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Institut für Mechanik

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J. Neumann, K. Schweizerhof
Universität Karlsruhe, Institut für Mechanik
J. Riccius
Department of Aeronautics, London, UK

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Institut für Mechanik
Kaiserstr. 12, Geb. 20.30
76128 Karlsruhe
Tel.: +49 (0) 721/ 608-2071
Fax: +49 (0) 721/ 608-7990
E-Mail: ifm@uni-karlsruhe.de
www.ifm.uni-karlsruhe.de

ADAPTIVE ANALYSIS OF DYNAMICALLY LOADED SHELL STRUCTURES

J. Neumann[⋆], J. Riccius[†] and K. Schweizerhof [⋆]

[⋆]Institut für Mechanik
Universität Karlsruhe
Kaiserstraße 12, D-76128 Karlsruhe, Germany
e-mail: gs34@rz.uni-karlsruhe.de, web page: http://www.uni-karlsruhe.de/∼gs03/

[†]Department of Aeronautics
Prince Consort Road
London SW7 2AZ, United Kingdom
e-mail: j.riccius@ic.ac.uk

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Abstract. *In the adaptive analysis of dynamically loaded structures, both strain and kinetic energy, have to be considered. The error in the strain energy can be estimated by the well-known estimators as e.g. of Zienkiewicz/Zhu[1,2], which is based on superconvergence stress points. For the kinetic energy part such points are not available. Thus an L_2 - indicator with deteriorated velocities is suggested by Riccius/Schweizerhof[3]. The actual velocities are assumed to be the improved velocities on a coarser mesh. A hierarchical mesh-adaptation procedure and a single time step algorithm for the semidiscrete finite element analysis of linear elastodynamic problems are proposed. A critical point is the transfer of variables with shell elements using mixed interpolation of displacements and strains, as e.g. the assumed strain elements. Furthermore the choice of the error-measure (relative or absolute error) for dynamically loaded systems is rather important. Finally the ratio of the strain energy error vs. the kinetic energy error is discussed on some examples.*

1 INTRODUCTION

Reliable error estimates and mesh adaption procedures are the most important tools for h-adaptive analysis. If considered separately, the well-known error estimators of the strain energy error[1,2,4,5] could be used. However with dynamic loading, it is still necessary to measure the error in the kinetic energy, and it is not a straightforward issue. In addition within the semi-discretization scheme the error resulting from the time integration scheme has to be considered within an adaptive scheme, which is not discussed here, but is refered to Riccius[6], Riccius/Schweizerhof[7]. In h-adaptive schemes mesh adaptation can be performed either by remeshing or by hierarchical refinement/coarsening. The hierarchical approach – preferred here – has the advantage that the mapping of the quantities between the meshes – in particular coarsening – is performed more efficiently than a complete remeshing. In order to avoid mesh modifications in every time step an estimate on the optimal number of elements in the refined/coarsened mesh is proposed. Also the interpolation of strains in shell elements introduces difficulties into the mapping scheme after mesh modifications, which we overcome by a solution with prescribed displacements. Some numerical examples show the effects of the proposed schemes.

2 BASIC EQUATIONS

The partial differential equation for elasto-dynamic problems considering internal damping can be written as follows,

$$\rho\,\ddot{\boldsymbol{u}} + a\rho\,\dot{\boldsymbol{u}} = \boldsymbol{L}^T\boldsymbol{C}\boldsymbol{L}(\boldsymbol{u} + b\dot{\boldsymbol{u}}) + \boldsymbol{f} \quad \text{in } \Omega \times (0,T) \tag{1}$$

with the boundary conditions

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}_b(t) \quad \text{on}\,\Gamma_u \times (0,T) \tag{2}$$

$$\boldsymbol{G}\boldsymbol{C}\boldsymbol{L}\,(\boldsymbol{u}(\boldsymbol{x},t)|_{t=0} = \boldsymbol{f}_b(t) \quad \text{on}\,\Gamma_f \times (0,T) \tag{3}$$

and initial conditions

$$\boldsymbol{u}(\boldsymbol{x},t)|_{t=0} = \boldsymbol{u}_0(\boldsymbol{x}); \quad \dot{\boldsymbol{u}}(\boldsymbol{x},t)|_{t=0} = \dot{\boldsymbol{u}}_0(\boldsymbol{x}) \quad \text{in}\,\Omega\,. \tag{4}$$

The notation used is:

$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},t)\,,\ \dot{\boldsymbol{u}} = \frac{\partial \boldsymbol{u}}{\partial t}\,,\ \ddot{\boldsymbol{u}} = \frac{\partial^2 \boldsymbol{u}}{\partial t^2}\ldots$ unknown displacement/velocity/acceleration function

$\boldsymbol{f}\ldots$ time dependent external loading function

$\boldsymbol{C}\ldots$ material tensor, $\boldsymbol{L}\ldots$ strain-displacement operator

$\boldsymbol{G}\ldots$ projection operator dependent from the outward normal to the boundary

$\Omega\ldots$ considered domain; $\rho\ldots$ mass density

$t,\,T\ldots$ time, final time; $a,b\ldots$ Rayleigh-damping parameters.

After partial integration the well known weak form for the Finite Element discretization is obtained

$$\int_\Omega \boldsymbol{v} \cdot \rho \ddot{\boldsymbol{u}}\, d\Omega \;+\; a \int_\Omega \boldsymbol{v} \cdot \rho \dot{\boldsymbol{u}}\, d\Omega + \int_\Omega (\boldsymbol{L}\boldsymbol{v})^T \boldsymbol{C}\boldsymbol{L}\boldsymbol{u}\, d\Omega + b \int_\Omega (\boldsymbol{L}\boldsymbol{v})^T \boldsymbol{C}\boldsymbol{L}\dot{\boldsymbol{u}}\, d\Omega = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{f}\, d\Omega + \int_{\Gamma_f} \boldsymbol{v} \cdot \boldsymbol{f}_b\, d\Gamma_f \tag{5}$$

with:

$$\begin{aligned} \forall \boldsymbol{v} \;&\in\; V_0 = \{\boldsymbol{v} : \boldsymbol{v} \in W_2^1(\Omega); \boldsymbol{v} = 0 \text{ on } \Gamma_u\} \\ \forall \boldsymbol{u} \;&\in\; V_b = \{\boldsymbol{u} : \boldsymbol{u} \in W_2^1(\Omega); \boldsymbol{u} = \boldsymbol{u}_b \text{ on } \Gamma_u\}\,. \end{aligned}$$

Within a semi-discretization scheme and the FE method, the exact solution $\boldsymbol{u}$ resp. the test function $\boldsymbol{v}$ are approximated by:

$$\boldsymbol{u}_h(\boldsymbol{x}, t) = \boldsymbol{N}(\boldsymbol{x})\boldsymbol{d}(t)\,, \qquad \boldsymbol{v}_h(\boldsymbol{x}, t) = \boldsymbol{N}(\boldsymbol{x})\delta\boldsymbol{d}(t)\,. \tag{6}$$

After the spatial discretization a coupled set of ordinary differential equations is obtained:

$$\boldsymbol{M}\ddot{\boldsymbol{d}}(t) + \boldsymbol{C}\dot{\boldsymbol{d}}(t) + \boldsymbol{K}\boldsymbol{d}(t) = \boldsymbol{F}(t) \tag{7}$$

with

$$\boldsymbol{M} = \int_\Omega \boldsymbol{N}^T \rho \boldsymbol{N}\, d\Omega \quad \ldots \text{mass matrix} \tag{8}$$

$$\boldsymbol{K} = \int_\Omega (\boldsymbol{L}\boldsymbol{N})^T \boldsymbol{C}\boldsymbol{L}\boldsymbol{N}\, d\Omega \quad \ldots \text{stiffness matrix} \tag{9}$$

$$\boldsymbol{C} \approx \boldsymbol{C}_R = a\boldsymbol{M} + b\boldsymbol{K} \quad \ldots \text{damping matrix (Rayleigh damping)} \tag{10}$$

$$\boldsymbol{F} = \int_\Omega \boldsymbol{N}^T \boldsymbol{f}\, d\Omega + \int_{\Gamma_f} \boldsymbol{N}^T \boldsymbol{f}_b\, d\Gamma_f - \int_\Omega (\boldsymbol{B})^T \boldsymbol{C}\boldsymbol{B}(\boldsymbol{d}_b + b\dot{\boldsymbol{d}}_b)\, d\Omega \tag{11}$$

$\ldots$ forcing vector

This set of ordinary differential equations with constant coefficients in the linear regime is often solved by direct integration. Within adaptive schemes it is favorable to use one-step integration schemes, e.g. the Newmark method[8,9]. The total energy-norm of the solution $\|\boldsymbol{u}\|_{total,\Omega}$ at any time t can be defined as:

$$\|\boldsymbol{u}\|_{total,\Omega} = (\|\boldsymbol{u}\|^2_{strain,\Omega} + \|\dot{\boldsymbol{u}}\|^2_{kin,\Omega})^{\frac{1}{2}} = \left(\int_\Omega (\boldsymbol{L}\boldsymbol{u})^T \boldsymbol{C}\boldsymbol{L}\boldsymbol{u}\, d\Omega + \int_\Omega \rho \dot{\boldsymbol{u}}^2\, d\Omega \right)^{\frac{1}{2}}. \tag{12}$$

3 THE ADAPTIVE PROCEDURE

For static calculations only the strain energy norm of the solution is required to compute error estimates and to apply mesh refinements. For dynamic loading the kinetic energy norm is needed and for efficiency reasons mesh coarsening capabilities are mandatory. In addition, it is particularly important, how the adaptive process is controlled to avoid inefficient mesh modifications. Finally, the kinematic variables $\boldsymbol{d}(t), \dot{\boldsymbol{d}}(t), \ddot{\boldsymbol{d}}(t)$ of the old mesh have to be projected onto the new mesh. For the error of the time discretization it is refered to Riccius[6], Riccius/Schweizerhof[7]. There, time step modifications based on error estimates for the time integration scheme are discussed.

3.1 Spatial error estimation

Within the semi-discretization the spatial discretization can be separated from the time integration and both errors can be computed separately. Then, following eq. (12), the energy norm $\|\boldsymbol{e}\|_{total,\Omega}$ of the error $\boldsymbol{e} = \boldsymbol{u}_h - \boldsymbol{u}$ at time t is defined as follows:

$$\|\boldsymbol{e}\|_{total,\Omega} = (\|\boldsymbol{e}\|^2_{strain,\Omega} + \|\boldsymbol{e}\|^2_{kin,\Omega})^{\frac{1}{2}} = \left(\int_\Omega (\boldsymbol{L}\boldsymbol{e})^T \boldsymbol{C} \boldsymbol{L}\boldsymbol{e} \, d\Omega + \int_\Omega \rho \dot{\boldsymbol{e}}^2 \, d\Omega \right)^{\frac{1}{2}} . \tag{13}$$

Since the accurate solution and thus the accurate error are generally unknown, an estimate of the the kinetic energy norm of the error and the strain energy norm of the error have to be computed.

3.2 Estimation of the strain-energy error

The well-known error estimator of Zienkiewicz/Zhu[1,2], already used for adaptive analysis of statically loaded structures[3,10], can be used in an identical fashion to determine the strain energy error. In a similar fashion the residual error estimator including the jump of the residuals at the element boundaries following Babuška/Miller[4] could be taken.

3.3 Kinetic energy-norm of the error

Schemes similar to the computation of the strain energy error cannot be applied, as e. g. no 'better' velocity is known in any element and no velocity jumps exist along the element boundaries. Therefore, Wiberg/Li[11] use a least square-fit approximation in order to determine improved velocities $\dot{\boldsymbol{u}}^*_h$. These improved velocities are based on functions containing higher-order polynomials than the spatial FE form. Hence, the error in the velocities can be computed

$$\dot{\boldsymbol{e}} = \dot{\boldsymbol{u}}_h - \dot{\boldsymbol{u}}^*_h \, . \tag{14}$$

The disadvantage of this procedure is the fact, that by setting up a least square-fit problem, a set of 6 equations has to be solved for each patch. Furthermore, one has to consider 9 integration points for the calculation of the kinetic energy norm of the error $\|\boldsymbol{e}\|_{kin}$. To

reduce this effort, the concept of deteriorated velocities is proposed. The idea is based on the experience that with standard interpolations the mass approximation and as a consequence the kinetic energy is much more accurate than the strain energy. The deteriorated velocities are determined by averaging the current element mid-point velocities (denoted by • in Fig.1). The current nodal velocities (⊙) are now considered as improved velocities. Here, a nodal error-indicator is defined as the difference of deteriorated and

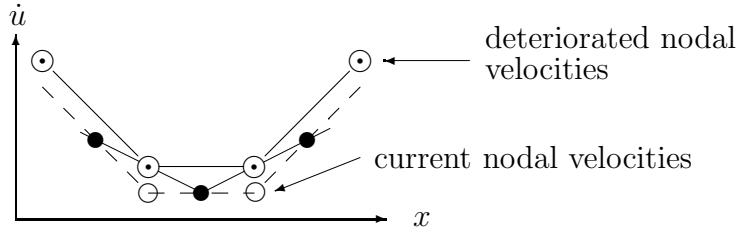


Figure 1: 1-D example for deteriorated velocities (—) from FE - velocities (- - -) by means of deteriorated nodal-velocities ⊙ , computed by averaging (—) the element midpoint velocities •

actual velocities:

$$\dot{e}_i^* = \dot{d}_i^* - \dot{d}_i \; ; \quad i = 1, \ldots, \mathcal{N}_{df} \, . \tag{15}$$

Using the error indicator vector $\dot{\boldsymbol{e}}^* = (\dot{e}_1^*, \dot{e}_2^*, \ldots, \dot{e}_{\mathcal{N}_{df}}^*)^T$, containing the velocity error at each node, the kinetic energy norm of the error at time t can now be defined and computed

$$\begin{aligned} \dot{\boldsymbol{e}}_h^* &= \boldsymbol{N} \dot{\boldsymbol{e}}^* \\ \|\boldsymbol{e}^*\|_{kin,\Omega} &= \int_\Omega \rho (\dot{\boldsymbol{e}}_h^*)^2 \, d\Omega \, . \end{aligned} \tag{16}$$

The disadvantage of this error measure is that the strain energy error is not computed on the same coarse mesh; thus simply summing up the error norms is not correct. Therefore the quality of the error estimator is compared to analytically computed correct solutions. The comparison on some numerical examples with rather high velocity gradients results in a correction factor $c_k = 0.365$, which has been applied to a class of examples with some success. Then the velocity error becomes

$$\dot{e}_i \approx c_k (\dot{d}_i^* - \dot{d}_i) \; ; \quad i = 1, \ldots, \mathcal{N}_{df} \, . \tag{17}$$

For more detailed discussions we refer to Riccius[6]. In addition some examples are presented in section 5 varying c_k to gain further information on a proper value.

3.4 Mesh generation

In general, two possibilities for mesh adaptation exist, algorithms based on remeshing and hierarchical strategies. The advantage of the first one is the fact, that an optimal mesh density with a minimum number of elements is mostly achieved. The disadvantage is the high effort necessary, in particular the time-consuming node search algorithms for the transfer of the kinematic quantities. Thus the second approach is chosen here in the form of the fission/fusion strategy of Belytschko/Wong/Plaskasz[12] for hierarchical meshes. Elements with a large error are refined. Element patches with small errors are fused into the parent element. This strategy ensures fully hierarchical meshes, which contain the initial coarsest mesh.

3.5 Refinement-/Coarsening control

At the beginning of an adaptive analysis upper and lower bounds for the error have to be given. The violation of one of the limits causes a mesh modification, i.e. if

$$\|\boldsymbol{e}\|_{total,\Omega} < \epsilon_{lower} \quad \text{or} \quad \|\boldsymbol{e}\|_{total,\Omega} > \epsilon_{upper}\,. \tag{18}$$

Which elements have to be modified, is now an open question. Once a mesh modification has to be performed, it is reasonable from an efficiency point of view to refine/coarsen in such a fashion, that the mesh can be taken for quite a few time steps without further modification. In order to achieve this a so-called prescribed error tolerance ϵ_{presc} is introduced. This is the new error after remeshing, $\epsilon_{presc} = \|\boldsymbol{e}\|_{total,\Omega'}$. In practical analyses this error tolerance should be chosen such that it is neither too close to the upper nor too close the lower bound.

3.5.1 Selection of elements for refinement and coarsening

In the following, an algorithm based on hierarchical mesh modification is presented. As the criterion for the mesh modification, the average error $\|\boldsymbol{e}\|_{total,average,\Omega'}$ of the new mesh is introduced. The conditions for refinement/coarsening can now be presented as follows:

1.) All elements E_i with energy norm of the error $\|\boldsymbol{e}\|_{total,E_i}$, fulfilling the inequality

$$\frac{1}{2}\|\boldsymbol{e}\|_{total,average,\Omega'} \;<\; \|\boldsymbol{e}\|_{total,E_i} \;\leq\; 2\|\boldsymbol{e}\|_{total,average,\Omega'} \tag{19}$$

remain unmodified.

2.) All elements E_i with energy norm of the error $\|\boldsymbol{e}\|_{total,E_i}$, which exceed an upper bound

$$\|\boldsymbol{e}\|_{total,E_i} \;>\; 2\|\boldsymbol{e}\|_{total,average,\Omega'}\,, \tag{20}$$

will be refined.

3.) Each patch of four elements $E_{i(1)}, \ldots, E_{i(4)}$, which was created in a former modification level by refinement of one element, will be combined to the former element if the energy norm of the errors $\|\boldsymbol{e}\|_{total,E_{i(1)}}, \ldots, \|\boldsymbol{e}\|_{total,E_{i(4)}}$, fulfills the inequality

$$\left(\sum_{j=1}^{4} \|\boldsymbol{e}\|^2_{total,E_{i(j)}} \right)^{\frac{1}{2}} < \|\boldsymbol{e}\|_{total,average,\Omega'} \,. \tag{21}$$

The question is now, how to compute $\|\boldsymbol{e}\|_{total,average,\Omega'}$?

First, it is assumed, that the new mesh is optimal. It implies, that the element error is identical over the domain of interest, i.e.

$$\|\boldsymbol{e}\|_{total,\Omega'} = \left(\sum_{i=1}^{\mathcal{N}'_E} \|\boldsymbol{e}\|^2_{total,E'_i} \right)^{\frac{1}{2}} = (\mathcal{N}'_E \|\boldsymbol{e}\|^2_{total,average,\Omega'})^{\frac{1}{2}} \,. \tag{22}$$

Using the prescribed error value ϵ_{presc} for the new mesh, the desired averaged error of the new mesh can be written as

$$\|\boldsymbol{e}\|_{total,average,\Omega'} = \frac{\epsilon_{presc}}{\sqrt{\mathcal{N}'_E}} \,. \tag{23}$$

Given the number of elements of the new mesh, the determination of the average energy norm would be straightforward. However, the number of elements of the new mesh is usually unknown. As a first idea, one can assume $\mathcal{N}'_E = \mathcal{N}_E$, i.e. refinement and coarsening are performed with similar intensity. However, this works only with meshes, which are already more or less optimal. A more general scheme including an estimate of the number of elements in the new mesh is described in the following. For uniform meshes the following relation between the mesh parameter h and the number of elements $\mathcal{N}_E$ can be assumed:

$$h^2 \sim \frac{1}{\mathcal{N}_E} \,. \tag{24}$$

Considering the convergence characteristics of FE with linear shape functions in the absence of singularities, it is known that

$$\|\boldsymbol{e}\|_{total,\Omega} \sim h \,. \tag{25}$$

Applying the relationships (24) and (25) to the old and to the new mesh yields

$$\mathcal{N}'_E \|\boldsymbol{e}\|^2_{total,\Omega'} = \mathcal{N}_E \|\boldsymbol{e}\|^2_{total,\Omega} \,, \tag{26}$$

and as the prescribed error value ϵ_{presc} should be identical to $\|\boldsymbol{e}\|_{total,\Omega'}$, we get

$$\mathcal{N}'_E = \mathcal{N}_E \left(\frac{\|\boldsymbol{e}\|_{total,\Omega}}{\epsilon_{presc}} \right)^2 \,. \tag{27}$$

The direct application of equation (27) would usually result in a rather large number of elements $\mathcal{N}_E^{'}$, because equation (27) is independent of the quality of the mesh. However, in reality two different meshes for the same problem with the same number of elements show different errors. The introduction of a new, optimal virtual mesh with the number of elements $\mathcal{N}_{E,opt}$ and the energy norm of error $\|\boldsymbol{e}\|_{total,\Omega,opt}$ is a possible remedy. Then equation (27) can be formulated with values from this virtual mesh,

$$\mathcal{N}_E^{'} = \mathcal{N}_{E,opt} \left(\frac{\|\boldsymbol{e}\|_{total,\Omega,opt}}{\epsilon_{presc}} \right)^2 , \quad \|\boldsymbol{e}\|_{total,average,opt} = \frac{1}{4} \|\boldsymbol{e}\|_{total,E_i} . \tag{28}$$

Next, an arbitrary value for the average error of this virtual mesh $\|\boldsymbol{e}\|_{total,average,opt}$ can be chosen. Further, it is known, that the energy norm of error is reduced by half in an uniform refinement step and the error of one element is reduced to a quarter of the former error. This is only valid for linear finite elements and is the reason for the factor 4 between lower and upper limit in eq. (19) and for the factor $\frac{1}{4}$ in eq. (28). In order to get the average error of the virtual optimal mesh, it may be necessary to refine an element more than once,

$$\|\boldsymbol{e}\|_{total,average,opt} = \left(\frac{1}{4} \right)^n \|\boldsymbol{e}\|_{total,E_i} \tag{29}$$

$$\text{then} \qquad n = \frac{\log \dfrac{\|\boldsymbol{e}\|_{total,E_i}}{\|\boldsymbol{e}\|_{total,average,opt}}}{\log 4} \tag{30}$$

The integer value n is an indicator of the number of refinement resp. coarsening (if n is negative) steps starting from the actual mesh to achieve the average error of the virtual mesh. Within a loop running over all elements $E_i; i = 1, \ldots, \mathcal{N}_E$ the 'estimated' number of elements $\mathcal{N}_{E,opt}$ and the energy norm of the error $\|\boldsymbol{e}\|_{total,\Omega,opt}$ of the optimal mesh is obtained:

$$\|\boldsymbol{e}\|^2_{total,\Omega,opt} = \|\boldsymbol{e}\|^2_{total,\Omega} - \sum_{i=1}^{\mathcal{N}_E} \left(1 - \left(\frac{1}{4} \right)^{n_i} \right) \|\boldsymbol{e}\|^2_{total,E_i} \tag{31}$$

$$\mathcal{N}_{E,opt} = \mathcal{N}_E + \sum_{i=1}^{N_E} \operatorname{sgn}(n_i) \cdot \left(4^{|n_i|} - 1 \right) . \tag{32}$$

An arbitrary choice of $\|\boldsymbol{e}\|_{total,average,opt}$ is allowed due to the fact that the product $\|\boldsymbol{e}\|^2_{total,\Omega,opt} \cdot \mathcal{N}_{E,opt}$ is constant, see eq. (26). The choice of a high value of $\|\boldsymbol{e}\|_{total,average,opt}$ would result in a large error $\|\boldsymbol{e}\|_{total,\Omega,opt}$ and fewer elements $\mathcal{N}_{E,opt}$. Based on equation (28) the value $\mathcal{N}_E^{'}$ can than be determined and the modification of the real mesh can be started.

Remark: At a first sight, this algorithm appears to be rather inefficient. However, with the

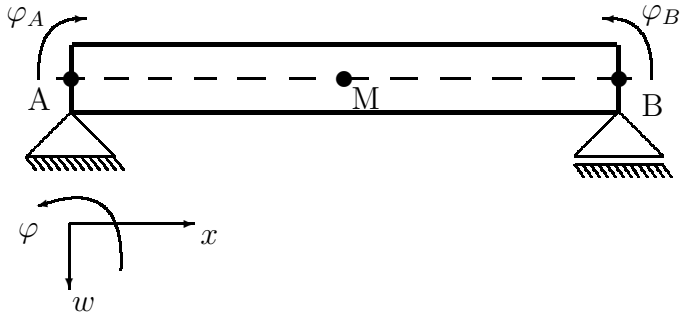
determination of $\mathcal{N}_E'$ and $\|\boldsymbol{e}\|_{total,\Omega,opt}$ the number of mesh modifications is considerably reduced compared to the choice of the average error of the actual mesh as a criterion which would cause a mesh modification in each time step.

3.6 Transfer/mapping of the variables between meshes

Due to the use of bilinear form functions for the elements, bilinear interpolation seems to be the obvious choice in a refinement step. However, bilinear interpolation of all nodal values $\boldsymbol{d}, \dot{\boldsymbol{d}}$ and $\ddot{\boldsymbol{d}}$ would yield a rather non-smooth solution. The additional application of a smoothing algorithm (Least-square fit) improves the quality of the solution, but a loss of energy is obtained. Then a further correction of $\boldsymbol{d}$ and $\dot{\boldsymbol{d}}$ is needed to achieve a reasonable result. For a detailed description of a useful algorithm reference is made to Riccius[6].

Using bilinear form functions for plates and shells with Reissner kinematics leads to further problems, as to prevent shear locking, the out of plane terms are independently interpolated; the so-called ANS elements, see Bathe/Dvorkin[13].

With bilinear interpolation of these quantities among meshes an energy increase is obtained, which is not admissible. The problem is briefly described in the following by the example of a Timoshenko beam loaded by prescribed edge rotations.



$$\begin{aligned} \varphi_A &= -0.5 \\ \varphi_B &= 0.5 \\ w_A &= w_A = 0.0 \end{aligned}$$

Figure 2: Timoshenko beam with prescribed edge rotations φ_A and φ_B

The well known Timoshenko-kinematics are $\varphi - w_{,x} + \gamma = 0$. φ is the cross section rotation, w the transversal displacement and γ is the shear strain. The total strain energy norm for the beam is

$$\|\boldsymbol{u}\|_{strain,\Omega} = \left(E\,I \int_A^B \varphi_{,x}^2 dx \,+\, G\,A \int_A^B (w_{,x} - \varphi)^2 dx \right)^{\frac{1}{2}} . \tag{33}$$

If linear shape functions are chosen for the interpolation of w and φ and the shear strains are assumed to be constant within the element with the value taken at the center M of the beam (similar to the ANS assumption), then no shear strains are activated under the

loading shown above,

$$\gamma \;=\; w_{,x} - \varphi_M \;=\; 0 - 0\,. \tag{34}$$

In the original mesh (Fig.2), the strain energy norm of the beam contains only bending terms without activating any shear strain energy. After refinement the situation is com-

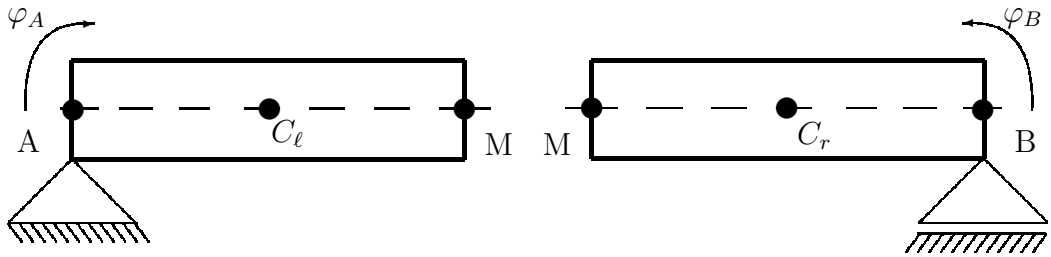


Figure 3: Situation after refinement

pletely different. If the nodal values φ and w are linearly interpolated, then each of the elements is only loaded by an edge rotation on one side. Using now also the constant shear strain interpolation, we obtain

$$\text{with} \qquad \varphi_{c_\ell} = \frac{1}{2}\varphi_A \quad \text{resp.} \quad \varphi_{c_r} = \frac{1}{2}\varphi_B\,, \text{ and } w_x = 0 \text{ for both elements}$$

the shear strains

$$\gamma_\ell = w_{,x} - \varphi_{c_\ell} = 0 - (\frac{1}{2}\cdot -0.5) = \frac{1}{4} \neq 0\,; \qquad \gamma_r = \frac{1}{4}\,.$$

As the bending energy is unmodified, the strain energy norm increases due to this 'artificially' activated shear strain. In order to remove this somehow artificial effect a global static solution of the new mesh is performed taking as loading the deformations – nodal displacements and rotations – of the nodes from the old mesh as prescribed values. Then the strain energy remains identical after refinement.

4 CHOICE OF THE ERROR MEASURE

For hyperbolic problems the question arises, what to choose as an error measure, as neither loading nor the behavior is identical at various times. Thus, dependent on the error measure, the meshes are overly fine or too coarse compared to the desired values. In static analyses the relative error is used, which can be written as the quotient of the total energy norm of error and 'exact' total energy norm of solution,

$$\|\boldsymbol{e}\|^{\%}_{total,\Omega} = \frac{\|\boldsymbol{e}\|_{total,\Omega}}{\sqrt{\|\boldsymbol{u}\|^2_{total,\Omega} + \|\boldsymbol{e}\|^2_{total,\Omega}}}\,.$$

The scaling by the total energy norm, which fairly rapidly converges in static adaptive analyses, does not remain constant for elastodynamic problems, as the energy decays e.g. in the presence of damping and increases, when external loading is applied. With an absolute error-measure, as e.g. the total energy norm of the error, the problem is in finding suitable error bounds. Because these bounds represent energy values, the energy-norm of the solution must be known in advance. However, the qualitative behavior of the energy norm of the error is nearly identical to the energy norm of the solution. On the basis of the example 'saddle roof', see Fig.4, error control is performed with the relative and the absolute error.

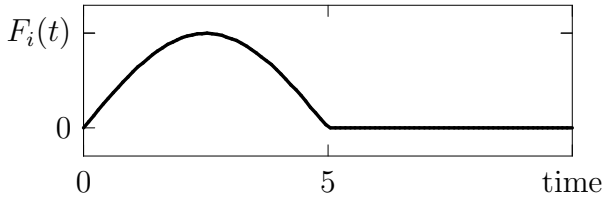
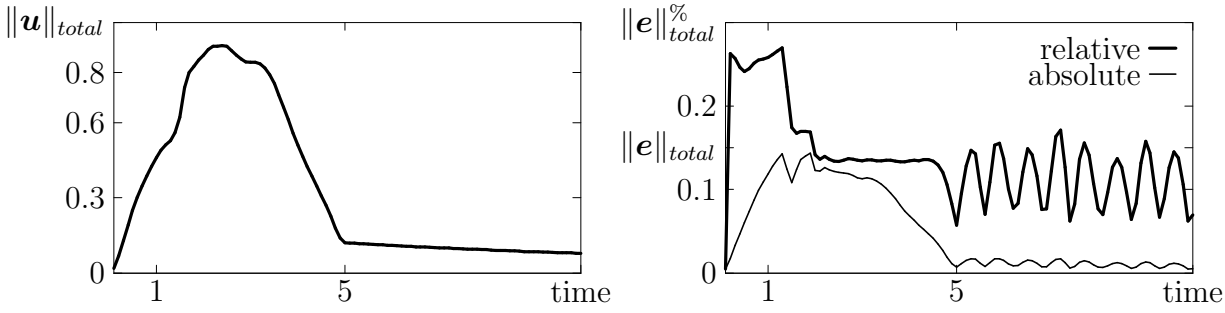


Figure 4: Saddle roof, Load function

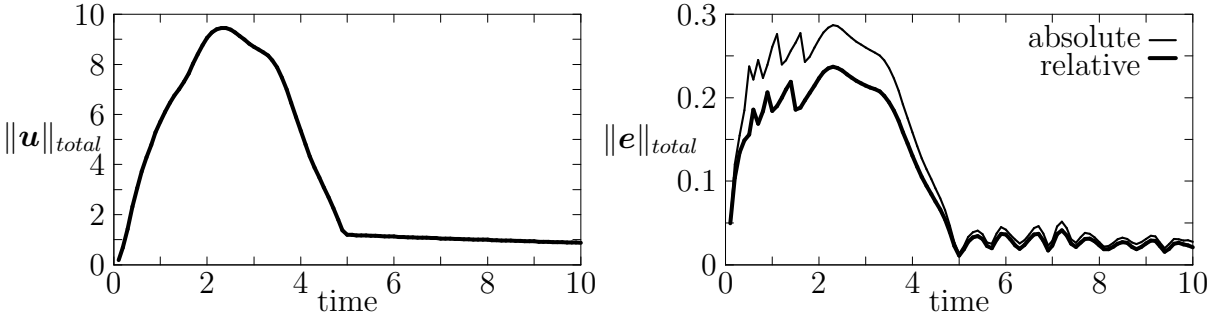
The data for this example are: linear elastic material, $E = 3.0\text{e+}07,\ \nu = 0.29,\ \rho = 7700$; shell thickness $t_{shell} = 0.17$; Rayleigh-damping, $a = 0.03, b = 0.003$. The geometry of the shell is given as: $z_{shell} = 0.5 * (x^2/20 - y^2/20)$. The time function for the external forces with the amplitudes $F_1 = 200$, $F_2 = 100$, $F_3 = 50$, all acting in z-direction, is displayed in Fig.4. For the FE analysis a shell element based on linear interpolation with shear interpolation, see Gebhardt[14], Hauptmann[15] is used. The mesh in Fig.4 was uniformly refined two times at the start of the analysis. The standard Newmark scheme with a time step size $\Delta t = 0.1$ is used. In Fig.5a) the curve for the energy norm of the solution is shown, whereas Fig.5b) contains curves for the relative and absolute error. As mentioned above, the relative error oscillates and the absolute error decreases in a similar fashion as the energy norm of the solution. Although Rayleigh-damping is involved, the relative error does not decrease. The main reason for this is the decrease of the energy of the system, such that the quotient (error/energy) does not decrease. In order to allow error control with the relative error-measure, the maximum energy norm of the solution $\|\boldsymbol{u}\|_{total,pre}$ is computed. In the adaptive process the relative error is then based on this maximum. To show the results the 'saddle roof' with the loads as above is taken as example. The external forces are scaled by a factor of 10.0. All other data are identical. The following error bounds are considered for the adaptive analysis; absolute error : $\epsilon_{low} = 0.0005$, $\epsilon_{presc} = 0.25$, $\epsilon_{upper} = 0.3$; relative error: $\epsilon_{low} = 0.005\%$, $\epsilon_{presc} = 2.5\%$, $\epsilon_{upper} = 3.2\%$. For the



a) b)

Figure 5: Saddle roof,
a) energy norm of the solution;b) energy norm of the error, absolute and relative

relative error control the energy norm of the solution, $\|\boldsymbol{u}\|_{max,pre} = 7.55$, on a coarse mesh is determined in a pre-simulation. In Fig.6a) the curve for the energy-norm of the solution with both error-measures is displayed. The identity is obvious. It is visible from



a) b)

Figure 6: Saddle roof, Comparison – error control by absolute resp. relative error
a) energy-norm of the solution b) absolute energy-norm of the error

Fig.6b), that the absolute error shows the similarities and the differences between the two error-measures. The difference is due to the fact that the energy-norm of the solution in the pre-simulation process was smaller. So the relative error increases fairly rapidly beyond the upper bound and causes mesh-refinements. Thus, the energy-norm of the error is smaller than for the absolute error. To increase the efficiency the energy-norm of the solution should be more exactly determined. This is usually not a problem as in realistic simulations the solution is performed not only once. Then the improved total energy-norms of the solution are available.
In Fig.7 the number of equations during the adaptive process is depicted. The maximum

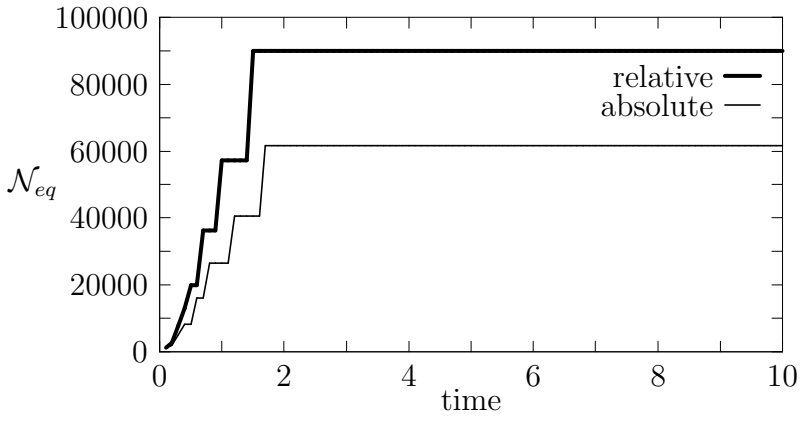


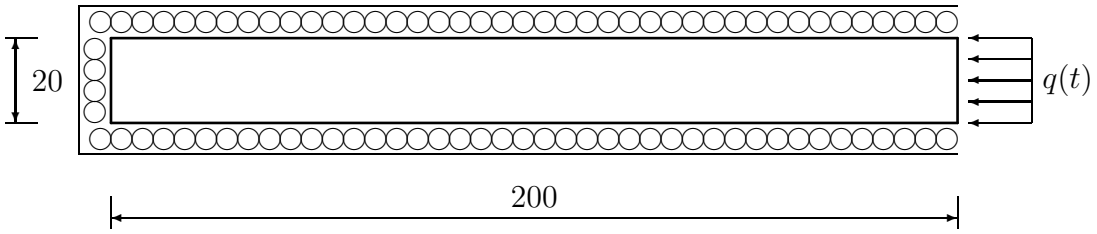
Figure 7: Number of equations with absolute and relative error control versus time

number is reached close to the time when the maximum energy is obtained. After this, the mesh is not coarsened due to the very small lower bound and the number of equations remains constant.

5 REMARKS TO THE KINETIC ENERGY NORM OF THE ERROR

As the kinetic energy norm of the error of the current mesh is computed on the basis of a somehow coarser mesh and then adjusted by a factor, the summation of the kinetic energy error and strain energy error seems to be rather arbitrary. Therefore, the effect of the kinetic energy norm of the error is discussed on two examples, representing two different types of problems.
First, a wave propagation problem[6,16] is considered. The material data are taken from Riccius[6]. As before, the standard Newmark scheme is used with a constant time step $\Delta t = 0.1$. Damping is neglected. With these parameters the energy norm of the solution reaches the value, $\|\boldsymbol{u}\|_{total} = 1.0$, in the time range $t \in [1:10]$. Thus relative and absolute error are identical, if this value is taken in the computation of the relative error. For comparison this example is analyzed with 4 different weighting factors $c_k = 0.0, 0.365, 4.0, 100.0$, see eq. (17), for the kinetic energy error. From Fig.9 and Fig.10 the influence of c_k is obvious; in addition the number of equations increases with increasing c_k. With $c_k = 0.365$ the energy norm of the adaptive numerical solution is identical to the analytical solution $\|\boldsymbol{u}\|_{total} = 1.0$. The minor oscillations in the curves in Fig.9a) are due to mesh modifications. From this example we conclude that a correct estimate of the kinetic error is needed when the loading introduces a high gradient in the velocity.

The second example is a clamped cantilever plate, see Fig.11, a vibration problem. At the right end a line load is applied. The material data are: $E = 2.1\text{e+}08$, $\nu = 0.29$, $\rho = 7700$; Rayleigh damping with $a = 0.01\,, b = 0.001$. The error bounds for the absolute error are: $\epsilon_{lower} = 0.0005, \epsilon_{presc} = 0.025, \epsilon_{upper} = 0.03$. For the standard Newmark scheme a



Material data: $E = 10000.0$, $\rho = 10.0$, $\nu = 0.3$, thickness $t = 1.0$

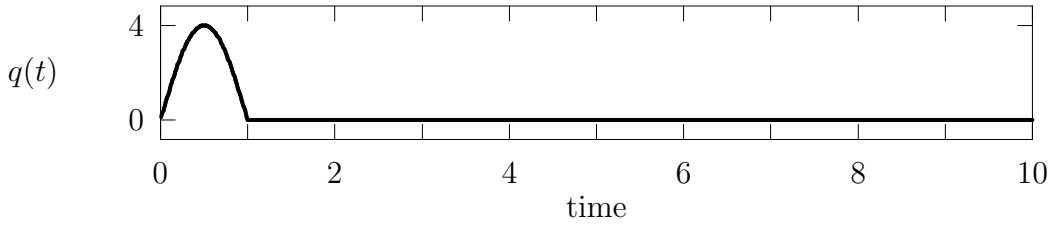


Figure 8: Rectangular domain, material data, loading

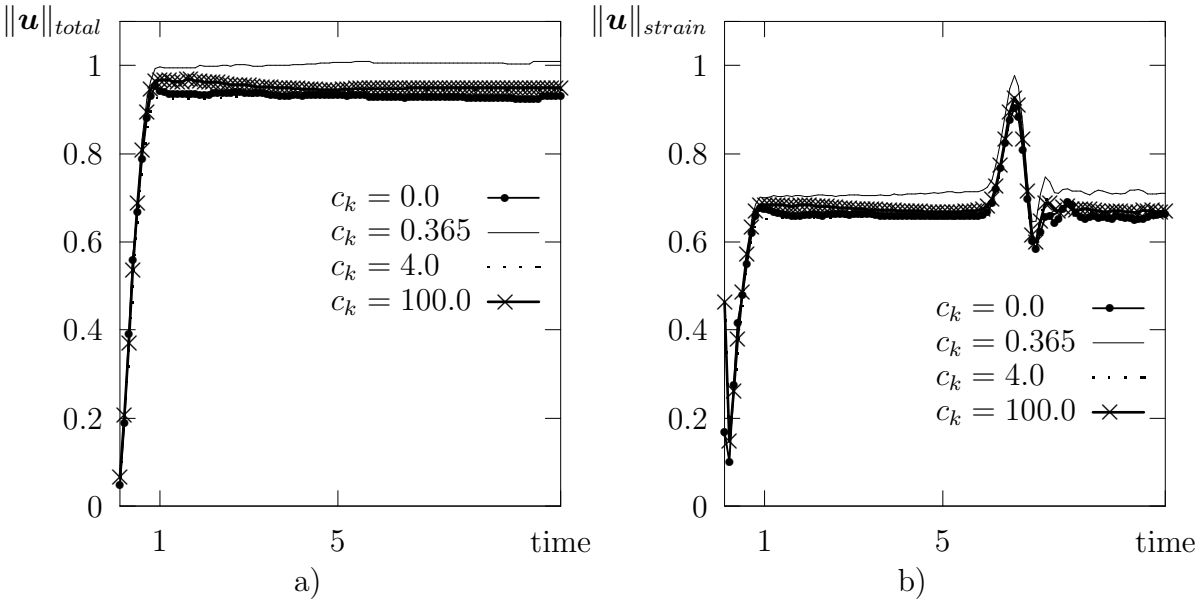


Figure 9: Rectangular domain with varying weighting factors c_k
a) energy norm of the solution b) strain energy norm of the solution

constant time step size $\Delta t = 0.05$ is chosen. The weighting factor c_k is varied between 0.0 and 100.0 as for the wave propagation problem. However, no difference in the energy norm of the solution is found, see Fig.12. A closer look reveals, see Fig.13, that at the start of the adaptive analysis the strain energy norm of the error grows faster than the kinetic energy norm of the error. As a consequence the mesh is refined only due to high

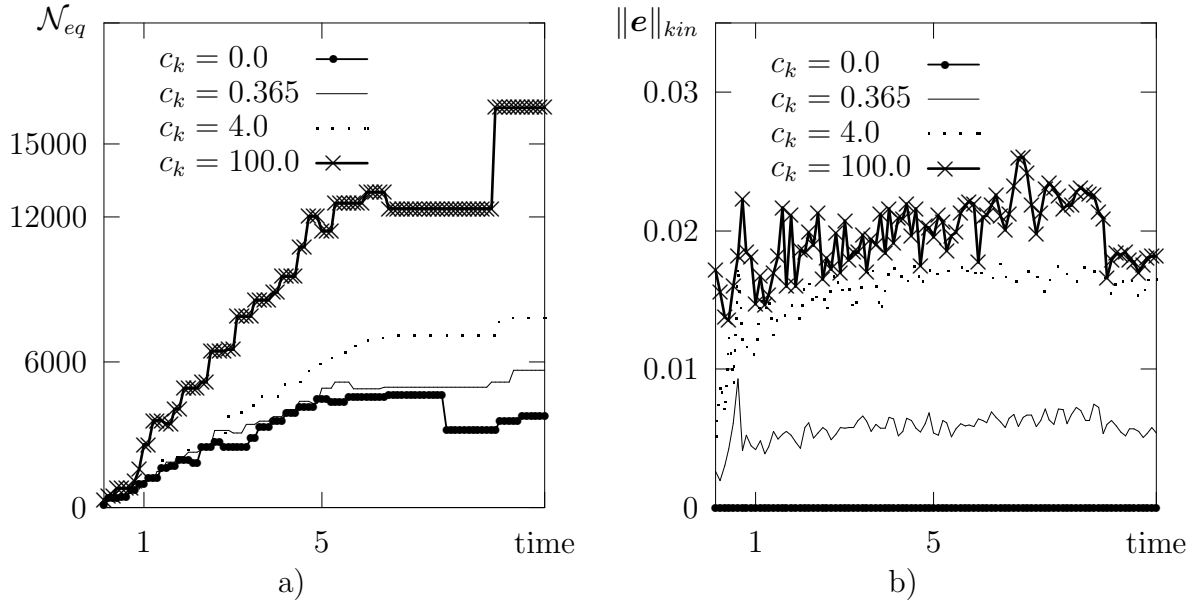


Figure 10: Rectangular domain with varying weighting factors c_k
a) Number of equations $\mathcal{N}_{eq}$ b) kinetic energy norm of the error

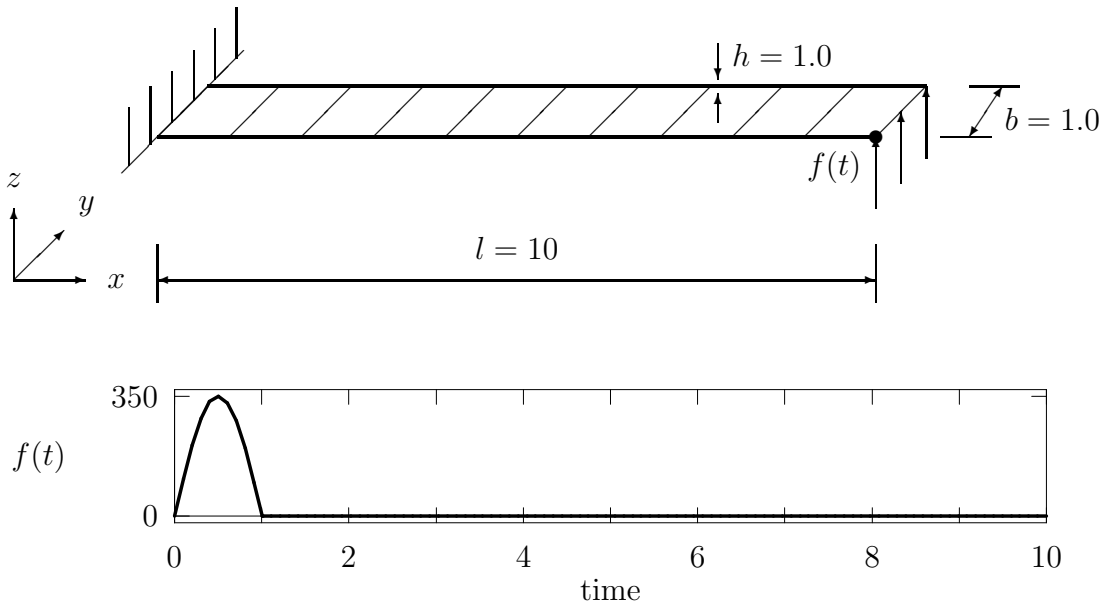


Figure 11: Clamped plate and load function

strain energy norm error in the time range $t \in [0 : 1]$. The kinetic energy norm of the error remains rather small. It is obvious that the kinetic energy is much better approximated than the strain energy with these meshes. The velocity gradient is rather small and thus the velocity distribution is well approximated by the discretization.

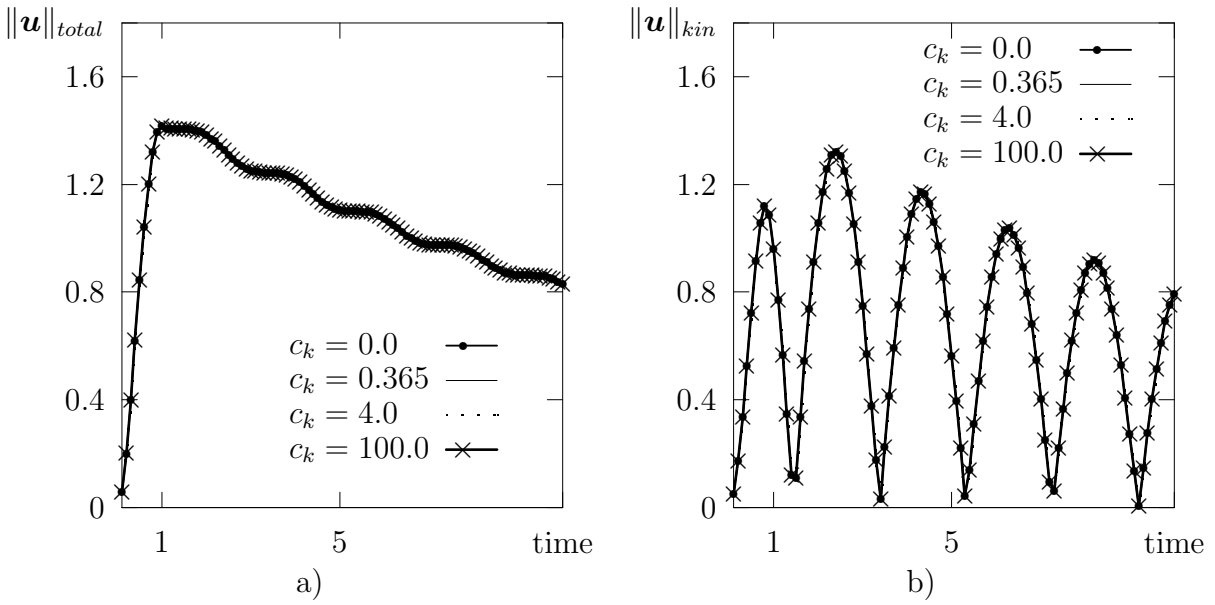


Figure 12: Clamped plate with different weighting factors c_k
a) energy norm of the solution b) kinetic energy norm of the solution

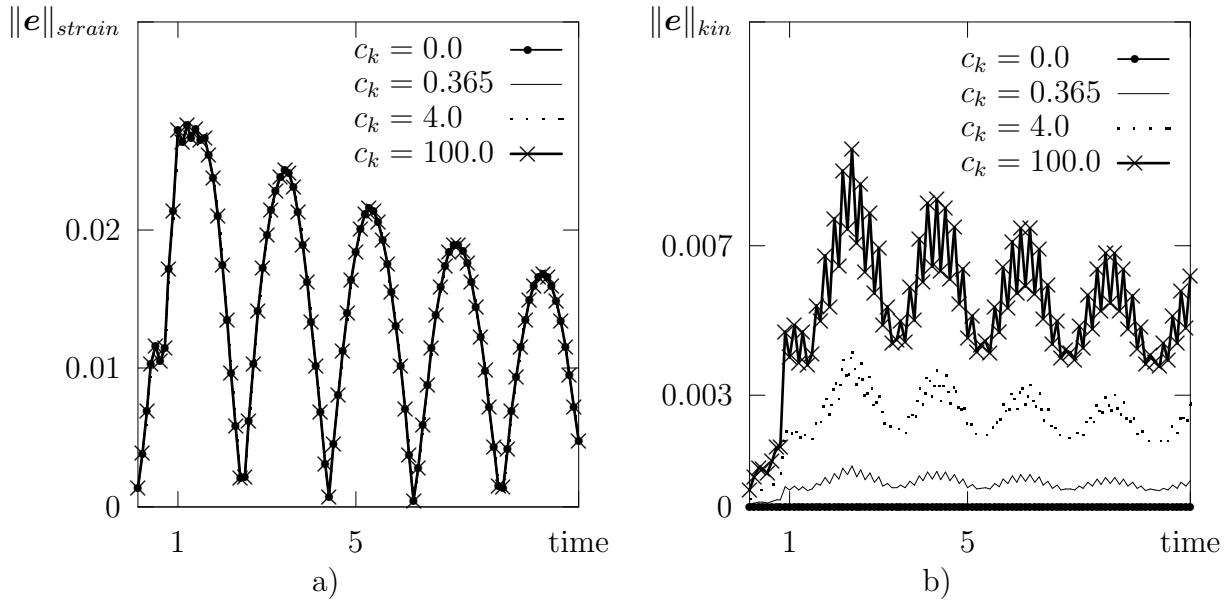


Figure 13: Clamped plate with different weighting factors c_k
a) strain energy norm of the error b) kinetic energy norm of the error

These two examples show, that the influence of the kinetic energy norm error is strongly problem-dependent. Moreover it is well known that the mass discretization is much more exact than the stiffness approximation. The expressions for the mass matrix do not

contain any derivative. An important aspect are also the initial conditions. For example, when initial velocities are applied the kinetic energy norm of error will grows heavily in the first time step and is the dominant error. But, often after the second resp. the following time steps, also the strain energy norm of error grows due to the larger deformations.

6 CONCLUSION

The main tools for adaptive dynamic analysis of elastic structures are presented. A simple post-processing error estimator for the kinetic part of the spatial discretization error was introduced. For the estimation of the time discretization error it is refered to Riccius/Schweizerhof[7] and to Zienkiewicz/Xie[17]. Analyzing an example the problems with a relative error measure for error control were demonstrated and a relative error with reference to a pre-calculation is introduced. An efficient error modification scheme based on the average error of an arbitrary virtual mesh was proposed. Also, the correct transfer of the kinematic variables for mixed shell elements is described. Finally, the influence of the estimation of the kinetic energy norm of the error on the total energy norm of the error and the adaptive analysis was shown on two examples.

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