Integral representations of projection functions

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Preface

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Contents

1 Introduction

The classical core of the geometry of convex bodies in $\mathbb{R}^d$, the so-called Brunn-Minkowski theory, is based on the notion of the mixed volume $V(K_1, \dots, K_d)$ of convex bodies $K_1, \dots, K_d$ (non-empty compact convex subsets of $\mathbb{R}^d$), and the central issues are uniqueness and extremality results. The volume $V_d(\alpha_1 K_1 + \ldots + \alpha_d K_d)$ of a linear combination (with non-negative coefficients) has a polynomial expansion in the variables $\alpha_1, \dots, \alpha_d$, and the mixed volumes are the coefficients of this polynomial. A special case is Steiner's formula

$$V_d(K + \rho B^d) = \sum_{i=0}^{d} \rho^{d-i} \kappa_{d-i} V_i(K), \quad \rho \geq 0,$$

where B^d is the unit ball in $\mathbb{R}^d$. The coefficients $V_0(K), \dots, V_d(K)$ are the intrinsic volumes $V_0(K), \dots, V_d(K)$ of the convex body K.

A local variant of Steiner's formula was introduced in 1937/1938 by Alexandrov and Fenchel-Jessen, the surface measures $S_0(K, \cdot), \dots, S_{d-1}(K, \cdot)$ of K. The surface measures are defined by a local variant of Steiner's formula,

$$V_d(M_\rho(K, \eta)) = \frac{1}{d} \sum_{j=0}^{1-d} \rho^{d-1-j} \binom{d-1}{j} S_j(K, \eta), \quad \rho \geq 0,$$

where η is a subset of the unit sphere S^{d-1}, and $M_\rho(K, \eta)$ is a local parallel set of K in the directions in η. The surface measures defined by this equation are measures on S^{d-1}. In the last decades other local variants have been introduced, for example the curvature measures by Federer, and the support measures by Schneider (which are measures on $\mathbb{R}^d$ and $\mathbb{R}^d \times S^{d-1}$, respectively).

The following equation is an integral representation of certain mixed volumes,

$$V(M, K, \dots, K) = \frac{1}{d} \int_{S^{d-1}} h(M, u) S_{d-1}(K, du), \tag{1.1}$$

where $h(M, \cdot)$ is the support function of M. Equation (1.1) relates surface measures and mixed volumes, and is fundamental for many uniqueness results. A special case is the representation of the projection function

$$V_{d-1}(K|\, v^\perp) = \frac{1}{2} \int_{S^{d-1}} |\langle v, u \rangle| S_{d-1}(K, du), \tag{1.2}$$

that expresses the $(d-1)$-dimensional volume of the orthogonal projection of K onto the hyperplane $L = v^\perp$ by means of the surface measure and the scalar product $\langle \cdot, \cdot \rangle$.

Equation (1.2) also makes a connection to spherical transforms, for example, the Radon transform.

$V_{d-1}(K|\cdot)$ is not the only projection function. More generally, for $j \in 1, \dots, d-1$, the function $V_j(K|\cdot)$ is a projection function of K, where $V_j(K|L)$ is the j-dimensional volume of $K|L$, and L ranges over $\mathcal{L}_j^d$, the Grassmann manifold of j-dimensional linear subspaces of $\mathbb{R}^d$. The only known integral representation is (1.2), i. e. for the case $j = d-1$ (where $\mathcal{L}_{d-1}^d$ and S^{d-1} are identified). However, if we consider only centrally symmetric convex bodies that fulfill a certain smoothness condition, there exists the integral representation

$$V_j(K|L) = \int_{\mathcal{L}_j^d} |\langle L, M\rangle| \varrho_j(K, dM), \quad j = 1, \dots, d-1,$$

with the so-called projection generating measure $\varrho_j(K, \cdot)$ on $\mathcal{L}_j^d$. However, it can be shown that such a representation cannot exist for general convex bodies, at least not with measures on $\mathcal{L}_j^d$. One of the goals of this thesis is to find integral representations of projection functions with measures on suitable flag manifolds.

In [1] Ambartzumian presents a so-called $\sin^2$-representation of the width function $w(K, u) = h(K, u) + h(K, -u)$ of convex bodies in $\mathbb{R}^3$,

$$h(K, u) + h(K, -u) = \int \sin^2(\alpha_{u,v,L})\, \mu(K, d(v, L)), \tag{1.3}$$

where $\mu(K, \cdot)$ is a measure on the flag manifold

$$\{(v, L) \in S^2 \times \mathcal{L}_1^3 : v \perp L\},$$

and $\alpha_{u,v,L}$ is a certain angle depending on u, v and L. These measures are defined for polytopes first, and then the existence of measures with property (1.3) is shown by approximating convex bodies with polytopes and using a compactness argument.

After some basic notions and results are presented in Chapter 2, we go on to generalize Ambartzumian's result to arbitrary dimensions d and arbitrary projection functions in Chapter 3. In that chapter we will also show that the measures $\mu_j(K, \cdot)$ that appear in the integral representations of the projection functions of K do not depend weakly continuously on K.

In order to obtain a representation with weakly continuous measures we introduce measures $\Theta_j^{(k)}(K, \cdot)$ on $\mathbb{R}^d \times S^{d-1} \times \mathcal{L}_k^d$ in the following way. For a convex body K, the invariant measure $\mu_k(M_\rho^{(k)}(K, \eta))$ of local parallel sets of k-flats has a polynomial expansion in the parameter ρ, where the measures $\Theta_j^{(k)}(K, \cdot)$ are the coefficients:

$$\mu_k(M_\rho^{(k)}(K, \eta)) = \frac{1}{d-k} \sum_{j=0}^{d-k-1} \rho^{d-k-j} \binom{d-k}{j} \Theta_j^{(k)}(K, \eta), \quad \rho \geq 0,$$

here η is a subset of $\mathbb{R}^d \times S^{d-1} \times \mathcal{L}_k^d$. These measures depend weakly continuously on K. The projection onto the second and third component yields measures $S_j^{(k)}$ on $S^{d-1} \times \mathcal{L}_k^d$

that are concentrated on the flag manifold

$$\{(u, L) \in S^{d-1} \times \mathcal{L}_k^d :\ u \perp L\}.$$

In Chapter 5 we will use these measures to prove integral representations of projections functions with measures depending weakly continuously on K.

2 Notation and preliminaries

2.1 Convex Geometry

In this section we present the basic notions and facts of convex geometry that we will need. Most of the notation follows the book of Schneider [9], and the book of Schneider and Weil [10]. A more in-depth introduction to convex geometry can also be found in the book by Schneider [9]. Where well-known facts are stated without proof, they can be found in one of these books.

2.1.1 Geometry

Our general setting is the Euclidean space $\mathbb{R}^d$ $(d \geq 1)$. Its scalar product and norm will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let B^d be the closed unit ball and let S^{d-1} be the unit sphere in $\mathbb{R}^d$.

For any subset $A \subset \mathbb{R}^d$ let $\operatorname{lin} A$ be the linear hull of A, and let $A^\perp$ be the (largest) subspace that is orthogonal to A. We designate the affine hull of A by $\operatorname{aff} A$.

Two linear subspaces $L, L' \subset \mathbb{R}^d$ are called parallel if $L \subset L'$ or $L' \subset L$. Two affine flats $E = L + x, E' = L' + x'$ for $x, x' \in \mathbb{R}^d$ are called parallel, if L and L' are parallel. Moreover, we speak of L as the linear subspace parallel to E. For any non-empty set $A \subset \mathbb{R}^d$ we denote the linear subspace parallel to its affine hull $\operatorname{aff} A$ by $L(A)$. The dimension of A is $\dim A := \dim L(A)$. As an exception, for a point $u \in \mathbb{R}^d$, we put $L(u) := \operatorname{lin}\{u\}$ for convenience.

For a set $A \subset \mathbb{R}^d$ and an (affine or linear) subspace $L \subset \mathbb{R}^d$ let $A|L$ be the orthogonal projection of A onto L, i. e. $A|L = (A + L^\perp) \cap L$. We also write $x|L$ for the orthogonal projection of any point x onto L.

For arbitrary sets $A, B \subset \mathbb{R}^d$ we define their Minkowski-sum as

$$A + B := \{a + b : \ a \in A, b \in B\},$$

and $x + B := \{x\} + B$ for $x \in \mathbb{R}^d$. The Minowski-difference $A \ominus B$ is defined as

$$A \ominus B := \{a \in A : \ a + B \subset A\}.$$

For $x, y \in \mathbb{R}^d$, we define $[x, y]$ as the segment connecting x and y. For compact sets $C_1, C_2 \subset \mathbb{R}^d$ we define the Hausdorff-distance

$$\tilde{d}(C_1, C_2) := \min\{\epsilon > 0 : C_2 \subset C_1 + \epsilon B^d, C_1 \subset C_2 + \epsilon B^d\}.$$

This distance defines a topology on the compact subsets of $\mathbb{R}^d$.

We denote the diameter of a compact set $C \subset \mathbb{R}^d$ by

$$D(C) := \max\{\|x - y\| : x, y \in C\}.$$

For $x \in \mathbb{R}^d$ and a compact set $C \subset \mathbb{R}^d$, we define the distance of x and C by

$$d(C, x) := \min\{\|x - y\| : y \in C\}.$$

In the special case $C = \{z\}$, we also write $d(z, x)$ instead of $d(\{z\}, x) = \|z - x\|$. For an affine flat $E \subset \mathbb{R}^d$ we define $d(E, C)$ and $d(E, x)$ analogously.

Let A be a subset of some larger set M. Then we define the indicator function

$$I_A : M \to \mathbb{R}, x \mapsto \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

When it is convenient, we write $I(x \in A)$ for $I_A(x)$.

For a set $A \subset \mathbb{R}^d$, $\operatorname{bd} A$ is the boundary of A, and $\operatorname{int} A$ is the interior of A. The closure of A is denoted by $\operatorname{cl} A$. $\operatorname{relbd} A$ is the relative boundary of A, i. e. the boundary of A with respect to $\operatorname{aff} A$. The relative interior $\operatorname{relint} A$ is defined analogously.

The Lebesgue measure on $\mathbb{R}^d$ will be called λ_d. If L is a k-dimensional affine subspace of $\mathbb{R}^d$, the Lebesgue measure on L will be denoted by λ_k^L. For convenience, we will sometimes write λ_k for λ_k^L if the subspace L is clear from the context. The spherical Lebesgue measure on S^{d-1} will be called ω_{d-1}, and the spherical Lebesgue measure on $S^{d-1} \cap L$ will be called ω_{k-1}^L. Again, we will write ω_{k-1} for ω_{k-1}^L if the subspace L is clear from the context. For a linear subspace $L' \subset \mathbb{R}^d$ of dimension $\dim L' < k$, we have $\omega_{k-1}^L(L') = 0$ and $\lambda_k^L(L') = 0$ for all k-flats $L \in \mathcal{L}_k^d$ containing L'. Therefore, the expressions $\omega_{k-1}^{L'}$ and $\lambda_k^{L'}$ denote the zero measure on L'.

We define the constants

$$\begin{aligned} \kappa_d &:= \lambda_d(B^d) &&= \frac{\pi^{d/2}}{\Gamma(n/2+1)}, \\ \sigma_d &:= \omega_{d-1}(S^{d-1}) = d\kappa_d. \end{aligned}$$

The Lebesgue measure is invariant under rotations, translations, and reflections. The spherical Lebesgue measure is invariant under rotations and reflections.

We introduce the binomial coefficients,

$$\binom{n}{k} := \frac{n!}{k!(n-k)!},$$

where n and k are non-negative integer numbers, and $k \leq n$. Moreover, for $n < 0$, $k < 0$, or $k > n$, we put $\binom{n}{k} := 0$.

For a topological space X, we designate the set of all Borel subsets of X by $\mathcal{B}(X)$. Measurability will always be with respect to this σ-algebra, and all measures will be Borel measures.

2.1.2 Flats

Let $\mathcal{L}_k^d$ be the Grassmann manifold of k-dimensional linear subspaces of $\mathbb{R}^d$, and let $\mathcal{E}_k^d$ be the set of k-dimensional affine subspaces of $\mathbb{R}^d$. An element L of $\mathcal{E}_k^d$ or $\mathcal{L}_k^d$ is called a k-flat. If the dimension of L is clear from the context, it will simply be called a flat. We now follow the steps of Schneider and Weil [10] to introduce topologies on $\mathcal{L}_k^d$ and $\mathcal{E}_k^d$.

We first need a topology on the group of rotations in $\mathbb{R}^d$, which is denoted by SO_d. Each rotation $\vartheta \in SO_d$ is represented with respect to the standard basis of $\mathbb{R}^d$ by an orthogonal matrix $M(\vartheta)$ whose determinant is 1. The mapping $\mu : \vartheta \mapsto M(\vartheta)$ is an isomorphism from the group SO_d onto the group $SO(d)$ of orthogonal (d,d)-matrices with determinant 1. $SO(d)$ can be interpreted as a bounded subset of $\mathbb{R}^{d^2}$. With respect to the topology of $\mathbb{R}^{d^2}$ the set $SO(d)$ is closed, and therefore compact. The mapping $(M,N) \mapsto MN^{-1}$ from $SO(d) \times SO(d)$ into $SO(d)$ is continuous, and the same holds for the mapping $(M,x) \mapsto Mx$ from $SO(d) \times \mathbb{R}^d$ into $\mathbb{R}^d$. Now we can use μ^{-1} to carry the topology of $SO(d)$ over to SO_d. Thus SO_d is a compact topological group with countable basis, and SO_d acts continuously on $\mathbb{R}^d$.

For a fixed k-flat $L \in \mathcal{L}_k^d$, we define the function

$$\beta_k : SO_d \to \mathcal{L}_k^d, \quad \vartheta \mapsto \vartheta L.$$

The topology we use for $\mathcal{L}_k^d$ is the finest topology such that β_k is continuous.

Similarly to the mapping μ, we define the mapping $\gamma : (x,\vartheta) \mapsto t_x \circ \vartheta$ from $\mathbb{R}^d \times SO_d$ onto the group G_d of rigid motions in $\mathbb{R}^d$ (t_x denotes translation by x). γ is used to introduce a topology on G_d. We define the function

$$\gamma_k : L^{\perp} \times SO_d \to \mathcal{E}_k^d, \quad (x,\vartheta) \mapsto \vartheta(L+x),$$

and the topology for $\mathcal{E}_k^d$ is the finest topology such that γ_k is continuous. The topologies on $\mathcal{L}_k^d$ and $\mathcal{E}_k^d$ do not depend on the choice of L.

It can then be shown that $\mathcal{L}_k^d$ and $\mathcal{E}_k^d$ are locally compact with a countable basis. SO_d acts continously and transitively on L_k^d, and G_d acts continuously and transitively on $\mathcal{E}_k^d$.

There is a unique probability measure ν_k on $\mathcal{L}_k^d$ that is invariant under rotations. More generally, let $L \in \mathcal{L}_k^d$, and let $0 \leq j \leq d$. We define ν_j^L to be the invariant probability measure on the topological space $\mathcal{L}_k^L$ of all j-dimensional linear subspaces of $\mathbb{R}^d$ that contain (or are contained in) L.

Let $A^k = \gamma_k([0,1]^{d-k} \times SO_d)$, where $[0,1]^{d-k}$ is a $(d-k)$-dimensional unit cube in $L^{\perp}$. There exists exactly one invariant measure μ_k on $\mathcal{E}_k^d$ such that $\mu_k(A^k) = 1$. This measure does not depend on the choice of L.

Let $E \in \mathcal{E}_k^d$ and $E' \in \mathcal{E}_j^d$ be two flats. They are called in general relative position if $\dim(E+E') = \min(d, k+j)$.

Let $E \in \mathcal{E}_k^d$. The set of all j-flats in $\mathcal{E}_j^d$ that are not in general relative position to E is a set of μ_j-measure zero.

A flat $H \in \mathcal{E}_{d-1}^d$ is called a hyperplane.

The relative position of two k-flats $E, F \in \mathcal{L}_k^d$ defines a number $|\langle E, F\rangle|$ in the following way:

$$|\langle E, F\rangle| := \lambda_k^F(C|F), \tag{2.1}$$

where $C \subset E$ is a measurable set with $\lambda_k^E(F) = 1$. Typically, one would use a unit cube in E for the set C, i.e. $C = [0, u_1] + \ldots + [0, u_k]$, where $u_1, \ldots, u_k$ is an orthonormal basis of E. The number $|\langle E, F\rangle|$ can also be defined in another way, which shows that (2.1) does not depend on the choice of C. Let $\pi : E \to F$ be the orthogonal projection onto F. The Jacobian of π at any point $x \in E$ is

$$c := \begin{vmatrix} \langle u_1, v_1\rangle & \cdots & \langle u_k, v_1\rangle \\ \vdots & \ddots & \vdots \\ \langle u_1, v_k\rangle & \cdots & \langle u_k, v_k\rangle \end{vmatrix},$$

where $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ are orthonormal bases of E and F, respectively. We see that this determinant does not depend on x, and therefore

$$\begin{aligned} \lambda_k^F(C|F) &= \int_F I_{\pi(C)}(x)\, d\lambda_k^F(x) \\ &= \int_E I_C(x)\, |c|\; d\lambda_k^E(x) \\ &= |c|\, \lambda_k^E(C). \end{aligned}$$

The right hand side is $|c|$, and thus $|\langle E, F\rangle| = |c|$, independently from C. It is also clear that $|\langle E, F\rangle| = |\langle F, E\rangle|$.

Now assume $E \in \mathcal{L}_k^d$, $F \in \mathcal{L}_j^d$, and $0 \le k < j \le d$. We extend the definition of $|\langle\cdot,\cdot\rangle|$ to this case. If $\dim(E|F) < k$, we put $|\langle F, E\rangle| := |\langle E, F\rangle| := 0$. If $\dim(E|F) = k$ (which is equivalent to E and $F^\perp$ being in general relative position), we put

$$|\langle F, E\rangle| := |\langle E, F\rangle| := |\langle E, E|F\rangle| = \lambda_k^{E|F}(C|F),$$

where C is any measurable subset of E with $\lambda_k^E(C) = 1$. For arbitrary flats E, F, the relation $|\langle E, F\rangle| = |\langle E^\perp, F^\perp\rangle|$ holds.

2.1.3 Convex bodies

A set $A \subset \mathbb{R}^d$ is convex if for all $x, y \in A$ we also have $[x, y] \subset A$. The convex hull $\operatorname{conv} A$ of a subset A of $\mathbb{R}^d$ is the smallest convex subset of $\mathbb{R}^d$ containing A.

A convex body K is a compact non-empty convex subset of $\mathbb{R}^d$. The set of all convex bodies is denoted by $\mathcal{K}$.

A convex body P that is the intersection of finitely many halfspaces is called a polytope. This is equivalent to P being the convex hull of a finite non-empty set. The set of all polytopes is a dense subset of $\mathcal{K}$, i.e. for any convex body K there exists a sequence (P_i) of polytopes such that

$$K = \lim_{i\to\infty} P_i$$

(with respect to the Hausdorff distance).

Let H be a hyperplane. If K is a convex body such that $H \cap K \neq 0$ and K is contained in a closed half-space bounded by H, H is called a supporting hyperplane of K. If furthermore $u \in S^{d-1}$ with $L(H) = u^\perp$, and K is not contained in the halfspace that contains $H + u$, u is called an outer normal of the supporting hyperplane, and we write $H_u = H_u(K)$ for this hyperplane. The set $K_u := K \cap H_u$ is called the support set of K in the direction u.

The support sets of a polytope P are called faces of P. P is also considered a face of itself. A k-dimensional face is called a k-face ($0 \leq k \leq d$). In particular, $(d-1)$-faces are facets of P, and 0-faces are vertices of P. The set of all k-faces of P is denoted by $\mathcal{F}_k(P)$. Each $x \in \operatorname{bd} P$ lies in the relative interior of exactly one face of P. For faces F of polytopes, we introduce the special notation $F^\perp := L(F)^\perp$. A face F of a polytope P and a flat E are in general relative position if $L(F)$ and E are in general relative position. If all faces of P are in general relative position to E, we say that P and E are in general relative position.

Let $K \in \mathcal{K}$. For $x \in \operatorname{bd} K$ we define the normal cone of K at x,

$$N(K, x) := \{u \in \mathbb{R}^d \setminus \{0\} : x \in K_{u/\|u\|}\}.$$

A related spherical set is $n(K, x) := N(K, x) \cap S^{d-1}$. For a polytope P with a face F we put $N(P, F) := N(P, x)$ and $n(P, F) := n(P, x)$, where x is any point in $\operatorname{relint} F$. These definitions do not depend on the choice of x.

We also define the exterior angle $\gamma(P, F)$ of P at a k-dimensional face F,

$$\gamma(P, F) := \frac{1}{\sigma_{d-k}} \omega_{d-k-1}(n(P, F)).$$

The sum of the exterior angles at the vertices of a polytope is 1,

$$\sum_{F \in \mathcal{F}_0(P)} \gamma(P, F) = 1.$$

We now investigate the case $j := \dim P < d$. We identify $\operatorname{aff} P$ with $\mathbb{R}^j$, and we designate the exterior angle of P at F in $\mathbb{R}^j$ by

$$\gamma^{(j)}(P, F) = \frac{1}{\sigma_{j-k}} \omega_{j-k-1}(n(P, F) \cap \operatorname{aff} P).$$

We then have

$$\gamma^{(j)}(P, F) = \gamma(P, F),$$

and consequently we speak of the exterior angle of P at F without mentioning the dimension of the surrounding space.

For a linear subspace $L \subset \mathbb{R}^d$ we put

$$\gamma_L(P, F) := \gamma((L + x) \cap P, (L + x) \cap F), \tag{2.2}$$

where x is any point in the relative interior of F. This definition does not depend on the choice of $x \in \operatorname{relint} F$.

For any convex body $K \in \mathcal{K}$ we define the metric projection $p(K,x)$ of $x \in \mathbb{R}^d$ as the point z of K that is nearest to x, i. e. for which $d(K,x) = \|z-x\|$. The metric projection is continuous in both components.

An additive mapping $\psi : \mathcal{K} \to M$ (where M is an abelian group) fulfills

$$\psi(K \cup K') + \psi(K \cap K') = \psi(K) + \psi(K')$$

for all $K, K' \in \mathcal{K}$ for which $K \cup K', K \cap K'$ are convex bodies.

For $K \in \mathcal{K}$ we define its support function

$$h(K,\cdot) : S^{d-1} \to \mathbb{R}, u \mapsto \max\{\langle u,x\rangle : x \in K\}.$$

Any convex body K is determined by its support function $h(K,\cdot)$.

A convex body Z is a zonotope, if Z is the Minkowski sum of (centrally symmetric) line segments. Zonotopes are also characterized in the following way. A polytope Z is a zonotope if any two edges with the same direction have the same length, i. e.

$$F, F' \in \mathcal{F}_1(Z), L(F) = L(F') \Longrightarrow V_1(F) = V_1(F'), \tag{2.3}$$

and for any $F \in \mathcal{F}_1$

$$\bigcup_{F' \in \mathcal{F}_1(Z), L(F') = L(F)} n(Z,F') = F^{\perp} \cap S^{d-1} \tag{2.4}$$

holds.

A convex body that can by approximated by zonotopes is a zonoid. For zonoids there also exists a characterization result. The convex body Z is a zonoid if and only if its support function can be represented in the form

$$h(Z,u) = \int_{S^{d-1}} |\langle u,v\rangle| \, d\rho(v), \quad u \in S^{d-1}, \tag{2.5}$$

with some even measure ρ on S^{d-1}. (A measure ρ on S^{d-1} is even, if for all Borel sets $\eta \subset S^{d-1}$ we have $\rho(\eta) = \rho(-\eta)$.)

2.1.4 Support measures of convex bodies

In this section we present the support measures of convex bodies (see Schneider [9], Section 4.2).

For a Borel set $\eta \subset \mathbb{R}^d \times S^{d-1}$ and $K \in \mathcal{K}$ we denote by $M_\rho(K,\eta)$ the local parallel set

$$M_\rho(K,\eta) := \{x \in \mathbb{R}^d : 0 < d(K,x) \leq \rho \text{ and } (p(K,x), u(K,x)) \in \eta\},$$

where $u(K,x)$ is the direction of the segment connecting $p(K,x)$ and x, i.e. $u(K,x) = \frac{x-p(K,x)}{\|x-p(K,x)\|}$. $M_\rho(K,\eta)$ is a Borel set. It turns out that its Lebesgue measure has a polynomial expansion in the following way,

$$\lambda_d(M_\rho(K,\eta)) = \frac{1}{d}\sum_{j=0}^{d-1} \rho^{d-j}\binom{d}{j}\Theta_j(K,\eta), \tag{2.6}$$

where $\Theta_0(K,\cdot),\dots,\Theta_{d-1}(K,\cdot)$ are finite Borel measures on $\mathbb{R}^d \times S^{d-1}$. These measures are concentrated on the normal bundle of K,

$$\mathrm{Nor}(K) = \{(x,u) \in \mathbb{R}^d \times S : x \in \mathrm{bd}\, K, u \in n(K,x)\},$$

and they depend additively and weakly continuously on $K \in \mathcal{K}$, i.e. for $K_i \to K$ we have $\Theta_j(K_i,\cdot) \xrightarrow{w} \Theta_j(K,\cdot)$, $j = 0,\dots,d-1$, for $i \to \infty$.

For $j = 0,\dots,d-1$, the j-th curvature measure is defined by

$$C_j(K,\cdot) := \Theta_j(K,\cdot \times S^{d-1}), \tag{2.7}$$

the projection of $\Theta_j(K,\cdot)$ onto its first component ($j = 0,\dots,d-1$). Occasionally, a different normalization is used,

$$\psi_j(K,\cdot) := \frac{1}{d\kappa_{d-j}}\binom{d}{j}C_j(K,\cdot).$$

The projection of $\Theta_j(K,\cdot)$ onto the second component is

$$S_j(K,\cdot) := \Theta_j(K,\mathbb{R}^d \times \cdot),$$

the j-th surface area measure of K.

The total measures of the curvature measures are given by

$$V_j(K) := \psi_j(K,\mathbb{R}^d), \quad j = 0,\dots,d-1,$$

where $V_j(K)$ is the j-th intrinsic volume of K. Additionally, we put $V_d(K) := \lambda_d(K)$, and $V_j(\emptyset) := 0$, $j = 0,\dots,d$. If $K \in \mathcal{K}$ is j-dimensional, we have $V_j(K) = \lambda_j^{\mathrm{aff}\,K}(K)$. For the intrinsic volumes the famous Steiner formula holds,

$$V_d(K+\rho B) = \sum_{i=0}^{d} \rho^{d-i}\kappa_{d-i}V_i(K), \quad \rho > 0. \tag{2.8}$$

If P is a polytope, then

$$V_j(P) = \sum_{F \in \mathcal{F}_j(P)} \gamma(F,P)V_j(F).$$

The intrinsic volumes have the following geometrical interpretations for general convex bodies: V_d is the volume, V_{d-1} is half the surface area, V_1 is the mean width (up to a constant depending on d), and $V_0 \equiv 1$.

Let $0 \leq j \leq k \leq d$, and let $K \in \mathcal{K}$. Then Crofton's formula holds,

$$\int_{\mathcal{E}_k^d} V_j(K \cap E)\, d\mu_k(E) = \alpha_{djk} V_{d+j-k}(K),$$

where

$$\alpha_{djk} = \frac{\binom{k}{j} \kappa_k \kappa_{d+j-k}}{\binom{d}{k-j} \kappa_j \kappa_d}.$$

2.2 Measure Theory

In this section, some of the results are given without a proof. In this case, the proof can be found, e. g., in Bauer [3].

Let X be a topological space. We know that the σ-algebra generated by the open subsets of X is the Borel σ-algebra $\mathcal{B}(X)$.

It can be difficult to find out if a given system $\mathcal{D}$ of subsets of X is a σ-algebra. This problem can sometimes be solved with the help of Dynkin systems.

Definition 1. *Let $\mathcal{D}$ be system of subsets of some set X. $\mathcal{D}$ is a* Dynkin system (in X), *if*

$$X \in \mathcal{D},$$
$$D \in \mathcal{D} \Rightarrow X \setminus D \in \mathcal{D},$$

and if, for each sequence (D_n) of pairwise disjoint sets in $\mathcal{D}$, the union $\bigcup_{n=1}^{\infty} D_n$ lies also in $\mathcal{D}$.

The following lemma states under which conditions a Dynkin system is a σ-algebra.

Lemma 2. *A Dynkin system $\mathcal{D}$ is a σ-algebra if and only if $D_1 \cap D_2 \in \mathcal{D}$ for all $D_1, D_2 \in \mathcal{D}$.*

Definition 3. *Let X be a locally compact space with countable basis. Let $\phi, \phi_1, \phi_2, \ldots$ be finite measures on X. The sequence (ϕ_i) is weakly convergent to ϕ if*

$$\lim_{n\to\infty} \int f\, d\phi_n = \int f\, d\phi$$

holds for all continuous, real-valued and bounded functions f on X.

We also denote weak convergence of (ϕ_i) to ϕ by

$$\phi_i \xrightarrow{w} \phi \quad \text{for } i \to \infty.$$

We will use weak convergence mainly in the following context. Let $\phi(K, \cdot)$ be a finite measure on X which depends on a convex body K. This measure is *weakly continuously* in K if $\mathcal{K}_i \to K$ implies $\phi(K_i, \cdot) \xrightarrow{w} \phi(K, \cdot)$ $(i \to \infty)$ for convex bodies $K, K_1, K_2, \ldots$.

In convex geometry, we often encounter measures that have a polynomial expansion with respect to some parameter ρ. An example is equation (2.6), where the Lebesgue measure $\lambda_d(M_\rho(K,\eta))$ of a local parallel set (at distance ρ) of the convex body K is expanded into a polynomial in ρ. The coefficients, in this case, are the support measures.

We now give some properties for a general class of measures that have polynomial expansions. The underlying method has been used, for example, by Schneider [9] and Fallert [4]. It is well-known that a polynomial p of degree n (and therefore its coefficients) are determined by its values $p(x_i)$ for $n+1$ pairwise distinct real numbers $x_1,\dots,x_{n+1}$. The following Lemma gives us some more information about the relationship of $p(x_1),\dots,p(x_{n+1})$ and the coefficients of p.

Lemma 4. *Let*

$$p(x) = \sum_{i=0}^{n} a_i x^i \tag{2.9}$$

be a real polynomial of degree n. Then there are coefficients $b_{i,j}$, $0 \le i,j \le n$, depending on i,j and n only, such that

$$a_i = r^{-i} \sum_{j=0}^{n} b_{i,j}\, p((j+1)r), \quad i = 0,\dots,n,$$

for every positive real constant r.

Proof. Let $r > 0$. We use the polynomials

$$L_j(x) := \prod_{i=0, i\neq j}^{n} \frac{x-(i+1)}{(j+1)-(i+1)} =: \sum_{i=0}^{n} b_{i,j} x^i$$

to write $\sum_{i=0}^{n} a_i x^i = p(x) = \sum_{j=0}^{n} L_j(x/r) p((j+1)r)$. Comparing coefficients yields $a_i x^i = \sum_{j=0}^{n} b_{i,j}(x/r)^i p((j+1)r)$, and for $x = 1$ we get $a_i = r^{-i}\sum_{j=0}^{n} b_{i,j} p((j+1)r)$. □

As a corollary we get the following result of Fallert [4], Satz 32.

Corollary 5. *Let $j \in \{0,\dots,d\}$. Then there exists a constant $c_{j,d}$ that depends only on j and d, such that for any convex body $K \subset \mathbb{R}^d$ and any ball B' with radius $r > 0$ the following inequality holds:*

$$\psi_j(K,B') \le c_{j,d} r^j.$$

Proof. We apply (2.9) to the polynomial

$$\lambda_d(M_\epsilon(K, B' \times S^{d-1})) = \sum_{i=0}^{d} \epsilon^i \kappa_i \psi_{d-i}(K,B'),$$

which follows from (2.6) and the definition (2.7) of the curvature measures. This yields

$$\psi_{d-i}(K,B') = \frac{1}{\kappa_{d-i}} \sum_{j=0}^{d} b_{i,j}\, r^{-i} \lambda_d(M_{(j+1)r}(K, B' \times S^{d-1})).$$

Now $M_{(j+1)r}(K, B' \times S^{d-1}) \subset M_{(d+1)r}(K, B' \times S^{d-1}) \subset B' + (d+1)rB$, and therefore

$$V(U_{(j+1)r}(K, B')) \leq (d+2)^d \kappa_d r^d.$$

The assertion now follows easily. □

We can also apply Lemma 4 to measures that depend on a parameter that is a convex body. If such a measure has a polynomial expansion, and depends weakly on the convex body it belongs to, the measures that are the coefficients of the polynomial also depend weakly on the convex body. This and some more results are stated in the following Lemma.

Lemma 6. *For each $\epsilon > 0$ and each $K \in \mathcal{K}$ let $\varrho_\epsilon(K, \cdot)$ be a finite Borel measure on a locally compact Hausdorff space X with countable basis. Let $\mathcal{M}$ be a dense subset of $\mathcal{K}$, and for each $K \in \mathcal{M}$ let $\phi_0(K, \cdot), \dots, \phi_n(K, \cdot)$ be finite Borel measures on X. For $K \in \mathcal{M}$ assume the following polynomial expansion,*

$$\varrho_\epsilon(K, \cdot) = \sum_{i=0}^{n} \epsilon^i \phi_i(K, \cdot).$$

Moreover, let ϕ_ϵ be weakly continuous in K. Then $\phi_0(K, \cdot), \dots, \phi_k(K, \cdot)$ can be expanded to be measures depending on $K \in \mathcal{K}$ (i. e. not only on $K \in \mathcal{M}$) such that they are also weakly continuous in K.

Proof. From Lemma 4 we get the polynomial expansion

$$\phi_i(K, \cdot) = r^{-i} \sum_{j=0}^{n} b_{i,j} \varrho_{(j+1)r}(K, \cdot), \qquad r > 0, \quad i = 0, \dots, n,$$

which holds for $K \in \mathcal{M}$. For $r = 1$ this equation gives

$$\phi_i(K, \cdot) = \sum_{j=0}^{n} b_{i,j} \varrho_{j+1}(K, \cdot). \tag{2.10}$$

We now use equation (2.10) to define the (for the moment possibly signed) measures $\phi_0(K, \cdot), \dots, \phi_n(K, \cdot)$ for all $K \in \mathcal{K}$. Clearly, for $K \in \mathcal{M}$ this definition coincides with the original measures. We now show that for arbitrary $K \in \mathcal{K}$ we also get a (positive) measure $\phi_i(K, \cdot)$. Let $K_l \in \mathcal{M}$ such that $K_l \to K$ for $l \to \infty$. Then $\phi_i(K_l, \cdot)$ is the weak limit of measures,

$$\phi_i(K_l, \cdot) \xrightarrow{w} \phi_i(K, \cdot), \quad l \to \infty,$$

and therefore a (positive) measure itself ($i \in \{0, \dots, n\}$). Moreover, as $\phi_i(K, \cdot)$ is a sum of weakly continuous measures, $\phi_i(K, \cdot)$ also depends weakly continuously on K ($i = 0, \dots, n$). □

We now consider another inequality. In contrast to corollary 5, we do not consider the measure of a ball of radius ϵ, but the measure of (a projection of) spherical images of zonal sets.

Definition 7. *Let* $k \in \{0, \dots, d\}$, $\epsilon \geq 0$ *and* $L \in \mathcal{L}_k^d$. *Then*

$$Z(L, \epsilon) := \{u \in S^{d-1} : \|u|L^\perp\| \leq \epsilon\} = \{u \in S^{d-1} : \|u|L\| \geq \sqrt{1 - \epsilon^2}\} \tag{2.11}$$

is called a zonal set (with respect to L and ϵ).

Theorem 8. *Let* $k \in \{0, \dots, d\}$, $0 \leq \epsilon \leq 1$ *and* $L \in \mathcal{L}_k^d$. *Let* $K \subset \mathbb{R}^d$ *be a convex body, and*

$$K_\epsilon := \{x \in \operatorname{bd} K : n(K, x) \cap Z(L, \epsilon) \neq \emptyset\}.$$

Then for $0 \leq \epsilon \leq 1$

$$K_\epsilon|L \subset \{x \in K|L : d(\operatorname{relbd} K|L, x) \leq \epsilon D(K)\}. \tag{2.12}$$

Moreover, a constant $c_k(K) > 0$ *exists that depends on* k *and* K *only, such that for* $0 \leq \epsilon \leq 1$

$$\lambda_k(\operatorname{cl}(K_\epsilon|L)) \leq \epsilon c_k(K). \tag{2.13}$$

Proof. For $\epsilon = 0$, the left hand side of (2.12) is $\operatorname{relbd} K|L$, which clearly is a subset of the set on the right hand side. $\operatorname{relbd} K|L$ is a set of dimension less than k, and thus (2.13) holds. On the other hand, for $\epsilon = 1$, the right hand side of (2.12) is $K|L$, and the left hand side $K_\epsilon|L$ is a subset thereof, as $K_\epsilon \subset K$. Equation (2.13) also holds, as $c_k(K)$ can be chosen to be greater than $V_k(K|L)$.

Thus we assume $0 < \epsilon < 1$ for the rest of the proof. We start with (2.12). We may assume without loss of generality that $\dim K|L = k$. The set on the right hand side is

$$\{x \in K|L : \|x - z\| < \epsilon D(K) \text{ for some } z \in \operatorname{relbd} K|L\}.$$

Thus, it suffices to show that for any $x \in \operatorname{bd} K$ that has an outer normal $u \in Z(L, \epsilon)$, a point $z \in \operatorname{relbd} K|L$ exists such that $d := \|x|L - z\| < \epsilon D(K)$. We assume without loss of generality that $x = 0$ and $\epsilon = \|u|L^\perp\|$. Let $v \in S^{d-1} \cap L$ such that $u|L = \sqrt{1 - \epsilon^2} v$. We put $E := \operatorname{lin}\{u, v\}$, $g := \operatorname{lin}\{v\}$. Let H_u, H_v be the supporting hyperplanes of K with outer normals u, v, respectively. $x \in H_u$ is a support point of K. Let $y \in H_v \cap K$ be another support point of K. From $H_v = g^\perp$ we get that $y|g$ lies in the relative boundary of $K|g$ with outer normal v (in g). As $x|g \in K|L$ and $g \subset L$, the line segment $[x|g, y|g]$ contains some $z \in \operatorname{relbd} K|L$. Then $x|g = x|L$ yields

$$d = \|x|L - z\| \leq \|x|g - y|g\|.$$

$x|E$ and $y|E$ are points in the relative boundary of the (2-dimensional) convex body $K|E$, and they have outer normals u and v, respectively. This follows from the fact that $u^\perp$ and $v^\perp$ contain $E^\perp$, and from $x \in H_u$ and $y \in H_v$. The convexity of $K|E$ then implies

$$\epsilon \cdot \| x|E - y|E \| \ \geq \ \| x|g - y|g \|.$$

The orthogonal projection onto E is a contraction, which implies

$$\|(x - y)|E\| \leq \|x - y\| \leq D(K).$$

Altogether these inequalities imply

$$\epsilon \cdot D(K) \geq d,$$

which finishes the proof of (2.12)

It is clear that $\mathrm{cl}(K_\epsilon|L)$ is a measurable set, and that

$$\mathrm{cl}(K_\epsilon|L) \subset \{x \in K|L : d(\mathrm{relbd}(K|L), x) \leq \epsilon D(K)\} =: K'.$$

If K is a polytope, for each $x \in K'$ there is a face $F \in \mathcal{F}_{k-1}(K|L)$ such that $d(F, x) \leq \epsilon D(K)$. Thus

$$K' = \bigcup_{F \in \mathcal{F}_{k-1}(K|L)} F + \epsilon D(K) \cdot [0, -u(F)],$$

where $u(F)$ is the outer normal of $K|L$ at F that lies in L. Consequently,

$$\begin{aligned}
\lambda_k(K') &\leq \epsilon D(K) \cdot \sum_{F \in \mathcal{F}_{k-1}(K|L)} V_k(F + [0, -u(F)]) \\
&= \epsilon D(K) \cdot \sum_{F \in \mathcal{F}_{k-1}(K|L)} V_{k-1}(F) \\
&= \epsilon D(K) \cdot V_{k-1}(K|L).
\end{aligned}$$

If K is an arbitrary convex body, we can approximate K by a series (P_i) of polytopes. We clearly have $P_i' \to K'$ and therefore $\lambda_k(P_i') \to \lambda_k(K')$ for $i \to \infty$. On the other hand, $P_i|L \to K|L$ and $V_{k-1}(P_i|L) \to V_{k-1}(K|L)$ for $i \to \infty$. Thus the inequality

$$\lambda_k(K') \leq \epsilon D(K) \cdot V_{k-1}(K|L)$$

holds for arbitrary $K \in \mathcal{K}$. We can now choose $c_k(K)$ to be any upper bound of $D(K)V_{k-1}(K|L)$ for $L \in \mathcal{L}_k^d$. □

We cite the following result of Santalò [8] as a Lemma for later use.

Lemma 9. *There are real constants $a_{d,k,m}$ for $k, m \in \{0, \ldots, d\}$ such that the following properties hold. Let $F \in \mathcal{L}_m^d$ and let $f : \mathcal{L}_k^d \to \mathbb{R}$ be an integrable function. If $k + m \geq d$ then the following equation holds,*

$$\int_{\mathcal{L}_k^d} f(L)\, d\nu_k(L) = a_{d,k,m} \int_{\mathcal{L}_{k+m-d}^F} \int_{\mathcal{L}_k^{L'}} |\langle L^\perp, F\rangle|^{k+m-d} f(L)\, d\nu_k^{L'}(L)\, d\nu_{k+m-d}^F(L'). \tag{2.14}$$

If $k + m \leq d$, the following equation holds,

$$\int_{\mathcal{L}_k^d} f(L)\, d\nu_k(L) = a_{d,d-k,d-m} \int_{\mathcal{L}_{k+m}^F} \int_{\mathcal{L}_k^{L'}} |\langle L^\perp, F\rangle|^{d-k-m} f(L)\, d\nu_k^{L'}(L)\, d\nu_{k+m}^F(L').$$

Proof. We first consider the case $k+m>d$. The result then is equation (14.40) in Santalò [8], where we apply the formula $|\langle L^\perp, L^\perp|F\rangle| = |\langle L^\perp, F\rangle|$ whenever L, F are in general relative position. Otherwise, $|\langle L^\perp, F\rangle| = 0$, which fits Santalò's formula.

For $k+m=d$ the equations hold with $a_{d,k,d-k}=1$.

If $k+m<d$, we orthogonalize the variable of integration and get

$$\int_{\mathcal{L}_k^d} f(L)\, d\nu_k(L) = \int_{\mathcal{L}_{d-k}^d} f(L^\perp)\, d\nu_{d-k}(L).$$

We observe that $(d-k)+(d-m)>d$, and apply (2.14) to $F^\perp \in \mathcal{L}_{d-m}^d$ instead of $F \in \mathcal{L}_m^d$, yielding

$$\begin{aligned}
&\int_{\mathcal{L}_{d-k}^d} f(L^\perp)\, d\nu_{d-k}(L) \\
&\quad = a_{d,d-k,d-m} \int_{\mathcal{L}_{d-m-k}^{F^\perp}} \int_{\mathcal{L}_{d-k}^{L'}} |\langle L^\perp, F^\perp\rangle|^{d-k-m} f(L^\perp)\, d\nu_{d-k}^{L'}(L)\, d\nu_{d-k-m}^{F^\perp}(L') \\
&\quad = a_{d,d-k,d-m} \int_{\mathcal{L}_{k+m}^{F}} \int_{\mathcal{L}_{d-k}^{L'^\perp}} |\langle L^\perp, F^\perp\rangle|^{d-k-m} f(L^\perp)\, d\nu_{d-k}^{L'^\perp}(L)\, d\nu_{k+m}^{F}(L') \\
&\quad = a_{d,d-k,d-m} \int_{\mathcal{L}_{k+m}^{F}} \int_{\mathcal{L}_{k}^{L'}} |\langle L, F^\perp\rangle|^{d-k-m} f(L^\perp)\, d\nu_{k}^{L'}(L)\, d\nu_{k+m}^{F}(L').
\end{aligned}$$

We apply the equation $|\langle L, F^\perp\rangle| = |\langle L^\perp, F\rangle|$, which yields the assertion. □

2.3 Grassmannians

We will later study functions in $L^2(\mathcal{L}_k^d)$, i. e. real-valued square integrable functions on the Grassmannian $\mathcal{L}_k^d$. Many of these functions will be defined using the function

$$E \mapsto |\langle E, F\rangle|, \quad E \in \mathcal{L}_k^d,$$

where F is some fixed k-flat. We will now state some results related to the relative position of E, F that are needed later.

Lemma 10. *Let $1 \le k \le d$, $0 \le m \le d$, and let $F \in \mathcal{L}_k^d$, $L \in \mathcal{L}_m^d$. Let $u_1 \in F \cap S^{d-1}$ be such that $\|u_1|L\|$ is maximal (or minimal). Then $u_1|L \perp (u_1^\perp \cap F)|L$, i. e. for any $u_2 \in F, u_2 \perp u_1$, we have $u_1|L \perp u_2|L$. Moreover, the set*

$$U := \{u \in F : \|u|L\| = \|u_1|L\| \cdot \|u\|\}$$

is a linear subspace of $\mathbb{R}^d$.

Proof. We assume that an $u_2 \in u_1^\perp \cap F$ exists such that $u_1|L \not\perp u_2|L$, and subsequently show that in this case $\|u_1|L\|$ can neither be maximal nor minimal.

We assume without loss of generality that $u_2 \in F \cap S^{d-1}$. Orthonormal vectors $v_1, v_2 \in L$ exist such that $u_1|L = av_1, u_2|L = bv_1 + cv_2$ for some real numbers a, b, c (and

$a, b \neq 0$ follows). For any $\alpha \in [-1,1]$ the vector $u(\alpha) := \alpha u_1 + \sqrt{1-\alpha^2}u_2$ is an element of $F \cap S^{d-1}$. Let $f(\alpha)$ be the squared norm of $u(\alpha)|L$, i. e.

$$\begin{aligned} f(\alpha) &= \|v(\alpha)|L\|^2 \\ &= \|(\alpha a + \sqrt{1-\alpha^2}b)v_1 + \sqrt{1-\alpha^2}cv_2\|^2 \\ &= (\alpha a + \sqrt{1-\alpha^2}b)^2 + (1-\alpha^2)c^2 \\ &= \alpha^2 a^2 + (1-\alpha^2)b^2 + (1-\alpha^2)c^2 + 2\alpha\sqrt{1-\alpha^2}ab \\ &= b^2 + c^2 + \alpha^2(a^2 - b^2 - c^2) + 2\alpha\sqrt{1-\alpha^2}ab. \end{aligned}$$

On $(-1,1)$ the derivative of f is given by

$$f'(\alpha) = 2\alpha(a^2 - b^2 - c^2) + 2\sqrt{1-\alpha^2}ab - \frac{\alpha^2}{\sqrt{1-\alpha^2}}ab.$$

We first consider the case $ab > 0$. The summand $2\alpha(a^2-b^2-c^2)+2\sqrt{1-\alpha^2}ab$ is bounded on $(-1,1)$, and thus $f'(\alpha)$ tends to $-\infty$ for $\alpha \to \pm 1$. The continuity of f implies the existence of some $\alpha_0, \alpha_1 \in (-1,1)$ such that $f(\alpha_0) > f(1)$ and $f(\alpha_1) < f(-1)$. We then have

$$\|u(\alpha_1)|L\| < \|-u_1|L\| = \|u_1|L\| < \|u(\alpha_0)|L\|.$$

Therefore $\|u_1|L\|$ is neither maximal nor minimal.

In the case $ab < 0$ a similar argument shows that $\|u_1|L\|$ is neither maximal nor minimal.

Because ab cannot be zero, this contradicts our assumption, and the assertion holds.

It remains to show that U is a linear subspace. If $\dim U \leq 1$, then obviously $U = [u_1]$, a linear subspace of $\mathbb{R}^d$. Otherwise, let $u_2 \in F \cap S^{d-1}$ be such that $\|u_1|L\| = \|u_2|L\| =: c$, and such that u_1, u_2 are linearly independent. Let u_1, u_2' be an orthonormal basis of $[u_1, u_2]$. It follows that $u_1|L$ is orthogonal to $u_2'|L$. For $\alpha_1, \alpha_2 \in \mathbb{R}$ we have

$$\|(\alpha_1 u_1 + \alpha_2 u_2')|L\|^2 = \alpha_1^2 c^2 + \alpha_2^2\|u_2'|L\|^2. \tag{2.15}$$

In particular, there are $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $u_2 = \alpha_1 u_1 + \alpha_2 u_2'$, $\alpha_1^2 + \alpha_2^2 = 1$ and $\alpha_2 \neq 0$. We therefore have

$$c^2 = \|u_2|L\|^2 = \alpha_1^2 c^2 + (1-\alpha^2)\|u_2'|L\|^2,$$

which implies $\|u_2'|L\| = c$. Equation (2.15) yields $\|u|L\| = c\|u\|$ for all $u \in [u_1, u_2]$. Thus all linear combinations of elements of U lie in U, i. e. U is a linear subspace. □

We can now construct an orthonormal basis of F in the following manner. We choose $u_1 \in F \cap S^{d-1}$ such that $\|u_1|L\|$ is maximal. Then we choose $u_2 \in F \cap S^{d-1} \cap u_1^\perp$ such that $\|u_2|L\|$ is maximal. In general, choose $u_i \in F \cap S^{d-1} \cap u_1^\perp \cap \ldots \cap u_{i-1}^\perp$ for $i = 1, \ldots, k$ with the corresponding maximality property. Thus $u_1, \ldots, u_k$ is an orthonormal basis of F. This motivates the following definition.

Definition 11. *Let $k, m \in \{0, \ldots, d\}$, and let $F \in \mathcal{L}_k^d, L \in \mathcal{L}_m^d$. For each i from 1 to k, choose*

$$u_i \in F \cap S^{d-1} \cap \bigcap_{j=1}^{i-1} u_j^\perp,$$

such that

$$\beta_i := \|u_i|L\|$$

*is maximal. The numbers $\beta_1, \ldots, \beta_k$ are the β-*numbers of F (with respect to L)*, and*

$$\beta_L(F) := (\beta_1, \ldots, \beta_k) \tag{2.16}$$

*is the β-*vector of F (with respect to L)*. Moreover, $\{u_1, \ldots u_k\}$ is called an L-*ONB of F.

Remark. The β-vector of F does not depend on the choice of $u_1, \ldots, u_k$. From Lemma 10 we know that

$$U := \{u \in F : \|u|L\| = \beta_1 \|u\|\}$$

is a linear subspace. This means that the vectors $u_1, \ldots, u_{\dim U}$ are an orthonormal basis of U, regardless of their choice, and $\beta_1 = \ldots = \beta_{\dim U}$. Thus $\beta_{\dim U+1}, \ldots, \beta_k$ do not depend on the choice of $u_1, \ldots, u_{\dim U}$.

Lemma 12. *Let $k, m \in \{0, \ldots, d\}$, and let $F \in \mathcal{L}_k^d, L \in \mathcal{L}_m^d$. Let $u_1, \ldots, u_k$ be an L-ONB of F. Then $u_1|L, \ldots, u_k|L, u_1|L^\perp, \ldots, u_k|L^\perp$ are orthogonal.*

Proof. We use induction on k for the proof. If $k = 0$, the empty set is the only L-ONB of F, and the assertion holds.

Now let $k \geq 1$ and assume we know the result for $k - 1$. $u_2, \ldots, u_k$ is an L-ONB of $F \cap u_1^\perp$. By induction, we know that $u_2|L, \ldots, u_k|L, u_2|L^\perp, \ldots, u_k|L^\perp$ are orthogonal. From Lemma 10 we know that $u_1|L$ is orthogonal to $u_2|L, \ldots, u_k|L$, because $\|u_1|L\|$ is maximal. Moreover, $\|u_1|L^\perp\| = \sqrt{1 - \|u_1|L\|^2}$ is minimal, and Lemma 10 implies $u_1|L^\perp \perp u_2|L^\perp, \ldots, u_k|L^\perp$. Finally, it is trivially clear that $u_1|L \perp u_1|L^\perp, \ldots, u_k|L^\perp$ and $u_1|L^\perp \perp u_1|L, \ldots, u_k|L$. Altogether we get that $u_1|L, \ldots, u_k|L, u_1|L^\perp, \ldots, u_k|L^\perp$ are orthogonal. □

Corollary 13. *Let $0 \leq k \leq m \leq d$ and let $F \in \mathcal{L}_k^d, L \in \mathcal{L}_m^d$. Let $\beta := \beta_L(F)$ be the β-vector of F w. r. t. L. Then*

$$|\langle F, L \rangle| = \beta_1 \cdot \ldots \cdot \beta_k. \tag{2.17}$$

Proof. Let $u_1, \ldots, u_k$ be an L-ONB of F. Then $[0, u_1] + \ldots + [0, u_k]$ is a unit cube in F, and its image under orthogonal projection onto L ist $[0, \beta_1 v_1] + \ldots + [0, \beta_k v_k]$ for some ONB $v_1, \ldots, v_m$ of L. The k-volume of this image is $\beta_1 \cdot \ldots \cdot \beta_k$. □

We now discuss how the β-numbers of F determine the relative position of F and L. By this, we mean that F is determined by these numbers, up to an orthogonal transform under which L and $L^\perp$ are invariant subspaces of $\mathbb{R}^d$.

For example, if we select a fixed plane L in $\mathbb{R}^3$, the real number $|\langle L, F\rangle|$ determines the position of the plane $F \subset \mathbb{R}^3$ relative to L. As in this case $\beta_1 = 1$, the number $\beta_2 = |\langle L, F\rangle|$ determines the position of F relative to L. $|\langle F, L\rangle|$ does clearly not determine the relative position of F and L for the general case of k-flats in $\mathbb{R}^d$. However, $\beta_L(F)$ determines the position of F relative to L.

Lemma 14. *Let $0 \leq k \leq m \leq d$ and $F \in \mathcal{L}_k^d, L \in \mathcal{L}_m^d$. Then $\beta_L(F) = (\beta_1, \dots, \beta_k)$ determines the relative position of F and L in the following sense. If $\beta_L(F_1) = \beta_L(F_2)$ for two k-flats F_1 and F_2, then an orthogonal transform ρ exists such that L and $L^\perp$ are invariant under ρ, and $\rho F_1 = F_2$.*

Proof. Let $v_1, \dots, v_m$ be an ONB of L, and let $v_{m+1}, \dots, v_d$ be an ONB of $L^\perp$. We put

$$w_i := \beta_i v_i + \sqrt{1-\beta_i^2}\, v_{d+1-i}, \quad i = 1, \dots, k.$$

Below, we will construct an orthogonal transform ρ under which L and $L^\perp$ are invariant, such that $\rho u_i = w_i$ for an L-ONB of F, i.e. $\rho F = \operatorname{lin}\{w_1, \dots, w_k\} =: W$. The linear subspace W depends on $\beta_L(F)$ only. Thus, if $\beta_L(F_1) = \beta_L(F_2)$, there are orthogonal transforms ρ_1, ρ_2 under which L and $L^\perp$ are invariant, such that $\rho_1 F_1 = W = \rho_2 F_2$. It then follows $\rho_2^{-1}\rho_1 F_1 = F_2$, and the assertion holds.

Let $u_1, \dots, u_k$ be an L-ONB of F. From Lemma 12 we know that $\{u_1|L, \dots, u_k|L\}$ is an orthogonal subset of L. Clearly, there is an orthogonal transform ρ_1 that leaves $L^\perp$ fixed, such that $(\rho_1 u_i)|L = \beta_i v_i$ for $i = 1, \dots, k$. (Note that for $i \geq m$ we have $v_i \notin L$. However, $\beta_i = 0$ is implied in this case, and the equation holds.)

Analogously, Lemma 12 also implies that $\{u_1|L^\perp, \dots, u_k|L^\perp\}$ is an orthogonal subset of $L^\perp$. There is an orthogonal transfrom ρ_2 that leaves L fixed, such that $(\rho_2 u_i)|L^\perp = \sqrt{1-\beta_i^2}\, v_{d+i-k}$ for $i = 1, \dots, k$. (Note that for $i \leq m + k - d$ we have $v_{d+i-k} \notin L^\perp$. However, $\beta_i = 1$ is implied in this case, and the equation holds.)

$\rho := \rho_1 \rho_2$ is an orthogonal transform under which L and $L^\perp$ are invariant subspaces. We have

$$\rho u_i = \beta_i v_i + \sqrt{1-\beta_i^2}\, v_{d+1-i} = w_i. \tag{2.18}$$

Thus ρF depends on $\beta_L(F)$ only, and the assertion follows as stated above. □

We now compute the spherical Lebesgue measure of the image of a set under a certain mapping, which we will use in the next chapter.

Lemma 15. *Let $0 \leq k \leq d$ and let $F, L \in \mathcal{L}_k^d$ be such that $|\langle F, L\rangle| \neq 0$. Let a mapping $\pi_L : F \setminus \{0\} \to L$ be defined by*

$$\pi_L : x \mapsto \frac{x|L}{\|x|L\|}.$$

Then for all Borel subsets η of $F \cap S^{d-1}$

$$\omega_{k-1}(\pi_L(\eta)) = |\langle F, L\rangle| \int_\eta \frac{1}{\|u|L\|^k}\, d\omega_{k-1}(u). \tag{2.19}$$

PROOF. We put $C := \{\alpha x : \alpha \in [0,1], x \in \pi_L(\eta)\}$ and

$$D := \{x \in F : x|L \in C\} = \{\alpha x : x \in \eta, 0 \leq \alpha \leq \frac{1}{\|x|L\|}\} \subset F.$$

The orthogonal projection from F onto L is injective, and thus

$$\begin{aligned}
\omega_{k-1}(\pi_L(\eta)) &= k\lambda_k(C) \\
&= k|\langle F, L\rangle|\lambda_k(D) \\
&= k|\langle F, L\rangle| \int_\eta \int_0^{1/\|u|L\|} r^{k-1}\, dr\, d\omega_{k-1}(u) \\
&= |\langle F, L\rangle| \int_\eta \frac{1}{\|u|L\|^k}\, d\omega_{k-1}(u).
\end{aligned}$$

□

3 Integral representations of projection functions I

In this chapter we present an integral representation of projection functions of convex bodies. More precisely, we will associate a measure $\mu_k(P,\cdot)$ with a convex polytope P. This measure is a Borel measure on $S^{d-1}\times\mathcal{L}_k^d$, which we abbreviate by

$$S^{d-1,k} := S^{d-1}\times\mathcal{L}_k^d. \tag{3.1}$$

We will also associate a function f_L on $S^{d-1,k}$ with $L\in\mathcal{L}_k^d$, such that

$$V_k(P|L) = \int f_L(\cdot)\,d\mu_k(P,\cdot). \tag{3.2}$$

For general convex bodies, there also exist measures $\mu_k(K,\cdot)$ such that (3.2) holds. However, in Example 20 we will see that $\mu_k(K,\cdot)$ does not depend weakly continuously on K.

In later chapters, we will define measures depending weakly continuously on K, which allow an integral representation of projection functions in the form (3.2).

3.1 Ambartzumian's integral representation of the width function in $\mathbb{R}^3$

The following theorem is Ambartzumian's $\sin^2$-representation of (the width function of) convex bodies in $\mathbb{R}^3$. The width function of a convex body K is given by

$$w(K,u) := h(K,u)+h(K,-u),\quad u\in S^{d-1}.$$

For a centrally symmetric convex body M we have $w(M,\cdot)=2h(M,\cdot)$, and hence the support function of M is determined by the width function of M. Thus a centrally symmetric convex body is determined by its width function. Therefore, a representation of the width function of a centrally symmetric convex body M can be considered a representation of M itself.

Note that this does not hold for general convex bodies. For any convex body K, we can define the convex body $M := \frac{1}{2}(K-K)$, which is centrally symmetric and has the same width function as K, i. e. $w(M,\cdot)=w(K,\cdot)$.

The width function of K in direction u is the distance of the supporting hyperplanes of K with outer normals u and $-u$. This is also the length of the line segment of $\operatorname{lin}\{u\}$ between these hyperplanes. This segment is $K|\operatorname{lin}\{u\}$. Thus, the width function $w(K,u)$ is the same as the projection function $V_1(K|\operatorname{lin}\{u\})$.

Theorem 16. (Ambartzumian [1]) *Let $K \subset \mathbb{R}^3$ be a convex body. Then a Borel measure $\mu_1(K,\cdot)$ on $S^{2,1}$ exists such that*

$$V_1(K|L) = \int_{S^{2,1}} \frac{\langle L, F\rangle^2}{\|u|L^\perp\|^2} \mu_1(K,\, d(u,F)), \quad L \in \mathcal{L}_1^3 \tag{3.3}$$

for all $L \in \mathcal{L}_1^3$. For polytopes P, the measures defined by

$$\mu_1(P,\eta) = \frac{1}{2\pi} \sum_{F\in\mathcal{F}_1(P)} V_k(F) \int_{n(P,F)} I(\,(u,F)\in\eta\,)\, d\omega_1^{S^2\cap F^\perp}(u) \tag{3.4}$$

have this property.

Remark. The measure $\mu_1(P,\cdot)$ defined by (3.4) is concentrated on $\{(u,F)\in S^{2,1} : u \perp F\}$, and the same holds for the measure $\mu_1(K,\cdot)$ used in (3.3). If $u \in L$, we have $L^\perp \subset u^\perp$. We also have $F \subset u^\perp$ for almost all (u,F), which means that $L^\perp$ and F are not in general relative position. More precisely, we have $F|L \subset u^\perp \cap L$, which is a set of dimension not greater than $k-1$. This means that the enumerator of the integrand in (3.3) is 0. By putting $\frac{\langle L,F\rangle^2}{\|u|L^\perp\|^2} := 0$ for $u \in L$ we get $\frac{\langle L,F\rangle^2}{\|u|L^\perp\|^2} = 0$ whenever F and $L^\perp$ are not in general relative position.

We do not give a direct proof of Theorem 16 here. However, Theorem 18 includes Theorem 16 as a special case.

To explain why this representation has the name $\sin^2$-representation, we define the function f_1^3 by

$$f_1^3 : \mathcal{L}_1^3 \times \mathcal{L}_1^3 \times S^2 \to \mathbb{R}, (L,F,u) \mapsto \begin{cases} \langle u^\perp \cap F^\perp, u^\perp \cap L^\perp\rangle^2, & |\langle L,F\rangle| \neq 0, \\ 0, & |\langle L,F\rangle| = 0. \end{cases}$$

If $|\langle L,F\rangle| = 0$, then $f_1^3(L,F,u) = 0 = \frac{\langle L,F\rangle^2}{\|u|L^\perp\|}$, as explained above. On the other hand, if $|\langle L,F\rangle| \neq 0$, then $u \notin L$ follows from $u \in F^\perp$. We choose an orthonormal basis u, u' of $F^\perp$, i. e. u' is an orthonormal basis of $F^\perp \cap u^\perp$. Then

$$\begin{aligned} |\langle L,F\rangle| &= |\langle F^\perp, L^\perp\rangle| \\ &= \lambda_2(([0,u]+[0,u'])|L^\perp) \\ &= \lambda_1([0,u|L^\perp]) \cdot \lambda_1([0,u'|(L^\perp \cap u^\perp)]) \\ &= \|u|L^\perp\| |\langle u^\perp \cap F^\perp, u^\perp \cap L^\perp\rangle|, \end{aligned}$$

yielding

$$\begin{aligned} \frac{\langle L,F\rangle^2}{\|u|L^\perp\|^2} &= \frac{\|u|L^\perp\|^2 \langle u^\perp \cap F^\perp, u^\perp \cap L^\perp\rangle^2}{\|u|L^\perp\|^2} \\ &= \langle u^\perp \cap F^\perp, u^\perp \cap L^\perp\rangle^2 \\ &= f_1^3(L,F,u). \end{aligned}$$

Thus, (3.3) can be expressed as

$$V_1(K|L) = \int_{S^{2,1}} f_1^3(L, F, u)\mu_1(K, d(u, F)), \quad L \in \mathcal{L}_1^3.$$

The squared sine of the angle between F and $u^\perp \cap L$ is $\langle u^\perp \cap F^\perp, u^\perp \cap L^\perp\rangle^2$. Therefore, this formula is called $\sin^2$-representation of the width function of convex bodies.

3.2 Integral representations of arbitrary projection functions

We will give an integral representation for polytopes first, and then generalize this formula to arbitrary convex bodies. Similarly to the last section, we will encounter a fraction of form $\frac{\langle F,L\rangle^2}{\|u|L^\perp\|^{d-k}}$, where $L, F \in \mathcal{L}_k^d$, and $u \in F^\perp$. The denominator becomes 0 for $u \in L$ only, and we will see that this happens for a set with measure 0 only. As above, the fact that $|\langle F, L\rangle| = 0$ for F, L not in general relative position motivates the definiton of $\frac{\langle F,L\rangle^2}{\|u|L^\perp\|^{d-k}} = 0$ whenever F, L are not in general relative position.

Lemma 17. *Let P be a convex polytope in $\mathbb{R}^d$ and $0 \leq k < d$. Then the projection function $L \mapsto V_k(P|L), L \in \mathcal{L}_k^d$ is given by*

$$V_k(P|L) = \frac{1}{\sigma_{d-k}} \sum_{F\in\mathcal{F}_k(P)} V_k(F) \int_{n(P,F)} \frac{\langle F, L\rangle^2}{\|u|L^\perp\|^{d-k}} \, d\omega_{d-k-1}(u). \tag{3.5}$$

Proof. First of all, as L is k-dimensional, the volume $V_k(P|L)$ is the Lebesgue measure on L of $P|L$. By definition, this means

$$\begin{aligned} V_k(P|L) &= \lambda_k(P|L) \\ &= \int_L I(x \in P|L)\, d\lambda_k(x) \\ &= \int_L I((L^\perp + x) \cap P \neq \emptyset)\, d\lambda_k(x). \end{aligned}$$

Now we can split the integrand into a sum ranging over the k-faces of P. The integrand vanishes, if $(L^\perp + x) \cap P = \emptyset$, and is 1 otherwise, i. e. if $(L^\perp + x) \cap P$ is a polytope. This polytope lies in an appropriately translated version of $L^\perp$, and therefore

$$\sum_{F\in\mathcal{F}_0((L^\perp+x)\cap P)} \gamma_{L^\perp}(P, F) = V_0((L^\perp + x) \cap P) = 1,$$

by the definition (2.2) of $\gamma_{L^\perp}(P, F)$. If $(L^\perp + x) \cap P = \emptyset$, the sum is empty, and therefore vanishes.

Replacing the indicator function with this sum, we get

$$V_k(P|L) = \int_L \sum_{F\in\mathcal{F}_0((L^\perp+x)\cap P)} \gamma_{L^\perp}(P, F)\, d\lambda_k(x).$$

Each $F \in \mathcal{F}_0((L^\perp + x) \cap P)$ is of the form $L^\perp \cap F'$ for some face F' of P, where the dimension of any such F' can obviously be at most $d-(d-k)=k$. In fact, if its dimension is less than k, x lies in the projection $F'|L$, a set of dimension $k-1$ or less. The union of all these sets has λ_k measure 0, as P has only finitely many faces. Therefore, in the integral we can replace $\sum_{F\in\mathcal{F}_0((L^\perp+x)\cap P)} \gamma_{L^\perp}(P,F)$ with $\sum_{F'\in\mathcal{F}_k(P),(L^\perp+x)\cap F'\neq\emptyset} \gamma_{L^\perp}(P,F')$. This gives us

$$\begin{aligned} V_k(P|L) &= \int_L \sum_{F'\in\mathcal{F}_k(P)} \gamma_{L^\perp}(P,F) I((L^\perp + x)\cap F' \neq \emptyset)\, d\lambda_k(x) \\ &= \sum_{F\in\mathcal{F}_k(P)} \gamma_{L^\perp}(P,F) \int_L I((L^\perp + x)\cap F \neq \emptyset)\, d\lambda_k(x) \\ &= \sum_{F\in\mathcal{F}_k(P)} \gamma_{L^\perp}(P,F) V_k(F|L). \end{aligned}$$

We now use the definition of $\gamma_{L^\perp}$ to expand

$$\gamma_{L^\perp}(P,F) = \frac{1}{\sigma_{d-k}} \omega_{d-k-1}(n_{L^\perp}(P,F)).$$

The set $n_{L^\perp}(P,F)$ is the image of $n(P,F)$ under the mapping $\pi_{L^\perp} : x \mapsto (x|L^\perp)/\|x|L^\perp\|$. From Lemma 15 we know that for $L^\perp, F$ in general relative position we have

$$\omega_{d-k-1}(\pi_{L^\perp}(n(P,F)) = \int_{n(P,F)} \frac{|\langle F^\perp, L^\perp\rangle|}{\|u|L^\perp\|^{d-k}}\, d\omega_{d-k-1}(u).$$

Together with $V_k(F|L) = |\langle F, L\rangle| V_k(F)$ and $|\langle F^\perp, L^\perp\rangle| = |\langle F, L\rangle|$ we get

$$V_k(P|L) = \frac{1}{\sigma_{d-k}} \sum_{F\in\mathcal{F}_k(P)} V_k(F) \int_{n(P,F)} \frac{\langle F, L\rangle^2}{\|u|L^\perp\|^{d-k}}\, d\omega_{d-k-1}(u). \tag{3.6}$$

If $L^\perp, F$ are not in general relative position, we have $|\langle F, L\rangle| = 0$, and (3.6) also holds. $\square$

We now present an alternative form of (3.5). We choose an orthonormal basis $u_1 = u, u_2, \dots, u_{d-k}$ of $F^\perp$. Then

$$\begin{aligned} |\langle F, L\rangle| &= |\langle F^\perp, L^\perp\rangle| \\ &= \lambda_{d-k}([0, u_1|L^\perp] + ([0,u_2] + \ldots + [0, u_{d-k}])|L^\perp]) \\ &= \lambda_1(u, u|L^\perp) \cdot \lambda_{d-k-1}(([0,u_2] + \ldots + [0,u_{d-k}])|(L^\perp \cap u^\perp)) \\ &= \|u|L^\perp\| \cdot |\langle u^\perp \cap F^\perp, u^\perp \cap L^\perp\rangle|, \end{aligned}$$

yielding

$$\frac{\langle F, L\rangle^2}{\|u|L^\perp\|^{d-k}} = \frac{\langle u^\perp \cap F^\perp, u^\perp \cap L^\perp\rangle^2}{\|u|L^\perp\|^{d-k-2}}$$

in the case $|\langle F, L\rangle| \neq 0$. We define the function f_k^d by

$$f_k^d : \mathcal{L}_k^d \times \mathcal{L}_k^d \times S^{d-1} \to \mathbb{R}, \quad (L, F, u) \mapsto \begin{cases} \frac{\langle u^\perp \cap F^\perp, u^\perp \cap L^\perp\rangle^2}{\|u|L^\perp\|^{d-k-2}}, & |\langle F, L\rangle| \neq 0, \\ 0, & |\langle F, L\rangle| = 0. \end{cases}$$

With this function we can express (3.5) as

$$V_k(P|L) = \frac{1}{\sigma_{d-k}} \sum_{F \in \mathcal{F}_k(P)} V_k(F) \int_{n(P,F)} f_k^d(F, L, u)\, d\omega_{d-k-1}(u). \tag{3.7}$$

One special case is $k = d-2$, giving

$$f_{d-2}^d(L, F, u) = \begin{cases} \langle u^\perp \cap F^\perp, u^\perp \cap L^\perp\rangle^2, & |\langle F, L\rangle| \neq 0, \\ 0, & |\langle F, L\rangle| = 0. \end{cases}$$

Applying this to $d = 3$, we can see at this point that (3.3) holds for polytopes, with a measure defined by (3.4).

Remark. It is easy to see that f_{d-2}^d is continuous at (L, F, u) if $u \notin L$. However, for $u \in L$, this is not the case, as the following example shows.

We consider the special case $d = 3$, and we put

$$F = \mathrm{lin}\{\begin{pmatrix}1\\0\\0\end{pmatrix}\}, u = \begin{pmatrix}0\\1\\0\end{pmatrix} \in F^\perp, L = \mathrm{lin}\{\begin{pmatrix}\sin\omega\\ \cos\omega\\ 0\end{pmatrix}\}, \quad \omega \in \mathbb{R}.$$

For $\omega = 0$, we have $f_1^3(L, F, u) = 0$, as $u \in L$. For $\omega \in (-\pi/2, \pi/2) \setminus \{0\}$ we have $|\langle F, L\rangle| \neq 0$, and thus

$$\begin{aligned} f_1^3(L, F, u) &= |\langle F^\perp \cap u^\perp, L^\perp \cap u^\perp\rangle| \\ &= |\langle \mathrm{lin}\{\begin{pmatrix}0\\0\\1\end{pmatrix}\}, \mathrm{lin}\{\begin{pmatrix}0\\0\\1\end{pmatrix}\}\rangle| \\ &= 1. \end{aligned}$$

Similarly, it is possible to show that f_1^3 takes any value in $[0, 1]$ in any neighbourhood of $(\mathrm{lin}\{u\}, F, u)$, and thus cannot be made continuous by changing its value for $|\langle F, L\rangle| = 0$.

An extension of Lemma 17 to arbitrary convex bodies is stated in the following Theorem.

Theorem 18. *Let $0 \leq k < d$ and let $K \subset \mathbb{R}^d$ be a convex body. Then a Borel measure $\mu_k(K, \cdot)$ on $S^{d-1,k}$ exists such that*

$$V_k(K|L) = \int_{S^{d-1,k}} f_k^d(F, L, u) \mu_k(K, d(u, F)), \quad L \in \mathcal{L}_k^d. \tag{3.8}$$

If K is a polytope, the measure defined by

$$\mu_k(K,\eta) = \frac{1}{\sigma_{d-k}} \sum_{F\in\mathcal{F}_k(K)} V_k(F) \int_{n(K,F)} I((u,L(F))\in\eta)\, d\omega_{d-k-1}(u) \tag{3.9}$$

has this property.

Proof. If K is a polytope, the measure $\mu_k(K,\cdot)$ defined by (3.9) has the desired property. This follows directly from Lemma 17.

Now let K be an arbitrary convex body. Let (P_i) be a sequence of polytopes converging to K. These measures are defined on the compact space $S^{d-1}\times\mathcal{L}_k^d$, and the sequence of their total measures,

$$\begin{aligned}\mu_k(P_i,S^{d-1,k}) &= \frac{1}{\sigma_{d-k}} \sum_{F\in\mathcal{F}_k(P_i)} V_k(F)\gamma(P_i,F)\\ &= V_k(P_i),\end{aligned}$$

converges to $V_k(K)$. Thus there exists an upper bound for the total measures. Therefore, a weakly convergent subsequence of $(\mu_k(P_i,\cdot))$ exists (see Bauer [3], Satz 46.3). We assume without loss of generality that (P_i) is such a subsequence, and we let $\mu_k(K,\cdot)$ be the weak limit of $(\mu_k(P_i,\cdot))$.

In view of the definition of zonal sets (2.11) we define a function

$$f_\epsilon(u,F) := \frac{\langle F,L\rangle^2}{\max(\epsilon,\|u|L^\perp\|)^{d-k}},$$

which is continuous on $S^{d-1,k}$, and increases when ϵ decreases. Moreover, the integrand of (3.8) is the limit $f := \lim_{\epsilon\to 0} f_\epsilon$. The Theorem of monotone convergence (see Bauer [3], Satz 11.4) shows

$$\int_{S^{d-1,k}} f(u,F)\mu_k(K,\, d(u,F)) = \lim_{\epsilon\to 0}\int_{S^{d-1,k}} f_\epsilon(u,F)\mu_k(K,\, d(u,F)),$$

and the continuity of f_ϵ implies (using the weak convergence of the measures $\mu_k(P_i,\cdot)$)

$$\begin{aligned}\int_{S^{d-1,k}} f_\epsilon(u,F)\mu_k(K,\, d(u,F)) &= \lim_{i\to\infty}\int_{S^{d-1,k}} f_\epsilon(u,F)\mu_k(P_i,\, d(u,F))\\ &\le \lim_{i\to\infty}\int_{S^{d-1,k}} f(u,F)\mu_k(P_i,\, d(u,F))\\ &= \lim_{i\to\infty} V_k(P_i|L)\\ &= V_k(K|L).\end{aligned}$$

It remains to show

$$\int_{S^{d-1,k}} f(u,F)\, d\mu_k(K,\, d(u,F)) \ge V_k(K|L).$$

In view of Theorem 8 it becomes clear that for $0 < \epsilon < 1$

$$\begin{aligned}\int_{S^{d-1,k}} f(u,F)\mu_k(K,\, d(u,F)) &\geq \int_{S^{d-1,k}} f_\epsilon(u,F)\mu_k(K,\, d(u,F)) \\ &= \lim_{i\to\infty}\int_{S^{d-1,k}} f_\epsilon(u,F)\mu_k(P_i,\, d(u,F)) \\ &\geq \lim_{i\to\infty}\left(V_k(P_i|L) - \lambda_k(\operatorname{cl} P_{i,\epsilon}|L)\right) \\ &\geq \lim_{i\to\infty} V_k(P_i|L) - \epsilon c_k(K) \\ &\xrightarrow{\epsilon\to 0} V_k(K|L),\end{aligned}$$

where $c_k(K)$ is some constant depending on k and K, and $P_{i,\epsilon}$ is defined as in Theorem 8:

$$P_{i,\epsilon} = \{x \in \operatorname{bd} P_i :\, n(P_i,x) \cap Z(L,\epsilon) \neq \emptyset\}.$$

Altogether we get the assertion. □

One application of the measures $\mu_k(K,\cdot)$ is the characterization of zonoids in the set of centrally symmetric convex bodies. Again, Ambartzumian has proved the result for $d = 3$, and we use a similar argument here.

Theorem 19. *Let Z be a centrally symmetric convex body. Z is a zonoid if and only if there exists a measure $\mu_1(Z)$ on $S^{d-1,1}$ that satisfies* (3.8) *and has the form*

$$\mu_1(Z,\eta) = \int_{\mathcal{L}_1^d}\int_{F^\perp\cap S^{d-1}} I((u,F)\in\eta)\, d\omega_{d-2}(u)\rho(Z,\, dF) \tag{3.10}$$

for some finite measure $\rho(Z,\cdot)$ on $\mathcal{L}_1^d$.

Proof. Let Z be a zonotope. We put

$$\mathcal{F}_1'(Z) := \{F \in \mathcal{L}_1^d :\, \exists F' \in \mathcal{F}_1(Z) \text{ such that } L(F') = F\}.$$

For $F \in \mathcal{F}_1'(Z)$ let $l(F)$ be the common length of all edges of Z in direction F (see (2.3)). We apply (2.4) to (3.9) and get

$$\begin{aligned}\mu_1(Z,\eta) &= \frac{1}{\sigma_{d-1}}\sum_{F\in\mathcal{F}_1(Z)} V_k(F)\int_{n(K,F)} I((u,L(F))\in\eta)\, d\omega_{d-2}(u) \\ &= \frac{1}{\sigma_{d-1}}\sum_{F\in\mathcal{F}_1'(Z)} l(F)\int_{F^\perp\cap S^{d-1}} I((u,F)\in\eta)\, d\omega_{d-2}(u),\end{aligned}$$

for all Borel sets η, and hence $\mu_1(Z,\cdot)$ is of the form (3.10) (and ρ is the sum of one point measures).

Now let Z be a zonoid. We can approximate Z by zonotopes. The construction in the proof of Theorem 18 yields a measure $\mu_1(Z,\cdot)$ with the desired properties. However, (2.5) yields the existence of a measure ρ satisfying (3.10) directly.

On the other hand, let Z be a centrally symmetric convex body for which $\mu_1(Z,\cdot)$ has the form (3.10). We apply (3.8), and carrying out the inner integration we get for $L \in \mathcal{L}_1^d$

$$\begin{aligned} V_1(Z|L) &= \int_{\mathcal{L}_1^d} \int_{F^\perp \cap S^{d-1}} f_1^d(L,F,u)\, d\omega_{d-2}(u) \rho(Z,\, dF) \\ &= \sigma_{d-1} \int_{\mathcal{L}_1^d} |\langle F, L\rangle| \rho(Z,\, dF). \end{aligned}$$

(Here we used that

$$\int_{F^\perp \cap S^{d-1}} \frac{1}{\|u|L^\perp\|^{d-1}}\, d\omega_{d-2}$$

is the $(d-1)$-volume of

$$M := \{x \in F^\perp : \|x|L^\perp\| \le 1\}.$$

Now $M|L^\perp = B^d \cap L^\perp$, and thus $V_{d-1}(M) = |\langle F^\perp, L^\perp\rangle|\sigma_{d-1} = |\langle F, L\rangle|\sigma_{d-1}$.)

Thus for $x \in S^{d-1}$ we have

$$h(Z,u) = \frac{1}{2} V_1(Z|L(u)) = \frac{\sigma_{d-1}}{2} \int_{\mathcal{L}_1^d} |\langle F, L(u)\rangle| \rho(Z,\, dF),$$

and (2.5) yields that Z is a zonoid. □

3.3 A counterexample

In (3.9) we have given a definition for the measures $\mu_k(P,\cdot)$ for polytopes P. By approximating a convex body K with polytopes $P_i \to K$ $(i \to \infty)$ in Theorem 18, $\mu_k(K,\cdot)$ was defined to be the weak limit of a converging subsequence of $(\mu_k(P_i,\cdot))$. Naturally, the question arises if $\mu_k(K,\cdot)$ depends weakly continuously on K. The following example shows that this is not even the case for $d=3, k=1$.

Example 20.

We are going to approximate the unit ball $B \subset \mathbb{R}^3$ by two sequences (P_n) and (Q_n) of polytopes. We then show that the sequences of associated measures $(\mu_1(P_n,\cdot))$ and $(\mu_1(Q_n,\cdot))$ converge weakly, but to different measures $\mu_1'(B,\cdot)$ and $\mu_1''(B,\cdot)$, respectively. This means that $\mu_1(B,\cdot)$ can not be defined such that $K \mapsto \mu_1(K,\cdot)$ is weakly continuous at B.

In fact, we will show that the limit measure $\mu_1'(B,\cdot)$ of $(\mu_1(P_n,\cdot))$ is not rotation invariant, i. e. there exists a rotation ρ such that $\mu_1'(B,\rho\eta) \ne \mu_1'(B,\eta)$ for some Borel set $\eta \subset S^2$. We can then put $Q_n := \rho^{-1}P_n$, and thus $\mu_1''(B,\eta) = \mu_1'(B,\rho\eta) \ne \mu_1'(B,\eta)$, i. e. $\mu_1'(B,\cdot) \ne \mu_1''(B,\cdot)$. To show that $\mu_1(B,\eta)$ is not rotation invariant, it suffices to show that $\tilde{\mu}(\cdot) := \mu_1'(B, S^2 \times \cdot)$ is not rotation invariant. $\tilde{\mu}$ is a measure on $\mathcal{L}_1^3$. We know that the only (up to scaling) finite rotation invariant measure on $\mathcal{L}_1^3$ is ν_1. Therefore it suffices to show that a Borel set $\eta \subset \mathcal{L}_1^3$ exists such that $\tilde{\mu}(\eta) > 0$ and $\nu_1(\eta) = 0$.

For $n \in \mathbb{N}$ we define

$$P_n := \operatorname{conv}\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S^2 : x = i2^{-n}, y = j2^{-n}, i, j \in \mathbb{Z}\}.$$

We obviously have $\lim_{n\to\infty} P_n = B$.

There exists a subsequence of $(\mu_k(P_n, \cdot))$ that converges weakly. Without loss of generality, we assume that $(\mu_k(P_n, \cdot))$ converges weakly, and define

$$\tilde{\mu}(\cdot) := \lim_{n\to\infty} \mu_k(P_n, S^2 \times \cdot),$$

a measure on $\mathcal{L}_1^3$.

We put

$$\eta := \{\operatorname{lin}\{\begin{pmatrix} x \\ y \\ z \end{pmatrix}\} : x = y = 1 \vee (x = 1, y = 0) \vee (x = 0, y = 1)\}.$$

Obviously $\nu_1(\eta) = 0$. It remains to show that $\tilde{\mu}(\eta) > 0$. For $i^2 + j^2 \leq 2^n$ we put

$$p(n, i, j) := (i2^{-n}, j2^{-n}, \sqrt{1 - (i^2 + j^2)2^{-2n}}),$$

which is a vertex of P_n. Consider the set

$$e(n, i', j') := [p(n, i, j), p(n, i + i', j + j')] \subset P_n, \quad i', j' \geq 0.$$

It is clear that all edges (or 1-faces) of P_n have this form. $\mu_1(P_n, S^2 \times \cdot)$ is concentrated on the set $\{L(F) : F \in \mathcal{F}_1(P_n)\}$. The line $L(e(n, i', j'))$ is in η for $i' = j' = 1$ or $i' = 1, j' = 0$ or $i' = 0, j' = 1$. We now show that $e(n, i', j')$ can be an edge only in these cases.

We define a function parametrizing $e(n, i', j')$,

$$f(t) := tp(n, i + i', j + j') + (1 - t)p(n, i, j).$$

In particular, if $f(1/i')$ lies in the interior of P_n, $e(n, i', j')$ lies not completely in the boundary of P_n, and thus it is no edge. For symmetry reasons, it suffices to show that $f(1/i')$ lies in the interior of P_n for $i, j \geq 0$ and $0 < j' < i'$.

The first and second component of $f(1/i')$ are $(1 + i)2^{-n}$ and

$$\frac{1}{i'}(j + j')2^{-n} + \frac{i' - 1}{i'} j2^{-n} = \frac{j' + i'j}{2}^{-n},$$

respectively. These are also the first and second component of

$$q(n, i', j') := j'/i' \cdot p(n, i + 1, j + 1) + (1 - j'/i') \cdot p(n, i + 1, j),$$

a point on the segment connecting $p(n, i + 1, j)$ and $p(n, i + 1, j + 1)$. We will show that $f(1/i')$ lies in the relative interior of the segment s connecting $q(n, i', j')$ and its reflection $\tilde{q}$ about $\mathrm{lin}(e_1, e_2)$ (which is also a point of P_n). It then follows that $f(1/i')$ lies in the interior of P_n, because s lies not in the boudary of

$$\mathrm{conv}\{q(n,i',j'), \tilde{q}, p(n,i,j), p(n,i+2,j), p(n,i+1,j+1), p(n,i+1,j-1)\} \subset P_n.$$

To show that $f(1/i')$ lies in the relative interior of s, it suffices to show that the third component of $f(1/i')$ is less than the third component of $q(n,i',j')$. This number is

$$z_1 := \frac{j'}{i'}\sqrt{1-((i+1)^2+(j+1)^2)2^{-2n}} + (1-\frac{j'}{i'})\sqrt{1-((i+1)^2+j^2)2^{-2n}},$$

and the third component of $f(1/i')$ is

$$z_2 := \frac{1}{i'}\sqrt{1-((i+i')^2+(j+j')^2)2^{-2n}} + (1-\frac{1}{i'})\sqrt{1-(i^2+j^2)2^{-2n}}.$$

The partial derivatives of z_1, z_2 with respect to j' are

$$\begin{aligned}
\frac{\partial}{\partial j'}z_1 &= \frac{1}{i'}\left(\sqrt{1-((i+1)^2+(j+1)^2)2^{-2n}} - \sqrt{1-((i+1)^2+j^2)2^{-2n}}\right) \\
&= -\frac{1}{i'}\frac{\left(1-((i+1)^2+j^2)2^{-2n}\right) - \left(1-((i+1)^2+(j+1)^2)2^{-2n}\right)}{\sqrt{1-((i+1)^2+j^2)2^{-2n}} + \sqrt{1-((i+1)^2+(j+1)^2)2^{-2n}}} \\
&= -\frac{1}{i'}\frac{(2j+1)2^{-2n}}{\sqrt{1-((i+1)^2+j^2)2^{-2n}} + \sqrt{1-((i+1)^2+(j+1)^2)2^{-2n}}} \\
&> \frac{-(j+j')2^{-2n}}{i'\sqrt{1-((i+i')^2+(j+j')^2)2^{-2n}}},
\end{aligned}$$

and

$$\frac{\partial}{\partial j'}z_2 = \frac{-(j+j')2^{-2n}}{i'\sqrt{1-((i+i')^2+(j+j')^2)2^{-2n}}},$$

respectively. As $\frac{\partial}{\partial j'}z_1 > \frac{\partial}{\partial j'}z_2$, it would suffice to show $z_1 > z_2$ for $j'=1$. Instead, we show the equivalent inequality $i'z_1 > i'z_2$. The partial derivatives of $i'z_1, i'z_2$ with respect to i' are

$$z_3 := \frac{\partial}{\partial i'}(i'z_1) = \sqrt{1-((i+1)^2+j^2)2^{-2n}},$$

and

$$z_4 := \frac{\partial}{\partial i'}(i'z_2) = \sqrt{1-(i^2+j^2)2^{-2n}} - \frac{(i+i')2^{-2n}}{\sqrt{1-((i+i')^2+(j+j')^2)2^{-2n}}}$$

respectively. The difference of z_3 and z_4 is

$$\begin{aligned}
z_3 - z_4 &= \sqrt{1-((i+1)^2+j^2)2^{-2n}} - \sqrt{1-(i^2+j^2)2^{-2n}} \\
&\quad + \frac{(i+i')2^{-2n}}{\sqrt{1-((i+i')^2+(j+j')^2)2^{-2n}}} \\
&= -\frac{2^{-2n}(2i+1)}{\sqrt{1-((i+1)^2+j^2)2^{-2n}}+\sqrt{1-(i^2+j^2)2^{-2n}}} \\
&\quad + \frac{(i+i')2^{-2n}}{\sqrt{1-((i+i')^2+(j+j')^2)2^{-2n}}} \\
&> -\frac{2^{-2n}(2i+1)}{2\sqrt{1-((i+1)^2+j^2)2^{-2n}}} + \frac{(i+i')2^{-2n}}{\sqrt{1-((i+1)^2+j^2)2^{-2n}}} \\
&= \frac{(i'-\frac{1}{2})2^{-2n}}{2\sqrt{1-((i+1)^2+j^2)2^{-2n}}} \\
&> 0.
\end{aligned}$$

Now all that remains to show $z_1 > z_2$ for all $i' > j' \geq 1$ is to show that $z_1 \geq z_2$ holds for $i' = j' = 1$. In this case we have

$$z_1 = \sqrt{1-((i+1)^2+(j+1)^2)2^{-2n}} = z_2,$$

and the assertion follows. $\square$

We have seen that the measures $\mu_k(K,\cdot)$ can be used for an integral representation of projections functions. For polytopes P, the measure $\mu_k(P,\cdot)$ is given by (3.9). For general convex bodies K, the existence of such a measure has been established by approximation of K by polytopes. In general, the resulting measure depends on the chosen approximation, and not only on K. In particular, there can be no definition of $\mu_k(K,\cdot)$ such that (3.9) holds for polytopes and $\mu_k(K,\cdot)$ depends weakly continuously on K.

In the next chapter we will define a measure depending weakly continuously on K. In Chapter 5 we will prove an integral representation of projection functions using this measure.

4 Generalized support measures

We now give a generalization of the support measures introduced in Section 2.1.4. Historically, there have been a number of generalizations and variants of the classical surface area and curvature measures.

In 1937/1938, the surface area measures $S_j(K, \cdot)$ (on S^{d-1}) were introduced by Alexandrov and Fenchel-Jessen. Later, the curvature measures (on $\mathbb{R}^d$) by Federer followed. The support measures (on $\mathbb{R}^d \times S^{d-1}$) by Schneider, which we introduced in Section 2.1.4, are a generalization of both the surface are measures and the curvature measures. A generaliztation of the curvature measures to the manifold of k-flats touching a convex body was investigated by Weil in [11] and [12]. Kropp [6] studied corresponding generalizations of the surface area measures. In the following we present a common generalization of the measures of Weil and of Kropp.

Instead of local parallel sets of K that consist of points $x \in \mathbb{R}^d$ with $d(K, x) < \rho$, where ρ is some positive real number, we consider local parallel sets of k-flats E with $d(K, E) < \rho$. For this, we use the method applied by Schneider [9] for the support measures. As we shall show, the invariant measure of these local parallel sets has a polynomial expansion in ρ, and the generalized curvature are the coefficients of this polynomial.

We mention that there are further generalizations of support measures by Rataj and Zähle [7] and Hug [5].

4.1 Local parallel sets of flats

Let $0 \leq k < d$, let K be a convex body, and let $\rho > 0$. For every affine k-flat $E \in \mathcal{E}_k^d$ we consider the set of points of E with minimal distance to K. If there is a unique point of E with minimal distance to K, we call this point $l(K, E)$. In this case, we write $p(K, E)$ for the metric projection of $l(K, E)$ onto K, i. e. $p(K, E) = p(K, l(K, E))$. Clearly $p(K, E)$ is the unique nearest point of K to E. We then have $d(E, K) = d(l(K, E), K) = \|l(K, E) - p(K, E)\|$.

Sometimes it is convenient to consider only the set of k-flats $E \in \mathcal{E}_k^d$ that have a unique point of minimal distance to K, and do not intersect K. We call this set $K^{(k)}$:

$$K^{(k)} := \{E \in \mathcal{E}_k^d : E \cap K = \emptyset, \text{ point of minimal distance is unique}\}.$$

In fact, $K^{(k)}$ comprises almost all k-flats not intersecting K.

Lemma 21. *Let $0 \leq k \leq d$, and let K be a convex body. Then*

$$\mu_k(\{\mathcal{E}_k^d : E \cap K = \emptyset\} \setminus K^{(k)}) = 0.$$

Proof. Any flat that does not intersect K and for which the nearest point is not unique must be parallel to some line segment in the boundary of K. From Schneider [9], Corollary 2.3.11, it thus follows that the set of all k-dimensional flats that neither intersect K nor have a unique nearest point to K has μ_k-measure 0. □

Schneider considers parallel sets of a convex body K. These sets consist of points whose distance from K is less than some real number ρ. A generalization are "parallel" sets that consist of (affine) k-flats whose distance from K is less than ρ. Also, the direction of the shortest segment connecting K and the flat plays an important role. It must be an outer normal of the points $x \in K$ that minimize the distance $d(E, x)$.

Definition 22. *Let $0 \leq k \leq d$ and let K be a convex body. For $E \in K^{(k)}$ we define the direction of the shortest segment connecting K and E as*

$$u(K, E) := \frac{l(K, E) - p(K, E)}{d(K, E)},$$

and we define the set $K_\rho^{(k)}$ of parallel k-flats *of positive distance not greater than $\rho \in \mathbb{R}$ as*

$$K_\rho^{(k)} := \{E \in K^{(k)} : 0 < d(K, E) \leq \rho\}.$$

Remark. The set $K_\rho^{(k)}$ is a measurable set.

Proof. The set of all k-flats intersecting a convex body is a closed set and thus measurable. Therefore the sets

$$\{E \in \mathcal{L}_k^d : E \cap K + \rho B \neq \emptyset\} \text{ and } \{E \in \mathcal{L}_k^d : E \cap K \neq \emptyset\}$$

are measurable. $K_\rho^{(k)}$ is – up to the flats with non-unique nearest point to K – the difference of these sets. □

Lemma 23. *Let $K \in \mathcal{K}$. Then the functions $p(K, \cdot), l(K, \cdot), u(K, \cdot) : K^{(k)} \to \mathbb{R}^d$ and $d(\cdot, \cdot) : \mathcal{K} \times \mathcal{L}_k^d \to \mathbb{R}^d$ are continuous.*

Proof. We start with the continuity of $d(\cdot, \cdot)$. Let L be a fixed k-flat and let (K_i) be a sequence of convex bodies converging to K. Then $|d(K_i, L) - d(K, L)| \leq \tilde{d}(K_i, K) \to 0$ for $i \to \infty$. On the other hand, let (E_i) be a sequence of k-flats in $\mathcal{L}_k^d$ that converges to some $E \in \mathcal{L}_k^d$, i. e. for a fixed k-flat L there exist converging sequences (ρ_i) of rotations and (x_i) of points in $L^\perp$ such that $E_i = \rho_i(L + x_i)$. It follows that

$$d(K, E_i) = d(K, \rho_i(L + x_i)) = d(\rho_i^{-1}K - x_i, L) \to d(\rho^{-1}K - x, L) = d(K, E), \quad i \to \infty,$$

where x is the limit of (x_i) and ρ is the limit of (ρ_i).

We next show that $p(K, \cdot)$ is continuous. Due to the compactness of K, it suffices to show that any accumulation point of $(p(K, E_i))$ coincides with $p(K, E)$. Let x be such an accumulation point, i. e. a subsequence (E_{i_j}) exists such that $p(K, E_{i_j}) \to x \in K$

$(j \to \infty)$. Now $d(p(K, E_{i_j}), E_{i_j}) \leq d(p(K, E), E_{i_j})$, and the limit for $j \to \infty$ yields $d(x, E) \leq d(K, E)$. The uniqueness property of $p(K, E)$ implies $x = p(K, E)$.

As the sequence $l(K, E_i)$ is clearly bounded, we can use a similar argument for the continuity of l. Consider a subsequence (E_{i_j}) such that $l(K, E_{i_j})$ converges to $y \in \mathbb{R}^d$ $(j \to \infty)$. Both y and $l(K, E)$ lie on E, thus $[y, l(K, E)] \subset E$. As $d(K, y) = d(K, l(K, E))$ by continuity of d, the uniqueness property of l implies $y = l(K, E)$.

Now we have that p, l, d are continuous, and $d(K, \cdot)$ is positive on $K^{(k)}$. The continuity of u follows from the definition. □

Lemma 24. *Let (K_i) be a converging sequence of convex bodies with limit $K \in \mathcal{K}$. Let $0 \leq k < d$ and let*

$$L \in K^{(k)} \cap \bigcap_{i=1}^{\infty} K_i^{(k)}.$$

Then

$$p(K_i, L) \to p(K, L),\ l(K_i, L) \to l(K, L),\ u(K_i, L) \to u(K, L).$$

Proof. The convergence of (K_i) implies the existence of some (compact) ball that contains K_i, $i \in \mathbb{N}$, and K. To show $p(K_i, L) \to p(K, L)$ it thus suffices to show that any accumulation point of the sequence $(p(K_i, L))$ is $p(K, L)$. The continuity of d implies that any accumulation point of $(p(K_i, L))$ has the same distance from L, namely $d(K, L)$. Because L is in $K^{(k)}$, there exists exactly one such point in K.

Orthogonal projections are continuous, and thus $l(K_i, L) = p(K_i, L)|L$ implies that $l(K_i, L) \to l(K, L)$, $i \to \infty$. Thus $(l(K_i, L) - (p(K_i, L))$ converges. $(d(K_i, L))$ converges also, and is positive with positive limit. Thus $u(K_i, L) \to u(K, L)$, $i \to \infty$ by the definition of u. □

The measures we will now introduce are defined on

$$\Sigma^{(k)} := \mathbb{R}^d \times S^{d-1} \times \mathcal{L}_k^d.$$

Let this set be equipped with the product topology. The measures will be concentrated on the set of triples (x, u, L), where x lies in the boundary of K, u is an outer normal of K at x, and L is a linear subspace of $\mathbb{R}^n$ that is orthogonal to u.

Let $K \in \mathcal{K}$ be fixed. From Lemma 23 we know that $E \mapsto p(K, E)$ and $E \mapsto u(K, E)$ are continuous on $\mathcal{E}_k^d$. The mapping $E \mapsto L(E)$ is obviously also continuous. Thus

$$f_\rho^{(k)} : \begin{cases} K_\rho^{(k)} \to \Sigma^{(k)} \\ E \mapsto (p(K, E), u(K, E), L(E)) \end{cases}$$

is a continuous function.

This allows us to define a Borel measure as the image of μ_k under $f_\rho^{(k)}$.

Definition 25. *For $K \in \mathcal{K}$ and $\rho > 0$ we define the Borel measure*

$$\mu_\rho^{(k)}(K, \cdot) := \mu_k((f_\rho^{(k)})^{-1}(\cdot)).$$

Thus $\mu_\rho^{(k)}(K,\cdot)$ is a finite measure on $\mathcal{B}(\Sigma^{(k)})$, and the measure of a Borel set $\eta \in \mathcal{B}(\Sigma^{(k)})$ is the μ_k-measure of a local parallel set of supporting k-flats,

$$\begin{aligned} M_\rho^{(k)}(K,\eta) &:= \left(f_\rho^{(k)}\right)^{-1}(\eta) \\ &= \{E \in K_\rho^{(k)} : (p(K,E), u(K,E), L(E)) \in \eta\}. \end{aligned}$$

While $\mu_\rho^{(k)}(K,\eta)$ has been defined for fixed K, we can also consider the convex body K as variable. In analogy to the classic case of the curvature measures, we investigate some properties of $\mu_\rho^{(k)}(K,\cdot)$ with respect to K, in particular continuity, measurability and additivity.

Lemma 26. *Let (K_i) be a sequence of convex bodies converging to $K \in \mathcal{K}$. Then the measures $\mu_\rho^{(k)}(K_i,\cdot)$ converge weakly to $\mu_\rho^{(k)}(K,\cdot)$,*

$$\mu_\rho^{(k)}(K_i,\cdot) \xrightarrow{w} \mu_\rho^{(k)}(K,\cdot) \qquad (k \to \infty).$$

Proof. Let A be the set of all k-flats that are parallel to a 1-face of some K_i, i. e.

$$A := \{E \in \mathcal{E}_k^d : \exists i \in \mathbb{N}, F \in \mathcal{F}_1(K_n) \text{ such that } F||E\}.$$

Note that A is a set of measure 0, which once more follows from Schneider [9], Corollary 2.3.11. Let $\eta \subset \Sigma^{(k)}$ be an open subset. Let $E \in M_\rho^{(k)}(K,\eta)\setminus A$ be an arbitrary flat with $d(K,E) < \rho$. For almost all i, K_i and E do not intersect. Therefore $d(K_i,E) \to d(K,E)$ and $(p(K_i,E), u(K_i,E)) \to (p(K,E), u(K,E))$ for $i \to \infty$. It follows that for almost all i the inequality $d(K_i,E) < \rho$ holds, and $(p(K_i,E), u(K_i,E), L(E)) \in \eta$. Thus, for almost all $i \in \mathbb{N}$ we have $E \in M_\rho^{(k)}(K_i,\eta)$. We get

$$(M_\rho^{(k)}(K,\eta) \setminus A) \cap \{E \in \mathcal{E}_k^n : d(K,E) < \rho\} \subset \liminf_{i\to\infty} M_\rho^{(k)}(K_i,\eta),$$

and thus

$$\begin{aligned} \mu_\rho^{(k)}(K,\eta) &= \mu_k(M_\rho^{(k)}(K,\eta)) \\ &= \mu_k\left((M_\rho^{(k)}(K,\eta) \setminus A) \cap \{E \in \mathcal{E}_k^n : d(K,E) < \rho\}\right) \\ &\le \mu_k\left(\liminf_{i\to\infty} M_\rho^{(k)}(K_i,\eta)\right) \\ &\le \liminf_{i\to\infty} \mu_k(M_\rho^{(k)}(K_i,\eta)) \\ &= \liminf_{i\to\infty} \mu_\rho^{(k)}(K_i,\eta). \end{aligned}$$

For the second equality we used two facts. From Crofton's formula we know

$$\int_{\mathcal{E}_k^d} I(M \cap E \neq \emptyset)\, d\mu_k(E) = \alpha_{d0k} V_{d-k}(M)$$

for all convex bodies M. The intrinsic volumes are continuous on $\mathcal{K}$, in particular $\lim_{r\to\rho} V_{d-k}(K+rB) = V_{d-k}(K+\rho B)$. Thus $\mu_k(\{E \in \mathcal{E}_k^d : d(K,E) = \rho\}) = 0$.

The same reasoning shows

$$\mu_\rho^{(k)}(K_i, \Sigma^{(k)}) \to \mu_\rho^{(k)}(K, \Sigma^{(k)}), \quad i \to \infty,$$

which completes the proof. □

Lemma 27. *For arbitrary $\eta \in \mathcal{B}(\Sigma^{(k)})$ the mapping $\mu_\rho^{(k)}(\cdot,\eta) : \mathcal{K} \to \mathbb{R}$ is measurable.*

Proof. We already know from the previous proof that for open sets η the mapping $\mu_\rho^{(k)}(\cdot,\eta)$ is lower half-continuous and therefore measurable. We show that the set $\mathcal{D}$ of all η for which $\mu_\rho^{(k)}(\cdot,\eta)$ is measurable is a Dynkin system. $\mu_\rho^{(k)}(\cdot,\Sigma^{(k)})$ is obviously measurable. For $\eta \in \mathcal{D}$ we have $M_\rho^{(k)}(K,\eta) \subset M_\rho^{(k)}(K,\Sigma^{(k)})$ and

$$M_\rho^{(k)}(K, \Sigma^{(k)} \setminus \eta) = M_\rho^{(k)}(K, \Sigma^{(k)}) \setminus M_\rho^{(k)}(K, \eta),$$

which yields

$$\mu_\rho^{(k)}(K, \Sigma^{(k)} \setminus \eta) = \mu_\rho^{(k)}(K, \Sigma^{(k)}) - \mu_\rho^{(k)}(K, \eta)$$

for all $K \in \mathcal{K}$. Thus $\Sigma^{(k)} \setminus \eta \in \mathcal{D}$. Let (η_i) be a sequence of pairwise disjoint elements of $\mathcal{D}$. Then we have for all $K \in \mathcal{K}$

$$\mu_\rho^{(k)}\Big(K, \bigcup_{i=0}^{\infty} \eta_i\Big) = \sum_{i=1}^{\infty} \mu_\rho(K, \eta_1),$$

since $\mu_\rho^{(k)}(K,\cdot)$ is a measure. This yields $\bigcup_{i=1}^\infty \eta_i \in \mathcal{D}$. Thus $\mathcal{D}$ is a Dynkin system containing the open sets.

Lemma 2 now yields that $\mathcal{D}$ contains the σ-algebra generated by the open sets. Thus we have measurability for each Borel set η. □

Lemma 28. *For each $\eta \in \Sigma^{(k)}$ the function $\mu_\rho^{(k)}(\cdot,\eta)$ is additive.*

Proof. Let $K, M \in \mathcal{K}$ satisfy $K \cup M \in \mathcal{K}$. Let $E \in K^{(k)} \cap M^{(k)}$. We put $y := p(K,E), z := p(M,E)$.

We now consider the case $p(K \cup M, E) = y$. As $K \cup M$ is convex, $[y,z]$ is a subset of $K \cup M$. Therefore an $a \in [y,z] \cap K \cap M$ exists. The mapping

$$t \mapsto d(tz + (1-t)y, E)$$

is convex and has a minimum on $[0,1]$ at $t = 0$. This implies $d(y,E) \le d(a,E) \le d(z,E)$. As $z = p(M,E)$, we have $d(a,E) \ge d(z,E)$, and thus $d(a,E) = d(z,E)$ (and $a, z \in M$). The uniqueness of the nearest point now implies $z = a \in K \cap M$. We get

$$d(K \cup M, E) = d(K,E), \quad d(K \cap M, E) = d(M,E)$$

and

$$u(K \cup M, E) = u(K, E), \quad u(K \cap M, E) = u(M, E).$$

Thus

$$\begin{aligned} E \in M_\rho^{(k)}(K \cup M, \eta) &\iff E \in M_\rho^{(k)}(K, \eta), \\ E \in M_\rho^{(k)}(K \cap M, \eta) &\iff E \in M_\rho^{(k)}(M, \eta). \end{aligned}$$

In the other case, $p(K \cup M, E) = z$, a similar reasoning shows

$$\begin{aligned} E \in M_\rho^{(k)}(K \cup M, \eta) &\iff E \in M_\rho^{(k)}(M, \eta), \\ E \in M_\rho^{(k)}(K \cap M, \eta) &\iff E \in M_\rho^{(k)}(K, \eta). \end{aligned}$$

This means that for almost all E we have the identity

$$\begin{aligned} &I(E \in M_\rho^{(k)}(K \cup M, \eta)) + I(E \in M_\rho^{(k)}(K \cap M, \eta)) \\ &\qquad = I(E \in M_\rho^{(k)}(K, \eta)) + I(E \in M_\rho^{(k)}(M, \eta)). \end{aligned}$$

Integration of this identity with respect to μ_k yields the assertion. □

4.2 k-support measures

The measure $\mu_\rho^{(k)}(K, \cdot)$ is a polynomial in ρ, i. e. it is a sum of measures that are homogeneous of different degrees in ρ. To show this, we start by considering the case of a polytope $P \in \mathcal{K}$. For each $E \in P_\rho^{(k)}$ the nearest point $p(P, E)$ lies in the relative interior of a uniquely determined face F of P. For a given face F of P we now compute the measure of the set

$$A := M_\rho^{(k)}(P, \eta) \cap p(P, \cdot)^{-1}(\operatorname{relint} F)$$

for a Borel subset η of $\Sigma^{(k)}$ and $\rho > 0$. If $\dim F \geq d - k$, the nearest point of F to a k-flat E is either not unique, or not in the relative interior, or E and F intersect. In each case E is not in A, so the set A is empty. Therefore we concentrate on the case $m := \dim F < d - k$. Then (assuming $0 \in F$ without loss of generality, as μ_k is translation invariant)

$$\begin{aligned} \mu_k(A) &= \int_{\mathcal{E}_k^d} I(E \in P^{(k)}) \cdot I(p(P, E) \in \operatorname{relint} F) \cdot I(0 < d(P, E) \leq \rho) \\ &\quad \times I((p(P, E), u(P, E), L(E)) \in \eta) \, d\mu_k(E) \\ &= \int_{\mathcal{L}_k^d} \int_{L^\perp} I(L + y \in P^{(k)}) \cdot I(p(P, L + y) \in \operatorname{relint} F) \cdot I(0 < d(F, L + y) \leq \rho) \\ &\quad \times I((p(F, L + y), u(F, L + y), L) \in \eta) \, d\lambda_{d-k}(y) \, d\nu_k(L) \end{aligned} \tag{4.1}$$

For each $L \in \mathcal{L}_k^d$ in general relative position to F we define $L_1 := L(F)|L^\perp = (L(F)+L)\cap L^\perp$ and $L_2 := L_1^\perp \cap L^\perp = L(F)^\perp \cap L^\perp$. We have $L_1 \perp L_2$ and the direct decomposition $L_1 \oplus L_2 = L^\perp$. Moreover, for $y_1 \in L_1$ und $y_2 \in L_2$ we have

$$(L + y_1 + y_2)|L(F) = (L + y_1)|L(F), \qquad (L + y_1)|L^\perp = \{y_1\},$$

and

$$u(F, L + y_1 + y_2) = \frac{y_2}{\|y_2\|}, \qquad d(F, L + y_1 + y_2) = \|y_2\|,$$

whenever $p(F, L + y_1 + y_2) \in \operatorname{relint} F$. Thus

$$\begin{aligned} & p(P, L + y_1 + y_2) \in \operatorname{relint} F \\ \Longleftrightarrow \quad & p(F, L + y_1 + y_2) \in \operatorname{relint} F,\ u(F, L + y_1 + y_2) \in n(P, F) \\ \Longleftrightarrow \quad & y_1 \in \operatorname{relint} F|L^\perp, y_2 \in N(P, F) \end{aligned}$$

and, in this case,

$$0 < d(F, L + y_1 + y_2) \le \rho \iff 0 < \|y_2\| \le \rho.$$

On the other hand, if L, F are not in general relative position, $L + y \notin P^{(k)}$ for all $y \in L^\perp$. Therefore, the inner integral of (4.1) is zero.

In each case, the inner integral of (4.1) is

$$\begin{aligned} & \int_{L_1} \int_{L_2} I(L + y_1 + y_2 \in P^{(k)}) \cdot I(p(P, L + y_1 + y_2) \in \operatorname{relint} F) \\ & \quad \times I(0 < d(F, L + y_1 + y_2) \le \rho) \\ & \quad \times I((p(F, L + y_1 + y_2), u(F, L + y_1 + y_2), L) \in \eta)\, d\lambda_{d-k-m}(y_2)\, d\lambda_m(y_1) \\ = & \int_{L_1} \int_{L_2} I(L + y_1 + y_2 \in P^{(k)}) \cdot I(y_1 \in \operatorname{relint} F|L^\perp) \cdot I(0 < \|y_2\| \le \rho) \\ & \quad \times I(y_2 \in N(P, F)) \cdot I((p(F, L + y_1), y_2/\|y_2\|, L) \in \eta)\, d\lambda_{d-k-m}(y_2)\, d\lambda_m(y_1) \end{aligned}$$

If $y_2 \in N(P, F)$ and $L + y_1 + y_2 \notin P^{(k)}$, then $L + y_1 + y_2$ must have more than one nearest point to F. This means that F and L are not in general relative position, i. e. the dimension of $F|L^\perp$ is less than m, and $\lambda_m^{F|L^\perp}$ is the zero measure. Thus we get that the inner integral of (4.1) is

$$\frac{1}{d-k-m}\rho^{d-k-m} \int_{F|L^\perp} \int_{L^\perp \cap n(P,F)} I((p(F, L + y_1), y_2, L) \in \eta)\, d\omega_{d-k-m-1}(y_2)\, d\lambda_m(y_1).$$

Altogether, we have

$$\begin{aligned} \mu_k(A) = \frac{1}{d-k-m}\rho^{d-k-m} \int_{\mathcal{L}_k^d} \int_{F|L^\perp} \int_{L^\perp \cap n(P,F)} & I((p(F, L + y_1), y_2, L) \in \eta) \\ & \times\, d\omega_{d-k-m-1}(y_2)\, d\lambda_m(y_1)\, d\nu_k(L). \end{aligned}$$

The last expression is translation invariant, and we need no longer assume $0 \in F$.
We can now take the sum over all faces F of P, and get

$$\begin{aligned}\mu_\rho(k)(P,\eta) &= \sum_{F \text{ is face of } P} \mu_k(M_\rho^{(k)}(P,\eta) \cap p(P,\cdot)^{-1}(\operatorname{relint} F)) \\ &= \sum_{m=0}^{d-k-1} \sum_{F \in \mathcal{F}_m(P)} \frac{1}{d-k-m} \rho^{d-k-m} \int_{\mathcal{L}_k^d} \int_{F|L^\perp} \int_{L^\perp \cap n(P,F)} \\ &\quad \times I((p(F, L+y_1), y_2, L) \in \eta)\, d\omega_{d-k-m-1}(y_2)\, d\lambda_m(y_1)\, d\nu_k(L).\end{aligned}$$

Thus we get the polynomial representation

$$\mu_\rho^{(k)}(P,\eta) = \frac{1}{d-k} \sum_{m=0}^{d-k-1} \rho^{d-k-m} \binom{d-k}{m} \Theta_m^{(k)}(P,\eta), \tag{4.2}$$

where the coefficients are measures as follows. For $m = 0, \dots, d-k-1$ we define

$$\begin{aligned}\Theta_m^{(k)}(P,\eta) := \binom{d-k-1}{m}^{-1} &\sum_{F \in \mathcal{F}_m(P)} \int_{\mathcal{L}_k^d} \int_{F|L^\perp} \int_{L^\perp \cap n(P,F)} \\ &\times I((p(F, L+y_1), y_2, L) \in \eta)\, d\omega_{d-k-m-1}(y_2)\, d\lambda_m(y_1)\, d\nu_k(L).\end{aligned} \tag{4.3}$$

This polynomial representation can be extended to arbitrary convex bodies.

Theorem 29. *For each convex body $K \in \mathcal{K}$ and each $k \in \{0, \dots, d-1\}$ there exist finite measures $\Theta_0^{(k)}(K,\cdot), \dots, \Theta_{d-k-1}^{(k)}(K,\cdot)$ on $\mathcal{B}(\Sigma^{(k)})$, such that for each $\eta \in \mathcal{B}(\Sigma^{(k)})$ and each $\rho > 0$ the measure $\mu_\rho^{(k)}(K,\eta)$ of the local parallel set of k-flats, $M_\rho^{(k)}(K,\eta)$, is given by*

$$\mu_\rho^{(k)}(K,\eta) = \frac{1}{d-k} \sum_{m=0}^{d-k-1} \rho^{d-k-m} \binom{d-k}{m} \Theta_m^{(k)}(K,\eta). \tag{4.4}$$

The mapping $K \mapsto \Theta_m^{(k)}(K,\cdot)$ is weakly continuous, i. e.

$$K_i \to K \Rightarrow \Theta_m^{(k)}(K_i,\cdot) \xrightarrow{w} \Theta_m^{(k)}(K,\cdot), \qquad (i \to \infty),$$

and additive, i. e. if $K_1, K_2, K_1 \cup K_2 \in \mathcal{K}$, then

$$\Theta_m^{(k)}(K_1 \cup K_2,\cdot) + \Theta_m^{(k)}(K_1 \cap K_2,\cdot) = \Theta_m^{(k)}(K_1,\cdot) + \Theta_m^{(k)}(K_2,\cdot).$$

For each $\eta \in \mathcal{B}(\Sigma^{(k)})$ the function $K \mapsto \Theta_m^{(k)}(K,\eta)$ is measurable on $\mathcal{K}$.

Proof. If K is a polytope, (4.4) has been established in (4.2). From Lemma 26 we know that $\mu_\rho^{(k)}(K,\cdot)$ is a measure depending weakly on $K \in \mathcal{K}$. The set of polytopes is dense in $\mathcal{K}$. Lemma 6 thus implies that (4.4) can be extended to all convex bodies, and that the measures $\Theta_m^{(k)}(K,\eta)$ depend weakly continuous on K. Moreover, the polynomial expansion

$$\Theta_m^{(k)}(K,\eta) = \sum_{l=1}^{d-k} a_{l,k}\mu_l^{(k)}(K,\eta) \tag{4.5}$$

holds for some real coefficients $a_{l,k}$. Therefore additivity and measurability follow from Lemma 26 and Lemma 28. □

We call the measure $\Theta_m^{(k)}(K,\cdot)$ the *m-th support measure of parallel k-flats* of K. It is concentrated on

$$\mathrm{Nor}_k(K) := \{(x,u,L) \in \Sigma^{(k)} : x \in \mathrm{bd}\, K, u \in n(K,x), L \perp u\},$$

the *k-th generalized normal bundle of* K. This follows because $(x,u,L) \in M_\rho(k)(K,\eta)$ implies $(x,u,L) = (p(K,E), u(K,E), L(E))$ for some $E \in \mathcal{E}_k^d$. Thus $\mu_\rho^{(k)}(K,\eta) = \mu_\rho^{(k)}(K,\eta \cap \mathrm{Nor}_k(K))$ and therefore $\Theta_m^{(k)}(K,\eta) = \Theta_m^{(k)}(K,\eta\cap\mathrm{Nor}_k(K))$ by (4.5).

For the measures $\Theta_m^{(k)}(K,\cdot)$ we have polynomial expansions analogous to (4.4) for $\mu_\rho^{(k)}(K,\cdot)$. Still analogous to Schneider, we define the mapping $t_\rho : \Sigma^{(k)} \to \Sigma^{(k)}$ by $t_\rho(x,u,L) := (x+\rho u, u, L)$.

Theorem 30. *Let $0 \le k \le d-1, K \in \mathcal{K}, \eta \in \mathcal{B}(\Sigma^{(k)}), \rho > 0$ and $0 \le m \le d-k-1$. Then*

$$\Theta_m^{(k)}(K+\rho B^d, t_\rho(\eta)) = \sum_{j=0}^{m} \rho^j \binom{m}{j} \Theta_{m-j}^{(k)}(K,\eta).$$

Proof. Let $E \in K^{(k)}$ and $d(K,E) > \rho$. Then $p(K+\rho B^d, E) = p(K,E) + \rho u(K,E), u(K+\rho B^d, E) = u(K,E)$ and $d(K+\rho B^d, E) = d(K,E) - \rho$. Except for a set of measure zero, for $\lambda > 0$ the local parallel set of k-flats $M_{\rho+\lambda}^{(k)}(K,\eta)$ is the disjoint union $M_\rho^{(k)}(K,\eta) \cup M_\lambda^{(k)}(K+\rho B^d, t_\rho(\eta))$. Thus the equation

$$\mu_{\rho+\lambda}^{(k)}(K,\eta) = \mu_\rho^{(k)}(K,\eta) + \mu_\lambda^{(k)}(K+\rho B^d, t_\rho(\eta)).$$

holds. All that remains to do now is to insert the polynomial expansion (4.4) and compare the coefficients. □

Often, will need a special case, or, to be more precise, the projection onto the second and third component.

Definition 31. *Let $k,m \in \{0,\dots,d-1\}$ be such that $k+m \le d-1$, and let K be a convex body. We call*

$$S_m^{(k)}(K,\cdot) := \Theta_m^{(k)}(K, \mathbb{R}^d \times \cdot)$$

the m-th k-surface area measure of K. *These measures are defined on*

$$S^{d-1,k} := S^{d-1} \times \mathcal{L}_k^d.$$

For later reference, we state these measures for polytopes explicitly. Let P be a polytope. Then

$$S_m^{(k)}(P,\eta) := \binom{d-k-1}{m}^{-1} \sum_{F\in\mathcal{F}_m(P)} \int_{\mathcal{L}_k^d} V_m(F|L^\perp) \int_{L^\perp\cap n(P,F)} \times I((u,L)\in\eta)\, d\omega_{d-k-m-1}(u)\, d\nu_k(L). \tag{4.6}$$

The following Lemma gives an alternative representation of $S_m^{(k)}(P,\cdot)$ in the case of polytopes. In fact, for polytopes P, $S_m^{(k)}(P,\cdot)$ is a sum of measures on the normal cones of the m-faces of P.

Lemma 32. *Let $0\le k,m<d$ with $k+m\le d-1$, and let P be a convex polytope. Let f be a non-negative measurable function on $S^{d-1,k}$. Then*

$$\int_{S^{d-1,k}} f(u,L)\, S_m^{(k)}(P,\, d(u,L)) = \binom{d-k-1}{m}^{-1} \sum_{F\in\mathcal{F}_m(P)} V_m(F)\cdot I(F,f),$$

where

$$I(F,f) = \alpha_{d,k,m} \int_{n(P,F)} \int_{\mathcal{L}_k^{F^\perp\cap u^\perp}} \int_{\mathcal{L}_k^{L'+L(F)}} |\langle F,L^\perp\rangle|^{d-k-m+1} f(u,L)\, d\nu_k^{L'+L(F)}(L)\, d\nu_k^{F^\perp\cap u^\perp}(L')\, d\omega_{d-m-1}(u), \tag{4.7}$$

and $\alpha_{d,k,m}$ is a constant depending on d, k and m only.

Proof. From (4.6) we get that the integral of f is

$$\int_{S^{d-1,k}} f(u,L)\, S_m^{(k)}(P,\, d(u,L)) = \binom{d-k-1}{m}^{-1} \sum_{F\in\mathcal{F}_m(P)} V_m(F)\cdot I(F,f),$$

where

$$I(F,f) = \int_{\mathcal{L}_k^d} |\langle F,L^\perp\rangle| \int_{L^\perp\cap n(P,F)} f(u,L)\, d\omega_{d-k-m-1}(u)\, d\nu_k(L). \tag{4.8}$$

Applying Lemma 9 to the outer integral of (4.8), we get that $I(F,f)$ is proportional to

$$\int_{\mathcal{L}_{k+m}^{L(F)}} \int_{\mathcal{L}_k^{L'}} |\langle F,L^\perp\rangle|\; \tilde f(L)\; |\langle L,F^\perp\rangle|^{d-k-m}\, d\nu_k^{L'}(L)\, d\nu_{k+m}^{L(F)}(L'), \tag{4.9}$$

where

$$\tilde f(L) = \int_{L^\perp\cap n(P,F)} f(u,L)\, d\omega_{d-k-m-1}(u).$$

As F is an m-flat, any flat $L'\in\mathcal{L}_{k+m}^F$ is the sum of $L(F)$ and a k-flat orthogonal to F. Therefore, (4.9) is equal to

$$\int_{\mathcal{L}_k^{F^\perp}} \int_{\mathcal{L}_k^{L'+L(F)}} |\langle F,L^\perp\rangle|^{d-k-m+1}\; \tilde f(L)\;\, d\nu_k^{L'+L(F)}(L)\, d\nu_k^{F^\perp}(L').$$

If $|\langle F, L^\perp\rangle| \neq 0$, the intersection of L and $L(F)$ is $\{0\}$. In this case, $L + L(F) = L' + L(F)$. For the orthogonal spaces we have $L^\perp \cap F^\perp = L'^\perp \cap F^\perp$. In particular, we have $L^\perp \cap n(P,F) = L'^\perp \cap n(P,F)$, yielding

$$|\langle F, L^\perp\rangle|^{d-k-m+1}\ \tilde{f}(L) = \int_{L'^\perp \cap n(P,F)} |\langle F, L^\perp\rangle|^{d-k-m+1}\ f(u,L)\, d\omega_{d-k-m-1}(u)$$

for all L. Thus (4.9) becomes

$$\int_{\mathcal{L}_k^{F^\perp}} \int_{\mathcal{L}_k^{L'+L(F)}} \int_{L'^\perp \cap n(P,F)} |\langle F, L^\perp\rangle|^{d-k-m+1}\ f(u,L) \\ d\omega_{d-k-m-1}(u)\, d\nu_k^{L'+L(F)}(L)\, d\nu_k^{F^\perp}(L').$$

We now can apply Fubini's Theorem, yielding that $I(F,f)$ is proportional to

$$\int_{\mathcal{L}_k^{F^\perp}} \int_{L'^\perp \cap n(P,F)} \int_{\mathcal{L}_k^{L'+L(F)}} |\langle F, L^\perp\rangle|^{d-k-m+1}\ f(u,L) \\ d\nu_k^{L'+L(F)}(L)\, d\omega_{d-k-m-1}(u)\, d\nu_k^{F^\perp}(L').$$

Instead of k-flats in $F^\perp$, we now integrate about the orthognoal $d-k-m$-flats in $F^\perp$, yielding

$$\int_{\mathcal{L}_{d-k-m}^{F^\perp}} \int_{L' \cap n(P,F)} \int_{\mathcal{L}_k^{L'^\perp}} |\langle F, L^\perp\rangle|^{d-k-m+1}\ f(u,L) \\ d\nu_k^{L'^\perp}(L)\, d\omega_{d-k-m-1}(u)\, d\nu_{d-k-m}^{F^\perp}(L').$$

The two outer integrations are on $d-k-m$-spaces respectively 1-spaces in $F^\perp$. Therefore we can apply a Theorem of Schneider and Weil [10], Satz 6.1.1, to show that $I(F,f)$ is proportional to

$$\int_{n(P,F)} \int_{\mathcal{L}_{d-k-m}^{F^\perp \cap u^\perp}} \int_{\mathcal{L}_k^{L'^\perp}} |\langle F, L^\perp\rangle|^{d-k-m+1}\ f(u,L) \\ d\nu_k^{L'^\perp}(L)\, d\nu_{d-k-m}^{F^\perp}(L')\, d\omega_{d-m-1}(u).$$

Replacing the integration about $d-k-m$-flats with an integration about k-flats again, we get that $I(F,f)$ is proportional to

$$\int_{n(P,F)} \int_{\mathcal{L}_k^{F^\perp \cap u^\perp}} \int_{\mathcal{L}_k^{L'+L(F)}} |\langle F, L^\perp\rangle|^{d-k-m+1} f(u,L) \\ d\nu_k^{L'+L(F)}(L)\, d\nu_k^{F^\perp \cap u^\perp}(L')\, d\omega_{d-m-1}(u), \tag{4.10}$$

which completes the proof. □

We now give an expression for the constants $\alpha_{d,k,m}$ by considering a special case. Let F be a polytope with $\dim F = m$ and let $f \equiv 1$. Then, from (4.8), we get

$$\begin{aligned} I(F,f) &= \int_{\mathcal{L}_k^d} |\langle F, L^\perp\rangle| \int_{L^\perp \cap F^\perp \cap S^{d-1}} d\omega_{d-k-m-1}(u)\, d\nu_k(L) \\ &= \sigma_{d-k-m} \int_{\mathcal{L}_k^d} |\langle F, L^\perp\rangle|\, d\nu_k(L). \end{aligned}$$

On the other hand, (4.10) is in this case

$$\int_{F^\perp \cap S^{d-1}} \int_{\mathcal{L}_k^{F^\perp \cap u^\perp}} \int_{\mathcal{L}_k^{L'+L(F)}} |\langle F, L^\perp\rangle|^{d-k-m+1} \\ d\nu_k^{L'+L(F)}(L)\, d\nu_k^{F^\perp \cap u^\perp}(L')\, d\omega_{d-m-1}(u).$$

Clearly, the inner integral is independent from L', and the expression simplifies to

$$\sigma_{d-m} \int_{\mathcal{L}_k^{F'}} |\langle F, L^\perp \cap F'\rangle|^{d-k-m+1}\, d\nu_k^{F'}(L),$$

where $F' \in \mathcal{L}_{k+m}^d$ is any $(k+m)$-flat containing F. Thus we get

$$\alpha_{d,k,m} = \frac{\sigma_{d-k-m}}{\sigma_{d-m}} \cdot \frac{\int_{\mathcal{L}_k^d} |\langle F, L^\perp\rangle|\, d\nu_k(L)}{\int_{\mathcal{L}_k^{k+m}} |\langle L, L'\rangle|^{d-k-m+1}\, d\nu_k^{k+m}(L)},$$

where $L' \in \mathcal{L}_k^{k+m}$ is an arbitrary k-flat.

We now apply a Theorem of Schneider and Weil [10], Satz 4.2.2, to a convex polytope K with $\dim K = m$ and $V_m(K) = 1$. We get

$$\int_{\mathcal{L}_{d-k}^d} |\langle F, L\rangle|\, d\nu_{d-k}(L) = \beta_{d(d+m-(d-k))m} = \frac{\binom{d-m}{k}\kappa_{d-m}\kappa_{d-k}}{\binom{d}{d-k-m}\kappa_k\kappa_d},$$

with the definition of β in Schneider and Weil [10], Satz 4.1.1. Altogether,

$$\alpha_{d,k,m} = \frac{\sigma_{d-k-m}\binom{d-m}{k}\kappa_{d-k}}{(d-m)\binom{d}{d-k-m}\kappa_k\kappa_d \int_{\mathcal{L}_k^{k+m}} |\langle L, L'\rangle|^{d-k-m+1}\, d\nu_k^{k+m}(L)}. \tag{4.11}$$

The most useful special case is $k = d-m-1$ (meaning F is a $(d-k-1)$-face, and we consider k-flats). We state the result for this case in the following Corollary.

Corollary 33. *Let $0 \le k < d$, let $f : S^{d-1,k} \to \mathbb{R}$ be a non-negative measurable function, and let P be a polytope. Then*

$$\int_{S^{d-1,k}} f(u,L) S_{d-k-1}^{(k)}(P, d(u,L)) = \alpha_{d,k,d-k-1} \sum_{F \in \mathcal{F}_{d-k-1}(P)} V_{d-k-1}(F) \\ \times \int_{n(P,F)} \int_{\mathcal{L}_k^{u^\perp}} \langle L, F^\perp \cap u^\perp\rangle^2 f(u,L)\, d\nu_k^{u^\perp}(L)\, d\omega_k(u), \tag{4.12}$$

and

$$\alpha_{d,k,d-k-1} = \frac{2\kappa_{d-k}\binom{d-1}{k}}{\kappa_k \sigma_d}.$$

Proof. We apply (4.11) to $m = d - k - 1$, and get

$$\begin{aligned}
\alpha_{d,k,d-k-1} &= \frac{\sigma_1 \binom{k+1}{k} \kappa_{d-k}}{(k+1)\binom{d}{1}\kappa_k\kappa_d \int_{\mathcal{L}_k^{d-1}} |\langle L, L'\rangle|^2 \, d\nu_k^{d-1}(L)} \\
&= \frac{2\kappa_{d-k}}{d\kappa_k\kappa_d \int_{\mathcal{L}_k^{d-1}} \langle L, L'\rangle^2 \, d\nu_k^{d-1}(L)}.
\end{aligned}$$

We will later show independently in Corollary 39 that

$$\int_{\mathcal{L}_k^{d-1}} \langle L, L'\rangle^2 \, d\nu_k^{d-1}(L) = \binom{d-1}{k}^{-1},$$

giving

$$\alpha_{d,k,d-k-1} = \frac{2\kappa_{d-k}\binom{d-1}{k}}{\kappa_k \sigma_d}.$$

On the other hand, (4.10) becomes

$$\int_{n(P,F)} \int_{\mathcal{L}_k^{F^\perp \cap u^\perp}} \int_{\mathcal{L}_k^{L'+L(F)}} \langle F, L^\perp\rangle^2 f(u,L) \, d\nu_k^{L'+L(F)}(L) \, d\nu_k^{F^\perp \cap u^\perp}(L') \, d\omega_k(u),$$

Now $\dim(F^\perp \cap u^\perp) = d-(d-k-1)-1 = k$, which means that $L' = F^\perp \cap u^\perp$, and $\nu_k^{F^\perp \cap u^\perp}$ is a one-point measure. Moreover, $|\langle F, L^\perp\rangle| = |\langle L, F^\perp\rangle| = |\langle L, F^\perp \cap u^\perp\rangle|$. Therefore $I(F,f)$ is proportional to

$$\int_{n(P,F)} \int_{\mathcal{L}_k^{u^\perp}} \langle L, F^\perp \cap u^\perp\rangle^2 f(u,L) \, d\nu_k^{u^\perp}(L) \, d\omega_k(u),$$

and the constant of proportionality is $\alpha_{d,k,d-k-1}$. □

We now return to the volume $V_k(P|L)$ of the projection of a polytope P onto a k-flat L. In fact, equation (4.12) allows us to make a connection between the measure $S_k^{(d-k-1)}(P,\cdot)$ of a polytope P, and the projection function $V(P|\cdot)$ of P. The following Corollary states this result. It represents $V_k(P|L)$ as an integral of a function g_L with respect to $S_k^{(d-k-1)}(P,\cdot)$. In the next chapter we will then show that such a function g_L indeed exists.

The function f_k^d that appears in the next corollary was defined in Section 3.2.

Corollary 34. *Let $0 \le k < d$, let P be a polytope and let $g_L : S^{d-1,d-k-1} \to \mathbb{R}$ be an integrable function satisfying*

$$\int_{\mathcal{L}_{d-k-1}^{u^\perp}} \langle E, F^\perp \cap u^\perp\rangle^2 g_L(u,E) \, d\nu_{d-k-1}^{u^\perp}(E) = \frac{\sigma_{d-k}}{\alpha_{d,d-k-1,k}} \cdot f_k^d(L,F,u)$$

for all $F \in \mathcal{L}_k^d$ *and* $u \in S^{d-1} \cap F^\perp$*. Then the following equation for projection functions holds,*

$$V_k(P|L) = \int_{S^{d-1,d-k-1}} g_L(u,E) S_k^{(d-k-1)}(P,\, d(u,E)). \tag{4.13}$$

Proof. This follows from a comparsion of (3.7) and (4.12), where in the latter equation the roles of k and $d-k-1$ have to be exchanged. $\square$

5 Integral representations of projection functions II

5.1 The integral equation

We consider convex bodies K in $\mathbb{R}^d$ and their projection functions $L \mapsto V_k(K|L)$, $L \in \mathcal{L}_k^d$ $(0 \le k \le d)$. We know from (4.13) that there are weakly continuous measures $S_k^{(d-k-1)}(K,\cdot)$ on $S^{d-1,d-k-1} = S^{d-1} \times \mathcal{L}_{d-k-1}^d$ such that (if K is a polytope)

$$\int g_L(u,E) S_k^{(d-k-1)}(K,\, d(u,E)) = V_k(K|L),$$

where g_L is any function satisfying

$$\begin{aligned} \int_{\mathcal{L}_{d-k-1}^{u^\perp}} \langle E, F^\perp \cap u^\perp \rangle^2 g_L(u,E)\, d\nu_{d-k-1}^{u^\perp}(E) &= \frac{\sigma_{d-k}}{\alpha_{d,d-k-1,k}} \cdot f_k^d(L,F,u) \\ &= \begin{cases} \dfrac{\sigma_{d-k}}{\alpha_{d,d-k-1,k}} \cdot \dfrac{\langle u^\perp \cap F^\perp, u^\perp \cap L^\perp \rangle^2}{\|u|L^\perp\|^{d-k-2}}, & |\langle F, L\rangle| \neq 0, \\ 0, & |\langle F, L\rangle| = 0, \end{cases} \end{aligned} \tag{5.1}$$

for all $F \in \mathcal{L}_k^d$ and $u \in S^{d-1} \cap F^\perp$. However, we have not yet established the existence of such a function g_L. In this chapter we will give such a function explicitly.

We simplify the notation for integrals with respect to a normalized invariant measure ν in the following way. Instead of $d\nu(E)$ we write dE, i. e. we leave out the measure ν.

It is clear that we can consider the problem to find a function g_L satisfying (5.1) as the problem to find a function $f_{L'}$ in $u^\perp$ (or, equivalently, $\mathbb{R}^{d-1}$), satisfying

$$\int_{\mathcal{L}_{k'}^{d'}} \langle E, F' \rangle^2 f_{L'}(E)\, dE = \langle F', L' \rangle^2,$$

where $d' = d-1$, $k' = d-k-1$, $F' = F^\perp \cap u^\perp$, $L' = L^\perp \cap u^\perp$. If such an $f_{L'}$ exists, we can put

$$g_L(u,E) = \begin{cases} \dfrac{\sigma_{d-k}}{\alpha_{d,d-k-1,k}\|u|L^\perp\|^{d-k-2}} \cdot f_{L^\perp \cap u^\perp}(E^\perp \cap u^\perp), & u \notin L, \\ 0, & u \in L, \end{cases} \tag{5.2}$$

to get a function g_L satisfying (5.1). From now on, we will only consider this second problem. Therefore, we will write d instead of d' etc.

Question. *For $L \in \mathcal{L}_k^d$, does a function $f_L : \mathcal{L}_k^d \to \mathbb{R}$ exist that satisfies the equation*

$$\int_{\mathcal{L}_k^d} \langle E, F\rangle^2 f_L(E)\, dE = \langle F, L\rangle^2 \tag{5.3}$$

for all $F \in \mathcal{L}_k^d$?

We first give an alternative form of (5.3). We define an integral operator on $\mathcal{C}(\mathcal{L}_k^d)$, the set of continuous real-valued functions on $\mathcal{L}_k^d$.

$$\Psi : \mathcal{C}(\mathcal{L}_k^d) \to \mathcal{C}(\mathcal{L}_k^d), f \mapsto \Psi(f), \text{ where } \Psi(f)(F) = \int_{\mathcal{L}_k^d} \langle E, F\rangle^2 f(E)\, dE. \tag{5.4}$$

Then we need to find a function f_L such that

$$\Psi(f_L)(F) = \langle F, L\rangle^2.$$

From now on, we will only consider the case $d \geq 2k$. The remaining cases can be solved using orthogonalization, as for $d < 2k$ we have $2(d-k) \leq d$, and thus

$$\begin{aligned}
\int_{\mathcal{L}_k^d} \langle E, F\rangle^2 f_{L^\perp}(E^\perp)\, dE &= \int_{\mathcal{L}_{d-k}^d} \langle E, F^\perp\rangle^2 f_{L^\perp}(E)\, dE \\
&= \langle F^\perp, L^\perp\rangle^2 \\
&= \langle F, L\rangle^2,
\end{aligned}$$

if $f_{L^\perp}$ is a solution for the first case.

We further introduce a scalar product $(\cdot,\cdot)$ on $\mathcal{C}(\mathcal{L}_k^d)$,

$$(f, g) := \int_{\mathcal{L}_k^d} f(E) g(E)\, dE. \tag{5.5}$$

This definition allows us to derive some properties of Ψ.

Lemma 35. *The operator Ψ is self-adjoint with respect to $(\cdot,\cdot)$.*

Proof. With the definition of Ψ and the definition of the scalar product (5.5), we get by an application of Fubini's Theorem

$$\begin{aligned}
(\Psi(f), g) &= \int_{\mathcal{L}_k^d} \int_{\mathcal{L}_k^d} \langle E', E\rangle^2 f(E')\, dE'\, g(E)\, dE \\
&= \int_{\mathcal{L}_k^d} \int_{\mathcal{L}_k^d} \langle E', E\rangle^2 f(E') g(E)\, dE'\, dE \\
&= \int_{\mathcal{L}_k^d} \int_{\mathcal{L}_k^d} \langle E', E\rangle^2 g(E)\, dE\, f(E')\, dE' \\
&= (f, \Psi(g)).
\end{aligned}$$

□

From linear algebra we know that the eigenspaces of self-adjoint linear operators are orthogonal to each other. For later use, we state this as a Corollary.

Corollary 36. *Eigenspaces of Ψ belonging to different eigenvalues are orthogonal.*

5.2 A multilinear function on pairs of matrices

As stated in the last section, we now assume $0 \leq k \leq d/2$, and $L \in \mathcal{L}_k^d$. Let $e_1, \ldots, e_d$ be an orthonormal basis of $\mathbb{R}^d$ such that $\operatorname{lin}\{e_1, \ldots, e_k\} = L$. (We can assume without loss of generality that $e_1, \ldots, e_d$ is the standard basis of $\mathbb{R}^d$.) From now on, J, I and M will denote subsets of $\{1, \ldots, d\}$ with k elements. In a sum of the form $\sum_I f(I)$, I ranges over all those subsets. J will always be the set $\{1, \ldots, k\}$. Let

$$E_I := \operatorname{lin}\{e_i \,|\, i \in I\}.$$

In particular, we have $E_J = L$.

For k-flats E, L, we have

$$|\langle E, L\rangle| = |\det \begin{pmatrix} \langle x_1, y_1\rangle & \cdots & \langle x_1, y_k\rangle \\ \vdots & \ddots & \vdots \\ \langle x_k, y_1\rangle & \cdots & \langle x_k, y_k\rangle \end{pmatrix}|,$$

where $x_1, \ldots, x_k$ is any orthonormal basis of E, and $y_1, \ldots, y_k$ is any orthonormal basis of L. We now give a generalization of this definition. For arbitrary vectors $x_1, \ldots, x_k$, $y_1, \ldots, y_k$ we consider the determinant of the matrix whose entries are $\langle x_i, y_j\rangle$.

Definition 37. *Let $X, Y \in \mathbb{R}^{d\times k}$ be $(d \times k)$-matrices with columns $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$, respectively. Then*

$$\langle X, Y\rangle := \begin{vmatrix} \langle x_1, y_1\rangle & \cdots & \langle x_1, y_k\rangle \\ \vdots & \ddots & \vdots \\ \langle x_k, y_1\rangle & \cdots & \langle x_k, y_k\rangle \end{vmatrix}. \tag{5.6}$$

Moreover, let $x'_1, \ldots, x'_k$ be the rows of X. We define X_I to be the $(k \times k)$-matrix of the rows x'_i with $i \in I$.

For $k = 1$, this is the standard scalar product in $\mathbb{R}^d$. For $k > 1$, (5.6) does not define a scalar product. In this case, $\langle \cdot, \cdot\rangle$ is not bilinear, but *multilinear* in both components. For example, if $x_1, \ldots, x_k$ are linearly dependent, $\langle X, Y\rangle = 0$ for all Y. It is also easy to see that $\langle \cdot, \cdot\rangle$ is symmetric, as $\langle X, Y\rangle = \det(X^\top Y) = \det(Y^\top X) = \langle Y, X\rangle$.

Now we consider the case of orthonormal columns, i. e. the columns $x_1, \ldots, x_k$ of X form an orthonormal basis of its linear hull, and the same holds for Y. We note that, up to the sign, $\langle X, Y\rangle$ depends on $E := \operatorname{lin}\{x_1, \ldots, x_k\}$ and $F := \operatorname{lin}\{y_1, \ldots, y_k\}$ only. In fact, we have $|\langle X, Y\rangle| = |\langle E, F\rangle|$, and the sign of $\langle X, Y\rangle$ depends on the orientation of the bases $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$. In particular, $\langle E, F\rangle^2 = \langle X, Y\rangle^2$, and the most important special case is

$$\langle E_I, E\rangle^2 = |X_I|^2. \tag{5.7}$$

Lemma 38. *Let X and Y be $(d \times k)$-matrices. Then*

$$\langle X, Y\rangle = \sum_I |X_I|\,|Y_I|. \tag{5.8}$$

Proof. We assume for the moment that the columns of X and Y are elements of $\{e_1, \ldots, e_d\}$. We further assume that the columns of X (and those of Y) are linearly independent. Then $\langle X, Y\rangle = 0$ if X and Y do not have the same columns (not necessarily at the same positions). In this case, for each I we have $|X_I| = 0$ or $|Y_I| = 0$, and the assertion holds. If X and Y have the same columns (not necessarily at the same positions), there is a permutation $\sigma \in S_k$ such that the i-th column of X is the $\sigma(i)$-th column of Y. If we apply σ to the columns of the matrix in (5.6), that matrix turns into the unit matrix. Thus $\langle X, Y\rangle$ is 1 if σ is even, and -1 if σ is odd. On the other hand, $|X_I|$ (and $|Y_I|$) does not vanish if and only if I is the set of the indices of the unit vectors that form the columns of X (and Y). For this I, the columns of Y_I are a permutation of the columns of X_I. This permutations is the same σ as above. Thus $|X_I|$ has the same sign as $|Y_I|$ if and only if σ is even, implying that $|X_I|\,|Y_I|$ is 1 if σ is even, and -1 if σ is odd.

The multilinearity of $\langle \cdot, \cdot \rangle$ as well as of $|\cdot_I|\,|\cdot_I|$ now implies the assertion. □

Now let X be a matrix with orthonormal columns again. The special case $X = Y$ gives

$$\sum_I \langle E, E_I\rangle^2 = \sum_I |X_I|^2 = \langle X, X\rangle = 1. \tag{5.9}$$

For the next corollary, we recall that $J = \{1, \ldots, k\}$, and thus $L = E_J = \operatorname{lin}\{e_1, \ldots, e_k\}$.

Corollary 39. *Let $0 \le k \le d$. Then*

$$\int_{\mathcal{L}_k^d} \langle E, L\rangle^2 \, dE = \binom{d}{k}^{-1}. \tag{5.10}$$

Proof. (5.9) and the invariance of ν_k imply

$$\begin{aligned} 1 &= \int_{\mathcal{L}_k^d} \sum_I \langle E, E_I\rangle^2 \, dE \\ &= \sum_I \int_{\mathcal{L}_k^d} \langle E, E_I\rangle^2 \, dE \\ &= \sum_I \int_{\mathcal{L}_k^d} \langle E, L\rangle^2 \, dE \\ &= \binom{d}{k} \int_{\mathcal{L}_k^d} \langle E, L\rangle^2 \, dE \end{aligned}$$

□

5.3 Integrals on Grassmannians

Now we compute more constants that arise as integrals of functions containing powers of $\langle E, E_I\rangle^2$. They will be needed later for the solution of the integral equation.

Definition 40. *Let $0 \le j \le k$ and $2k - j \le d$. The constant $c^{(d)}_{k,j}$ is defined by*

$$c^{(d)}_{k,j} := \int_{\mathcal{L}^d_k} \langle E, E_J\rangle^2 \langle E, E_{\{1,\dots,j,k+1,\dots,2k-j\}}\rangle^2 \, dE.$$

It is clear that $E_{\{1,\dots,j,k+1,\dots,2k-j\}}$ could be replaced by E_I for any set I with $|I \cap J| = j$. More generally, we have

$$\int_{\mathcal{L}^d_k} \langle E, E_M\rangle^2 \langle E, E_I\rangle^2 \, dE = c^{(d)}_{k,|M\cap I|}. \tag{5.11}$$

The following considerations will allow us to calculate $c^{(d)}_{k,j}$ explicitly.

Proposition 41. *Let $d \notin I, M$. Then*

$$\int_{\mathcal{L}^d_k} \langle E, E_I\rangle^2 \langle E, E_M\rangle^2 \, dE = H_{d,k} \int_{\mathcal{L}^{d-1}_k} \langle E, E_I\rangle^2 \langle E, E_M\rangle^2 \, dE \tag{5.12}$$

with

$$H_{d,k} = \frac{(d-k)(d-k+2)}{d(d+2)}.$$

Proof. We note that

$$A \mapsto \int_{\mathcal{L}^d_k} 1_{E|e_d^\perp \in A} \langle E, e_d^\perp\rangle^4 \, dE, \quad A \in \mathcal{B}(\mathcal{L}^{e_d^\perp}_k),$$

is a measure on $\mathcal{L}^{e_d^\perp}_k$ that is invariant with respect to rotations in $e_d^\perp$, and thus it is a multiple of $\nu^{e_d^\perp}_k$. Writing $F := \operatorname{lin}\{e_1, \dots, e_{d-k}\}$, we calculate the factor to

$$\begin{aligned}
\int_{\mathcal{L}^d_k} \langle E, e_d^\perp\rangle^4 \, dE &= \int_{SO_d} \langle \rho F^\perp, e_d^\perp\rangle^4 \, d\rho \\
&= \int_{SO_d} \langle F, \rho e_d\rangle^4 \, d\rho \\
&= \frac{1}{\sigma_d} \int_{S^{d-1}} \|u|F\|^4 \, du
\end{aligned} \tag{5.13}$$

We further use spherical cylinder coordinates to compute

$$\begin{aligned}
\int_{S^{d-1}} \|u|F\|^4 \, du &= \int_{S^{d-1}\cap e_d^\perp} \int_{-1}^{1} (1-t^2)^{(d-3)/2} \|(te_d + \sqrt{1-t^2}u)|F\|^4 \, dt \, du \\
&= \frac{\Gamma(1/2)\Gamma((d+3)/2)}{\Gamma((d+4)/2)} \int_{S^{d-1}\cap e_d^\perp} \|u|F\|^4 \, du.
\end{aligned}$$

Applying this recursively k times, we get

$$\int_{S^{d-1}} \|u|L^\perp\|^4 \, du = \pi^{(d-k)/2} \frac{\Gamma((d-k+1+3)/2)}{\Gamma((d+4)/2)} \int_{S^{d-1}\cap e_d^\perp \cap \dots \cap e_{d-k+1}^\perp} \|u|F\|^4 \, du, \tag{5.14}$$

and the latter integrand is identical to 1. Thus, applying (5.14) to (5.13), we get

$$\begin{aligned}\int_{\mathcal{L}_k^d} \langle E, e_d^\perp \rangle^4 \, dE &= \frac{\sigma_{d-k}}{\sigma_d} \pi^{(d-k)/2} \frac{\Gamma((d-k+4)/2)}{\Gamma((d+4)/2)} \\ &= \frac{(d-k)(d-k+2)}{d(d+2)}.\end{aligned}$$

Moreover, as $e_d \perp E_I$ and $e_d \perp E_M$, it follows

$$\begin{aligned}\langle E, E_I \rangle^2 &= \langle E, e_d^\perp \rangle^2 \langle E|e_d^\perp, E_I \rangle^2, \\ \langle E, E_M \rangle^2 &= \langle E, e_d^\perp \rangle^2 \langle E|e_d^\perp, E_M \rangle^2,\end{aligned}$$

and thus

$$\begin{aligned}\int_{\mathcal{L}_k^d} \langle E, E_I \rangle^2 \langle E, E_M \rangle^2 \, dE &= \int_{\mathcal{L}_k^d} \langle E|e_d^\perp, E_I \rangle^2 \langle E|e_d^\perp, E_M \rangle^2 \langle E, e_d^\perp \rangle^4 \, dE \\ &= \frac{(d-k)(d-k+2)}{d(d+2)} \int_{\mathcal{L}_k^{e_d^\perp}} \langle E, E_I \rangle^2 \langle E, E_M \rangle^2 \, dE,\end{aligned}$$

which finishes the proof. □

This allows us to give a recursion formula for $c_{k,j}^{(d)}$. First of all, we define $c_{k,j} := c_{k,j}^{(2k)}$ for $k > 0$, and put $c_{0,0} := 1$. We get

$$c_{k,j}^{(d)} = H_{d,k} \cdots H_{2k+1,k} c_{k,j} = \frac{(d-k)!(d-k+2)!}{k!(k+2)! d^{d-2k} (d+2)^{d-2k}} c_{k,j}. \tag{5.15}$$

Note that the factor does not depend on j.

For $j \geq 1$ we can use (5.12) to reduce the dimensions of the spaces involved in $c_{k,j}$. An application of this formula gives $c_{k,j} = H_{2k,k} c_{k,j}^{(2k-1)}$. Orthogonalizing, i.e. considering $(2k-1-k)$-flats instead of k-flats, gives $c_{k,j} = H_{2k,k} c_{k-1,j-1}^{(2k-1)}$. Another reduction of dimensions finally gives

$$\begin{aligned}c_{k,j} &= H_{2k,k} H_{2k-1,k-1} c_{k-1,j-1} \\ &= \frac{k(k+2)}{2k(2k+2)} \cdot \frac{k(k+2)}{(2k-1)(2k+1)} \cdot c_{k-1,j-1} \\ &= \frac{(k+2)^2 k}{16\,(k+1)(k+1/2)(k-1/2)} \cdot c_{k-1,j-1}.\end{aligned} \tag{5.16}$$

For the case $j = 0$ we apply (5.9) to (5.10), and get

$$\begin{aligned}
\binom{2k}{k}^{-1} &= \int_{\mathcal{L}_k^{2k}} \langle E_J, E\rangle^2 \, dE \\
&= \int_{\mathcal{L}_k^{2k}} \langle E_J, E\rangle^2 \sum_I \langle E, E_I\rangle^2 \, dE \\
&= \sum_I c_{k,|I\cap J|} \\
&= \sum_{j=0}^{k} \binom{k}{j}\binom{k}{k-j} c_{k,j}
\end{aligned}$$

The factor $\binom{k}{j}$ is the number of ways to choose j elements of I that belong to $J = \{1, \dots, k\}$, and $\binom{k}{k-j}$ is the number of ways to choose the other $k - j$ elements of I, which must lie in $\{k+1, \dots, 2k\}$. This yields

$$c_{k,0} = \binom{2k}{k}^{-1} - \sum_{j=1}^{k} \binom{k}{j}^2 c_{k,j}. \tag{5.17}$$

Now we are ready to prove an explicit formula for the $c_{k,j}^{(d)}$. As $c_{k,j}^{(d)}$ is a multiple of $c_{k,j}$ (and we have given the factor in (5.15)), it suffices to give explicit values of $c_{k,j}$. We already know $c_{0,0} = 1$ from the definition. For $k > 0$ the following Lemma states the result.

Lemma 42. *Let $0 \le j \le k$ and let $k > 0$. Then*

$$c_{k,j} = \binom{k+2-j}{2}^{-1} \frac{(k+2)^2}{8} \binom{2k+1}{k}^{-1} \binom{2k-1}{k}^{-1}. \tag{5.18}$$

Proof. For $j = k$, we apply (5.16) k times recursively to $c_{k,k}$. We note that

$$\prod_{i=1}^{k} (2i+1)(2i-1) = \frac{(2k+1)!(2k-1)!}{\prod_{i=1}^{k}(2i) \cdot \prod_{i=1}^{k-1}(2i)} = \frac{(2k+1)!(2k-1)!}{2^{2k-1}k!(k-1)!},$$

and (5.18) follows:

$$\begin{aligned}
c_{k,k} &= \frac{\big((k+2)!/2\big)^2 k!}{4^k (k+1)! \prod_{i=1}^{k}(2i+1)(2i-1)} \\
&= \frac{((k+2)!)^2 k! k! (k-1)!}{8(k+1)!(2k+1)!(2k-1)!} \\
&= \frac{(k+2)^2}{8} \frac{k!(k+1)!}{(2k+1)!} \frac{k!(k-1)!}{(2k-1)!}.
\end{aligned}$$

Thus we have shown (5.18) for $j = k$.

We now use induction on k. For $k = 1$, equation (5.18) is easily verified.

We assume that (5.18) holds for $k-1$ and $j \in \{0, \dots, k-2\}$, and show (5.18) for k and $j \in \{0, \dots, k-1\}$. (Note that there is no assumption for the base case $k = 1$.)

We first consider the case $j \geq 1$, and leave the case $j = 0$ for later. From (5.16) we know

$$c_{k,j} = H_{2k,k} H_{2k-1,k-1} c_{k-1,j-1}.$$

We apply the induction hypothesis and get

$$c_{k,j} = \binom{(k-1)+2-(j-1)}{2}^{-1} H_{2k,k} H_{2k-1,k-1} c_{k-1,k-1}$$

Applying (5.16) once more yields

$$c_{k,j} = \binom{k+2-j}{2}^{-1} c_{k,k},$$

which is the desired result.

It remains to compute $c_{k,0}$. We have

$$\binom{2k}{k}^{-1} = \frac{k!k!}{(2k)!} = 8\frac{(2k+1)!(2k-1)!(k+1)!}{(k+2)!(k+2)!(k-1)!(2k)!} \cdot c_{k,k}.$$

and

$$\begin{aligned} \binom{k}{j}^2 c_{k,j} &= \binom{k}{j}^2 \binom{k+2-j}{2}^{-1} c_{k,k} \\ &= \binom{k+2}{j}\binom{k}{j}\binom{k+2}{2}^{-1} c_{k,k}. \end{aligned}$$

Thus (5.17) implies

$$\begin{aligned} c_{k,0} &= \binom{2k}{k}^{-1} - \binom{k+2}{2}^{-1} \sum_{j=1}^{k} \binom{k+2}{j}\binom{k}{j} c_{k,k} \\ &= \left(8\frac{(2k+1)!(2k-1)!(k+1)!}{(k+2)!(k+2)!(k-1)!(2k)!} - \binom{2k+2}{k}\binom{k+2}{2}^{-1} + \binom{k+2}{2}^{-1} \right) c_{k,k}. \end{aligned}$$

(Here we have used the well-known fact $\sum_{j=0}^{k} \binom{k+2}{j}\binom{k}{j} = \binom{2k+2}{k}$.) To prove the desired result, we have to show that

$$8\frac{(2k+1)!(2k-1)!(k+1)!}{(k+2)!(k+2)!(k-1)!(2k)!} = \binom{2k+2}{k}\binom{k+2}{2}^{-1},$$

which is verified easily. □

Lemma 43. *Let $F \in \mathcal{L}_k^d$. Then*

$$\int_{\mathcal{L}_k^d} \langle E, F\rangle^2 \langle E, E_M\rangle^2 \, dE = \sum_I c^{(d)}_{k,|I\cap M|} \langle F, E_I\rangle^2 = \sum_{j=0}^{k} c^{(d)}_{k,j} \sum_{I:|I\cap M|=j} \langle F, E_I\rangle^2. \tag{5.19}$$

Proof. For any orthonormal basis $x_1, \ldots, x_k$ of E let X denote the matrix that has the columns $x_1, \ldots, x_k$. Similarly, let Y be a matrix whose columns are an orthonormal basis of F. Then $\langle E, E_I\rangle^2 = |X_I|^2$, and from (5.8) we know

$$\langle E, F\rangle^2 = \langle X, Y\rangle^2 = \left(\sum_I |X_I||Y_I|\right)^2.$$

Thus

$$\begin{aligned}
&\sigma_d \cdots \sigma_{d-k+1} \int_{\mathcal{L}_k^d} \langle E, F\rangle^2 \langle E, E_M\rangle^2 \, dE \\
&\quad = \int_{S^{d-1}} \cdots \int_{S^{d-1}\cap x_1^\perp \cap \ldots \cap x_{k-1}^\perp} \langle X, Y\rangle^2 |X_M|^2 \, d\omega_{d-k}(x_k) \ldots \, d\omega_{d-1}(x_1) \\
&\quad = \sum_{I,I'} \int_{S^{d-1}} \cdots \int_{S^{d-1}\cap x_1^\perp \cap \ldots \cap x_{k-1}^\perp} |Y_I||Y_{I'}||X_I|\,|X_{I'}|\,|X_M|^2 \, d\omega_{d-1}(x_1) \ldots \, d\omega_{d-k}(x_k) \\
&\quad = \sum_I |Y_I|^2 \int_{S^{d-1}} \cdots \int_{S^{d-1}\cap x_1^\perp \cap \ldots \cap x_{k-1}^\perp} |X_I|^2 |X_M|^2 \, d\omega_{d-k}(x_k) \ldots \, d\omega_{d-1}(x_1) \\
&\qquad + \sum_{I,I':I\neq I'} |Y_I||Y_{I'}| \int_{S^{d-1}} \cdots \int_{S^{d-1}\cap x_1^\perp \cap \ldots \cap x_{k-1}^\perp} |X_I|\,|X_{I'}|\,|X_M|^2 \, d\omega_{d-k}(x_k) \ldots \, d\omega_{d-1}(x_1).
\end{aligned}$$

In the last expression, the first sum equals

$$\sigma_d \cdots \sigma_{d-k+1} \sum_I \langle F, E_I\rangle^2 c^{(d)}_{k,|I\cap M|},$$

i. e. it is the desired result. We have to show that the second sum is 0. In fact, for fixed $I \neq I'$, we may assume without loss of generality $d \in I, d \notin I'$. Moreover, ω_{d-1} is invariant under reflection about $e_d^\perp$. We will now denote by $\tilde{X}$ the matrix X whose columns are reflected about $e_d^\perp$. Then $|\tilde{X}_M|^2 = |X_M|^2$, $|\tilde{X}_{I'}| = |X_{I'}|$, and $|\tilde{X}_I| = -|X_I|$. This lets us evaluate the integral in the summand that belongs to I and I' as

$$\begin{aligned}
&\int_{S^{d-1}} \cdots \int_{S^{d-1}\cap x_1^\perp \cap \ldots \cap x_{k-1}^\perp} |X_I|\,|X_{I'}|\,|X_M|^2 \, d\omega_{d-k}(x_k) \ldots \, d\omega_{d-1}(x_1) \\
&\qquad = \int_{S^{d-1}} \cdots \int_{S^{d-1}\cap x_1^\perp \cap \ldots \cap x_{k-1}^\perp} |\tilde{X}_I|\,|\tilde{X}_{I'}|\,|\tilde{X}_M|^2 \, d\omega_{d-k}(x_k) \ldots \, d\omega_{d-1}(x_1) \\
&\qquad = -\int_{S^{d-1}} \cdots \int_{S^{d-1}\cap x_1^\perp \cap \ldots \cap x_{k-1}^\perp} |X_I|\,|X_{I'}|\,|X_M|^2 \, d\omega_{d-k}(x_k) \ldots \, d\omega_{d-1}(x_1),
\end{aligned}$$

i. e. it must vanish. □

From Lemma 43 we see that for a function $f : \mathcal{L}_k^d \to \mathbb{R}$ of the form

$$f(E) = \sum_I \alpha_I \langle E, E_I \rangle^2$$

with real constants α_I, we have

$$\Psi(f)(F) = \sum_I \beta_I \langle F, E_I \rangle^2,$$

with some real constants β_I. If we can choose the α_I in such a way that $\beta_J = 1$ and $\beta_I = 0$ for $I \neq J$, we have a solution for the integral equation (5.3). We will follow this approach in section 5.5. There we will get a system of linear equations for the α_I. For given d and k, it is in fact possible to compute the coefficients, i. e. to find a solution for (5.3). However, using only the system of linear equations, there seems to be no easy way to show the existence of a solution for all d and k. Therefore, we use a different approach to show the existence of a solution independently.

5.4 Eigenfunctions of the integral operator

To motivate the following considerations, assume that we knew that some functions were eigenfunctions of the integral operator Ψ defined in (5.4). Assume further that we can express $\langle \cdot, L \rangle^2$ as a linear combination of these eigenfunctions. (Note that every rotation of an eigenfunction is an eigenfunction, too.) If in this sum we divide every summand by its eigenvalue, we get a function satisfying the integral equation (5.3).

Consequently, in this section we will give some eigenfunctions. We then show that $\langle \cdot, L \rangle^2$ is a linear combination of these eigenfunctions.

We recall that in this chapter the set I is always a subset of $\{1, \ldots, d\}$ with k elements. When selecting a subset of all such sets I (for example in the range of a sum), we will often not state this condition explicitly, or abbreviate it by $|I| = k$.

Definition 44. *For $1 \leq n \leq k$ we define*

$$G_n := \{2, 4, \ldots, 2n\}, \tag{5.20}$$

$$\mathcal{G}_n := \{I : |I| = k, |I \cap \{1, 2\}| = |I \cap \{3, 4\}| = \ldots = |I \cap \{2n-1, 2n\}| = 1\}, \tag{5.21}$$

and

$$f_n(E) = \sum_{I \in \mathcal{G}_n} (-1)^{|I \cap G_n|} \langle E, E_I \rangle^2.$$

Moreover, we define

$$G_0 := \emptyset,$$
$$\mathcal{G}_0 := \{I : |I| = k\}.$$

(Note that the definitions of G_0 and $\mathcal{G}_0$ formally coincide with (5.20) and (5.21) for $n = 0$, respectively. However, in particular the definition of $\mathcal{G}_0$ is much clearer if given explicitly.)

Lemma 45. *The functions f_n are eigenfunctions of the integral operator Ψ, i.e.*

$$\int_{\mathcal{L}_k^d} \langle E, F\rangle^2 f_n(E)\, dE = \alpha_n f_n(F). \tag{5.22}$$

Proof. The left hand side of (5.22) is

$$\int_{\mathcal{L}_k^d} \langle E, F\rangle^2 f_n(E)\, dE = \sum_{I\in\mathcal{G}_n} (-1)^{|I\cap G_n|} \int_{\mathcal{L}_k^d} \langle E, F\rangle^2 \langle E, E_I\rangle^2\, dE \tag{5.23}$$

We know from (5.19) that each summand on the right hand side of (5.23) results in a linear combination of $\langle F, E_I\rangle^2$ (where I, as stated above, ranges over all subsets of $\{1, \ldots, d\}$ with k elements). Therefore

$$\int_{\mathcal{L}_k^d} \langle E, F\rangle^2 f_n(E)\, dE = \sum_{I\colon |I|=k} \beta_I \langle F, E_I\rangle^2$$

for some β_I. We have to show that $\beta_I = 0$ for $I \notin \mathcal{G}_n$, and that there exists an α_n such that $\beta_I = (-1)^{|I\cap G_n|}\alpha_n$ for all $I \in \mathcal{G}_n$.

Consider the case $I \notin \mathcal{G}_n$. This means that there is some $i \in \{1, \ldots, n\}$ such that $|I \cap \{2i-1, 2i\}| \neq 1$. Without loss of generality we assume $|I \cap \{1, 2\}| \neq 1$. Equation (5.19) yields

$$\begin{aligned}
\beta_I &= \sum_{M\in\mathcal{G}_n} (-1)^{|M\cap G_n|} c^{(d)}_{k,|I\cap M|} \\
&= \sum_{M\in\mathcal{G}_n, 1\in M} (-1)^{|M\cap G_n|} c^{(d)}_{k,|I\cap M|} + \sum_{M\in\mathcal{G}_n, 2\in M} (-1)^{|M\cap G_n|} c^{(d)}_{k,|I\cap M|} \\
&= \sum_{M\in\mathcal{G}_n, 1\in M} (-1)^{|M\cap G_n|} c^{(d)}_{k,|I\cap M|} + \sum_{M\in\mathcal{G}_n, 1\in M} (-1)^{|((M\cup\{2\})\setminus\{1\})\cap G_n|} c^{(d)}_{k,|I\cap((M\cup\{2\})\setminus\{1\})|}.
\end{aligned}$$

We now simplify the second sum. The set G_n contains 2, but not 1. Thus, for all M containing 1 (and therefore not 2), we have

$$|((M \cup \{2\}) \setminus \{1\}) \cap G_n| = |M \cap G_n| + 1,$$

yielding

$$(-1)^{|((M\cup\{2\})\setminus\{1\})\cap G_n|} = -(-1)^{|M\cap G_n|}.$$

We have to consider the two cases $I\cap\{1, 2\} = \emptyset$ and $I\cap\{1, 2\} = \{1, 2\}$. In the first case, $I\cap((M\cup\{2\})\setminus\{1\}) = I\cap M$. In the second case, $I\cap((M\cup\{2\})\setminus\{1\}) = ((I\cap M)\cup\{2\})\setminus\{1\}$. In both cases, $|I \cap ((M \cup \{2\}) \setminus \{1\})| = |I \cap M|$ follows. Altogether, we get

$$\begin{aligned}
\beta_I &= \sum_{M\in\mathcal{G}_n, 1\in M} (-1)^{|M\cap G_n|} c^{(d)}_{k,|I\cap M|} + \sum_{M\in\mathcal{G}_n, 1\in M} -(-1)^{|M\cap G_n|} c^{(d)}_{k,|I\cap M|} \\
&= 0.
\end{aligned}$$

In the case $I \in \mathcal{G}_n$ let $\pi \in S_d$ be a permutation for which the sets $\{1,2\}, \ldots, \{2n-1, 2n\}$ are invariant, and for which $\pi(I) = G_k$. Note that $M \in \mathcal{G}_n$ if and only if $\pi(M) \in \mathcal{G}_n$, which means that π is a bijection of $\mathcal{G}_n$.

Let $i \in \{1, \ldots, n\}$. We have $2i \in G_n$. Moreover, $2i$ is in I if and only if $\pi(2i) = 2i$. Thus, $2i \in \pi(G_n)$ if and only if $2i \in I$. Let $M \in \mathcal{G}_n$. Then $2i \in M \cap \pi(G_n)$ if and only if $2i \in M \cap I$, and $2i-1 \in M \cap \pi(G_n)$ if and only if $2i-1 \in M \cap I$. It follows that

$$(-1)^{|M \cap \pi(\{2i\})|} = -(-1)^{|M \cap \{2i\}|}(-1)^{|I \cap \{2i\}|},$$

and hence

$$(-1)^{|M \cap \pi(G_n)|} = (-1)^n (-1)^{|M \cap G_n|}(-1)^{|I \cap G_n|}.$$

Thus (5.19) yields

$$\begin{aligned} \beta_I &= \sum_{M \in \mathcal{G}_n} (-1)^{|\pi^{-1}(M) \cap G_n|} c^{(d)}_{k,|I \cap \pi^{-1}(M)|} \\ &= \sum_{M \in \mathcal{G}_n} (-1)^{|M \cap \pi(G_n)|} c^{(d)}_{k,|\pi(I) \cap M|} \\ &= (-1)^{|I \cap G_n|}(-1)^n \sum_{M \in \mathcal{G}_n} (-1)^{|M \cap G_n|} c^{(d)}_{k,|G_k \cap M|}. \end{aligned}$$

Therefore the constants

$$\alpha_n := (-1)^n \sum_{M \in \mathcal{G}_n} (-1)^{|M \cap G_n|} c^{(d)}_{k,|G_k \cap M|}, \quad n = 0, \ldots, k, \tag{5.24}$$

fulfill (5.22), and the proof is complete. □

We can even give the eigenvalues α_n in a more explicit form,

$$\begin{aligned} \alpha_n &= (-1)^n \sum_{M \in \mathcal{G}_n} (-1)^{|M \cap G_n|} c^{(d)}_{k,|G_k \cap M|} \\ &= (-1)^n \sum_{j=0}^{k} c^{(d)}_{k,j} \sum_{M \in \mathcal{G}_n, |M \cap G_k| = j} (-1)^{|M \cap G_n|} \\ &= (-1)^n \sum_{j=0}^{k} c^{(d)}_{k,j} \sum_{l=0}^{n} (-1)^l \left|\{M \in \mathcal{G}_n : |M \cap G_n| = l, |M \cap (G_k \setminus G_n)| = j-l\}\right| \end{aligned}$$

We now determine the number of elements of

$$A := \{M \in \mathcal{G}_n : |M \cap G_n| = l, |M \cap (G_k \setminus G_n)| = j-l\}.$$

For M to be in A, we have $\binom{n}{l}$ ways to choose the l elements of M which must lie in G_n. ($n-l$ more elements of M are determined by this choice.) We then have $\binom{k-n}{j-l}$ ways to choose $j-l$ additional elements, which lie in $G_k \setminus G_n$. Finally we have

$\binom{d-k-n}{k+l-n-j}$ ways to choose the remaining $k+l-n-j$ elements of M, which must lie in $\{1,\ldots,d\} \setminus (\{1,\ldots,2n\} \cup G_k)$. Thus the set A has $\binom{n}{l}\binom{k-n}{j-l}\binom{d-k-n}{k+l-n-j}$ elements, and

$$\alpha_n = (-1)^n \sum_{j=0}^{k} c_{k,j}^{(d)} \sum_{l=0}^{n} (-1)^l \binom{n}{l}\binom{k-n}{j-l}\binom{d-k-n}{k+l-n-j}.$$

As mentioned before, rotations of the f_n are eigenfunctions, too. In particular, the functions given by

$$f_n^{(a_1,b_1),\ldots,(a_n,b_n)}(E) := \sum_{I\colon |I|=k, |I\cap\{a_1,b_1\}|=\ldots=|I\cap\{a_n,b_n\}|=1} (-1)^{|I\cap\{b_1,\ldots,b_n\}|} \langle E, E_I\rangle^2,$$

where $a_1,\ldots,a_n,b_1,\ldots,b_n \in \{1,\ldots,d\}$ are pairwise distinct, are eigenfunctions with eigenvalue α_n.

Let $A=\{a_1,\ldots,a_n\} \subset \{1,\ldots,d\}, B \subset \{1,\ldots,d\}$ be disjoint sets with n elements. Sums of rotations of a fixed f_n are eigenfunctions., and thus the function

$$f_n^{A,B} := \sum_{\pi: A\to B, \pi \text{ bijective}} f_n^{(a_1,\pi(a_1)),\ldots,(a_n,\pi(a_n))}$$

is an eigenfunction (and does not depend on the order of $a_1,\ldots,a_n$). Another eigenfunction is

$$f_n^A := \sum_{B\colon B\subset\{1,\ldots,d\}\setminus A, |B|=n} f_n^{A,B}.$$

$f_n^A(E)$ is certainly a linear combination of the functions $E \mapsto \langle E, E_I\rangle^2$, where I ranges over all subsets of $\{1,\ldots,d\}$ with k elements. We will now determine the coefficients. If $|I\cap A| = j$, we know that $\langle E, E_I\rangle^2$ appears in $f_n^{A,B}$ if and only if $I\setminus A \subset B$ and that it then appears $j!(n-j)!$ times. This is the case if and only if $n-j$ of the elements of B are in $I\setminus A$ (which has $k-j$ elements), and j of its elements are in $\{1,\ldots,d\}\setminus(I\cup A)$ (which has $d-k-n+j$ elements). Moreover, whenever $\langle E, E_I\rangle^2$ appears, it has the sign $(-1)^{|I\cap B|} = (-1)^{n-j}$. Therefore,

$$f_n^A(E) = \sum_{j=0}^{n} (-1)^{n-j} \gamma_{n,j} \sum_{I\colon |I\cap A|=j} \langle E, E_I\rangle^2, \tag{5.25}$$

where

$$\gamma_{n,j} := j!(n-j)! \binom{k-j}{n-j}\binom{d-k-n+j}{j}.$$

We have

$$f_n^A(E) = \sum_{j=0}^{n} (-1)^{n-j} \gamma_{n,j} \sum_{S\colon S\subset A, |S|=j} \sum_{I\colon I\cap A=S} \langle E, E_I\rangle^2. \tag{5.26}$$

We now want to give another form for the inner sum in the last equation, namely

$$\sum_{I\colon I\cap A=S} \langle E, E_I\rangle^2 = \sum_{i=0}^{n-j} \sum_{T\colon S\subset T\subset A, |T|=j+i} (-1)^i \sum_{I\colon T\subset I} \langle E, E_I\rangle^2. \tag{5.27}$$

We will prove this equation by applying the inclusion-exclusion principle. We put $\{a_1, \dots, a_{n-j}\} := A \setminus S$, and we define the sets

$$\mathcal{M}_i := \{I : S \cup \{a_i\} \subset I\}, \quad i = 1, \dots, n-j.$$

Note that

$$\bigcup_{i=1}^{n-j} \mathcal{M}_i = \{I : S \subset I\} \setminus \{I : I \cap A = S\}.$$

(This equation holds for $n - j = 0$, too.) Moreover, for $1 \leq i \leq n - j$,

$$\bigcap_{l_1 < \dots < l_i} \mathcal{M}_{l_i} = \{I : S \cup \{a_{l_1}, \dots, a_{l_i}\} \subset I\}.$$

We now apply the inclusion-exclusion principle to the function $I \mapsto \langle E, E_I \rangle^2$ and the sets $\mathcal{M}_1, \dots, \mathcal{M}_{n-j}$, which yields

$$\begin{aligned} \sum_{I:\, S \subset I, I \cap A \neq S} \langle E, E_I \rangle^2 &= \sum_{i=1}^{n-j} (-1)^{i-1} \sum_{l_1 < \dots < l_i} \sum_{I:\, S \cup \{a_{l_1}, \dots, a_{l_i}\} \subset I} \langle E, E_I \rangle^2 \\ &= \sum_{i=1}^{n-j} (-1)^{i-1} \sum_{T: S \subset T \subset A, |T| = j+i} \langle E, E_I \rangle^2. \end{aligned} \tag{5.28}$$

The left hand side of (5.28) is

$$\begin{aligned} \sum_{I:\, S \subset I, I \cap A \neq S\}} \langle E, E_I \rangle^2 &= \sum_{I:\, S \subset I} \langle E, E_I \rangle^2 - \sum_{I:\, I \cap A = S} \langle E, E_I \rangle^2 \\ &= (-1)^0 \sum_{T:\, S \subset T \subset A, |T| = j+0} \sum_{I:\, T \subset I} \langle E, E_I \rangle^2 - \sum_{I:\, I \cap A = S} \langle E, E_I \rangle^2. \end{aligned}$$

Comparing this to the right hand side of (5.28) and rearranging for $\sum_{I:\, I \cap A = S} \langle E, E_I \rangle^2$ yields (5.27).

We now apply (5.27) to (5.26) and get

$$f_n^A(E) = \sum_{j=0}^{n} (-1)^{n-j} \gamma_{n,j} \sum_{S \subset A, |S| = j} \sum_{i=0}^{n-j} \sum_{T:\, S \subset T \subset A, |T| = j+i} (-1)^i \sum_{I:\, T \subset I} \langle E, E_I \rangle^2.$$

If $|T| = m$ with $T \subset A$, the coefficient of $\sum_{I:\, T \subset I} \langle E, E_I \rangle^2$ in this equation is

$$\sum_{j=0}^{n} (-1)^{n-j} \gamma_{n,j} \sum_{S:\, S \subset T, |S| = j} (-1)^{m-j} = \sum_{j=0}^{m} (-1)^{n+m} \gamma_{n,j} \binom{m}{j}.$$

So we get, after rearranging for $m = |T|$,

$$f_n^A(E) = \sum_{m=0}^{n} (-1)^{n+m} \sum_{T:\, T \subset A, |T| = m} \sum_{j=0}^{m} \gamma_{n,j} \binom{m}{j} \sum_{I:\, T \subset I} \langle E, E_I \rangle^2. \tag{5.29}$$

If f is a function on $\mathcal{L}_k^d$ and $\rho \in SO(d)$, then the rotated function ρf is defined by $\rho f(E) = f(\rho^{-1}E), E \in \mathcal{L}_k^d$.

Definition 46. *The linear space spanned by f_n and rotations thereof is called*

$$H^{d,k,n} := \mathrm{lin}\{\rho f_n : \rho \in SO(d)\}.$$

$H^{d,k,n}$ is a subspace of the space of continuous (real-valued) functions on $\mathcal{L}_k^d$.

For the next Lemma, the reader is reminded that I is always a subset of $\{1, \dots, d\}$, and we always have $|I| = k$.

Lemma 47. *Let $A \subset \{1, \dots, d\}$ and $l = |A| \leq k$. Then for $g_l^A : \mathcal{L}_k^d \to \mathbb{R}$ defined by*

$$g_l^A(E) := \sum_{I:\, A \subset I} \langle E, E_I \rangle^2 \tag{5.30}$$

we have

$$g_l^A \in \sum_{n=0}^{l} H^{d,k,n}. \tag{5.31}$$

Proof. The proof is by induction on l. For $l = 0$ we have $g_0^\emptyset(E) = \sum_I \langle E, E_I \rangle^2 = f_0(E)$, thus $g_0^\emptyset \in H^{d,k,0}$.

If we know (5.31) for $0, 1, \dots, l$, the case $l + 1$ can be proven using (5.29) as follows.

$$f_{l+1}^A = \sum_{m=0}^{l+1} (-1)^{l+1+m} \sum_{T:\, T \subset A, |T|=m} \sum_{j=0}^{m} \gamma_{l+1,j} \binom{m}{j} g_m^T,$$

and solving for $g_{l+1}^T = g_{l+1}^A$ we get

$$g_{l+1}^A = \frac{1}{\sum_{j=0}^{l+1} \gamma_{l+1,j} \binom{l+1}{j}} \left(f_{l+1}^A - \sum_{m=0}^{l} (-1)^{l+1+m} \sum_{T:\, T \subset A, |T|=m} \sum_{j=0}^{m} \gamma_{l+1,j} \binom{m}{j} g_m^T \right)$$

f_{l+1}^A is in $H^{d,k,l+1} \subset \sum_{n=0}^{l+1} H^{d,k,n}$ by definition. The rest of the right hand side is the sum of functions of the form g_m^T, where $m \leq l$, and we know

$$g_m^T \in \sum_{n=0}^{m} H^{d,k,n} \subset \sum_{n=0}^{l+1} H^{d,k,n}.$$

Therefore g_{l+1}^A must be in $\sum_{n=0}^{l+1} H^{d,k,n}$ also, which completes the proof. □

Corollary 48. *The function $\langle \cdot, L \rangle^2 = g_k^J$ has a representation*

$$\langle \cdot, L \rangle^2 = \sum_{l=0}^{k} h_l,$$

where

$$h_l \in H^{d,k,l}, \qquad 0 \leq l \leq k.$$

The eigenvalue of h_l is α_l, of course. If the eigenvalues $\alpha_0, \dots, \alpha_k$ are pairwise distinct, the sum in (5.31) is a direct sum. As Ψ is self-adjoint, in this case it would even be a sum of orthogonal spaces.

For the construction of a function satisfying (5.3) it is not necessary that $\alpha_0, \dots, \alpha_k$ are pairwise distinct. However, we will see that we need that $\sum_{i\colon \alpha_i = \alpha_l} h_i \neq 0$ implies $\alpha_l \neq 0$.

Lemma 49. *Let $0 \leq l \leq k$ such that $\sum_{i\colon \alpha_i = \alpha_l} h_i \neq 0$. Then $\alpha_l \neq 0$.*

Proof. Let $\tilde{k} := |\{\alpha_1, \dots, \alpha_k\}|$ and $\{\tilde{\alpha}_1, \dots, \tilde{\alpha}_{\tilde{k}}\} = \{\alpha_1, \dots, \alpha_k\}$. Moreover, let $\tilde{l}$ be the natural number such that $\tilde{\alpha}_{\tilde{l}} = \alpha_l$.

The functions

$$\tilde{h}_i := \sum_{l : \alpha_l = \tilde{\alpha}_i} h_l, \qquad 1 \leq i \leq \tilde{k}$$

satisfy

$$\sum_{i=1}^{\tilde{k}} \tilde{h}_i = \langle \cdot, L \rangle^2$$

and

$$\int_{\mathcal{L}_k^d} \langle \cdot, F \rangle^2 \tilde{h}_i(E)\, dE = \tilde{\alpha}_i \tilde{h}_i(F), \qquad 1 \leq i \leq \tilde{k}.$$

Note that for $i \neq j$ we have $\tilde{\alpha}_i \neq \tilde{\alpha}_j$. Corollary 36 implies that the eigenspaces of Ψ in which $\tilde{h}_i$ and $\tilde{h}_j$ lie must be orthogonal. We use this fact and the special case $F = L$ to get

$$\begin{aligned}
\tilde{\alpha}_{\tilde{l}} \tilde{h}_{\tilde{l}}(L) &= \int_{\mathcal{L}_k^d} \langle E, L \rangle^2 \tilde{h}_{\tilde{l}}(E)\, dE \\
&= \int_{\mathcal{L}_k^d} \sum_{j=1}^{\tilde{k}} \tilde{h}_j(E) \tilde{h}_{\tilde{l}}(E)\, dE \\
&= \sum_{j=1}^{\tilde{k}} \int_{\mathcal{L}_k^d} \tilde{h}_{\tilde{l}}(E) \tilde{h}_j(E)\, dE \\
&= \sum_{j=1}^{\tilde{k}} (\tilde{h}_{\tilde{l}}, \tilde{h}_j) \\
&= (\tilde{h}_{\tilde{l}}, \tilde{h}_{\tilde{l}}) \\
&> 0.
\end{aligned}$$

This clearly implies $\alpha_l = \tilde{\alpha}_{\tilde{l}} \neq 0$. □

We are now in a position to state that a function satisfying (5.3) exists.

Theorem 50. *There exists a function satisfying the integral equation (5.3) in the space spanned by $\{\langle \cdot, E_I \rangle^2 : I \subset \{1, \dots, d\}, |I| = k\}$.*

PROOF. We consider the function

$$h := \sum_{l=0,\alpha_l \neq 0}^{k} \frac{1}{\alpha_l} h_l.$$

Then

$$\begin{aligned}
\Psi(h) &= \sum_{l=0,\alpha_l \neq 0}^{k} \frac{1}{\alpha_l} \Psi(h_l) \\
&= \sum_{l=0,\alpha_l \neq 0}^{k} \frac{1}{\alpha_l} \alpha_l h_l \\
&= \sum_{l=0,\alpha_l \neq 0}^{k} h_l.
\end{aligned}$$

Lemma 49 implies $\sum_{l=0,\alpha_l=0}^{k} h_l = 0$, and we get

$$\Psi(h) = \sum_{l=0}^{k} h_l = \langle \cdot, L \rangle^2.$$

□

We can give an even more explicit result. First of all, we need more notation. We define three sequences of functions, which we need for the explicit representation and its proof.

Definition 51. *For $i \in 0, \ldots, k$ let*

$$\begin{aligned}
p_i &:= \sum_{A: A \subset J, |A| = i} f_i^A, \\
q_i &:= \sum_{I: |I \cap J| = i} \langle \cdot\,, E_I \rangle^2, \\
g_i &:= \sum_{A: A \subset J, |A| = i} g_i^A.
\end{aligned} \tag{5.32}$$

The functions q_i, which are, in a sense, the simplest of these functions, will be used for the representation. We start by showing some relations between these sequences of functions.

Lemma 52. *For $0 \le i \le k$ we have*

$$p_i(E) = \sum_{A:A\subset J,|A|=i} \sum_{j=0}^{i} (-1)^{i-j}\gamma_{i,j} \sum_{I:|I\cap A|=j} \langle E, E_I\rangle^2 \tag{5.33}$$

$$= \sum_{m=0}^{k} \left(\sum_{j=0}^{i} (-1)^{i-j}\gamma_{i,j} \binom{m}{j}\binom{k-m}{i-j} \right) q_m(E), \tag{5.34}$$

$$g_i(E) = \sum_{A:A\subset J,|A|=i} \sum_{I:A\subset I} \langle E, E_I\rangle^2, \tag{5.35}$$

and p_i is an eigenfunction of Ψ with eigenvalue α_i.

PROOF. (5.33) follows from (5.25) and the definition of p_i. (5.35) follows from (5.30) and the definition of g_i. For (5.34) we write (5.33) as

$$p_i(E) = \sum_{j=0}^{i} (-1)^{i-j}\gamma_{i,j} \sum_{A:A\subset J,|A|=i} \sum_{I:|I\cap A|=j} \langle E, E_I\rangle^2.$$

For how many A does a fixed I satisfy the condition of the last sum, $|I \cap A| = j$? If $|I \cap J| = m$, there are $\binom{m}{j}$ possibilities for the elements of A that lie in I, and $\binom{k-m}{i-j}$ possibilities for the elements of A in $J \setminus I$. This means

$$\sum_{A:A\subset J,|A|=i} \sum_{I:|I\cap A|=j} \langle E, E_I\rangle^2 = \sum_{m=0}^{k} \binom{m}{j}\binom{k-m}{i-j} q_m(E).$$

Substituting this into the last equation and rearranging we get (5.34). p_i is an eigenfunction, as f_i^A is an eigenfunction for each A, and the eigenvalue must obviously be the same. □

Lemma 53. *For $i \in \{0, \dots, k\}$ we have*

$$g_i, p_i \in \operatorname{lin}\{q_0, \dots, q_k\}, \quad g_i \in \operatorname{lin}\{p_0, \dots, p_i\}.$$

PROOF. $p_i \in \operatorname{lin}\{q_0, \dots, q_k\}$ follows directly from (5.34). An immediate consequence is $g_i \in \operatorname{lin}\{q_0, \dots, q_k\}$, if $g_i \in \operatorname{lin}\{p_0, \dots, p_i\}$, which is all that remains to show. We will use induction on i.

For $i = 0$ we have $g_0(E) = \sum_{I:\emptyset\subset I} \langle E, E_I\rangle^2 = 1$ from (5.35) and (5.9). On the other hand, from (5.34) we get

$$\begin{aligned} p_0(E) &= \sum_{m=0}^{k} \left((-1)^{0-0}\gamma_{0,0} \binom{m}{0}\binom{k-m}{0-0} \right) q_m(E) \\ &= \sum_{m=0}^{k} q_m(E) \\ &= \sum_{I} \langle E, E_I\rangle^2 \\ &= 1. \end{aligned}$$

This shows $g_0 \in \operatorname{lin}\{p_0\}$. We use (5.29) once again to get

$$p_i(E) = \sum_{m=0}^{i} (-1)^{i+m} \sum_{j=0}^{m} \gamma_{i,j} \binom{m}{j} \sum_{A:A\subset J,|A|=i} \ \sum_{T:T\subset A,|T|=m} \ \sum_{I:T\subset I} \langle E, E_I\rangle^2.$$

For how many A does a fixed $T \subset J$ occur in this sum? The $i-m$ elements of $A \setminus T$ can be chosen from among the $k-m$ elements of $J \setminus T$, therefore T occurs $\binom{k-m}{i-m}$ times. As $\sum_{T:T\subset J,|T|=m} \sum_{I:T\subset I} \langle E, E_I\rangle^2 = g_m(E)$, we have altogether

$$p_i = \sum_{m=0}^{i} \left((-1)^{i+m} \sum_{j=0}^{m} \gamma_{i,j} \binom{m}{j} \binom{k-m}{i-m} \right) g_m. \tag{5.36}$$

The coefficient of g_i is not 0, and thus we can solve this equation for g_i and see that g_i is a linear combination of $p_i, g_0, \ldots, g_{i-1}$. From the induction we know that $g_j \in \operatorname{lin}\{p_0, \ldots, p_j\} \subset \operatorname{lin}\{p_0, \ldots, p_{i-1}\}$ for $j < i$, showing that $g_i \in \operatorname{lin}\{p_0, \ldots, p_i\}$. □

Lemma 54. *The sets $\{p_0, \ldots, p_l\}$, $\{q_0, \ldots, q_k\}$ and $\{g_0, \ldots, g_k\}$ are bases of the same $(k+1)$-dimensional linear subspace of continuous functions of $\mathcal{L}_k^d$.*

PROOF. We start with the linear independence of $q_0, \ldots, q_k$. For this, it suffices to show that for $i \in \{0, \ldots, k\}$ there is an $E(i) \subset \mathcal{L}_k^d$ such that $q_j(E(i)) = 0$ for $j \neq i$ and $q_i(E(i)) = 1$. Such a flat is given by $E(i) = E_M$ with $M = \{1, \ldots, i, k+1, \ldots, 2k-i\}$, because

$$\langle E_M, E_I\rangle^2 = \begin{cases} 1, & M = I \\ 0, & M \neq I \end{cases},$$

and $I = M$ occurs in (5.32) only for $i = |M \cap J|$ (and then exactly once).

Lemma 53 gives

$$\operatorname{lin}\{g_0, \ldots, g_k\} \subset \operatorname{lin}\{p_0, \ldots, p_k\} \subset \operatorname{lin}\{q_0, \ldots, q_k\}.$$

It remains to show $\operatorname{lin}\{q_0, \ldots, q_k\} \subset \operatorname{lin}\{g_0, \ldots, g_k\}$. The summands of g_i in (5.35) are of the form $\langle \cdot, E_I\rangle^2$ with $|I \cap J| \geq i$. These summands appear in q_j for $j \geq i$ only. Together with the linear independence of $q_0, \ldots, q_k$ this shows that $g_0, \ldots, g_k$ are linearly independent. Thus $\operatorname{lin}\{q_0, \ldots, q_k\}$ and $\operatorname{lin}\{g_0, \ldots, g_k\}$ are two $(k+1)$-dimensional linear spaces, one of them containing the other. Therefore, these subspaces must be identical, and the assertion follows. □

Theorem 55. *There exists exactly one solution of* (5.3) *of the form*

$$f_L = \sum_{i=0,\alpha_i\neq 0}^{k} a_i q_i = \sum_{i=0,\alpha_i\neq 0}^{k} a_i \sum_{I:|I\cap J|=i} \langle \cdot, E_I\rangle^2. \tag{5.37}$$

PROOF. $g_k = \langle \cdot, E_J\rangle^2$ can be written as a unique linear combination $\sum_{i=0}^{k} \tilde{a}_i p_i$. As in Lemma 49, we see that $\alpha_i = 0$ implies $\tilde{a}_i = 0$. Thus, we have $\Psi(\sum_{i=0,\alpha_i\neq 0}^{k} \frac{\tilde{a}_i}{\alpha_i} p_i) = g_k$. Each p_i can be expressed as a unique linear combination $p_i = \sum_{j=0}^{k} a'_{i,j} q_j$. Thus (5.37) with $a_j = \sum_{i=0,\alpha\neq 0}^{k} \frac{\tilde{a}_i}{\alpha_i} a'_{i,j}$ gives the unique solution to (5.3). □

5.5 Symmetry approach

We now give a method to compute the coefficients a_i in (5.37) directly. The form we assume for f_L is

$$f_L = \sum_{i=0}^{k} a_i^{(d,k)} q_i, \tag{5.38}$$

where $a_j^{(d,k)}$ $(0 \leq j \leq k)$ are real constants. Under this assumption, we have from (5.19)

$$\begin{aligned}
\int_{\mathcal{L}_k^d} \langle E, F\rangle^2 f_L(E)\, dE &= \sum_{i=0}^{k} a_i^{(d,k)} \sum_{I\colon |I\cap J|=i} \int_{\mathcal{L}_k^d} \langle E, F\rangle^2 \langle E, E_I\rangle^2\, dE \\
&= \sum_{i=0}^{k} a_i^{(d,k)} \sum_{I\colon |I\cap J|=i} \sum_{M} c_{k,|M\cap I|}^{(d)} \langle F, E_M\rangle^2 \\
&= \sum_{i=0}^{k} a_i^{(d,k)} \sum_{I\colon |I\cap J|=i} \sum_{m=0}^{k} \sum_{M\colon |M\cap J|=m} c_{k,|M\cap I|}^{(d)} \langle F, E_M\rangle^2 \\
&= \sum_{i=0}^{k} \sum_{m=0}^{k} a_m^{(d,k)} \sum_{I\colon |I\cap J|=i} \sum_{M\colon |M\cap J|=m} c_{k,|M\cap I|}^{(d)} \langle F, E_I\rangle^2 .
\end{aligned}$$

For symmetry reasons,

$$d_{i,m}^{(d,k)} := \sum_{|M\cap J|=m} c_{k,|M\cap I|}^{(d)}$$

depends only on d, k, and $i = |I \cap J|$, $m = |M \cap J|$. Using the definition of q_i, we get

$$\Psi(f_L) = \sum_{i=0}^{k} q_i \sum_{m=0}^{k} a_m^{(d,k)} d_{i,m}^{(d,k)}.$$

Thus f_L satisfies (5.3) if the $a_i^{(d,k)}$ satisfy the following system of linear equations.

$$D^{(d,k)} \begin{pmatrix} a_0^{(d,k)} \\ a_1^{(d,k)} \\ \vdots \\ a_{k-1}^{(d,k)} \\ a_k^{(d,k)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{5.39}$$

Here $D^{(d,k)}$ is the matrix with coefficient $d_{i,m}^{(d,k)}$ in row i, column m. Moreover, from the definition of $d_{i,m}^{(d,k)}$ it follows

$$\begin{aligned}
d_{i,m}^{(d,k)} &= \sum_{M:|M\cap J|=m} c_{k,|M\cap I|}^{(d)} \\
&= \sum_{j=0}^{k} c_{k,j}^{(d)} \sum_{M:|M\cap J|=m,|M\cap I|=j} 1 \\
&= \sum_{j=0}^{k} c_{k,j}^{(d)} \sum_{l=0}^{k} \binom{i}{l}\binom{k-i}{m-l}\binom{k-i}{j-l}\binom{d-2k+i}{k+l-m-j}.
\end{aligned}$$

In particular, for $k=2$ the matrix $D^{(d,2)}$ is

$$D^{(d,2)} = c_{2,0}^{(d)} \cdot \begin{pmatrix} \frac{(-1+d)\,d}{2} & 2\,d & 1 \\ \frac{(-3+d)\,d}{2} & -1+3\,d & 2 \\ \frac{(-3+d)\,(-2+d)}{2} & 4\,(-2+d) & 6 \end{pmatrix}.$$

Now we can easily find a function satisfying (5.3) for $k=2$ (and $d \geq 4$).

$$f_L = \left(\frac{2}{(d+1)d} f_L^{(0)} - \frac{1}{d+1} f_L^{(1)} + f_L^{(2)}\right) \cdot \frac{d}{3(d+2)c_{2,0}^{(d)}}.$$

Using the same technique, we can compute the function f_L for any k (and $d \geq 2k$). This works only if (5.39) has a solution. From Theorem 55 we know that such a solution exists.

We have used the computer algebra system Maple to compute the coefficients for $k \leq 50$, yielding

$$a_j^{(d,k)} = (-1)^{k-j} \binom{d+1}{k-j}^{-1} \frac{d+2-k}{(k+1)(d+2)c_{k,0}^{(d)}}.$$

This would hold in the general case, i. e. for all k, if the following Conjecture is true.

Conjecture 56. *For* $0 \leq i \leq k \leq d/2$

$$\frac{d+2-k}{(k+1)(d+2)} \binom{k+2}{2} \sum_{m=0}^{k} (-1)^{k-m} \binom{d+1}{k-m}^{-1} \sum_{j=0}^{k} \binom{k+2-j}{2}^{-1}$$
$$\sum_{l=0}^{k} \binom{i}{l}\binom{k-i}{m-l}\binom{k-i}{j-l}\binom{d-2k+i}{k+l-m-j} = \begin{cases} 0, & 0 \leq i < k \\ 1, & i = k \end{cases}.$$

In fact, for each $k \leq 50$, the solutions stated above are unique. This also means that the eigenvalues $\alpha_0, \ldots, \alpha_k$ are non-zero.

5.6 The projection function for general convex bodies

From Theorem 55 we know that exactly one solution of the integral equation (5.3) exists. Equation (5.2) yields a function $g_L(u,E)$ on $S^{d-1,k}$,

$$g_L(u,E) = \begin{cases} \dfrac{\sigma_{d-k}}{\alpha_{d,d-k-1,k}\|u|L^\perp\|^{d-k-2}} \cdot f_{L^\perp\cap u^\perp}(E^\perp \cap u^\perp), & u \notin L, \\ 0, & u \in L. \end{cases}$$

If this function is integrable, Corollary 34 states that for polytopes P we have the following integral representation of projection functions,

$$V_k(P|L) = \int_{S^{d-1,k}} g_L(u,E) S_k^{d-k-1}(P,\, d(u,E)). \tag{5.40}$$

The integrability of $g_L(u,E)$ is not clear, because $g_L(u,E)$ has both positive and negative summands, and is not bounded. The following Theorem states in which cases the integrability of $g_L(u,E)$, and therefore the representation (5.40) has been established.

Theorem 57. *Let $0 \leq k \leq d-1$, $L \in \mathcal{L}_k^d$, and let $K \in \mathcal{K}$ be a convex body. The integral representation of the projection function* (5.40) *holds in the following cases:*

(i) $k \geq d-2$,

(ii) K is a polytope in general relative position to $L^\perp$,

(iii) $k = 1$, and K is a polytope.

Proof. The function $(u,E) \mapsto f_{L^\perp\cap u^\perp}(E^\perp \cap u^\perp)$ is bounded. To show that g_L is integrable it therefore suffices to show that

$$(u,L) \mapsto h(u,L) := \begin{cases} \dfrac{1}{\|u|L^\perp\|^{d-k-2}}, & u \in L, \\ 0, & u \notin L, \end{cases}$$

is integrable.

For $k \geq d-2$ the function h is bounded by 1. The measure $S_k^{(d-k-1)}$ is finite. Thus h is integrable in case (i).

For case (ii) we note that $S_k^{(d-k-1)}$ is concentrated on the set

$$A := \bigcup_{F\in\mathcal{F}_k(K)} \{(u,L) \in S^{d-1,k} : u \in n(P,F)\}.$$

However, if $u \in n(P,F)$ for some $F \in \mathcal{F}_k(K)$, u is in $F^\perp$. The general relative position of F and $L^\perp$ implies that $u \notin L$. Therefore, h is bounded, and thus integrable in case (ii).

In case (iii), we have to show

$$\int_{S^{d-1,d-2}} h(u,L) S_1^{(d-2)}(K,\, d(u,L)) < \infty.$$

According to Corollary 33 and equation (4.12) we have to show that for any $F \in \mathcal{F}_1^d(K)$

$$\int_{n(P,F)} \int_{\mathcal{L}_{d-2}^{u^\perp}} \langle E, F^\perp \cap u^\perp \rangle^2 h(u,L)\, d\nu_{d-2}^{u^\perp}(E)\, d\omega_{d-2}(u) < \infty.$$

Because $h(u,L)$ does not depend on E, it suffices to show

$$\int_{S^{d-1}\cap F^\perp} h(u,L)\, d\omega_{d-2}(u) < \infty.$$

For F in general relative position to $L^\perp$, the integrand is bounded, and the asssertion is clear. Now assume that F and $L^\perp$ are not in general relative position. Then some $v \in S^{d-1} \cap F^\perp \cap L$ exists. Because $\dim L = 1$, we have $L = \mathrm{lin}(v)$. We use spherical cylinder coordinates to compute

$$\begin{aligned}
\int_{S^{d-1}\cap F^\perp} h(u,L)\, d\omega_{d-2}(u) &= \int_{S^{d-1}\cap F^\perp \cap v^\perp} \int_{-1}^{1} (1-t^2)^{\frac{d-4}{2}} h(tv + \sqrt{1-t^2}u, L)\, dt\, d\omega_{d-3}(u) \\
&= \int_{S^{d-1}\cap F^\perp \cap v^\perp} \int_{-1}^{1} (1-t^2)^{\frac{d-4}{2}} \frac{1}{\sqrt{1-t^2}^{d-3}}\, dt\, d\omega_{d-3}(u) \\
&= \sigma_{d-2} \int_{-1}^{1} (1-t^2)^{\frac{d-4-(d-3)}{2}}\, dt \\
&= \sigma_{d-2} \cdot \pi,
\end{aligned}$$

which finishes the proof of (iii). □

Up to now, we have considered the k-th intrinsic volume of the projection of some body K onto a k-flat L. We now consider the j-th intrinsic volume of the projection, where $0 \le j \le k$.

Theorem 58. *Let $0 \le k \le d-1$ and $L \in \mathcal{L}_k^d$. Let K be a convex body such that g_L is integrable by $S_j^{(d-k-1)}$, $j = 0, \dots, k$. Then*

$$V_j(K|L) = \int_{S^{d-1,k}} g_L(u,E) S_j^{(d-k-1)}(K, d(u,E)).$$

Proof. We apply Theorem 30 to sets of the form $\eta = \mathbb{R}^d \times \eta'$, yielding

$$S_k^{(d-k-1)}(K + \rho B^d, \eta') = \sum_{j=0}^{k} \rho^j \binom{k}{j} S_{k-j}^{(d-k-1)}(K, \eta'), \quad \forall \rho > 0. \tag{5.41}$$

It is clear that g_L is integrable by the measure on left hand side. Integrating g_L by the measures in (5.41) yields

$$V_k((K + \rho B^d)|L) = \sum_{j=0}^{k} \rho^j \binom{k}{j} v_{k-j},$$

where the coefficients are

$$v_j = \int_{S^{d-1,k}} g_L(u,E) S_j^{(d-k-1)}(K,\, d(u,E)), \quad j = 0, \ldots, k.$$

We note that $V_k((K+\rho B^d)|L) = V_k\big((K|L) + \rho(B^d|L)\big)$. A comparison with Steiner's formula (2.8) in the k-dimensional space L now yields

$$v_j = V_j(K|L),$$

which finishes the proof. $\square$

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