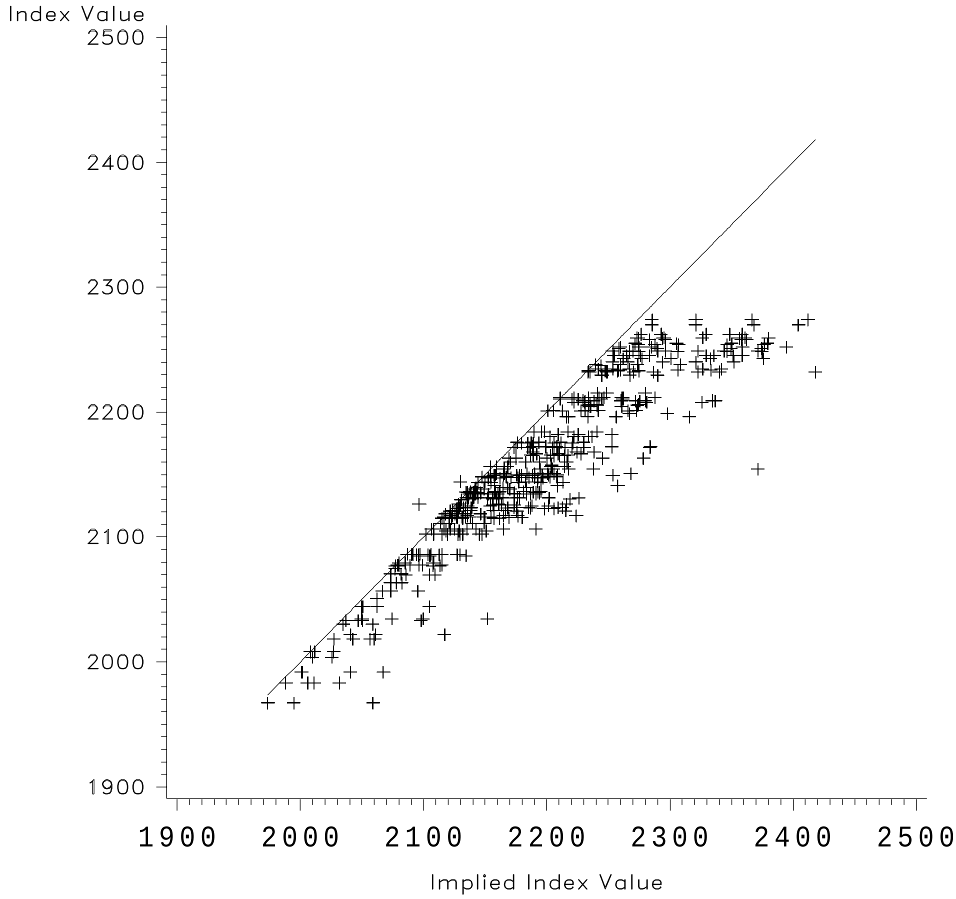


Figure 3:
Observed vs. implied DAX prices for puts

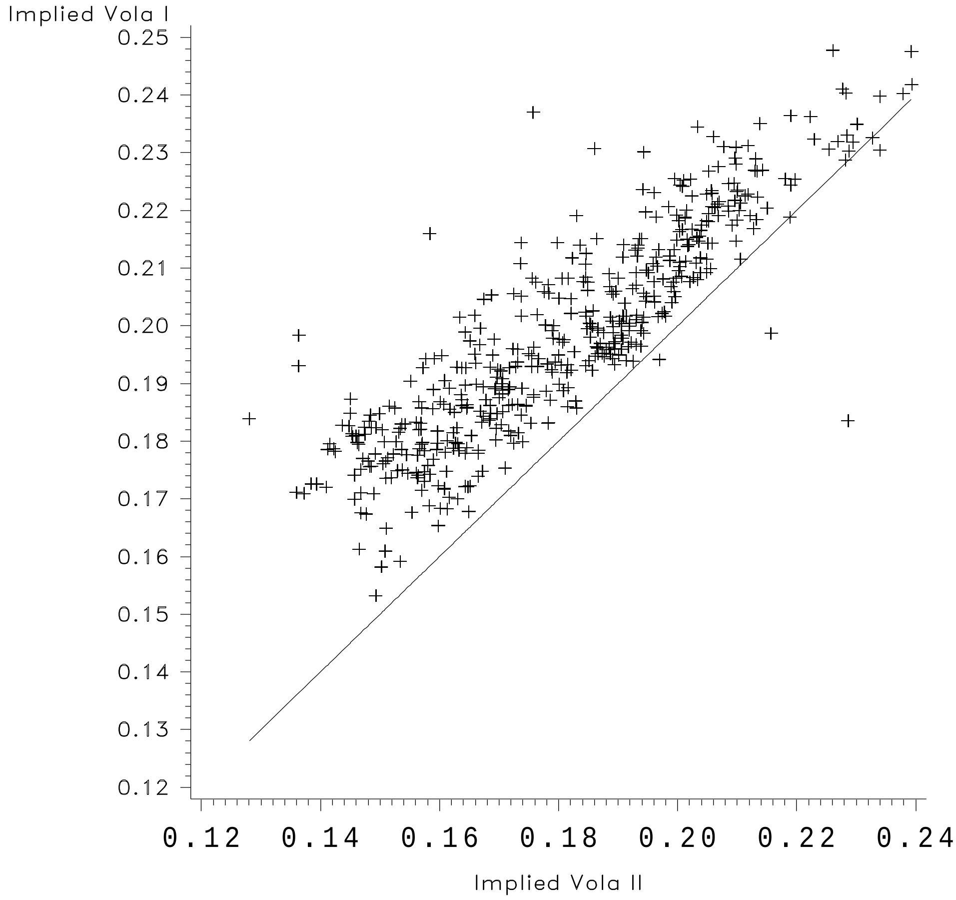


Figure 4:
Implied volatilities from Longstaff methods for calls

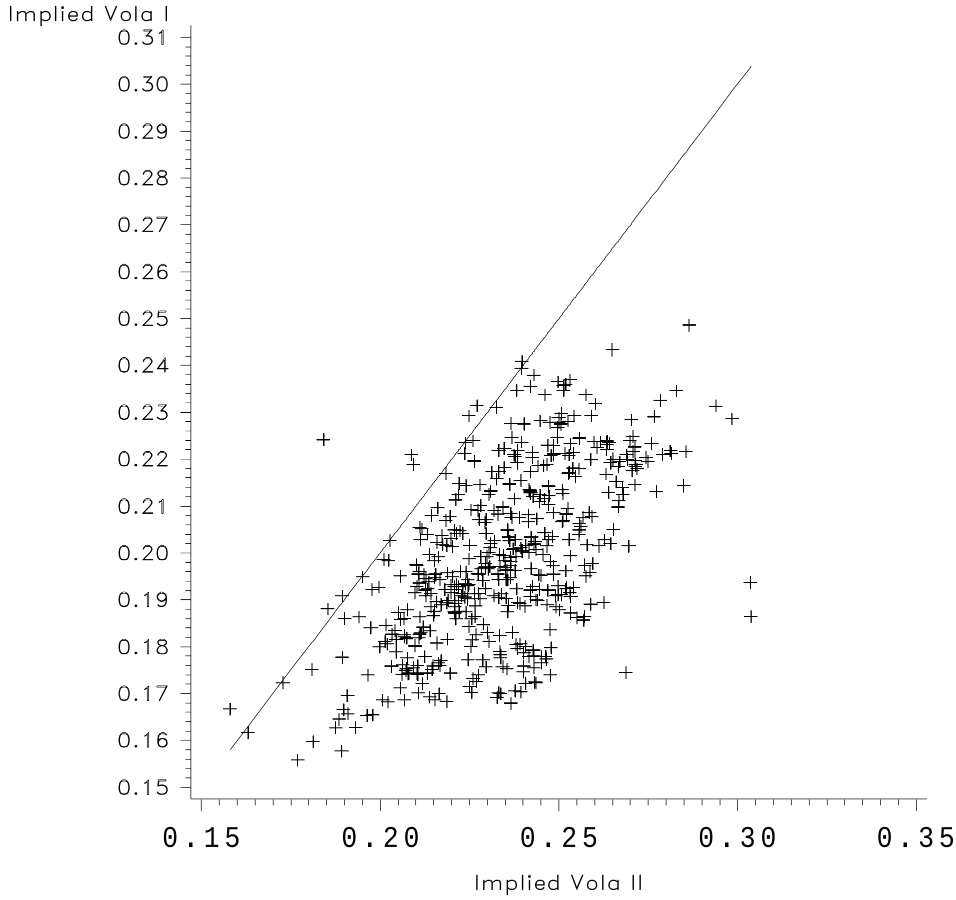


Figure 5:
Implied volatilities from Longstaff methods for puts

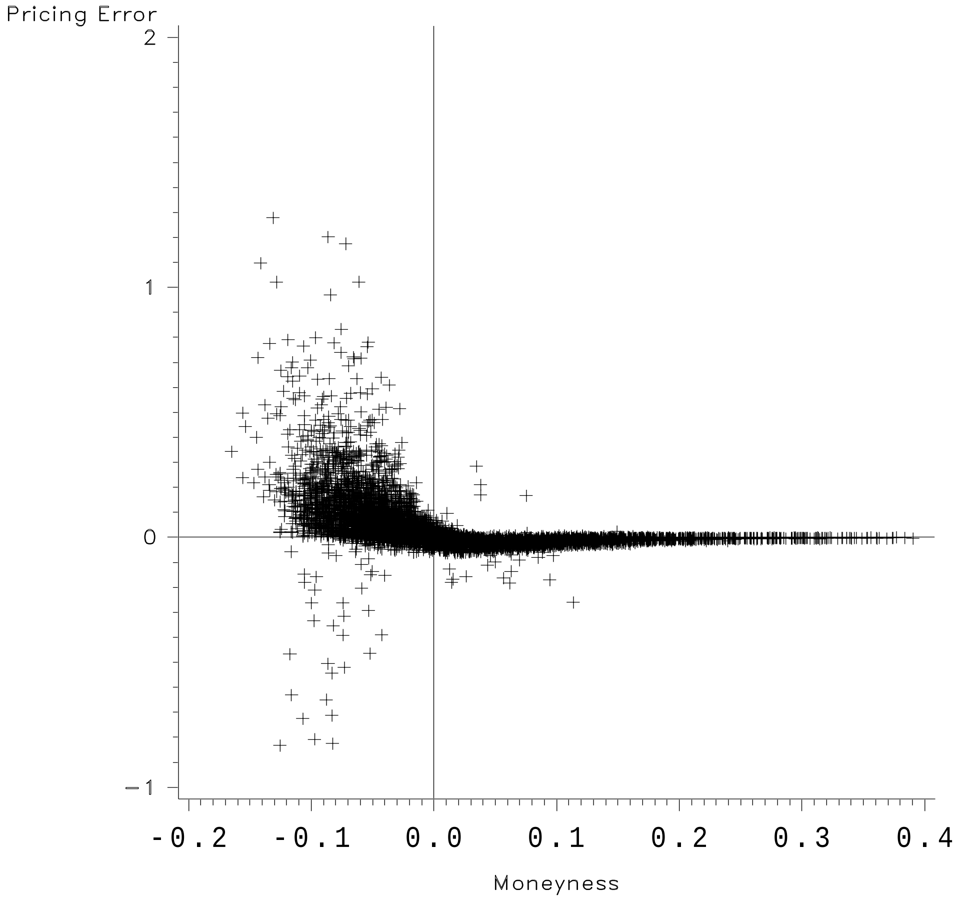


Figure 6:
Relative call pricing errors Longstaff I versus moneyness

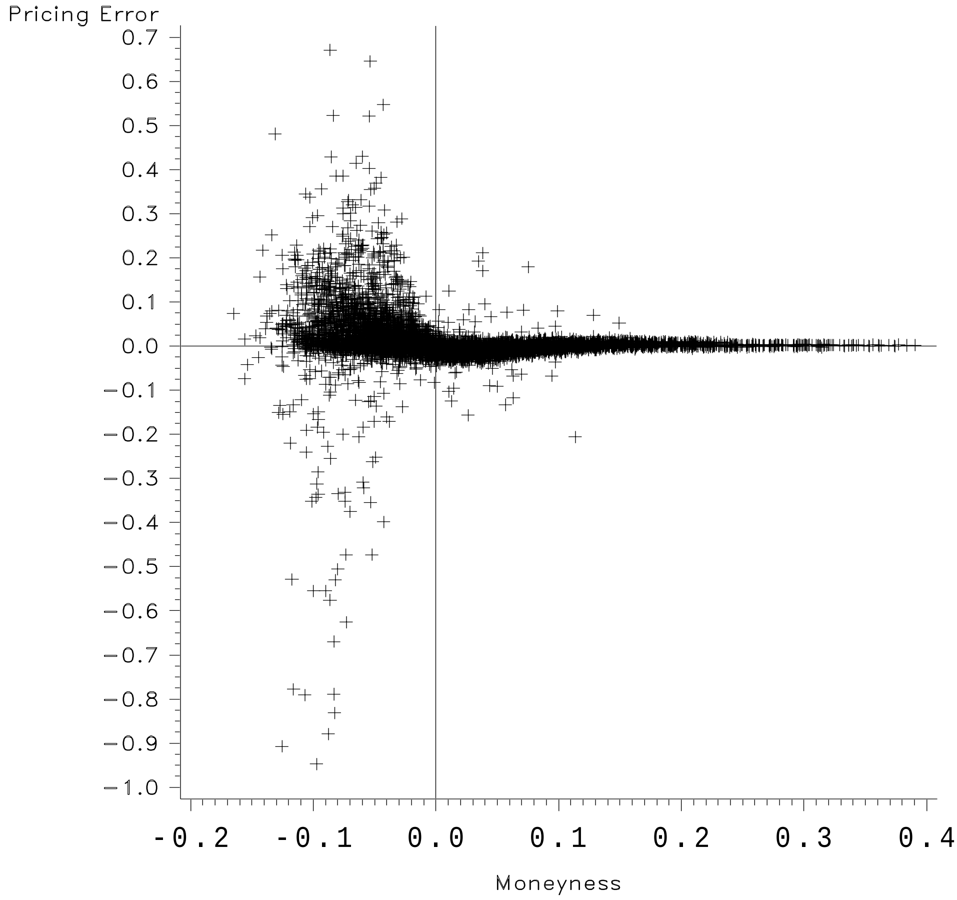


Figure 7:
Relative call pricing errors Longstaff II versus moneyness

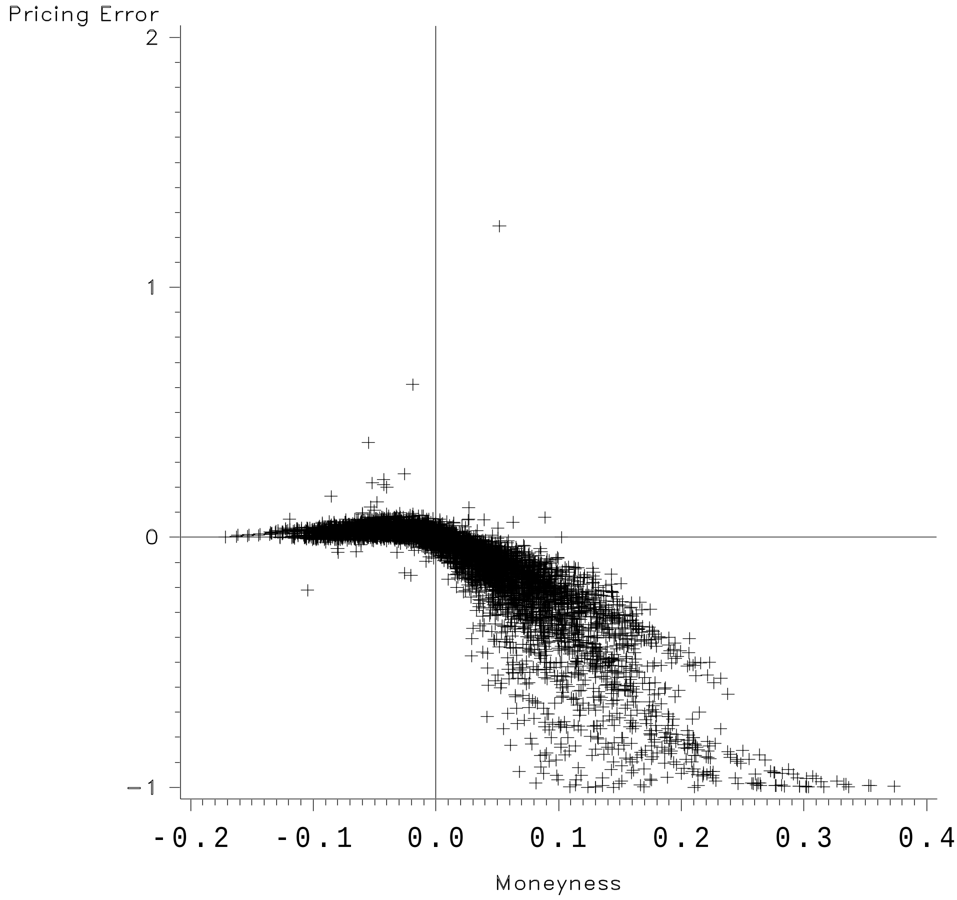


Figure 8:
Relative put pricing errors Longstaff I versus moneyness

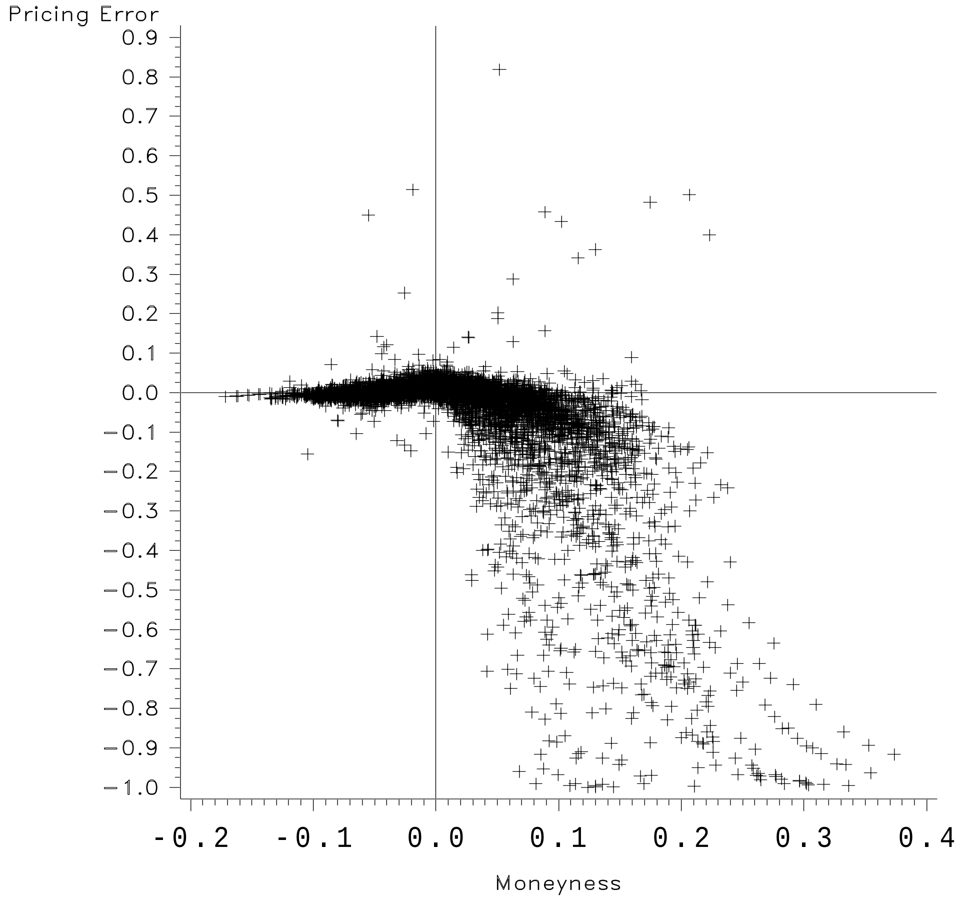


Figure 9:
Relative put pricing errors Longstaff II versus moneyness

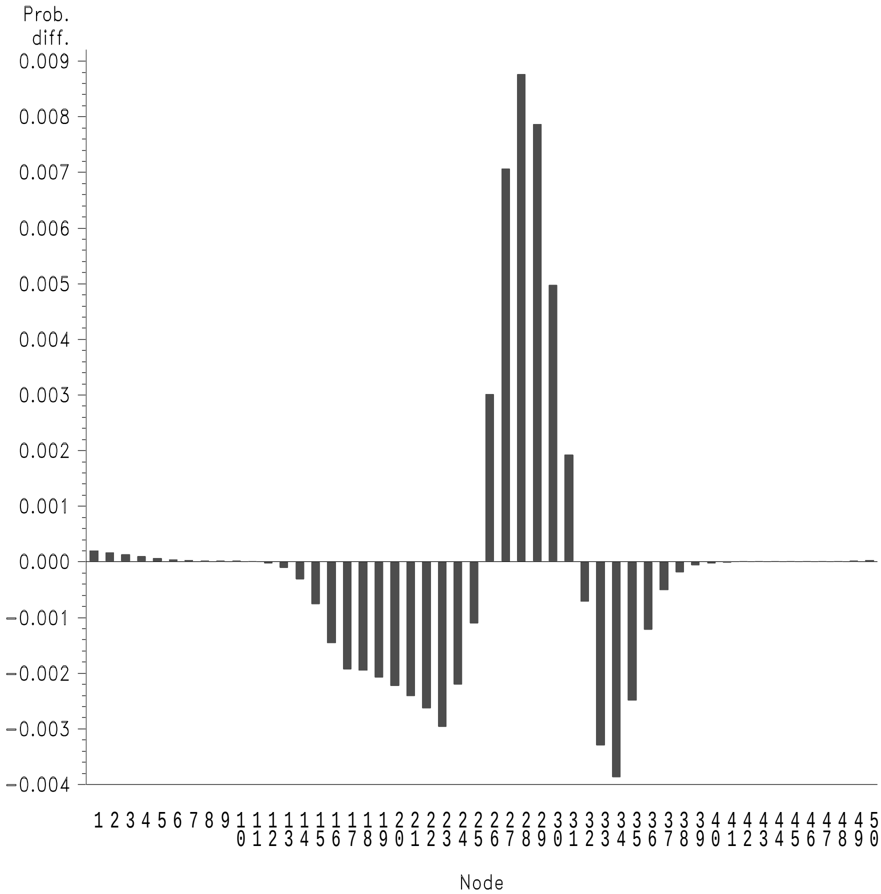


Figure 10:
Differences between implied and a priori probabilities for calls

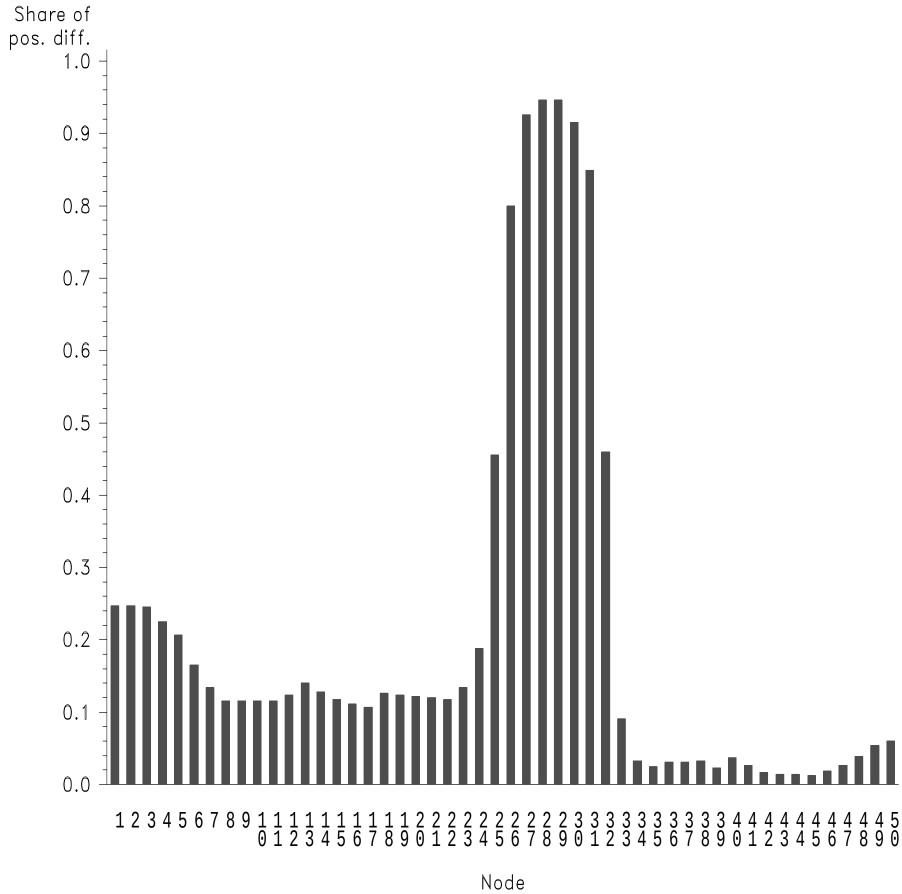


Figure 11:
Share of positive differences
between implied and a priori distribution for calls

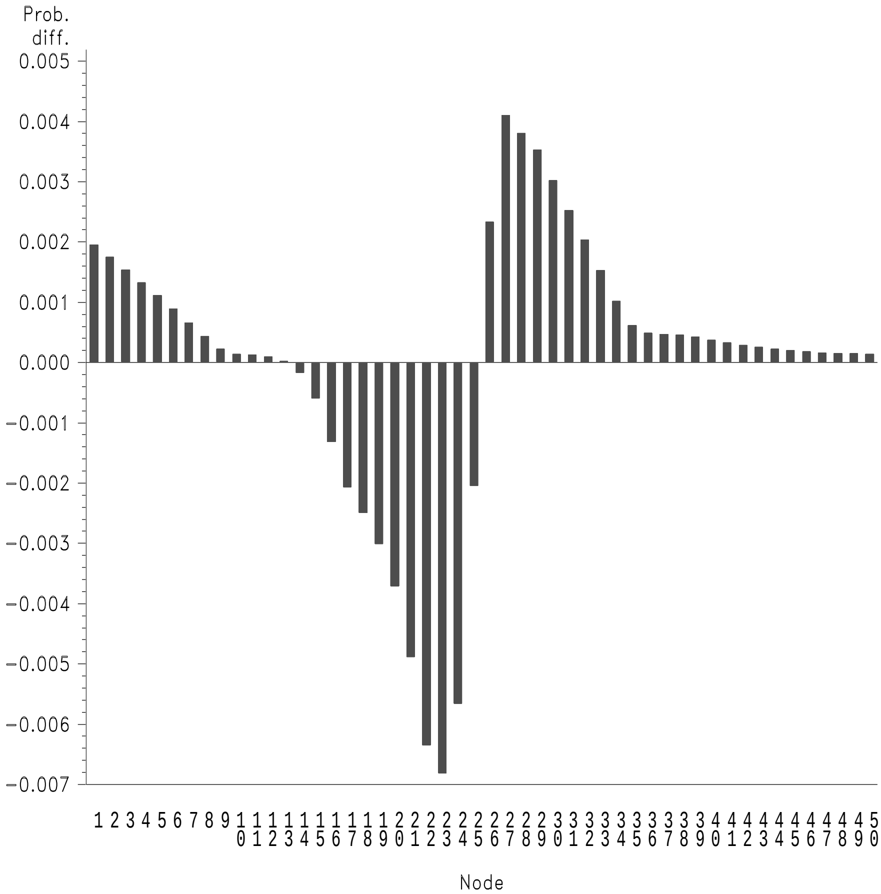


Figure 12:
Differences between implied and a priori probabilities for puts

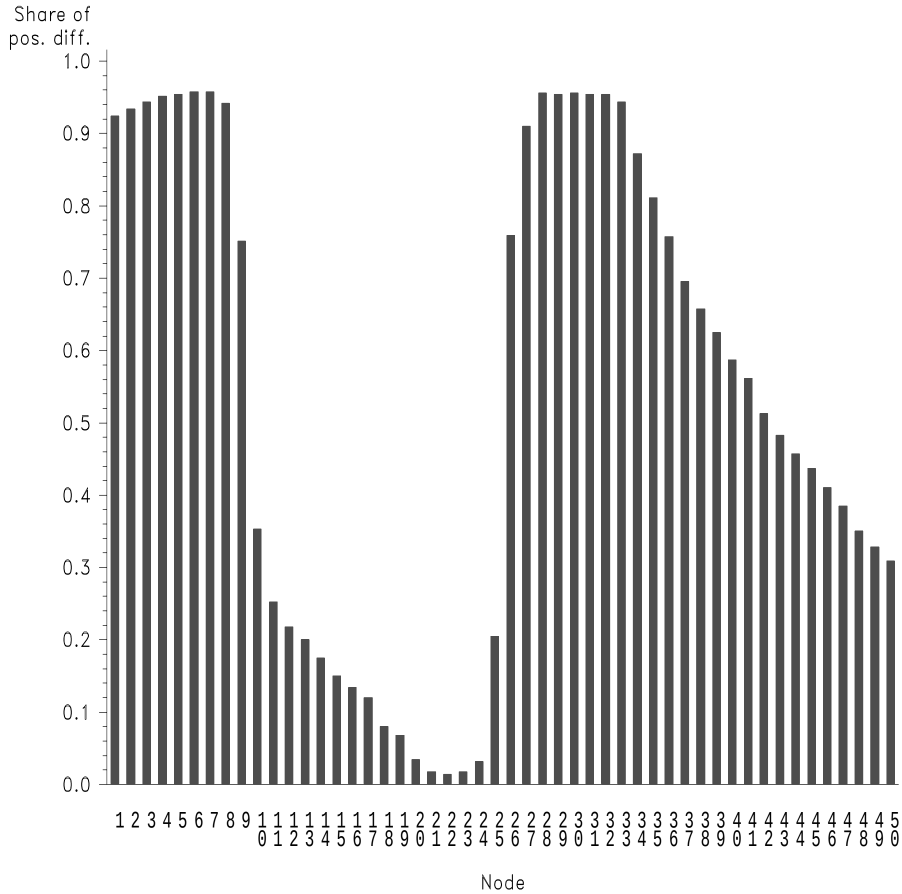


Figure 13:
Share of positive differences
between implied and a priori distribution for puts