On Computational Interpretations of the Modal Logic S4 IIIa. Termination, Confluence, Conservativity of $\lambda\mathtt{ev}Q$.

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Abstract

A language of constructions for minimal logic is the λ-calculus, where cut-elimination is encoded as β-reduction. We examine corresponding languages for the minimal version of the modal logic S4, with notions of reduction that encodes cut-elimination for the corresponding sequent system. It turns out that a natural interpretation of the latter constructions is a λ-calculus extended by an idealized version of Lisp's `eval` and `quote` constructs.

In this Part IIIa, we examine the termination and confluence properties of the $\lambda\mathtt{ev}Q$ and $\lambda\mathtt{ev}Q_H$-calculi. Most results are negative: the typed calculi do not terminate, the subsystems Σ and Σ_H that propagate substitutions, quotations and evaluations downwards do not terminate either in the untyped case, and the untyped $\lambda\mathtt{ev}Q_H$-calculus is not confluent. However, the typed versions of Σ and Σ_H do terminate, so the typed $\lambda\mathtt{ev}Q$-calculus is confluent. It follows that the typed $\lambda\mathtt{ev}Q$-calculus is a conservative extension of the typed λ_{S4}-calculus.

Part IIIb will cover the confluence of the typed $\lambda\mathtt{ev}Q_H$-calculus, which is not dealt with here.

1 Plan

Part IIIa is organized as follows. In Section 2, we examine the properties of $\lambda\mathtt{ev}Q$ and $\lambda\mathtt{ev}Q_H$ related to termination; in Section 3, we examine their confluence properties. And in Section 4, we use these results to show that, in the typed case, G induces an embedding of λ_{S4} inside $\lambda\mathtt{ev}Q$ that makes the latter conservative extensions of the former. This also holds for λ_{S4H} and $\lambda\mathtt{ev}Q_H$, provided that the latter is confluent in the typed case, a conjecture that will be the subject of part IIIb.

2 Termination

2.1 Termination

As far as termination is concerned, the answer is simple:

Theorem 2.1 (Melliès) *Neither the typed $\lambda\mathtt{ev}Q$-calculus nor the typed $\lambda\mathtt{ev}Q_H$-calculus terminates.*

Proof: These calculi both include the typed $\lambda\sigma_{\Uparrow}$-calculus at level 1, therefore Paul-André Melliès' counter-examples to termination in typed $\lambda\sigma_{\Uparrow}$ apply [Mel94, Mel95]. □

As a corollary, the untyped calculus does not terminate, as well. But this was even clearer, as the latter can simulate any reduction in the untyped λ-calculus.

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We may be tempted to try and repair this. So we might choose another λ-calculus with explicit substitutions that terminates, say Lescanne and Rouyer-Degli's $\lambda\upsilon$ [LRD94]; but it is only confluent on closed terms, and we need confluence on open terms to get confluence on terms where λ-bound variables occur. Then, we may choose or Muñoz' λ_ς-calculus [MH96], which is confluent, terminating and simulates β-normalization but not individual β-contraction steps. At the time of this writing, the holy grail of a confluent, strongly normalizing simply-typed λ-calculus with explicit substitutions that can simulate β-contraction is still to be found.

The next question is whether the rules in Σ, i.e. all rules but (β) and (β^ℓ), terminate. We shall need this in Section 3.1. The answer is simple in the untyped case:

Lemma 2.2 *The rules of Σ do not terminate in the untyped case.*

This still holds if we restrict the untyped language to terms built with $\mathtt{ev}^1_S$, $\mathtt{ev}^2_S$ and id^1 only, and use only rules $(\mathtt{ev}^1\mathtt{ev}^2)$ and $(\mathtt{ev}id^1)$.

Proof: The following loops:

$$\begin{array}{ll}
\mathtt{ev}^1_S(\mathtt{ev}^2_S id^1\ id^1)(\mathtt{ev}^2_S id^1\ id^1) & \\
\mathtt{ev}^1_S(\mathtt{ev}^1_S id^1(\mathtt{ev}^2_S id^1\ id^1))(\mathtt{ev}^1_S id^1(\mathtt{ev}^2_S id^1\ id^1)) & \text{by } (\mathtt{ev}^1\mathtt{ev}^2) \\
\longrightarrow \mathtt{ev}^1_S(\mathtt{ev}^2_S id^1\ id^1)(\mathtt{ev}^1_S id^1(\mathtt{ev}^2_S id^1\ id^1)) & \text{by } (\mathtt{ev}^1 id^1) \\
\longrightarrow \mathtt{ev}^1_S(\mathtt{ev}^2_S id^1\ id^1)(\mathtt{ev}^2_S id^1\ id^1) & \text{by } (\mathtt{ev}^1 id^1)
\end{array}$$

□

So much for the untyped case. We may then consider the following *semi-stratified* restriction of the calculus. This cannot claim to be really untyped, but at least it allows terms of type T to remain mostly untyped, and it gets around the counter-example of Lemma 2.2.

Definition 2.1 *The* semi-stratified $\lambda\mathtt{ev}Q$-terms *is the following sublanguage of $\lambda\mathtt{ev}Q$-terms. Terms s, t, u, v, w, ... are elements of the language $T \cup \bigcup_{i=0}^{+\infty} S^i$, where T is the language of* elementary terms *and S^i, $i \in \mathbb{N}$, is that of* explicit substitutions *or* stacks *at level i:*

$$\begin{array}{lll}
T & ::= & \mathcal{V} \mid \lambda\mathcal{V}\cdot T \mid TT \mid 1S^0 \mid \lambda^\ell T \mid T \star^\ell T \mid 1^\ell \mid \mathtt{ev}^\ell_T T S^{\ell-1} \mid T \circ^\ell_T S^\ell \mid Q^\ell_T T \\
S^0 & ::= & () \mid T \bullet S^0 \mid\uparrow S^0 \mid \mathtt{ev}^1_S S^1 S^0 \\
S^\ell & ::= & id^\ell \mid\uparrow^\ell\mid T \bullet^\ell S^\ell \mid\Uparrow^\ell S^\ell \mid \mathtt{ev}^i_S S^{\ell+1} S^{i-1}\ (1 \leq i \leq \ell+1) \\
 & & \mid S^\ell \circ^i_S S^i\ (1 \leq i \leq \ell) \mid Q^i_S S^{\ell-1}\ (1 \leq i \leq \ell-1)
\end{array}$$

modulo α-renaming, and ℓ ranges over all integers ≥ 1.

This restriction is natural, in the sense that we can prove the following properties (proofs omitted): the G-translation of every λ_{S4}-term is semi-stratified of sort T; The quotation function $u \mapsto u`\rho$ maps semi-stratified terms of sort T to semi-stratified terms of sort T, and semi-stratified terms of sort S^i to semi-stratified terms of sort S^{i+1}, for every $i \in \mathbb{N}$. The types preserve the semi-stratified sorts, in the sense that every typable $\lambda\mathtt{ev}Q$-term u is semi-stratified, that u is of sort T if its type is a term type and of sort S^j if its type is a metastack type of the form $\overline{\varsigma^{j-1} \overset{\square}{\Rightarrow} \varsigma}$ for some $j \geq 0$. Moreover, we can decide in polynomial time whether a term is semi-stratified, and what its unique sort is; this sort is preserved by the reduction rules, including the η-like rules.

Unfortunately:

Lemma 2.3 *The rules of Σ do not terminate in the semi-stratified case.*

Proof: The following loops:

$$\begin{array}{ll}
\mathtt{ev}^1_T(\mathtt{ev}^2_T 1^1\ id^1)(\mathtt{ev}^2_T 1^1\ id^1 \bullet ()) & \\
\longrightarrow \mathtt{ev}^1_T(\mathtt{ev}^1_T 1^1(\mathtt{ev}^2_T 1^1\ id^1 \bullet ()))(\mathtt{ev}^1_S id^1(\mathtt{ev}^2_T 1^1\ id^1 \bullet ())) & \text{by } (\mathtt{ev}^1\mathtt{ev}^2) \\
\longrightarrow \mathtt{ev}^1_T(1(\mathtt{ev}^2_T 1^1\ id^1 \bullet ()))(\mathtt{ev}^1_S id^1(\mathtt{ev}^2_T 1^1\ id^1 \bullet ())) & \text{by } (\mathtt{ev}1^1) \\
\longrightarrow \mathtt{ev}^1_T(\mathtt{ev}^2_T 1^1\ id^1)(\mathtt{ev}^1_S id^1(\mathtt{ev}^2_T 1^1\ id^1 \bullet ())) & \text{by } (1) \\
\longrightarrow \mathtt{ev}^1_T(\mathtt{ev}^2_T 1^1\ id^1)(\mathtt{ev}^2_T 1^1\ id^1 \bullet ()) & \text{by } (\mathtt{ev}id^1)
\end{array}$$

Moreover, all terms in the derivation are semi-stratified. □

Therefore, we believe that types are crucial in making Σ terminate. Since we also want to include the η-like rules, we define:

Definition 2.2 (Σ_H) *Let Σ_H be the set of rules in Σ plus group (H), i.e. all rules but (β) and (β^ℓ).*

From now on, we shall implicitly assume that all terms that we handle are typed, unless we say otherwise.

It turns out that showing that Σ_H terminates is difficult. As Lemma 2.2 shows, the type information is crucial. This explains why no classical termination argument for unsorted rewriting systems [Der87] applies. In particular, recursive path orderings fail even where they would seem to be applicable (in groups (D), (E), (F)), as we shall see in Section 2.2. Furthermore, the system is not left-linear (because of ($\eta\,\bullet$) and ($\eta\,\bullet\,\circ^\ell$)), it is not right-linear, it contains both collapsing and duplicating rules, in short it has no remarkable property that would make its study simpler.

Moreover, take the rules in group (B) at some level, say $\ell = 1$, and add rules ($\eta \Uparrow^1$), ($\eta\,\bullet^1$) and ($\eta\,\bullet\,\circ^1$). If we consider the restriction of this system where ($\eta \Uparrow^1$) is applied eagerly (just after (λ^1)), we get the σ-calculus [ACCL90], whose termination proofs are all difficult, to the exception of Zantema's [Zan94]. But Zantema's proof rests on transformations of the rewrite system that do not preserve types (or even semi-stratified sorts); but we have seen that types were essential to termination.

The proof that we show is intricate, and rather tedious. We proceed by showing that larger and larger systems of rules are terminating, beginning with some parts where the sort information is not yet indispensible.

2.2 Behaviour of Q^ℓ-Terms

Roughly, Σ_H can be separated in two parts: groups (A), (B), (C) and (H) (Figures 3 and 9, Part II) on the one hand, which propagate substitutions down terms at the same level; and groups (D), (E), (F) (Figure 4, Part II), which push terms of lower levels below terms of higher levels. We start by studying the latter.

In groups (D), (E) and (F), there are basically three kinds of operators: $\mathtt{ev}^\ell$ propagates down and decreases the exponents of operators that it goes through; Q^ℓ instead increases the exponents of operators; and $\circ^\ell$ leaves them unchanged. For example, the ($\circ^{\mathcal{L}}\circ^\ell$) rule ($(u \circ^{\mathcal{L}} v) \circ^\ell w \to (u \circ^\ell w) \circ^{\mathcal{L}} (v \circ^\ell w)$) pushes the $\circ^\ell$ operator with the lower exponent below the other one, leaving it unchanged.

Such rules are usually well handled by the *recursive path ordering* [Der87], which we now define. Recall that a *quasi-ordering* $\succeq$ is a reflexive and transitive relation, that its associated equivalence relation $\approx$ is defined by $u \approx v$ if and only if $u \succeq v$ and $v \succeq u$, and that its strict part $\succ$ is defined by $u \succ v$ if $u \succeq v$ and $v \not\succeq u$. Recall also that a (finite) *multiset* of objects in A is a map from objects in A to integers (their *multiplicities*), all but finitely many of which are 0. We let $\{\!|x_1, \ldots, x_n|\!\}$ be the multiset containing x_1, ..., x_n, counted with their multiplicities, and $\uplus$ denote multiset union. The *multiset extension* $>^{mul}$ of a strict ordering $>$ on a set A is defined as the transitive closure of the relation that rewrites $M \uplus \{\!|x|\!\}$ into $M \uplus M'$, where $x > x'$ for every $x' \in M'$. Consider now a set of first-order terms with a precedence (i.e., an ordering) on function symbols $\succeq$. Then it induces a recursive path ordering on terms $\succeq_{rpo}$, together with associated relations $\succ_{rpo}$ and $\approx_{rpo}$ as follows. Given two first-order terms $s = f(s_1, \ldots, s_m)$ and $t = g(t_1, \ldots, t_n)$, we have $s \succ_{rpo} t$ if and only if:

1. $s_i \succeq_{rpo} t$ for some i, $1 \leq i \leq m$,
2. or $f \succ g$ and $s \succ_{rpo} t_j$ for all j, $1 \leq j \leq n$,
3. or $f \approx g$ and $\{\!|s_1, \ldots, s_m|\!\} \succ_{rpo}^{mul} \{\!|t_1, \ldots, t_n|\!\}$.

Then, a rewrite system $\mathcal{R}$ over a set of first-order terms is terminating if and only if there exists a well-founded quasi-ordering $\succeq$ on the set of function symbols such that $t \succ_{rpo} u$ for every rule $t \to u$ in $\mathcal{R}$ [Der87].

Define the precedence by $\circ^\ell \succ \circ^{\mathcal{L}}$ whenever $\ell < \mathcal{L}$. Then ($\circ^{\mathcal{L}}\circ^\ell$) is a decreasing rule. Moreover, if we use only this rule, then the set of function symbols that can appear during a derivation is finite, so $\succ$ is well-founded: ($\circ^{\mathcal{L}}\circ^\ell$) terminates.

Unfortunately, all the rules of group (F) create possibly new function symbols, with higher and higher exponents, and which are therefore lower and lower with respect to $\succ$: $\succ$ is not well-founded. We repair this by applying a transformation to our terms, so that all exponent increments are encoded in advance (through the use of the functions q_i in the following definition):

Definition 2.3 *We adopt the following reading convention for $\lambda\mathtt{ev}Q$-terms. The $\lambda\mathtt{ev}Q$-terms are considered as first-order terms built with function symbols f^j, $j \geq 0$, where we take for granted that $1u$ and $\uparrow u$ stand for $1^0 \circ^0 u$ and $\uparrow^0 \circ^0 u$ respectively, consing $\bullet$ is $\bullet^0$, application is $\star^0$, λ-headers $\lambda x\cdot$ are some function symbols λ^0_x and variables x are constants (i.e., 0-ary functions) $x^0()$. The function symbols are then binary, unary or zero-ary (constants).*

Let q_i, $i \geq 1$, be the functions defined as follows:

$$q_i(\mathtt{ev}^j uv) = \begin{cases} \mathtt{ev}^{j+1}(q_i(u))(q_i(v)) & \text{if } i \leq j-1 \\ \mathtt{ev}^j(q_{i+1}(u))v & \text{if } j-1 < i \end{cases} \qquad q_i(u \circ^j v) = \begin{cases} q_i(u) \circ^{j+1} q_i(v) & \text{if } i \leq j \\ q_{i+1}(u) \circ^j v & \text{if } j < i \end{cases}$$

$$q_i(f^j(u,v)) = \begin{cases} f^{j+1}(q_i(u), q_i(v)) & \text{if } i \leq j \\ f^j(q_i(u), q_i(v)) & \text{if } j < i \end{cases} \qquad \text{for all binary operators } f \text{ other than } \mathtt{ev} \text{ or } \circ$$

$$q_i(\Uparrow^j u) = \begin{cases} \Uparrow^{j+1}(q_i(u)) & \text{if } i \leq j \\ \Uparrow^j(q_{i+1}(u)) & \text{if } j < i \end{cases}$$

$$q_i(f^j u) = \begin{cases} f^{j+1}(q_i(u)) & \text{if } i \leq j \\ f^j(q_i(u)) & \text{if } j < i \end{cases} \qquad \text{for all unary operators } f \text{ other than } \Uparrow$$

$$q_i(f^j) = \begin{cases} f^{j+1} & \text{if } i \leq j \\ f^j & \text{if } j < i \end{cases} \qquad \text{for } f \text{ constant}$$

where $j \geq 0$,

We define the following interpretation $[\![\_]\!]_q$ on terms:

$$\begin{array}{l} [\![Q^\ell u]\!]_q = Q^\ell(q_\ell([\![u]\!]_q)) \quad (\ell \geq 1) \\ [\![\Uparrow^\ell u]\!]_q = \Uparrow^\ell (q_\ell([\![u]\!]_q)) \quad (\ell \geq 1) \\ [\![u \circ^\ell v]\!]_q = q_\ell([\![u]\!]_q) \circ^\ell [\![v]\!]_q \\ [\![f^\ell(u_1, \ldots, u_m)]\!]_q = f^\ell([\![u_1]\!]_q, \ldots, [\![u_m]\!]_q) \end{array}$$

for all other operators f^ℓ, $\ell \geq 0$.

Whereas quoted terms are modified by using q_i, $\mathtt{ev}$-terms in a sense decrease the level of their first argument when it is high enough. To restore a balance, we therefore use q_{i+1} instead of q_i in the second case of the definition of q_i on $\mathtt{ev}$-terms. The seemingly tortuous case of $\circ^j$ terms is due to the fact that we wish $u \circ^j w$ to behave as $\mathtt{ev}^{j+1}(Q^j v)w$, which is reasonable because of rule ($\eta\mathtt{ev}^j$). And because of rule ($\eta \Uparrow^\ell$), we must do some similar to $\Uparrow^j$-terms.

To simplify the definition of q_i, we shall make an abuse of notations and write, for every f other than $\mathtt{ev}$, $\circ$ or $\Uparrow$, for every $j \geq 0$:

$$q_i(f^j(u_1, \ldots, u_m)) = \begin{cases} f^{j+1}(q_i(u_1), \ldots, q_i(u_m)) & \text{if } i \leq j \\ f^j(q_i(u_1), \ldots, q_i(u_m)) & \text{if } j < i \end{cases}$$

We shall also write $\overline{v}$ instead of the sequence $v_1, \ldots, v_m$.

Lemma 2.4 *For every $1 \leq \ell < \mathcal{L}$, $q_\ell \circ q_{\mathcal{L}-1} = q_\mathcal{L} \circ q_\ell$.*

Proof: Observe that $\circ$ is just ordinary composition of functions here, not an operator in the language. We prove that $(q_\ell \circ q_{\mathcal{L}-1})(u) = (q_\mathcal{L} \circ q_\ell)(u)$ for every $1 \leq \ell < \mathcal{L}$, by structural induction on u.

If $u = f^j(v_1, \ldots, v_m)$, f other than $\mathtt{ev}$, $\circ$ or $\Uparrow$, then we have three cases:

- $j < \ell$: $(q_\ell \circ q_{\mathcal{L}-1})(u) = q_\ell(f^j(\overline{q_{\mathcal{L}-1}(v)}))$ (because $j < \mathcal{L}-1$) $= f^j(\overline{(q_\ell \circ q_{\mathcal{L}-1})(v)})$ (because $j < \ell$) $= f^j(\overline{(q_\mathcal{L} \circ q_\ell)(v)})$ (by induction hypothesis); and $(q_\mathcal{L} \circ q_\ell)(u) = q_\mathcal{L}(f^j(\overline{q_\ell(v)})) = f^j(\overline{(q_\mathcal{L} \circ q_\ell)(v)})$ (because $j < \mathcal{L}$).
- $\ell \leq j < \mathcal{L}-1$: $(q_\ell \circ q_{\mathcal{L}-1})(u) = q_\ell(f^j(\overline{q_{\mathcal{L}-1}(v)})) = f^{j+1}(\overline{(q_\ell \circ q_{\mathcal{L}-1})(v)}) = f^{j+1}(\overline{(q_\mathcal{L} \circ q_\ell)(v)})$ (by induction hypothesis); and $(q_\mathcal{L} \circ q_\ell)(u) = q_\mathcal{L}(f^{j+1}(\overline{q_\ell(v)})) = f^{j+1}(\overline{q_\ell(v)})$ (because $j+1 < \mathcal{L}$).

- $\mathcal{L}-1 \leq j$: $(q_\ell \circ q_{\mathcal{L}-1})(u) = q_\ell(f^{j+1}(\overline{q_{\mathcal{L}-1}(v)})) = f^{j+2}(\overline{(q_\ell \circ q_{\mathcal{L}-1})(v)})$ (because $\ell \leq j \leq j+1$) $= f^{j+2}(\overline{(q_{\mathcal{L}} \circ q_\ell)(v)})$ (by induction hypothesis); and $(q_{\mathcal{L}} \circ q_\ell)(u) = q_{\mathcal{L}}(f^{j+1}(\overline{q_\ell(v)})) = f^{j+2}(\overline{(q_{\mathcal{L}} \circ q_\ell)(v)})$ (because $\mathcal{L} \leq j+1$).

Similarly when $u = \mathtt{ev}^j vw$:

- $j < \ell+1$: $(q_\ell \circ q_{\mathcal{L}-1})(u) = q_\ell(\mathtt{ev}^j(q_{\mathcal{L}}(v))w)$ (because $j-1 < \mathcal{L}-1$) $= \mathtt{ev}^j((q_{\ell+1} \circ q_{\mathcal{L}})(v))w$ (because $j-1 < \ell$) $= \mathtt{ev}^j((q_{\mathcal{L}+1} \circ q_{\ell+1})(v))w$ (by induction hypothesis); and $(q_{\mathcal{L}} \circ q_\ell)(u) = q_{\mathcal{L}}(\mathtt{ev}^j(q_{\ell+1}(v))w) = \mathtt{ev}^j((q_{\mathcal{L}+1} \circ q_{\ell+1})(v))w$.

- $\ell+1 \leq j < \mathcal{L}$: $(q_\ell \circ q_{\mathcal{L}-1})(u) = q_\ell(\mathtt{ev}^j(q_{\mathcal{L}}(v))w)$ (because $j-1 < \mathcal{L}-1$) $= \mathtt{ev}^{j+1}((q_\ell \circ q_{\mathcal{L}})(v))(q_\ell(w))$ (because $\ell \leq j-1$) $= \mathtt{ev}^{j+1}((q_{\mathcal{L}+1} \circ q_\ell)(v))(q_\ell(w))$ (by induction hypothesis); and $(q_{\mathcal{L}} \circ q_\ell)(u) = q_{\mathcal{L}}(\mathtt{ev}^{j+1}(q_\ell(v))(q_\ell(w))) = \mathtt{ev}^{j+1}((q_{\mathcal{L}+1} \circ q_\ell)(v))(q_\ell(w))$.

- $\mathcal{L} \leq j$: $(q_\ell \circ q_{\mathcal{L}-1})(u) = q_\ell(\mathtt{ev}^{j+1}(q_{\mathcal{L}-1}(v))(q_{\mathcal{L}-1}(w)))$ (because $\mathcal{L}-1 \leq j-1$) $= \mathtt{ev}^{j+2}((q_\ell \circ q_{\mathcal{L}-1})(v))((q_\ell \circ q_{\mathcal{L}-1})(w)))$ (because $\ell \leq \mathcal{L} \leq j$) $= \mathtt{ev}^{j+2}((q_{\mathcal{L}} \circ q_\ell)(v))((q_{\mathcal{L}} \circ q_\ell)(w))$ (by induction hypothesis); and $(q_{\mathcal{L}} \circ q_\ell)(u) = q_{\mathcal{L}}(\mathtt{ev}^{j+1}(q_\ell(v))(q_\ell(w))) = \mathtt{ev}^{j+1}((q_{\mathcal{L}} \circ q_\ell)(v))((q_{\mathcal{L}} \circ q_\ell)(w))$.

When $u = v \circ^j w$:

- $j < \ell$: $(q_\ell \circ q_{\mathcal{L}-1})(u) = q_\ell(q_{\mathcal{L}}(v) \circ^j w)$ (because $j < \mathcal{L}-1$) $= (q_{\ell+1} \circ q_{\mathcal{L}})(v) \circ^j w$ (because $j < \ell$) $= (q_{\mathcal{L}+1} \circ q_{\ell+1})(v) \circ^j w$ (by induction hypothesis, since $\ell+1 < \mathcal{L}$); and $(q_{\mathcal{L}} \circ q_\ell)(u) = q_{\mathcal{L}}(q_{\ell+1}(v) \circ^j w) = (q_{\mathcal{L}+1} \circ q_{\ell+1})(v) \circ^j w$.

- $\ell \leq j < \mathcal{L}-1$: $(q_\ell \circ q_{\mathcal{L}-1})(u) = q_\ell(q_{\mathcal{L}}(v) \circ^j w)$ (since $j < \mathcal{L}-1$) $= (q_\ell \circ q_{\mathcal{L}})(v) \circ^{j+1} q_\ell(w)$ (since $\ell \leq j$) $= (q_{\mathcal{L}+1} \circ q_\ell)(v) \circ^{j+1} q_\ell(w)$ (by induction hypothesis, since $\ell < \mathcal{L}+1$); and $(q_{\mathcal{L}} \circ q_\ell)(u) = q_{\mathcal{L}}(q_\ell(v) \circ^{j+1} q_\ell(w))$ (since $\ell \leq j$) $= (q_{\mathcal{L}+1} \circ q_\ell)(v) \circ^{j+1} q_\ell(w)$ (since $j+1 < \mathcal{L}$).

- $\mathcal{L}-1 \leq j$: $(q_\ell \circ q_{\mathcal{L}-1})(u) = q_\ell(q_{\mathcal{L}-1}(v) \circ^{j+1} q_{\mathcal{L}-1}(w))$ (because $\mathcal{L}-1 \leq j$) $= (q_\ell \circ q_{\mathcal{L}-1})(v) \circ^{j+2} (q_\ell \circ q_{\mathcal{L}-1})(w)$ (because $\ell \leq \mathcal{L} \leq j+1$) $= (q_{\mathcal{L}} \circ q_\ell)(v) \circ^{j+2} (q_{\mathcal{L}} \circ q_\ell)(w)$ (by induction hypothesis); and $(q_{\mathcal{L}} \circ q_\ell)(u) = q_{\mathcal{L}} q_\ell(v) \circ^{j+1} q_\ell(w)$ (since $\ell \leq \mathcal{L}-1 \leq j$) $= (q_{\mathcal{L}} \circ q_\ell)(v) \circ^{j+2} (q_{\mathcal{L}} \circ q_\ell)(w)$.

And similarly when $u = \Uparrow^\ell v$. □

Recall that a context $\mathcal{C}$ is a term with a unique distinguished occurrence called the hole and written $[]$. $\mathcal{C}[u]$ denotes the term obtained by replacing the hole by the term u. Recall that $u \longrightarrow v$ is and only if there is a rule $l \to r$ and a context $\mathcal{C}$ such that $u = \mathcal{C}[l]$ and $v = \mathcal{C}[r]$.

Applying the $[\![\_]\!]_q$ transformation to the rules in groups (D), (E) and (F) yield new rules, shown in Figure 1. Indeed:

Lemma 2.5 *For any rule $u \to v$ in (D), (E) or (F), $[\![u]\!]_q \to [\![v]\!]_q$ is an instance of some rule in (D′), (E′) or (F′) respectively (see Figure 1).*

Proof: By case analysis on the rule.

Rule $(f^{\mathcal{L}} \circ^\ell)$. The $[\![\_]\!]_q$-translation of the left-hand side is $q_\ell(f^{\mathcal{L}}([\![u_1]\!]_q, \ldots, [\![u_m]\!]_q)) \circ^\ell [\![w]\!]_q = f^{\mathcal{L}+1}(q_\ell[\![u_1]\!]_q, \ldots, q_\ell[\![u_m]\!]_q)$ (since $\ell \leq \mathcal{L}$), while the translation of the right-hand side is $f^{\mathcal{L}}(q_\ell[\![u_1]\!]_q \circ^\ell [\![w]\!]_q, \ldots, q_\ell[\![u_m]\!]_q \circ^\ell [\![w]\!]_q)$.

Rule $(\mathtt{ev}^{\mathcal{L}} \circ^\ell)$. The translated left-hand side is $q_\ell(\mathtt{ev}^{\mathcal{L}}([\![u]\!]_q, [\![v]\!]_q)) \circ^\ell [\![w]\!]_q = \mathtt{ev}^{\mathcal{L}+1}(q_\ell[\![u]\!]_q, q_\ell[\![v]\!]_q)$ (since $\ell \leq \mathcal{L}-1$), while the translation of the right-hand side is $\mathtt{ev}^{\mathcal{L}}(q_\ell[\![u]\!]_q \circ^\ell [\![w]\!]_q, q_\ell[\![v]\!]_q \circ^\ell [\![w]\!]_q)$.

Rule $(Q^{\mathcal{L}} \circ^\ell)$. The left-hand side translates to $q_\ell(Q^{\mathcal{L}}(q_{\mathcal{L}}[\![u]\!]_q)) \circ^\ell [\![w]\!]_q = Q^{\mathcal{L}+1}((q_\ell \circ q_{\mathcal{L}})[\![u]\!]_q) \circ^\ell [\![w]\!]_q$ (since $\ell \leq \mathcal{L}$) $= Q^{\mathcal{L}+1}((q_{\mathcal{L}+1} \circ q_\ell)[\![u]\!]_q) \circ^\ell [\![w]\!]_q$ (by Lemma 2.4, since $\ell < \mathcal{L}+1$). The right-hand side translates to $Q^{\mathcal{L}}(q_{\mathcal{L}}(q_\ell[\![u]\!]_q \circ^\ell [\![w]\!]_q)) = Q^{\mathcal{L}}((q_{\mathcal{L}+1} \circ q_\ell)[\![u]\!]_q \circ^\ell [\![w]\!]_q)$.

Rule $(\circ^{\mathcal{L}} \circ^\ell)$. Translating the left-hand side yields $q_\ell(q_{\mathcal{L}}[\![u]\!]_q \circ^{\mathcal{L}} [\![v]\!]_q) \circ^\ell [\![w]\!]_q = ((q_\ell \circ q_{\mathcal{L}})[\![u]\!]_q \circ^{\mathcal{L}+1} q_\ell[\![v]\!]_q) \circ^\ell [\![w]\!]_q$ (since $\ell \leq \mathcal{L}$) $= ((q_{\mathcal{L}+1} \circ q_\ell)[\![u]\!]_q \circ^{\mathcal{L}+1} q_\ell[\![v]\!]_q) \circ^\ell [\![w]\!]_q$ (by Lemma 2.4, since $\ell < \mathcal{L}+1$). And the right-hand side yields $q_{\mathcal{L}}(q_\ell[\![u]\!]_q \circ^\ell [\![w]\!]_q) \circ^{\mathcal{L}} (q_\ell[\![v]\!]_q \circ^\ell [\![w]\!]_q) = ((q_{\mathcal{L}+1} \circ q_\ell)[\![u]\!]_q \circ^\ell [\![w]\!]_q) \circ^{\mathcal{L}} (q_\ell[\![v]\!]_q \circ^\ell [\![w]\!]_q)$ (since $\ell < \mathcal{L}$).

The case of rule $(\Uparrow^{\mathcal{L}} \circ^\ell)$ is similar.

The cases of rules $(\mathtt{ev}^\ell f^{\mathcal{L}})$ and $(\mathtt{ev}^\ell \mathtt{ev}^{\mathcal{L}})$ are trivial.

$$
\left.\begin{array}{ll}
\llbracket f^{\mathcal{L}} \circ^{\ell} \rrbracket_q & f^{\mathcal{L}+1}(u_1, \ldots, u_m) \circ^{\ell} w \to f^{\mathcal{L}}(u_1 \circ^{\ell} w, \ldots, u_m \circ^{\ell} w) \\
\llbracket \mathtt{ev}^{\mathcal{L}} \circ^{\ell} \rrbracket_q & (\mathtt{ev}^{\mathcal{L}+1} uv) \circ^{\ell} w \to \mathtt{ev}^{\mathcal{L}}(u \circ^{\ell} w)(v \circ^{\ell} w) \\
\llbracket Q^{\mathcal{L}} \circ^{\ell} \rrbracket_q & (Q^{\mathcal{L}+1} u) \circ^{\ell} w \to Q^{\mathcal{L}}(u \circ^{\ell} w) \\
\llbracket \circ^{\mathcal{L}} \circ^{\ell} \rrbracket_q & (u \circ^{\mathcal{L}+1} v) \circ^{\ell} w \to (u \circ^{\ell} w) \circ^{\mathcal{L}} (v \circ^{\ell} w) \\
\llbracket \Uparrow^{\mathcal{L}} \circ^{\ell} \rrbracket_q & (\Uparrow^{\mathcal{L}+1} u) \circ^{\ell} w \to \Uparrow^{\mathcal{L}} (u \circ^{\ell} w)
\end{array}\right\} (D')
$$

$$
\left.\begin{array}{ll}
\llbracket \mathtt{ev}^{\ell} f^{\mathcal{L}} \rrbracket_q & \mathtt{ev}^{\ell}(f^{\mathcal{L}}(u_1, \ldots, u_m))w \to f^{\mathcal{L}-1}(\mathtt{ev}^{\ell} u_1 w, \ldots, \mathtt{ev}^{\ell} u_m w) \\
\llbracket \mathtt{ev}^{\ell} \mathtt{ev}^{\mathcal{L}} \rrbracket_q & \mathtt{ev}^{\ell}(\mathtt{ev}^{\mathcal{L}} uv)w \to \mathtt{ev}^{\mathcal{L}-1}(\mathtt{ev}^{\ell} uw)(\mathtt{ev}^{\ell} vw) \\
\llbracket \mathtt{ev}^{\ell} Q^{\mathcal{L}} \rrbracket_q & \mathtt{ev}^{\ell}(Q^{\mathcal{L}} u)w \to Q^{\mathcal{L}-1}(\mathtt{ev}^{\ell} uw) \\
\llbracket \mathtt{ev}^{\ell} \circ^{\mathcal{L}} \rrbracket_q & \mathtt{ev}^{\ell}(u \circ^{\mathcal{L}} v)w \to (\mathtt{ev}^{\ell} uw) \circ^{\mathcal{L}-1} (\mathtt{ev}^{\ell} vw) \\
\llbracket \mathtt{ev}^{\ell} \Uparrow^{\mathcal{L}} \rrbracket_q & \mathtt{ev}^{\ell}(\Uparrow^{\mathcal{L}} u)w \to \Uparrow^{\mathcal{L}-1} (\mathtt{ev}^{\ell} uw)
\end{array}\right\} (E')
$$

$$
\left.\begin{array}{ll}
\llbracket Q^{\ell} f^{\mathcal{L}} \rrbracket_q & Q^{\ell}(f^{\mathcal{L}}(u_1, \ldots, u_m)) \to f^{\mathcal{L}}(Q^{\ell} u_1, \ldots, Q^{\ell} u_m) \\
\llbracket Q^{\ell} \mathtt{ev}^{\mathcal{L}} \rrbracket_q & Q^{\ell}(\mathtt{ev}^{\mathcal{L}+1} uw) \to \mathtt{ev}^{\mathcal{L}+1}(Q^{\ell} u)(Q^{\ell} w) \\
\llbracket Q^{\ell} Q^{\mathcal{L}} \rrbracket_q & Q^{\ell}(Q^{\mathcal{L}} u) \to Q^{\mathcal{L}}(Q^{\ell} u) \\
\llbracket Q^{\ell} \circ^{\mathcal{L}} \rrbracket_q & Q^{\ell}(u \circ^{\mathcal{L}} v) \to (Q^{\ell} u) \circ^{\mathcal{L}} (Q^{\ell} v) \\
\llbracket Q^{\ell} \Uparrow^{\mathcal{L}} \rrbracket_q & Q^{\ell}(\Uparrow^{\mathcal{L}} u) \to \Uparrow^{\mathcal{L}} (Q^{\ell} u)
\end{array}\right\} (F')
$$

Figure 1: Translating rules by $\llbracket \_ \rrbracket_q$ ($1 \leq \ell < \mathcal{L}$, f other than $\mathtt{ev}$, $\circ$, $\Uparrow$, Q)

Rule ($\mathtt{ev}^{\ell} Q^{\mathcal{L}}$). The left-hand side translates to $\mathtt{ev}^{\ell}(Q^{\mathcal{L}}(q_{\mathcal{L}} \llbracket u \rrbracket_q)) \llbracket w \rrbracket_q$, while the right-hand side yields $Q^{\mathcal{L}-1}(q_{\mathcal{L}-1}(\mathtt{ev}^{\ell} \llbracket u \rrbracket_q \llbracket w \rrbracket_q)) = Q^{\mathcal{L}-1}(\mathtt{ev}^{\ell}(q_{\mathcal{L}} \llbracket u \rrbracket_q) \llbracket w \rrbracket_q)$ since $\ell - 1 < \mathcal{L} - 1$.

Rule ($\mathtt{ev}^{\ell} \circ^{\mathcal{L}}$). The translation of the left-hand side is $\mathtt{ev}^{\ell}(q_{\mathcal{L}} \llbracket u \rrbracket_q \circ^{\mathcal{L}} \llbracket v \rrbracket_q) \llbracket w \rrbracket_q$, while that of the right-hand side is $q_{\mathcal{L}-1}(\mathtt{ev}^{\ell} \llbracket u \rrbracket_q \llbracket w \rrbracket_q) \circ^{\mathcal{L}-1} (\mathtt{ev}^{\ell} \llbracket v \rrbracket_q \llbracket w \rrbracket_q) = (\mathtt{ev}^{\ell}(q_{\mathcal{L}} \llbracket u \rrbracket_q) \llbracket w \rrbracket_q) \circ^{\mathcal{L}-1} (\mathtt{ev}^{\ell} \llbracket v \rrbracket_q \llbracket w \rrbracket_q)$ since $\ell - 1 < \mathcal{L} - 1$.

The case of rule ($\mathtt{ev}^{\ell} \Uparrow^{\mathcal{L}}$) is similar.

Rule ($Q^{\ell} f^{\mathcal{L}}$). The left-hand side translates to $\llbracket Q^{\ell}(f^{\mathcal{L}-1}(\overline{u})) \rrbracket_q = Q^{\ell}(q_{\ell}(f^{\mathcal{L}-1}(\overline{\llbracket u \rrbracket_q}))) = Q^{\ell}(f^{\mathcal{L}}(\overline{q_{\ell}(\llbracket u \rrbracket_q)}))$ because $\ell \leq \mathcal{L} - 1$. And the right-hand side yields $\llbracket f^{\mathcal{L}}(Q^{\ell} u_1, \overline{Q^{\ell} u}) \rrbracket_q = f^{\mathcal{L}}(\overline{Q^{\ell}(q_{\ell}(\llbracket u \rrbracket_q))})$.

Rule ($Q^{\ell} \mathtt{ev}^{\mathcal{L}}$). The left-hand side translates to $\llbracket Q^{\ell}(\mathtt{ev}^{\mathcal{L}} uw) \rrbracket_q = Q^{\ell}(q_{\ell}(\mathtt{ev}^{\mathcal{L}} \llbracket u \rrbracket_q \llbracket w \rrbracket_q)) = Q^{\ell}(\mathtt{ev}^{\mathcal{L}+1}(q_{\ell}(\llbracket u \rrbracket_q))(q_{\ell}(\llbracket w \rrbracket_q)))$ (because $\ell \leq \mathcal{L} - 1$). And the right-hand side yields $\llbracket \mathtt{ev}^{\mathcal{L}+1}(Q^{\ell} u)(Q^{\ell} w) \rrbracket_q = \mathtt{ev}^{\mathcal{L}+1}(Q^{\ell}(q_{\ell}(\llbracket u \rrbracket_q)))(Q^{\ell}(q_{\ell}(\llbracket w \rrbracket_q)))$.

Rule ($Q^{\ell} Q^{\mathcal{L}}$). The left-hand side translates to $\llbracket Q^{\ell}(Q^{\mathcal{L}-1} u) \rrbracket_q = Q^{\ell}(q_{\ell}(Q^{\mathcal{L}-1}(q_{\mathcal{L}-1}(\llbracket u \rrbracket_q)))) = Q^{\ell}(Q^{\mathcal{L}}((q_{\ell} \circ q_{\mathcal{L}-1})(\llbracket u \rrbracket_q))) = Q^{\ell}(Q^{\mathcal{L}}((q_{\mathcal{L}} \circ q_{\ell})(\llbracket u \rrbracket_q)))$ by Lemma 2.4. And the right-hand side yields $\llbracket Q^{\mathcal{L}}(Q^{\ell} u) \rrbracket_q = Q^{\mathcal{L}}(q_{\mathcal{L}}(Q^{\ell}(q_{\ell}(\llbracket u \rrbracket_q)))) = Q^{\mathcal{L}}(Q^{\ell}((q_{\mathcal{L}} \circ q_{\ell})(\llbracket u \rrbracket_q)))$.

Rule ($Q^{\ell} \circ^{\mathcal{L}}$). The translation of the left-hand side is $Q^{\ell}(q_{\ell}(q_{\mathcal{L}-1} \llbracket u \rrbracket_q \circ^{\mathcal{L}-1} \llbracket v \rrbracket_q)) = Q^{\ell}((q_{\ell} \circ q_{\mathcal{L}-1}) \llbracket u \rrbracket_q \circ^{\mathcal{L}} \llbracket v \rrbracket_q)$ (since $\ell \leq \mathcal{L} - 1$) $= Q^{\ell}((q_{\mathcal{L}} \circ q_{\ell}) \llbracket u \rrbracket_q \circ^{\mathcal{L}} \llbracket v \rrbracket_q)$ by Lemma 2.4, since $\ell \leq \mathcal{L} - 1$. And the right-hand side translates to $q_{\mathcal{L}}(Q^{\ell}(q_{\ell} \llbracket u \rrbracket_q)) \circ^{\mathcal{L}} (Q^{\ell}(q_{\ell} \llbracket v \rrbracket_q)) = Q^{\ell}((q_{\mathcal{L}} \circ q_{\ell}) \llbracket u \rrbracket_q) \circ^{\mathcal{L}} (Q^{\ell}(q_{\ell} \llbracket v \rrbracket_q))$ (since $\ell < \mathcal{L}$).

The case of rule ($Q^{\ell} \Uparrow^{\mathcal{L}}$) is similar. □

Lemma 2.6 *We let the set of q-functions be the smallest set containing the identity function on $\lambda\mathtt{ev}Q$-terms and stable by composition with any q_i, $i \geq 1$.*

For every context $\mathcal{C}$, there is a context $\llbracket \mathcal{C} \rrbracket_q$ and a q-function $q_{\mathcal{C}}$ such that, for every term t, $\llbracket \mathcal{C}[t] \rrbracket_q = \llbracket \mathcal{C} \rrbracket_q [q_{\mathcal{C}}(t)]$.

Proof: First observe that (*) for any context $\mathcal{C}$, for any q-function q, there is a context $\mathcal{C}'$ and a q-function q' such that $q(\mathcal{C}[t]) = \mathcal{C}'[q'(t)]$ for any term t. Indeed, this is clear if q is the identity; when q is q_i for some i, then this is an easy structural induction on $\mathcal{C}$, using Definition 2.3; and otherwise, this is an easy induction on the length n of a given presentation of q as composition of n q_i functions.

Then we prove the lemma by structural induction on $\mathcal{C}$. If $\mathcal{C} = []$, we take $\llbracket \mathcal{C} \rrbracket_q = []$ and $q_{\mathcal{C}}$ equal to the identity function. If $\mathcal{C} = f^{\ell}(u_1, \ldots, u_{i-1}, \mathcal{C}_i, u_{i+1}, \ldots, u_m)$, where f is any operator but Q, $\circ$ or $\Uparrow$, where the u_j's, $1 \leq j \leq m$, $j \neq i$, are terms and $\mathcal{C}_i$ is a context, then $\llbracket \mathcal{C} \rrbracket_q = f^{\ell}(u_1, \ldots, u_{i-1}, \llbracket \mathcal{C}_i \rrbracket_q, u_{i+1}, \ldots, u_m)$ and $q_{\mathcal{C}} = q_{\mathcal{C}_i}$.

If $\mathcal{C} = Q^{\ell}(\mathcal{C}_1)$, then let $\mathcal{C}'$ be $\llbracket \mathcal{C}_1 \rrbracket_q$, q' be $q_{\mathcal{C}_1}$ (using the induction hypothesis), so that $\llbracket \mathcal{C}[t] \rrbracket_q = Q^{\ell}(q_{\ell}(\llbracket \mathcal{C}_1[t] \rrbracket_q)) = Q^{\ell}(q_{\ell}(\mathcal{C}'[q'(t)]))$. By remark (*), there is a context $\mathcal{C}''$ and a q-function q'' such that $q_{\ell}(\mathcal{C}'[q'(t)]) = \mathcal{C}_2'[q''(t)]$, and we let $\llbracket \mathcal{C} \rrbracket_q = Q^{\ell}(\mathcal{C}_2')$ and $q_{\mathcal{C}} = q''$.

R	$q_i(R)$	when:
$\llbracket f^{\mathcal{L}} \circ^{\ell} \rrbracket_q$	$\llbracket f^{\mathcal{L}+1} \circ^{\ell+1} \rrbracket_q$	$i \leq \ell$
	$\llbracket f^{\mathcal{L}+1} \circ^{\ell} \rrbracket_q$	$\ell < i \leq \mathcal{L}$
	$\llbracket f^{\mathcal{L}} \circ^{\ell} \rrbracket_q$	$\mathcal{L} < i$
$\llbracket \mathtt{ev}^{\mathcal{L}} \circ^{\ell} \rrbracket_q$	$\llbracket \mathtt{ev}^{\mathcal{L}+1} \circ^{\ell+1} \rrbracket_q$	$i \leq \ell$
	$\llbracket \mathtt{ev}^{\mathcal{L}+1} \circ^{\ell} \rrbracket_q$	$\ell < i \leq \mathcal{L}-1$
	$\llbracket \mathtt{ev}^{\mathcal{L}} \circ^{\ell} \rrbracket_q$	$\mathcal{L}-1 < i$
$\llbracket Q^{\mathcal{L}} \circ^{\ell} \rrbracket_q$	$\llbracket Q^{\mathcal{L}+1} \circ^{\ell+1} \rrbracket_q$	$i \leq \ell$
	$\llbracket Q^{\mathcal{L}+1} \circ^{\ell} \rrbracket_q$	$\ell < i \leq \mathcal{L}$
	$\llbracket Q^{\mathcal{L}} \circ^{\ell} \rrbracket_q$	$\mathcal{L} < i$
$\llbracket \circ^{\mathcal{L}} \circ^{\ell} \rrbracket_q$	$\llbracket \circ^{\mathcal{L}+1} \circ^{\ell+1} \rrbracket_q$	$i \leq \ell$
	$\llbracket \circ^{\mathcal{L}+1} \circ^{\ell} \rrbracket_q$	$\ell < i \leq \mathcal{L}$
	$\llbracket \circ^{\mathcal{L}} \circ^{\ell} \rrbracket_q$	$\mathcal{L} < i$
$\llbracket \Uparrow^{\mathcal{L}} \circ^{\ell} \rrbracket_q$	$\llbracket \Uparrow^{\mathcal{L}+1} \circ^{\ell+1} \rrbracket_q$	$i \leq \ell$
	$\llbracket \Uparrow^{\mathcal{L}+1} \circ^{\ell} \rrbracket_q$	$\ell < i \leq \mathcal{L}$
	$\llbracket \Uparrow^{\mathcal{L}} \circ^{\ell} \rrbracket_q$	$\mathcal{L} < i$
$\llbracket \mathtt{ev}^{\ell} f^{\mathcal{L}} \rrbracket_q$	$\llbracket \mathtt{ev}^{\ell+1} f^{\mathcal{L}+1} \rrbracket_q$	$i \leq \ell-1$
	$\llbracket \mathtt{ev}^{\ell} f^{\mathcal{L}+1} \rrbracket_q$	$\ell-1 < i \leq \mathcal{L}-1$
	$\llbracket \mathtt{ev}^{\ell} f^{\mathcal{L}} \rrbracket_q$	$\mathcal{L}-1 < i$
$\llbracket \mathtt{ev}^{\ell} \mathtt{ev}^{\mathcal{L}} \rrbracket_q$	$\llbracket \mathtt{ev}^{\ell+1} \mathtt{ev}^{\mathcal{L}+1} \rrbracket_q$	$i \leq \ell-1$
	$\llbracket \mathtt{ev}^{\ell} \mathtt{ev}^{\mathcal{L}+1} \rrbracket_q$	$\ell-1 < i \leq \mathcal{L}-2$
	$\llbracket \mathtt{ev}^{\ell} \mathtt{ev}^{\mathcal{L}} \rrbracket_q$	$\mathcal{L}-2 < i$
$\llbracket \mathtt{ev}^{\ell} Q^{\mathcal{L}} \rrbracket_q$	$\llbracket \mathtt{ev}^{\ell+1} Q^{\mathcal{L}+1} \rrbracket_q$	$i \leq \ell-1$
	$\llbracket \mathtt{ev}^{\ell} Q^{\mathcal{L}+1} \rrbracket_q$	$\ell-1 < i \leq \mathcal{L}-1$
	$\llbracket \mathtt{ev}^{\ell} Q^{\mathcal{L}} \rrbracket_q$	$\mathcal{L}-1 < i$

R	$q_i(R)$	when:
$\llbracket \mathtt{ev}^{\ell} \circ^{\mathcal{L}} \rrbracket_q$	$\llbracket \mathtt{ev}^{\ell+1} \circ^{\mathcal{L}+1} \rrbracket_q$	$i \leq \ell-1$
	$\llbracket \mathtt{ev}^{\ell} \circ^{\mathcal{L}+1} \rrbracket_q$	$\ell-1 < i \leq \mathcal{L}-1$
	$\llbracket \mathtt{ev}^{\ell} \circ^{\mathcal{L}} \rrbracket_q$	$\mathcal{L}-1 < i$
$\llbracket \mathtt{ev}^{\ell} \Uparrow^{\mathcal{L}} \rrbracket_q$	$\llbracket \mathtt{ev}^{\ell+1} \Uparrow^{\mathcal{L}+1} \rrbracket_q$	$i \leq \ell-1$
	$\llbracket \mathtt{ev}^{\ell} \Uparrow^{\mathcal{L}+1} \rrbracket_q$	$\ell-1 < i \leq \mathcal{L}-1$
	$\llbracket \mathtt{ev}^{\ell} \Uparrow^{\mathcal{L}} \rrbracket_q$	$\mathcal{L}-1 < i$
$\llbracket Q^{\ell} f^{\mathcal{L}} \rrbracket_q$	$\llbracket Q^{\ell+1} f^{\mathcal{L}+1} \rrbracket_q$	$i \leq \ell$
	$\llbracket Q^{\ell} f^{\mathcal{L}+1} \rrbracket_q$	$\ell < i \leq \mathcal{L}$
	$\llbracket Q^{\ell} f^{\mathcal{L}} \rrbracket_q$	$\mathcal{L} < i$
$\llbracket Q^{\ell} \mathtt{ev}^{\mathcal{L}} \rrbracket_q$	$\llbracket Q^{\ell+1} \mathtt{ev}^{\mathcal{L}+1} \rrbracket_q$	$i \leq \ell$
	$\llbracket Q^{\ell} \mathtt{ev}^{\mathcal{L}+1} \rrbracket_q$	$\ell < i \leq \mathcal{L}$
	$\llbracket Q^{\ell} \mathtt{ev}^{\mathcal{L}} \rrbracket_q$	$\mathcal{L} < i$
$\llbracket Q^{\ell} Q^{\mathcal{L}} \rrbracket_q$	$\llbracket Q^{\ell+1} Q^{\mathcal{L}+1} \rrbracket_q$	$i \leq \ell$
	$\llbracket Q^{\ell} Q^{\mathcal{L}+1} \rrbracket_q$	$\ell < i \leq \mathcal{L}$
	$\llbracket Q^{\ell} Q^{\mathcal{L}} \rrbracket_q$	$\mathcal{L} < i$
$\llbracket Q^{\ell} \circ^{\mathcal{L}} \rrbracket_q$	$\llbracket Q^{\ell+1} \circ^{\mathcal{L}+1} \rrbracket_q$	$i \leq \ell$
	$\llbracket Q^{\ell} \circ^{\mathcal{L}+1} \rrbracket_q$	$\ell < i \leq \mathcal{L}$
	$\llbracket Q^{\ell} \circ^{\mathcal{L}} \rrbracket_q$	$\mathcal{L} < i$
$\llbracket Q^{\ell} \Uparrow^{\mathcal{L}} \rrbracket_q$	$\llbracket Q^{\ell+1} \Uparrow^{\mathcal{L}+1} \rrbracket_q$	$i \leq \ell$
	$\llbracket Q^{\ell} \Uparrow^{\mathcal{L}+1} \rrbracket_q$	$\ell < i \leq \mathcal{L}$
	$\llbracket Q^{\ell} \Uparrow^{\mathcal{L}} \rrbracket_q$	$\mathcal{L} < i$

Figure 2: Applying q_i to rules in (D'), (E') and (F')

If $\mathcal{C} = \mathcal{C}_1 \circ^{\ell} u$ for some u, then let $\mathcal{C}'$ be $\llbracket \mathcal{C}_1 \rrbracket_q$, q' be $q_{\mathcal{C}'}$ (using the induction hypothesis), so $\llbracket \mathcal{C}[t] \rrbracket_q = q_\ell(\llbracket \mathcal{C}_1[t] \rrbracket_q) \circ^{\ell} \llbracket u \rrbracket_q = q_\ell(\mathcal{C}'[q'(t)]) \circ^{\ell} \llbracket u \rrbracket_q$. By remark (*), there is a context $\mathcal{C}''$ and a q-function q'' such that $q_\ell(\mathcal{C}'[q'(t)]) = \mathcal{C}''[q''(t)]$, and we let $\llbracket \mathcal{C} \rrbracket_q = \mathcal{C}'' \circ^{\ell} u$ and $q_{\mathcal{C}} = q''$.

And if $\mathcal{C} = u \circ^{\ell} \mathcal{C}_1$, then we let $\llbracket \mathcal{C} \rrbracket_q = q_\ell(u) \circ^{\ell} \llbracket \mathcal{C}_1 \rrbracket_q$ and $q_{\mathcal{C}} = q_{\mathcal{C}_1}$.

Finally, if $\mathcal{C} = \Uparrow^{\ell} \mathcal{C}_1$, then let $\mathcal{C}'$ be $\llbracket \mathcal{C}_1 \rrbracket_q$, q' be $q_{\mathcal{C}'}$ (using the induction hypothesis), so $\llbracket \mathcal{C}[t] \rrbracket_q = \Uparrow^{\ell} (q_\ell(\llbracket \mathcal{C}_1[t] \rrbracket_q)) = \Uparrow^{\ell} (q_\ell(\mathcal{C}'[q'(t)]))$. By remark (*), there is a context $\mathcal{C}''$ and a q-function q'' such that $q_\ell(\mathcal{C}'[q'(t)]) = \mathcal{C}''[q''(t)]$, and we let $\llbracket \mathcal{C} \rrbracket_q = \Uparrow^{\ell} \mathcal{C}''$ and $q_{\mathcal{C}} = q''$. □

Lemma 2.7 *For any rule $u \to v$ in (D'), (E') or (F'), for any context $\mathcal{C}$, $\llbracket \mathcal{C}[u] \rrbracket_q$ rewrites in one step to $\llbracket \mathcal{C}[v] \rrbracket_q$ by the rules of group (D'), (E') or (F') respectively (see Figure 1).*

Proof: First, if $u \to v$ is any rule R in Figure 1, then $q_i(u) \to q_i(v)$ is also an instance of some rule in Figure 1, which we shall call $q_i(R)$. Indeed, check the table in Figure 2, where f is any operator except $\mathtt{ev}$, Q or $\circ$.

It follows that for any q-function f, for any rule $u \to v$ in Figure 1, $f(u) \to f(v)$ is also an instance of some rule in Figure 1: this is by induction on the number of q_i's we compose to get f.

Now, let $\mathcal{C}$ be a context. By Lemma 2.6, $\llbracket \mathcal{C}[u] \rrbracket_q = \llbracket \mathcal{C} \rrbracket_q[q_{\mathcal{C}}(u)]$ and $\llbracket \mathcal{C}[v] \rrbracket_q = \llbracket \mathcal{C} \rrbracket_q[q_{\mathcal{C}}(v)]$. By the above, $q_{\mathcal{C}}(u) \to q_{\mathcal{C}}(v)$ is an instance of some rule in Figure 1. Therefore $\llbracket \mathcal{C}[u] \rrbracket_q$ rewrites to $\llbracket \mathcal{C}[v] \rrbracket_q$ in one step. □

Lemma 2.8 *The set of function symbols occurring in any derivation in the system of Figure 1 is finite.*

Proof: For every term u, let $F(u)$ be the set of function symbols f^i, where f^j is any function symbol occurring in u, and $i \leq j$. For any u, $F(u)$ is clearly finite. Check that, if u rewrites in one step to v, then

$F(v) \subseteq F(u)$. In any derivation $u_0 \longrightarrow u_1 \longrightarrow \ldots \longrightarrow u_k \longrightarrow \ldots$, the set of function symbols that occur is therefore $\bigcap_{i \geq 0} F(u_i) = F(u_0)$, which is finite. □

It follows:

Lemma 2.9 *Let $\succ$ be the precedence defined by $f^i \succ g^j$ if and only if $i < j$, and let $>_q$ denote $\succ_{rpo}$.*

For any rule $u \to v$ in group (F'), $u >_q v$. In particular, group (F) terminates.

Proof: Consider the $[\![Q^\ell f^{\mathcal{L}}]\!]_q$ rule. For every i, $1 \leq i \leq m$, $f^{\mathcal{L}}(u_1, \ldots, u_m) \succ_{rpo} u_i$ by clause 1 of the definition of $\succ_{rpo}$. By clause 3, it follows that $Q^\ell(f^{\mathcal{L}}(u_1, \ldots, u_m)) \succ_{rpo} Q^\ell u_i$. Since $Q^\ell \succ f^{\mathcal{L}}$, it follows by clause 2 of the definition of $\succ_{rpo}$ that $Q^\ell(f^{\mathcal{L}}(u_1, \ldots, u_m)) \succ_{rpo} f^{\mathcal{L}}(Q^\ell u_1, \ldots, Q u_m)$. The three other rules are treated similarly.

By Lemma 2.8, then, we can restrict ourselves to some fixed finite set of function symbols in any derivation in group (F'), on which $\succ$ is well-founded. It follows from the above that this system terminates. That group (F) terminates then follows from Lemma 2.7 (in fact, this translation preserves the lengths of derivations). □

2.3 Behaviour of $\mathtt{ev}^\ell$-Terms

In Section 2.2, we have not considered the rules in groups (D) and (E). At first glance, it seems that we could have used a similar trick to handle the decreasing of indices incurred by $\mathtt{ev}^\ell$ going down terms. This would require us to define $[\![\mathtt{ev}^\ell uv]\!]_q = \mathtt{ev}^\ell(e_\ell([\![u]\!]_q))[\![v]\!]_q$, with in particular $e_i(f^j(\overline{u})) = f^{j-1}(\overline{e_i(u)})$ if $i < j$ and $e_i(f^j(\overline{u})) = f^j(\overline{e_i(u)})$ if $i \geq j$, for any f but $\circ$, Q or $\mathtt{ev}$, and similar rules when f is $\circ$, Q or $\mathtt{ev}$. But then we would be forced to include rules such as $\mathtt{ev}^\ell(\mathtt{ev}^\ell uv)w \to \mathtt{ev}^\ell(\mathtt{ev}^\ell uw)(\mathtt{ev}^\ell vw)$ in the translated system of Figure 1, which is not $\succ_{rpo}$-decreasing. The problem lies in the fact that e_i confuses indices: $e_i(f^{i+1}(\overline{u})) = e_i(f^i(\overline{u})) = f^i(\overline{e_i(u)})$. An entirely different solution is called for.

Definition 2.4 *An infinite sequence s over some alphabet A is any total function from $\mathbb{N}$ to A.*

We write s_i the letter at position i in s, which is $s(i)$ by definition.

We denote by $s_{i..j}$ the finite sequence of all letters s_i, s_{i+1}, …, s_j; if $i > j$, we take by convention $s_{i..j}$ to be the empty word ϵ. We denote by $s_{i..\infty}$ the infinite sequence of all letters s_i, s_{i+1}, …

For any letter x, let x^ω be the infinite sequence consisting only of x. If w is a finite sequence and w' is an finite or infinite sequence, let $w \,.\, w'$ be the concatenation of w and w'. Concatenation is associative and has ϵ as unit element.

Definition 2.5 *Let Γ be the set of all infinite sequences γ of non-negative integers containing only finitely many non-zero integers. Every such sequence can be written as the concatenation of some finite sequence $\gamma_{0..k}$ and of 0^ω.*

For every $\ell \geq 0$, let C_ℓ (compose), E_ℓ (eval), K_ℓ (kwote) be functions from $\mathbb{N}$ to $\mathbb{N}$. Let also P_ℓ (pair) be functions from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$, L_ℓ (lambda) be functions from $\mathbb{N}$ to $\mathbb{N}$, F_ℓ (first), S_ℓ (second), U_ℓ (up), I_ℓ (identity) be functions from $\mathbb{N}$ to $\mathbb{N}$. Let finally δ be some fixed element of Γ.

We define the function $[\![\_]\!]_e\_$ from $\lambda\mathtt{ev}Q$-terms and elements of Γ to non-negative integers as follows. To save a few parentheses, we write $[\![u]\!]_e s \,.\, \gamma$ instead of $[\![u]\!]_e(s \,.\, \gamma)$; $C_\ell[\![u]\!]_e\gamma$ instead of $C_\ell([\![u]\!]_e\gamma)$, and similarly with E_ℓ and Q_ℓ (in this the γ part is assumed to extend as far right as possible); and parentheses are used to promote an integer n to a sequence (n) containing exactly the integer n.

$$
\begin{aligned}
[\![\mathtt{ev}^\ell uv]\!]_e\gamma &= [\![u]\!]_e\gamma_{0..\ell-1} \,.\, (E_\ell[\![v]\!]_e\gamma_{0..\ell-1} \,.\, \delta) \,.\, \gamma_{\ell..\infty} \\
[\![Q^\ell u]\!]_e\gamma &= [\![u]\!]_e\gamma_{0..\ell-1} \,.\, (K_\ell\gamma_\ell) \,.\, \gamma_{\ell+1..\infty} \\
[\![u \circ^\ell v]\!]_e\gamma &= [\![u]\!]_e\gamma_{0..\ell} \,.\, (C_\ell[\![u]\!]_e\gamma_{0..\ell} \,.\, \delta) \,.\, \gamma_{\ell+1..\infty} \\
[\![id^\ell]\!]_e\gamma &= \textstyle\sum_{i\geq 0, i\neq\ell}\gamma_i + I_\ell(\gamma_\ell) \\
[\![1^\ell]\!]_e\gamma &= \textstyle\sum_{i\geq 0, i\neq\ell}\gamma_i + F_\ell(\gamma_\ell) \\
[\![\uparrow^\ell]\!]_e\gamma &= \textstyle\sum_{i\geq 0, i\neq\ell}\gamma_i + S_\ell(\gamma_\ell) \\
[\![u \bullet^\ell v]\!]_e\gamma &= P_\ell([\![u]\!]_e\gamma, [\![v]\!]_e\gamma) \\
[\![u \star^\ell v]\!]_e\gamma &= [\![u \bullet^\ell v]\!]_e\gamma \\
[\![\lambda^\ell u]\!]_e &= L_\ell([\![u]\!]_e\gamma) \\
[\![\Uparrow^\ell u]\!]_e\gamma &= U_\ell([\![1^\ell \bullet^\ell (u \circ^\ell \uparrow^\ell)]\!]_e\gamma)
\end{aligned}
$$

Finally, we define $[\![u]\!]_{eq}\gamma$ as being $[\![[\![u]\!]_q]\!]_e\gamma$.

Say that a function f from $\mathbb{N}$ to $\mathbb{N}$ is *superlinear* if and only if $f(n) > n$ for every integer n. Finally, a function f from $A \times B$ to $\mathbb{N}$ is superlinear if and only if it is superlinear in each of its arguments separately.

Define the ordering $\geq$ on sequences pointwise, i.e. $\gamma \geq \gamma'$ if and only if $\gamma_i \geq \gamma'_i$ for every $i \geq 0$. Let $\gamma > \gamma'$ denote $\gamma \geq \gamma'$ and $\gamma \neq \gamma'$. Similarly, define $(a, b) \geq (a', b')$ by $a \geq a'$ and $b \geq b'$, and $(a, b) > (a', b')$ if and only if $(a, b) \geq (a', b')$ and $(a, b) \neq (a', b')$.

We say that a function f is *monotonic* if and only if $a > b$ implies $f(a) > f(b)$, where $>$ is defined on naturals, sequences, or couples appriopriately.

We extend the ordering $>$ to functions pointwise, i.e. $f > g$ if and only if $f(a) > g(a)$ for every a in the common domain of f and g. Then, any family $(f_\ell)_{\ell \geq 0}$ of functions is said to be *increasing* if and only if, for all $0 \leq i < j$, $f_i < f_j$.

We shall assume the following properties in the sequel:

(P1) For every $\ell \geq 0$, K_ℓ, I_ℓ, F_ℓ, S_ℓ, P_ℓ, L_ℓ, U_ℓ are superlinear.

(P2) For every $\ell \geq 0$, E_ℓ, K_ℓ, C_ℓ, I_ℓ, F_ℓ, S_ℓ, P_ℓ, L_ℓ, U_ℓ are monotonic.

(P3) $(E_\ell)_{\ell \geq 0}$, $(K_\ell)_{\ell \geq 0}$, $(C_\ell)_{\ell \geq 0}$, $(I_\ell)_{\ell \geq 0}$, $(F_\ell)_{\ell \geq 0}$, $(S_\ell)_{\ell \geq 0}$, $(P_\ell)_{\ell \geq 0}$, $(L_\ell)_{\ell \geq 0}$, $(U_\ell)_{\ell \geq 0}$ are increasing families of functions.

These properties are easy to verify. Take for instance $E_\ell(x) = K_\ell(x) = C_\ell(x) = I_\ell(x) = F_\ell(x) = S_\ell(x) = L_\ell(x) = U_\ell(x) = x + \ell + 1$, $P_\ell(x, y) = x + y + \ell + 1$.

Lemma 2.10 *For every term u, for every $i \in \mathbb{N}$, for every γ in , , $[\![u]\!]_e\gamma > \gamma_i$.*

Proof: By structural induction on u, using only property (P1) (superlinearity).

If $u = \mathtt{ev}^\ell vw$, then $[\![u]\!]_e\gamma = [\![v]\!]_e\gamma_{0..\ell-1} \,.\, (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \,.\, \delta) \,.\, \gamma_{\ell..\infty}$. For every $i < \ell$, the claim follows by the induction hypothesis, applied to v and index i. For every $i \geq \ell$, it follows by the induction hypothesis applied to v and index $i + 1$.

The argument is similar if $u = v \circ^\ell w$.

If $u = Q^\ell v$, then $[\![u]\!]_e\gamma = [\![v]\!]_e\gamma_{0..\ell-1} \,.\, (K_\ell\gamma_\ell) \,.\, \gamma_{\ell+1..\infty}$. For every $i \neq \ell$, the claim follows by induction hypothesis applied to v and index i. When $i = \ell$, $[\![u]\!]_e\gamma > K_\ell(\gamma_\ell)$ (by induction hypothesis) $> \gamma_\ell$ (by superlinearity of K_ℓ).

If $u = id^\ell$, then $[\![u]\!]_e\gamma = \sum_{j \geq 0, j \neq \ell} \gamma_j + I_\ell(\gamma_\ell)$. For every $i \neq \ell$, notice that since I_ℓ is superlinear, $I_\ell(\gamma_\ell) > \gamma_\ell \geq 0$, so $[\![u]\!]_e\gamma > \sum_{j \geq 0, j \neq \ell} \gamma_j \geq \gamma_i$. And when $i = \ell$, then $[\![u]\!]_e\gamma \geq I_\ell(\gamma_\ell) > \gamma_\ell$ by superlinearity of I_ℓ.

The argument is similar when $u = 1^\ell$ or $u = \uparrow^\ell$.

If $u = v \bullet^\ell w$ or $u = v \star^\ell w$, then $[\![u]\!]_e\gamma = P_\ell([\![v]\!]_e\gamma, [\![w]\!]_e\gamma)$. Since P_ℓ is superlinear in its first argument, say, then $[\![u]\!]_e\gamma > [\![v]\!]_e\gamma > \gamma_i$ by induction hypothesis.

The argument is similar if $u = \lambda^\ell v$.

If $u = \Uparrow^\ell v$, then $[\![u]\!]_e\gamma = U_\ell([\![1^\ell \bullet^\ell v \circ^\ell \uparrow^\ell]\!]_e\gamma) > [\![1^\ell \bullet^\ell v \circ^\ell \uparrow^\ell]\!]_e\gamma$ (by superlinearity of U_ℓ) $> \gamma_i$ (by induction hypothesis). $\square$

Lemma 2.11 *For every term u, for every γ in , , for every i in $\mathbb{N}$, the function $k \mapsto [\![u]\!]_e\gamma_{0..i-1} \,.\, (k) \,.\, \gamma_{i+1..\infty}$ is monotonic.*

Proof: By structural induction on u, using property (P2). This is clear if u is id^ℓ, 1^ℓ or $\uparrow^\ell$. In the sequel, we assume $m > n$.

If $u = \mathtt{ev}^\ell vw$, then there are two cases. If $i \geq \ell$, then:

$$
\begin{aligned}
&[\![u]\!]_e\gamma_{0..i-1} \,.\, (m) \,.\, \gamma_{i+1..\infty} \\
&= [\![v]\!]_e\gamma_{0..\ell-1} \,.\, (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \,.\, \delta) \,.\, \gamma_{\ell..i-1} \,.\, (m) \,.\, \gamma_{i+1..\infty} \\
&> [\![v]\!]_e\gamma_{0..\ell-1} \,.\, (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \,.\, \delta) \,.\, \gamma_{\ell..i-1} \,.\, (n) \,.\, \gamma_{i+1..\infty} \qquad \text{by induction hypothesis on } v \\
&= [\![u]\!]_e\gamma_{0..i-1} \,.\, (n) \,.\, \gamma_{i+1..\infty}
\end{aligned}
$$

And if $i < \ell$, then:

$$\begin{array}{ll}
[\![u]\!]_e\gamma_{0..i-1} \cdot (m) \cdot \gamma_{i+1..\infty} & \\
= [\![v]\!]_e\gamma_{0..i-1} \cdot (m) \cdot \gamma_{i+1..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..i-1} \cdot (m) \cdot \gamma_{i+1..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\infty} & \\
> [\![v]\!]_e\gamma_{0..i-1} \cdot (n) \cdot \gamma_{i+1..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..i-1} \cdot (m) \cdot \gamma_{i+1..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\infty} & \text{by induction hypothesis on } v \\
> [\![v]\!]_e\gamma_{0..i-1} \cdot (n) \cdot \gamma_{i+1..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..i-1} \cdot (n) \cdot \gamma_{i+1..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\infty} & \text{by induction hypothesis on } w \\
& \text{monotonicity of } E_\ell \text{ and} \\
& \text{induction hypothesis on } v \\
= [\![u]\!]_e\gamma_{0..i-1} \cdot (n) \cdot \gamma_{i+1..\infty} &
\end{array}$$

If $u = v \circ^\ell w$, the argument is similar, except that the two cases are now $i \geq \ell + 1$ and $i < \ell + 1$, and we use the fact that C_ℓ is monotonic.

If $u = Q^\ell v$, then we have three cases. If $i \geq \ell + 1$ or $i \leq \ell - 1$, then the claim follows directly by induction hypothesis on v. If $i = \ell$, then by monotonicity of K_ℓ, we have $K_\ell(m) > K_\ell(n)$ and the result follows by induction hypothesis.

If $u = v \bullet^\ell w$ or $u = v \star^\ell w$, then we have two cases. If $i \geq \ell + 1$, then:

$$\begin{array}{ll}
[\![u]\!]_e\gamma_{0..i-1} \cdot (m) \cdot \gamma_{i+1..\infty} & \\
= P_\ell([\![v]\!]_e\gamma_{0..i-1} \cdot (m) \cdot \gamma_{i+1..\infty}, [\![w]\!]_e\gamma_{0..i-1} \cdot (m) \cdot \gamma_{i+1..\infty}) & \\
> P_\ell([\![v]\!]_e\gamma_{0..i-1} \cdot (m) \cdot \gamma_{i+1..\infty}, [\![w]\!]_e\gamma_{0..i-1} \cdot (m) \cdot \gamma_{i+1..\infty}) & \text{by induction hypothesis} \\
& \text{and monotonicity of } P_\ell \\
= [\![u]\!]_e\gamma_{0..i-1} \cdot (n) \cdot \gamma_{i+1..\infty} &
\end{array}$$

If $u = \lambda^\ell v$, then the argument is similar, using the monotonicity of L_ℓ.

Finally, if $u = \Uparrow^\ell$, then this follows directly from the induction hypothesis and the monotonicity of U_ℓ. □

Lemma 2.12 *For any γ in , , for any rule $l \to r$ in Figure 1, $[\![l]\!]_e\gamma \geq [\![r]\!]_e\gamma$. Moreover, the inequality is strict except for rules in group (F').*

Proof: By case analysis. We start by examining the rules in group (F'):

- $[\![Q^\ell \mathtt{ev}^\mathcal{L}]\!]_q$:

$$\begin{array}{l}
[\![l]\!]_e\gamma = [\![Q^\ell(\mathtt{ev}^{\mathcal{L}+1} uv)]\!]_e\gamma \\
= [\![\mathtt{ev}^{\mathcal{L}+1} uv]\!]_e\gamma_{0..\ell-1} \cdot (K_\ell\gamma_\ell) \cdot \gamma_{\ell+1..\infty} \\
= [\![u]\!]_e\gamma_{0..\ell-1} \cdot (K_\ell\gamma_\ell) \cdot \gamma_{\ell+1..\mathcal{L}} \cdot (E_{\mathcal{L}+1}[\![v]\!]_e\gamma_{0..\ell-1} \cdot (K_\ell\gamma_\ell) \cdot \gamma_{\ell+1..\mathcal{L}} \cdot \delta) \cdot \gamma_{\mathcal{L}+1..\infty} \\
= [\![u]\!]_e\gamma_{0..\ell-1} \cdot (K_\ell\gamma_\ell) \cdot \gamma_{\ell+1..\mathcal{L}} \cdot (E_{\mathcal{L}+1}[\![Q^\ell v]\!]_e\gamma_{0..\mathcal{L}} \cdot \delta) \cdot \gamma_{\mathcal{L}+1..\infty} \\
= [\![Q^\ell u]\!]_e\gamma_{0....\mathcal{L}} \cdot (E_{\mathcal{L}+1}[\![Q^\ell v]\!]_e\gamma_{0..\mathcal{L}} \cdot \delta) \cdot \gamma_{\mathcal{L}+1..\infty} \\
= [\![\mathtt{ev}^{\mathcal{L}+1}(Q^\ell u)(Q^\ell v)]\!]_e\gamma = [\![r]\!]_e\gamma
\end{array}$$

- $[\![Q^\ell Q^\mathcal{L}]\!]_q$:

$$\begin{array}{l}
[\![l]\!]_e\gamma = [\![Q^\ell(Q^\mathcal{L} u)]\!]_e\gamma \\
= [\![Q^\mathcal{L} u]\!]_e\gamma_{0..\ell-1} \cdot (K_\ell\gamma_\ell) \cdot \gamma_{\ell+1..\infty} \\
= [\![u]\!]_e\gamma_{0..\ell-1} \cdot (K_\ell\gamma_\ell) \cdot \gamma_{\ell+1..\mathcal{L}-1} \cdot (K_\mathcal{L}\gamma_\mathcal{L}) \cdot \gamma_{\mathcal{L}+1..\infty} \\
= [\![Q^\ell u]\!]_e\gamma_{0..\mathcal{L}-1} \cdot (K_\mathcal{L}\gamma_\mathcal{L}) \cdot \gamma_{\mathcal{L}+1..\infty} \\
= [\![Q^\mathcal{L}(Q^\ell u)]\!]_e\gamma = [\![r]\!]_e\gamma
\end{array}$$

- $[\![Q^\ell \circ^\mathcal{L}]\!]_q$ is similar to the previous two (and follows from the intuition that we treat $u \circ^\ell v$ in the same way as $\mathtt{ev}^{\ell+1}(Q^\ell u)v$, changing $K_\ell E_\ell$ into C_ℓ).

- $[\![Q^\ell id^\mathcal{L}]\!]_q$:

$$\begin{array}{l}
[\![l]\!]_e\gamma = [\![Q^\ell id^\mathcal{L}]\!]_e\gamma = [\![id^\mathcal{L}]\!]_e\gamma_{0..\ell-1} \cdot (K_\ell\gamma_\ell) \cdot \gamma_{\ell+1..\infty} \\
= \sum_{i \neq \ell, i \neq \mathcal{L}} \gamma_i + K_\ell(\gamma_\ell) + I_\mathcal{L}(\gamma_\mathcal{L}) \\
> \sum_{i \neq \ell, i \neq \mathcal{L}} \gamma_i + \gamma_\ell + I_\mathcal{L}(\gamma_\mathcal{L}) \\
\quad \text{since } K_\ell \text{ is superlinear} \\
= [\![id^\mathcal{L}]\!]_e\gamma = [\![r]\!]_e\gamma
\end{array}$$

and similarly for $[\![Q^\ell 1^\mathcal{L}]\!]_q$ and $[\![Q^\ell \uparrow^\mathcal{L}]\!]_q$.

- $[\![Q^\ell \bullet^\mathcal{L}]\!]_q$:

$$\begin{aligned}
&[\![l]\!]_e\gamma = [\![Q^\ell(u \bullet^\mathcal{L} v)]\!]_e\gamma \\
&= [\![u \bullet^\mathcal{L} v]\!]_e\gamma_{0..\ell-1} \cdot (K_\ell\gamma_\ell) \cdot \gamma_{\ell+1..\infty} \\
&= P_\mathcal{L}([\![u]\!]_e\gamma_{0..\ell-1} \cdot (K_\ell\gamma_\ell) \cdot \gamma_{\ell+1..\infty}, [\![v]\!]_e\gamma_{0..\ell-1} \cdot (K_\ell\gamma_\ell) \cdot \gamma_{\ell+1..\infty}) \\
&= P_\mathcal{L}([\![Q^\ell u]\!]_e\gamma, [\![Q^\ell v]\!]_e\gamma) \\
&= [\![(Q^\ell u) \bullet^\mathcal{L} (Q^\ell v)]\!]_e\gamma = [\![r]\!]_e\gamma
\end{aligned}$$

and similarly for $[\![Q^\ell \star^\mathcal{L}]\!]_q$ and $[\![Q^\ell \lambda^\mathcal{L}]\!]_q$.

- $[\![Q^\ell \Uparrow^\mathcal{L}]\!]_q$ follows from the previous cases.

Then, group (E'). Be aware that the interpretations of $\mathtt{ev}^\ell$ and $\circ^\ell$ shift indices of the associated sequences.

- $[\![\mathtt{ev}^\ell\mathtt{ev}^\mathcal{L}]\!]_q$:

$$\begin{aligned}
&[\![l]\!]_e\gamma = [\![\mathtt{ev}^\ell(\mathtt{ev}^\mathcal{L} uv)w]\!]_e\gamma \\
&= [\![\mathtt{ev}^\mathcal{L} uv]\!]_e\gamma_{0..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\infty} \\
&= [\![u]\!]_e\gamma_{0..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\mathcal{L}-2} \\
&\quad \cdot (E_\mathcal{L}[\![v]\!]_e\gamma_{0..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\mathcal{L}-2} \cdot \delta) \cdot \gamma_{\mathcal{L}-1..\infty} \\
&> [\![u]\!]_e\gamma_{0..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\mathcal{L}-2} \\
&\quad \cdot (E_{\mathcal{L}-1}[\![v]\!]_e\gamma_{0..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\mathcal{L}-2} \cdot \delta) \cdot \gamma_{\mathcal{L}-1..\infty} \\
&\quad \text{since } (E_i)_{i\geq 0} \text{ is increasing} \\
&= [\![u]\!]_e\gamma_{0..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\mathcal{L}-2} \cdot (E_{\mathcal{L}-1}[\![\mathtt{ev}^\ell vw]\!]_e\gamma_{0..\mathcal{L}-2} \cdot \delta) \cdot \gamma_{\mathcal{L}-1..\infty} \\
&= [\![\mathtt{ev}^\ell uw]\!]_e\gamma_{0..\mathcal{L}-2} \cdot (E_{\mathcal{L}-1}[\![\mathtt{ev}^\ell vw]\!]_e\gamma_{0..\mathcal{L}-2} \cdot \delta) \cdot \gamma_{\mathcal{L}-1..\infty} \\
&= [\![\mathtt{ev}^{\mathcal{L}-1}(\mathtt{ev}^\ell uw)(\mathtt{ev}^\ell vw)]\!]_e\gamma = [\![r]\!]_e\gamma
\end{aligned}$$

- $[\![\mathtt{ev}^\ell Q^\mathcal{L}]\!]_q$:

$$\begin{aligned}
&[\![l]\!]_e\gamma = [\![\mathtt{ev}^\ell(Q^\mathcal{L} u)w]\!]_e\gamma \\
&= [\![Q^\mathcal{L} u]\!]_e\gamma_{0..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\infty} \\
&= [\![u]\!]_e\gamma_{0..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\mathcal{L}-2} \cdot (K_\mathcal{L}\gamma_{\mathcal{L}-1}) \cdot \gamma_{\mathcal{L}..\infty} \\
&> [\![u]\!]_e\gamma_{0..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\mathcal{L}-2} \cdot (K_{\mathcal{L}-1}\gamma_{\mathcal{L}-1}) \cdot \gamma_{\mathcal{L}..\infty} \\
&\quad \text{since } (K_i)_{i\geq 0} \text{ is increasing} \\
&= [\![\mathtt{ev}^\ell uw]\!]_e\gamma_{0..\mathcal{L}-2} \cdot (K_{\mathcal{L}-1}\gamma_{\mathcal{L}-1}) \cdot \gamma_{\mathcal{L}..\infty} \\
&= [\![Q^{\mathcal{L}-1}(\mathtt{ev}^\ell uw)]\!]_e\gamma = [\![r]\!]_e\gamma
\end{aligned}$$

- $[\![\mathtt{ev}^\ell \circ^\mathcal{L}]\!]_q$ is similar to the previous two (and follows from the intuition that we treat $u \circ^\ell v$ in the same way as $\mathtt{ev}^{\ell+1}(Q^\ell u)v$, changing $K_\ell E_{\ell+1}$ into C_ℓ).

- $[\![\mathtt{ev}^\ell id^\mathcal{L}]\!]_q$:

$$\begin{aligned}
&[\![l]\!]_e\gamma = [\![\mathtt{ev}^\ell id^\mathcal{L} w]\!]_e\gamma \\
&= [\![id^\mathcal{L}]\!]_e\gamma_{0..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\infty} \\
&= \textstyle\sum_{i\neq\mathcal{L}-1}\gamma_i + E_\ell([\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) + I_\mathcal{L}(\gamma_{\mathcal{L}-1}) \\
&> \textstyle\sum_{i\neq\mathcal{L}-1}\gamma_i + E_\ell([\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) + I_{\mathcal{L}-1}(\gamma_{\mathcal{L}-1}) \\
&\quad \text{since } (I_i)_{i\geq 0} \text{ is increasing} \\
&\geq \textstyle\sum_{i\neq\mathcal{L}-1}\gamma_i + I_{\mathcal{L}-1}(\gamma_{\mathcal{L}-1}) \\
&\quad \text{since } E_\ell \text{ is non-negative} \\
&= [\![id^{\mathcal{L}-1}]\!]_e\gamma = [\![r]\!]_e\gamma
\end{aligned}$$

and similarly for $[\![\mathtt{ev}^\ell 1^\mathcal{L}]\!]_q$ and $[\![\mathtt{ev}^\ell \uparrow^\mathcal{L}]\!]_q$.

- $[\![\mathtt{ev}^\ell \bullet^{\mathcal{L}}]\!]_q$:

$$\begin{array}{l}[\![\mathtt{ev}^\ell(u \bullet^{\mathcal{L}} v)w]\!]_e\gamma \\ = [\![u \bullet^{\mathcal{L}} v]\!]_e\gamma_{0..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\infty} \\ = P_{\mathcal{L}}([\![u]\!]_e\gamma_{0..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\infty}, [\![v]\!]_e\gamma_{0..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\infty}) \\ > P_{\mathcal{L}-1}([\![u]\!]_e\gamma_{0..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\infty}, [\![v]\!]_e\gamma_{0..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\infty}) \\ \quad \text{since } (P_i)_{i\geq 0} \text{ is increasing} \\ = P_{\mathcal{L}-1}([\![\mathtt{ev}^\ell uw]\!]_e\gamma, [\![\mathtt{ev}^\ell vw]\!]_e\gamma) \\ = [\![(\mathtt{ev}^\ell uw) \bullet^{\mathcal{L}-1} (\mathtt{ev}^\ell vw)]\!]_e\gamma = [\![r]\!]_e\gamma\end{array}$$

  The case of $[\![\mathtt{ev}^\ell \star^{\mathcal{L}}]\!]_q$ is the same, and that of $[\![\mathtt{ev}^\ell \lambda^{\mathcal{L}}]\!]_q$ is similar.

- $[\![\mathtt{ev}^\ell \Uparrow^{\mathcal{L}}]\!]_q$ follows from the previous cases.

Group (D') follows similarly, or by noticing that $u \circ^\ell v$ behaves as $\mathtt{ev}^{\ell+1}(Q^\ell u)v$, with $K_\ell E_{\ell+1}$ replaced by C_ℓ. □

Lemma 2.13 *If $[\![u]\!]_e\gamma > [\![v]\!]_e\gamma$ (resp. $\geq$) for every γ in , , then for every context $\mathcal{C}$, for every γ in , , $[\![\mathcal{C}[u]]\!]_e\gamma > [\![\mathcal{C}[v]]\!]_e\gamma$ (resp. $\geq$).*

Proof: We only treat the case of $>$, since the case of $\geq$ follows easily. The proof is by structural induction on $\mathcal{C}$. If $\mathcal{C} = []$, this is clear. Otherwise, we have several cases.

If $\mathcal{C} = \mathtt{ev}^j\mathcal{C}_\infty w$, then $[\![\mathcal{C}[u]]\!]_e\gamma = [\![\mathcal{C}_1[u]]\!]_e\gamma_{0..j-1} \cdot (E_j[\![w]\!]_e\gamma_{0..j-1} \cdot \delta) \cdot \gamma_{j..\infty} > [\![\mathcal{C}_1[v]]\!]_e\gamma_{0..j-1} \cdot (E_j[\![w]\!]_e\gamma_{0..j-1} \cdot \delta) \cdot \gamma_{j..\infty}$ (by induction hypothesis) $= [\![\mathcal{C}[v]]\!]_e\gamma$.

If $\mathcal{C} = \mathtt{ev}^j w\mathcal{C}_1$, then $[\![\mathcal{C}[u]]\!]_e\gamma = [\![w]\!]_e\gamma_{0..j-1} \cdot (E_j[\![\mathcal{C}_1[u]]\!]_e\gamma_{0..j-1} \cdot \delta) \cdot \gamma_{j..\infty}$ and v the claim follows by the induction hypothesis, monotonicity of E_j and Lemma 2.11.

If $\mathcal{C} = Q^j\mathcal{C}_1$, then the claim follows directly from the induction hypothesis. The cases where $\mathcal{C}$ has $\circ^j$ as top operator follows by similar arguments.

If $\mathcal{C} = \mathcal{C}_1 \bullet^j w$, then $[\![\mathcal{C}[u]]\!]_e\gamma = P_j([\![\mathcal{C}_1[u]]\!]_e\gamma, [\![w]\!]_e\gamma) > P_j([\![\mathcal{C}_1[v]]\!]_e\gamma, [\![w]\!]_e\gamma)$ (by induction hypothesis and monotonicity of P_j) $= [\![\mathcal{C}[v]]\!]_e\gamma$. Similarly when $\mathcal{C} = w \bullet^j \mathcal{C}_1$, or with $\star^j$ or λ^j instead of $\bullet^j$.

The case when $\mathcal{C} = \Uparrow^j \mathcal{C}_1$ follows again by similar arguments, noticing that it behaves just as $1^j \bullet^j (\mathcal{C}_1 \circ^j \uparrow^j)$.
□

Lemma 2.14 *Let $>_e$ (resp. $\geq_e$) be defined by $u >_e v$ (resp. $\geq_e$) if and only if for every γ in , , $[\![u]\!]_e\gamma > [\![v]\!]_e\gamma$ (resp. $\geq$).*

Let $>_{eq}$ be $(>_e, >_q)_{lex}$, i.e. the ordering defined by $u >_{eq} v$ if and only if $u >_e v$, or $u \geq_e v$ and $u >_q v$. Then, whenever u rewrites to v by some rule of groups (D'), (E') or (F'), then $u >_{eq} v$.

Let $\succ_{eq}$ be defined by $u \succ_{eq} v$ if and only $[\![u]\!]_q >_{eq} [\![v]\!]_q$. If u rewrites to v by some rule in groups (D), (E) or (F), then $u \succ_{eq} v$. Therefore, the rewrite system consisting of (D), (E) and (F) terminates.

Proof: If u rewrites to v by some rule of groups (D'), (E') or (F'), then there exists a context $\mathcal{C}$ and a rule $l \to r$ such that $u = \mathcal{C}[l]$ and $v = \mathcal{C}[r]$.

If this rule is in group (F'), then by Lemma 2.12, $l \geq_e r$. By Lemma 2.13, $u \geq_e v$. By Lemma 2.9, $u >_q v$. So $u >_{eq} v$.

If the rule is in (D') or (E'), then by Lemma 2.12, $l >_e r$. By Lemma 2.13, $u >_e v$, so $u >_{eq} v$.

Now by Lemmas 2.5 and 2.7, if u rewrites to v by some rule in groups (D), (E) or (F), then $[\![u]\!]_q$ rewrites to $[\![v]\!]_q$ by some rule in groups (D'), (E') or (F'), so $u \succ_{eq} v$.

Moreover, $\succ_{eq}$ is clearly well-founded for derivations (i.e., the intersection of $\succ_{eq}$ and the reduction preordering is well-founded, see [Der87]), so groups (D), (E) and (F) as a whole defined a terminating rewrite relation. □

2.4 Going Further

The interpretation $[\![\_]\!]_{eq}$ of the last section actually proves that more rules are in fact decreasing. We start with the following observation:

Lemma 2.15 *For every term u, for every $i \geq 1$, for every $n > 0$, $[\![q_i(u)]\!]_e \gamma_{0..i-1} \cdot (n) \cdot \gamma_{i..\infty} > [\![u]\!]_e \gamma$.*

Proof: By structural induction on u. We have several cases:

Case $u = \mathtt{ev}^j vw$. If $i \leq j-1$, then $q_i(u) = \mathtt{ev}^{j+1}(q_i(v))(q_i(w))$, so:

$$\begin{aligned}
&[\![q_i(u)]\!]_e \gamma_{0..i-1} \cdot (n) \cdot \gamma_{i..\infty} \\
&= [\![\mathtt{ev}^{j+1}(q_i(v))(q_i(w))]\!]_e \gamma_{0..i-1} \cdot (n) \cdot \gamma_{i..\infty} \\
&= [\![q_i(v)]\!]_e \gamma_{0..i-1} \cdot (n) \cdot \gamma_{i..j-1} \cdot (E_{j+1}[\![q_i(w)]\!]_e \gamma_{0..i-1} \cdot (n) \cdot \gamma_{i..j-1} \cdot \delta) \cdot \gamma_{j..\infty} \\
&> [\![v]\!]_e \gamma_{0..j-1} \cdot (E_{j+1}[\![q_i(w)]\!]_e \gamma_{0..i-1} \cdot (n) \cdot \gamma_{i..j-1} \cdot \delta) \cdot \gamma_{j..\infty} \\
&\qquad \text{by induction hypothesis} \\
&> [\![v]\!]_e \gamma_{0..j-1} \cdot (E_{j+1}[\![w]\!]_e \gamma_{0..j-1} \cdot \delta) \cdot \gamma_{j..\infty} \\
&\qquad \text{by induction hypothesis, monotonicity of } E_{j+1} \text{ and Lemma 2.11} \\
&> [\![v]\!]_e \gamma_{0..j-1} \cdot (E_j[\![w]\!]_e \gamma_{0..j-1} \cdot \delta) \cdot \gamma_{j..\infty} \\
&\qquad \text{since } (E_\ell)_{\ell \geq 0} \text{ is increasing and by Lemma 2.11} \\
&= [\![\mathtt{ev}^j vw]\!]_e \gamma = [\![u]\!]_e \gamma
\end{aligned}$$

If $i > j-1$, i.e. $i \geq j$, then $q_i(u) = \mathtt{ev}^j(q_{i+1}(v))w$, so:

$$\begin{aligned}
&[\![q_i(u)]\!]_e \gamma_{0..i-1} \cdot (n) \cdot \gamma_{i..\infty} \\
&= [\![\mathtt{ev}^j(q_{i+1}(v))w]\!]_e \gamma_{0..i-1} \cdot (n) \cdot \gamma_{i..\infty} \\
&= [\![q_{i+1}(v)]\!]_e \gamma_{0..j-1} \cdot (E_j[\![w]\!]_e \gamma_{0..j-1} \cdot \delta) \cdot \gamma_{j..i-1} \cdot (n) \cdot \gamma_{i..\infty} \\
&> [\![v]\!]_e \gamma_{0..j-1} \cdot (E_j[\![w]\!]_e \gamma_{0..j-1} \cdot \delta) \cdot \gamma_{j..\infty} \\
&\qquad \text{by induction hypothesis (note how indices were shifted)} \\
&= [\![\mathtt{ev}^j vw]\!]_e \gamma = [\![u]\!]_e \gamma
\end{aligned}$$

Case $u = Q^j v$. If $i \leq j$, then $q_i(u) = Q^{j+1}(q_i(v))$, so:

$$\begin{aligned}
&[\![q_i(u)]\!]_e \gamma_{0..i-1} \cdot (n) \cdot \gamma_{i..\infty} \\
&= [\![Q^{j+1}(q_i(v))]\!]_e \gamma_{0..i-1} \cdot (n) \cdot \gamma_{i..\infty} \\
&= [\![q_i(v)]\!]_e \gamma_{0..i-1} \cdot (n) \cdot \gamma_{i..j-1} \cdot (K_{j+1}\gamma_j) \cdot \gamma_{j+1..\infty} \\
&> [\![v]\!]_e \gamma_{0..j-1} \cdot (K_{j+1}\gamma_j) \cdot \gamma_{j+1..\infty} \\
&\qquad \text{by induction hypothesis} \\
&> [\![v]\!]_e \gamma_{0..j-1} \cdot (K_j\gamma_j) \cdot \gamma_{j+1..\infty} \\
&\qquad \text{since } (K_\ell)_{\ell \geq 0} \text{ is increasing and by Lemma 2.11} \\
&= [\![Q^j v]\!]_e \gamma = [\![u]\!]_e \gamma
\end{aligned}$$

If $i > j$, then $q_i(u) = Q^j(q_i(v))$, so:

$$\begin{aligned}
&[\![q_i(u)]\!]_e \gamma_{0..i-1} \cdot (n) \cdot \gamma_{i..\infty} \\
&= [\![Q^j(q_i(v))]\!]_e \gamma_{0..i-1} \cdot (n) \cdot \gamma_{i..\infty} \\
&= [\![q_i(v)]\!]_e \gamma_{0..j-1} \cdot (K_j\gamma_j) \cdot \gamma_{j+1..i-1} \cdot (n) \cdot \gamma_{i..\infty} \\
&> [\![v]\!]_e \gamma_{0..j-1} \cdot (K_j\gamma_j) \cdot \gamma_{j+1..\infty} \\
&\qquad \text{by induction hypothesis} \\
&= [\![Q^j v]\!]_e \gamma = [\![u]\!]_e \gamma
\end{aligned}$$

The case $u = v \circ^j w$ follows by similar considerations as the two previous cases.

Case $u = id^j$. If $i \leq j$, then:

$$\begin{aligned}
&[\![q_i(u)]\!]_e \gamma_{0..i-1} \cdot (n) \cdot \gamma_{i..\infty} \\
&= [\![id^{j+1}]\!]_e \gamma_{0..i-1} \cdot (n) \cdot \gamma_{i..\infty} \\
&= \textstyle\sum_{k \neq j} \gamma_i + I_{j+1}(\gamma_j) + n \\
&> \textstyle\sum_{k \neq j} \gamma_i + I_{j+1}(\gamma_j) && \text{since } n > 0 \\
&> \textstyle\sum_{k \neq j} \gamma_i + I_j(\gamma_j) && \text{since } (I_\ell)_{\ell \geq 0} \text{ is increasing} \\
&= [\![id^j]\!]_e \gamma = [\![u]\!]_e \gamma
\end{aligned}$$

If $i > j$, then:

$$\begin{array}{ll} [\![q_i(u)]\!]_e\gamma_{0..i-1} \,.\, (n) \,.\, \gamma_{i..\infty} & \\ = [\![id^j]\!]_e\gamma_{0..i-1} \,.\, (n) \,.\, \gamma_{i..\infty} & \\ = \sum_{k\neq j}\gamma_i + I_j(\gamma_j) + n & \\ > \sum_{k\neq j}\gamma_i + I_j(\gamma_j) & \text{since } n > 0 \\ = [\![id^j]\!]_e\gamma = [\![u]\!]_e\gamma & \end{array}$$

and similarly in the cases $u = 1^j$ and $u = \uparrow^j$.

Case $u = v \bullet^j w$. If $i \leq j$, then $q_i(u) = q_i(v) \bullet^{j+1} q_i(w)$, so:

$$\begin{array}{l} [\![q_i(u)]\!]_e\gamma_{0..i-1} \,.\, (n) \,.\, \gamma_{i..\infty} \\ = [\![q_i(v) \bullet^{j+1} q_i(w)]\!]_e\gamma_{0..i-1} \,.\, (n) \,.\, \gamma_{i..\infty} \\ = P_{j+1}([\![q_i(v)]\!]_e\gamma_{0..i-1} \,.\, (n) \,.\, \gamma_{i..\infty}, [\![q_i(w)]\!]_e\gamma_{0..i-1} \,.\, (n) \,.\, \gamma_{i..\infty}) \\ > P_{j+1}([\![v]\!]_e\gamma, [\![w]\!]_e\gamma) \\ \quad \text{by induction hypothesis and monotonicity of } P_{j+1} \\ > P_j([\![v]\!]_e\gamma, [\![w]\!]_e\gamma) \\ \quad \text{since } (P_\ell)_{\ell \geq 0} \text{ is increasing} \\ = [\![v \bullet^\ell w]\!]_e\gamma = [\![u]\!]_e\gamma \end{array}$$

If $i > j$, then:

$$\begin{array}{l} [\![q_i(u)]\!]_e\gamma_{0..i-1} \,.\, (n) \,.\, \gamma_{i..\infty} \\ = [\![q_i(v) \bullet^j q_i(w)]\!]_e\gamma_{0..i-1} \,.\, (n) \,.\, \gamma_{i..\infty} \\ = P_j([\![q_i(v)]\!]_e\gamma_{0..i-1} \,.\, (n) \,.\, \gamma_{i..\infty}, [\![q_i(w)]\!]_e\gamma_{0..i-1} \,.\, (n) \,.\, \gamma_{i..\infty}) \\ > P_j([\![v]\!]_e\gamma, [\![w]\!]_e\gamma) \\ \quad \text{by induction hypothesis and monotonicity of } P_j \\ = [\![v \bullet^\ell w]\!]_e\gamma = [\![u]\!]_e\gamma \end{array}$$

and similarly when $u = v \star^j w$ and $u = \lambda^j v$.

The case of $u = \Uparrow^j v$ follows similarly, or noticing that this case works as for $u = 1^j \bullet^j (v \circ^j \uparrow^j)$. □

It follows:

Lemma 2.16 *If s rewrites to t by rule $(\mathtt{ev}Q^\ell)$, then $s \succ_{eq} t$.*

Proof: $(\mathtt{ev}Q^\ell)$: let $s = \mathcal{C}[\mathtt{ev}^\ell(Q^\ell u)w]$, $t = \mathcal{C}[u]$. Then, $[\![s]\!]_q = [\![\mathcal{C}]\!]_q[q_\mathcal{C}(\mathtt{ev}^\ell(Q^\ell(q_\ell([\![u]\!]_q)))[\![w]\!]_q)]$ and $[\![t]\!]_q = [\![\mathcal{C}]\!]_q[q_\mathcal{C}([\![u]\!]_q)]$ (see Lemma 2.6). Now, $q_\mathcal{C}$ can be written in the form $q_{i_1} \circ q_{i_2} \circ \ldots \circ q_{i_p}$, and by Lemma 2.4 we may assume that $i_1 \leq i_2 \leq \ldots \leq i_p$ (otherwise we rewrite some $q_i \circ q_j$ where $i > j$ into $q_j \circ q_{i-1}$: this decreases the sum of all indices strictly, so the process must terminate). Let j be the greatest index such that $i_1 \leq \ldots \leq i_{j-1} \leq \ell - 1 \leq i_j \leq \ldots i_p$ (in particular, if $j \leq p$, then $\ell - 1 < i_j$). Then:

$$\begin{array}{l} [\![q_\mathcal{C}(\mathtt{ev}^\ell(Q^\ell(q_\ell([\![u]\!]_q)))[\![w]\!]_q)]\!]_e\gamma \\ = [\![(q_{i_1} \circ \ldots \circ q_{i_p})(\mathtt{ev}^\ell(Q^\ell(q_\ell([\![u]\!]_q)))[\![w]\!]_q)]\!]_e\gamma \\ = [\![\mathtt{ev}^{\ell+j-1}((q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j+1} \circ \ldots q_{i_p+1})(Q^\ell(q_\ell([\![u]\!]_q))))w']\!]_e\gamma \\ \quad \text{where } w' = (q_{i_1} \circ \ldots \circ q_{i_{j-1}})([\![w]\!]_q) \\ = [\![\mathtt{ev}^{\ell+j-1}(Q^{\ell+j-1}((q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j+1} \circ \ldots q_{i_p+1})(q_\ell([\![u]\!]_q))))w']\!]_e\gamma \\ = [\![\mathtt{ev}^{\ell+j-1}(Q^{\ell+j-1}((q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j+1} \circ \ldots q_{i_p+1} \circ q_\ell)([\![u]\!]_q)))w']\!]_e\gamma \\ = [\![\mathtt{ev}^{\ell+j-1}(Q^{\ell+j-1}((q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_\ell \circ q_{i_j} \circ \ldots q_{i_p})([\![u]\!]_q)))w']\!]_e\gamma \\ \quad \text{by Lemma 2.4 } p - j + 1 \text{ times} \\ = [\![\mathtt{ev}^{\ell+j-1}(Q^{\ell+j-1}((q_{\ell+j-1} \circ q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j} \circ \ldots q_{i_p})([\![u]\!]_q)))w']\!]_e\gamma \\ \quad \text{by Lemma 2.4 } j - 1 \text{ times (in the other direction)} \\ = [\![\mathtt{ev}^{\ell+j-1}(Q^{\ell+j-1}(q_{\ell+j-1}(q_\mathcal{C}([\![u]\!]_q))))w']\!]_e\gamma \\ = [\![Q^{\ell+j-1}(q_{\ell+j-1}(q_\mathcal{C}([\![u]\!]_q)))]\!]_e\gamma_{0..\ell+j-2} \,.\, (E_{\ell+j-1}[\![w']\!]_e\gamma_{0..\ell+j-2} \,.\, \delta) \,.\, \gamma_{\ell+j-1..\infty} \\ = [\![q_{\ell+j-1}(q_\mathcal{C}([\![u]\!]_q))]\!]_e\gamma_{0..\ell+j-2} \,.\, (K_{\ell+j-1}E_{\ell+j-1}[\![w']\!]_e\gamma_{0..\ell+j-2} \,.\, \delta) \,.\, \gamma_{\ell+j-1..\infty} \\ > [\![q_\mathcal{C}([\![u]\!]_q)]\!]_e\gamma \end{array}$$

by Lemma 2.15. By Lemma 2.13, it follows that $[\![[\![\mathcal{C}]\!]_q[q_\mathcal{C}(\mathtt{ev}^\ell(Q^\ell(q_\ell([\![u]\!]_q)))[\![w]\!]_q)]]\!]_e\gamma > [\![[\![\mathcal{C}]\!]_q[q_\mathcal{C}([\![u]\!]_q)]]\!]_e\gamma$, that is, $[\![s]\!]_e\gamma > [\![t]\!]_e\gamma$. □

We shall in the sequel assume the additional property:

(P4) for every $\ell \geq 0$, $E_{\ell+1} > C_\ell$.

which is verified by our proposal of Section 2.3. Then:

Lemma 2.17 *If s rewrites to t by rule $(\eta\mathtt{ev}^\ell)$, then $s \succ_{eq} t$.*

Proof: This works exactly as the rules in groups (D), (E) and (F). By the $[\![\_]\!]_q$ translation, the rule becomes $[\![\eta\mathtt{ev}^\ell]\!]_q$: $\mathtt{ev}^{\ell+1}(Q^\ell u)w \to u \circ^\ell w$. Moreover, $q_i([\![\eta\mathtt{ev}^\ell]\!]_q)$ is $[\![\eta\mathtt{ev}^{\ell+1}]\!]_q$ if $i \leq \ell$ and $[\![\eta\mathtt{ev}^\ell]\!]_q$ if $i > \ell$, so Lemma 2.7 extends to this case. Finally:

$$\begin{array}{l}
[\![\mathtt{ev}^{\ell+1}(Q^\ell u)w]\!]_e\gamma \\
= [\![Q^\ell u]\!]_e\gamma_{0..\ell} \cdot (E_{\ell+1}[\![w]\!]_e\gamma_{0..\ell} \cdot \delta) \cdot \gamma_{\ell+1..\infty} \\
= [\![u]\!]_e\gamma_{0..\ell-1} \cdot (K_\ell\gamma_\ell) \cdot (E_{\ell+1}[\![w]\!]_e\gamma_{0..\ell} \cdot \delta) \cdot \gamma_{\ell+1..\infty} \\
> [\![u]\!]_e\gamma_{0..\ell} \cdot (E_{\ell+1}[\![w]\!]_e\gamma_{0..\ell} \cdot \delta) \cdot \gamma_{\ell+1..\infty} \\
\quad \text{by superlinearity of } K_\ell \\
> [\![u]\!]_e\gamma_{0..\ell} \cdot (C_\ell[\![w]\!]_e\gamma_{0..\ell} \cdot \delta) \cdot \gamma_{\ell+1..\infty} \\
\quad \text{by (P4)} \\
= [\![u \circ^\ell w]\!]_e\gamma
\end{array}$$

□

Lemma 2.18 *If s rewrites to t by rule $(\eta \Uparrow^\ell)$, then $s \succ_{eq} t$.*

Proof: The proof is again similar. The $[\![\_]\!]_q$ translation yields the rule $[\![\eta \Uparrow^\ell]\!]_q$, which is just $(\eta \Uparrow^\ell)$ itself: $\Uparrow^\ell u \to 1^\ell \bullet^\ell (u \circ^\ell \uparrow^\ell)$. Moreover, $q_i([\![\eta \Uparrow^\ell]\!]_q)$ is $[\![\eta \Uparrow^{\ell+1}]\!]_q$ if $i \leq \ell$ and $[\![\eta \Uparrow^\ell]\!]_q$ if $i > \ell$, so Lemma 2.7 again extends to cover this case. Finally, by definition $[\![\Uparrow^\ell u]\!]_e\gamma = U_\ell([\![1^\ell \bullet^\ell (u\circ^\ell \uparrow^\ell)]\!]_e\gamma) > [\![1^\ell \bullet^\ell (u\circ^\ell \uparrow^\ell)]\!]_e\gamma$ because U_ℓ is superlinear. □

Lemma 2.19 *If s rewrites to t by rule $(\mathtt{ev}1^\ell)$ or $(\mathtt{ev} \uparrow^\ell)$, then $s \succ_{eq} t$.*

Proof: Notice because of our convention that $1u$ was an abbreviation for $1^0 \circ^0 u$ (resp. $\uparrow u$ of $\uparrow^0 \circ^0 u$), these rules can be written $\mathtt{ev}^\ell 1^\ell w \to 1^{\ell-1} \circ^{\ell-1} w$ and $\mathtt{ev}^\ell \uparrow^\ell w \to \uparrow^{\ell-1} \circ^{\ell-1} w$ respectively, for every $\ell \geq 1$. We deal with $(\mathtt{ev}1^\ell)$, as the other rule is similar.

By the $[\![\_]\!]_q$ translation, rule $(\mathtt{ev}1^\ell)$ becomes $[\![\mathtt{ev}1^\ell]\!]_q$: $\mathtt{ev}^\ell 1^\ell w \to 1^\ell \circ^{\ell-1} w$. Then, $q_i([\![\mathtt{ev}1^\ell]\!]_q)$ is $[\![\mathtt{ev}1^{\ell+1}]\!]_q$ if $i \leq \ell-1$, and $[\![\mathtt{ev}1^\ell]\!]_q$ if $i > \ell$, so again Lemma 2.7 extends to this case. Finally, $[\![\mathtt{ev}^\ell 1^\ell w]\!]_e\gamma = [\![1^\ell]\!]_e\gamma_{0..\ell-1} \cdot (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\infty} > [\![1^\ell]\!]_e\gamma_{0..\ell-1} \cdot (C_{\ell-1}[\![w]\!]_e\gamma_{0..\ell-1} \cdot \delta) \cdot \gamma_{\ell..\infty}$ (by property (P4) and Lemma 2.11) $= [\![1^\ell \circ^{\ell-1} w]\!]_e\gamma$. □

Lemma 2.20 *If s rewrites to t by rule $(\bullet^\ell)$, $(\mathtt{ev}\ \bullet^\ell)$, $(\star^\ell)$ or $(\mathtt{ev}\star^\ell)$, then $s \succ_{eq} t$.*

Proof: By the $[\![\_]\!]_q$ translation, rule $(\bullet^\ell)$ becomes $[\![\bullet^\ell]\!]_q$: $(u \bullet^{\ell+1} v) \circ^\ell w \to (u \circ^\ell w) \bullet^\ell (v \circ^\ell w)$. Then, $q_i([\![\bullet^\ell]\!]_q)$ is $[\![\bullet^{\ell+1}]\!]_q$ if $i \leq \ell$, and $[\![\bullet^\ell]\!]_q$ if $i > \ell$, so Lemma 2.7 again extends to this case. And:

$$\begin{array}{l}
[\![(u \bullet^{\ell+1} v) \circ^\ell w]\!]_e\gamma \\
= [\![u \bullet^{\ell+1} v]\!]_e\gamma_{0..\ell} \cdot (C_\ell[\![w]\!]_e\gamma_{0..\ell} \cdot \delta) \cdot \gamma_{\ell+1..\infty} \\
= P_{\ell+1}([\![u]\!]_e\gamma_{0..\ell} \cdot (C_\ell[\![w]\!]_e\gamma_{0..\ell} \cdot \delta) \cdot \gamma_{\ell+1..\infty}, [\![v]\!]_e\gamma_{0..\ell} \cdot (C_\ell[\![w]\!]_e\gamma_{0..\ell} \cdot \delta) \cdot \gamma_{\ell+1..\infty}) \\
> P_\ell([\![u]\!]_e\gamma_{0..\ell} \cdot (C_\ell[\![w]\!]_e\gamma_{0..\ell} \cdot \delta) \cdot \gamma_{\ell+1..\infty}, [\![v]\!]_e\gamma_{0..\ell} \cdot (C_\ell[\![w]\!]_e\gamma_{0..\ell} \cdot \delta) \cdot \gamma_{\ell+1..\infty}) \\
\quad \text{because } (P_i)_{i\geq 0} \text{ is increasing} \\
= P_\ell([\![u \circ^\ell w]\!]_e\gamma, [\![v \circ^\ell w]\!]_e\gamma) \\
= [\![(u \circ^\ell w) \bullet^\ell (v \circ^\ell w)]\!]_e\gamma
\end{array}$$

By the $[\![\_]\!]_q$ translation, rule $(\mathtt{ev}\ \bullet^\ell)$ becomes rule $[\![\mathtt{ev}\ \bullet^\ell]\!]_q$: $\mathtt{ev}^\ell(u \bullet^\ell v)w \to (\mathtt{ev}^\ell uw) \bullet^{\ell-1} (\mathtt{ev}^\ell vw)$. Lemma 2.7 again extends to this case, as $q_i([\![\mathtt{ev}\ \bullet^\ell]\!]_q)$ is $[\![\mathtt{ev}\ \bullet^{\ell+1}]\!]_q$ if $i \leq \ell - 1$ and $[\![\mathtt{ev}\ \bullet^\ell]\!]_q$ if $i > \ell - 1$. And:

$$
\begin{aligned}
&[\![\mathtt{ev}^\ell(u \bullet^\ell v)w]\!]_e\gamma \\
&= [\![u \bullet^\ell v]\!]_e\gamma_{0..\ell-1} \,.\, (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \,.\, \delta) \,.\, \gamma_{\ell..\infty} \\
&= P_\ell([\![u]\!]_e\gamma_{0..\ell-1} \,.\, (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \,.\, \delta) \,.\, \gamma_{\ell..\infty}, [\![v]\!]_e\gamma_{0..\ell-1} \,.\, (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \,.\, \delta) \,.\, \gamma_{\ell..\infty}) \\
&> P_{\ell-1}([\![u]\!]_e\gamma_{0..\ell-1} \,.\, (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \,.\, \delta) \,.\, \gamma_{\ell..\infty}, [\![v]\!]_e\gamma_{0..\ell-1} \,.\, (E_\ell[\![w]\!]_e\gamma_{0..\ell-1} \,.\, \delta) \,.\, \gamma_{\ell..\infty}) \\
&\qquad \text{because } (P_i)_{i\geq 0} \text{ is increasing} \\
&= P_{\ell-1}([\![\mathtt{ev}^\ell uw]\!]_e\gamma, [\![\mathtt{ev}^\ell vw]\!]_e\gamma) \\
&= [\![(\mathtt{ev}^\ell uw) \bullet^{\ell-1} (\mathtt{ev}^\ell vw)]\!]_e\gamma
\end{aligned}
$$

The cases of $(\star^\ell)$ and $(\mathtt{ev}\star^\ell)$ are identical. □

We can prove the following as well, although we won't really need it in the end:

Lemma 2.21 *If s rewrites to t by rule $(\circ^\ell)$ or $(Q\circ^\ell)$, then $s \succ_{eq} t$.*

Proof: Let's give the intuitive idea first. Basically, $\_ \circ^\ell \_$ is similar to $\mathtt{ev}^{\ell+1}(Q^\ell\_)\_$, with a few changes (replacing C_ℓ functions by E_ℓ, in particular).

In the first case, $(u \circ^\ell v) \circ^\ell w$ is similar to $\mathtt{ev}^{\ell+1}(Q^\ell(\mathtt{ev}^{\ell+1}(Q^\ell u)v))w$, which rewrites to $\mathtt{ev}^{\ell+1}(\mathtt{ev}^{\ell+2}(Q^\ell Q^\ell u)(Q^\ell v))w$ by rule $(Q^\ell \mathtt{ev}^{\ell+1})$, then to $\mathtt{ev}^{\ell+1}(\mathtt{ev}^{\ell+1}(Q^\ell Q^\ell u)w)(\mathtt{ev}^{\ell+1}(Q^\ell v)w)$ by rule $(\mathtt{ev}^{\ell+1}\mathtt{ev}^{\ell+2})$, then to $\mathtt{ev}^{\ell+1}(\mathtt{ev}^{\ell+1}(Q^{\ell+1}Q^\ell u)w)(\mathtt{ev}^{\ell+1}(Q^\ell v)w)$ by rule $(Q^\ell Q^{\ell+1})$, then to $\mathtt{ev}^{\ell+1}(Q^\ell u)(\mathtt{ev}^{\ell+1}(Q^\ell v)w)$ by rule $(\mathtt{ev}Q^\ell)$, and the latter is similar to $u \circ^\ell (v \circ^\ell w)$. So the argument for proving that $s >_{eq} t$ in this case will be a mix of the arguments for all the rules above.

In the second case, $Q^\ell u \circ^\ell w$ is similar to $\mathtt{ev}^{\ell+1}(Q^\ell Q^\ell u)w$, which rewrites by rule $(Q^\ell Q^{\ell+1})$ to $\mathtt{ev}^{\ell+1}(Q^{\ell+1}Q^\ell u)w$, then to $Q^\ell u$ by rule $(\mathtt{ev}Q^{\ell+1})$. Again, the argument for proving that $s >_{eq} t$ in this case will be a mix of the arguments for these two rules.

Here we go. Consider rule $(\circ^\ell)$ first, and let s be $\mathcal{C}[(u \circ^\ell v) \circ^\ell w]$, t be $\mathcal{C}[u \circ^\ell (v \circ^\ell w)]$. $[\![s]\!]_q = [\![\mathcal{C}]\!]_q[q_\mathcal{C}(q_\ell(q_\ell[\![u]\!]_q \circ^\ell [\![v]\!]_q) \circ^\ell [\![w]\!]_q)] = [\![\mathcal{C}]\!]_q[q_\mathcal{C}((q_\ell \circ q_\ell)[\![u]\!]_q \circ^{\ell+1} q_\ell[\![v]\!]_q) \circ^\ell [\![w]\!]_q)] = [\![\mathcal{C}]\!]_q[q_\mathcal{C}((q_{\ell+1} \circ q_\ell)[\![u]\!]_q \circ^{\ell+1} q_\ell[\![v]\!]_q) \circ^\ell [\![w]\!]_q)]$ by Lemma 2.4, and $[\![t]\!]_q = [\![\mathcal{C}]\!]_q[q_\mathcal{C}(q_\ell[\![u]\!]_q \circ^\ell (q_\ell[\![v]\!]_q \circ^\ell [\![w]\!]_q))]$. Let's write $q_\mathcal{C}$ as $q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j} \circ \ldots \circ q_{i_p}$, where j is the greatest index such that $i_1 \leq \ldots \leq i_{j-1} \leq \ell \leq i_j \leq \ldots \leq i_p$. Then:

$$
\begin{aligned}
&[\![q_\mathcal{C}((q_{\ell+1} \circ q_\ell)[\![u]\!]_q \circ^{\ell+1} q_\ell[\![v]\!]_q) \circ^\ell [\![w]\!]_q)]\!]_e\gamma \\
&= [\![(q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j+1} \circ \ldots \circ q_{i_p+1})((q_{\ell+1} \circ q_\ell)[\![u]\!]_q \circ^{\ell+1} q_\ell[\![v]\!]_q) \circ^{\ell+j-1} w']\!]_e\gamma \\
&\qquad \text{with } w' = (q_{i_1} \circ \ldots \circ q_{i_{j-1}})[\![w]\!]_q \\
&= [\![((q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j+2} \circ \ldots q_{i_p+2} \circ q_{\ell+1} \circ q_\ell)[\![u]\!]_q \circ^{\ell+j} v') \circ^{\ell+j-1} w']\!]_e\gamma \\
&\qquad \text{with } v' = (q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_\ell)[\![v]\!]_q \\
&= [\![((q_{\ell+j} \circ q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j+1} \circ \ldots q_{i_p+1} \circ q_\ell)[\![u]\!]_q \circ^{\ell+j} v') \circ^{\ell+j-1} w']\!]_e\gamma \\
&\qquad \text{using Lemma 2.4 } p \text{ times} \\
&= [\![(q_{\ell+j}(u') \circ^{\ell+j} v') \circ^{\ell+j-1} w']\!]_e\gamma \\
&\qquad \text{where } u' = (q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j+1} \circ \ldots q_{i_p+1} \circ q_\ell)[\![u]\!]_q \\
&= [\![q_{\ell+j}(u') \circ^{\ell+j} v']\!]_e\gamma_{0..\ell+j-1} \,.\, (C_{\ell+j-1}[\![w']\!]_e\gamma_{0..\ell+j-1} \,.\, \delta) \,.\, \gamma_{\ell+j..\infty} \\
&= [\![q_{\ell+j}(u')]\!]_e\gamma_{0..\ell+j-1} \,.\, (C_{\ell+j-1}[\![w']\!]_e\gamma_{0..\ell+j-1} \,.\, \delta) \\
&\qquad .\, (C_{\ell+j}[\![v']\!]_e\gamma_{0..\ell+j-1} \,.\, (C_{\ell+j-1}[\![w']\!]_e\gamma_{0..\ell+j-1} \,.\, \delta) \,.\, \delta) \,.\, \gamma_{\ell+j..\infty} \\
&> [\![u']\!]_e\gamma_{0..\ell+j-1} \,.\, (C_{\ell+j}[\![v']\!]_e\gamma_{0..\ell+j-1} \,.\, (C_{\ell+j-1}[\![w']\!]_e\gamma_{0..\ell+j-1} \,.\, \delta) \,.\, \delta) \,.\, \gamma_{\ell+j..\infty} \\
&\qquad \text{by Lemma 2.15} \\
&> [\![u']\!]_e\gamma_{0..\ell+j-1} \,.\, (C_{\ell+j-1}[\![v']\!]_e\gamma_{0..\ell+j-1} \,.\, (C_{\ell+j-1}[\![w']\!]_e\gamma_{0..\ell+j-1} \,.\, \delta) \,.\, \delta) \,.\, \gamma_{\ell+j..\infty} \\
&\qquad \text{because } (C_i)_{i\geq 0} \text{ is increasing and by Lemma 2.11}
\end{aligned}
$$

while:

$$
\begin{aligned}
&[\![q_\mathcal{C}(q_\ell[\![u]\!]_q \circ^\ell (q_\ell[\![v]\!]_q \circ^\ell [\![w]\!]_q))]\!]_e\gamma \\
&= [\![(q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j+1} \circ \ldots q_{i_p+1} \circ q_\ell)[\![u]\!]_q \circ^{\ell+j-1} (q_{i_1} \circ \ldots \circ q_{i_{j-1}})(q_\ell[\![v]\!]_q \circ^\ell [\![w]\!]_q)]\!]_e\gamma \\
&= [\![u' \circ^{\ell+j-1} (q_{i_1} \circ \ldots \circ q_{i_{j-1}})(q_\ell[\![v]\!]_q \circ^\ell [\![w]\!]_q)]\!]_e\gamma \\
&= [\![u' \circ^{\ell+j-1} ((q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_\ell)[\![v]\!]_q \circ^{\ell+j-1} (q_{i_1} \circ \ldots \circ q_{i_{j-1}})[\![w]\!]_q)]\!]_e\gamma \\
&= [\![u' \circ^{\ell+j-1} (v' \circ^{\ell+j-1} w')]\!]_e\gamma \\
&= [\![u']\!]_e\gamma_{0..\ell+j-1} \,.\, (C_{\ell+j-1}[\![v' \circ^{\ell+j-1} w']\!]_e\gamma_{0..\ell+j-1} \,.\, \delta) \,.\, \gamma_{\ell+j..\infty} \\
&= [\![u']\!]_e\gamma_{0..\ell+j-1} \,.\, (C_{\ell+j-1}[\![v']\!]_e\gamma_{0..\ell+j-1} \,.\, (C_{\ell+j-1}[\![w']\!]_e\gamma_{0..\ell+j-1} \,.\, \delta) \,.\, \delta) \,.\, \gamma_{\ell+j..\infty}
\end{aligned}
$$

It just remains to apply Lemma 2.13 to get $[\![s]\!]_{eq}\gamma > [\![t]\!]_{eq}\gamma$, i.e. $s \succ_{eq} t$.

Now on to rule $(Q\circ^\ell)$. Let s be $\mathcal{C}[Q^\ell u \circ^\ell w]$ and t be $\mathcal{C}[Q^\ell u]$. Then $[\![s]\!]_q = [\![\mathcal{C}]\!]_q[q_\mathcal{C}(q_\ell(Q^\ell(q_\ell[\![u]\!]_q)) \circ^\ell [\![w]\!]_q)] = [\![\mathcal{C}]\!]_q[q_\mathcal{C}(Q^{\ell+1}((q_\ell \circ q_\ell)[\![u]\!]_q)) \circ^\ell [\![w]\!]_q)] = [\![\mathcal{C}]\!]_q[q_\mathcal{C}(Q^{\ell+1}((q_{\ell+1} \circ q_\ell)[\![u]\!]_q)) \circ^\ell [\![w]\!]_q)]$ by Lemma 2.4, and $[\![t]\!]_q = [\![\mathcal{C}]\!]_q[q_\mathcal{C}(Q^\ell(q_\ell[\![u]\!]_q))]$. Let's write $q_\mathcal{C}$ as $q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j} \circ \ldots \circ q_{i_p}$, where j is the greatest index such that $i_1 \leq \ldots \leq i_{j-1} \leq \ell \leq i_j \leq \ldots \leq i_p$. Then:

$$
\begin{aligned}
& [\![q_\mathcal{C}(Q^{\ell+1}((q_{\ell+1} \circ q_\ell)[\![u]\!]_q) \circ^\ell [\![w]\!]_q)]\!]_e\gamma \\
&= [\![(q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j+1} \circ \ldots \circ q_{i_p+1})(Q^{\ell+1}((q_{\ell+1} \circ q_\ell)[\![u]\!]_q)) \circ^{\ell+j-1} w']\!]_e\gamma \\
&\quad \text{where } w' = (q_{i_1} \circ \ldots \circ q_{i_{j-1}})[\![w]\!]_q \\
&= [\![Q^{\ell+j}((q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j+1} \circ \ldots \circ q_{i_p+1} \circ q_{\ell+1} \circ q_\ell)[\![u]\!]_q) \circ^{\ell+j-1} w']\!]_e\gamma \\
&= [\![Q^{\ell+j}((q_{\ell+j} \circ q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j} \circ \ldots \circ q_{i_p} \circ q_\ell)[\![u]\!]_q) \circ^{\ell+j-1} w']\!]_e\gamma \\
&\quad \text{by Lemma 2.4 } p \text{ times} \\
&= [\![Q^{\ell+j}(q_{\ell+j}(u')) \circ^{\ell+j-1} w']\!]_e\gamma \\
&\quad \text{with } u' = (q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j} \circ \ldots \circ q_{i_p} \circ q_\ell)[\![u]\!]_q \\
&= [\![Q^{\ell+j}(q_{\ell+j}(u'))]\!]_e\gamma_{0..\ell+j-1} \cdot (C_{\ell+j-1}[\![w']\!]_e\gamma_{0..\ell+j-1} \cdot \delta) \cdot \gamma_{\ell+j..\infty} \\
&= [\![q_{\ell+j-1}(Q^{\ell+j-1}u')]\!]_e\gamma_{0..\ell+j-1} \cdot (C_{\ell+j-1}[\![w']\!]_e\gamma_{0..\ell+j-1} \cdot \delta) \cdot \gamma_{\ell+j..\infty} \\
&> [\![Q^{\ell+j-1}u']\!]_e\gamma_{0..\ell+j-2} \cdot (C_{\ell+j-1}[\![w']\!]_e\gamma_{0..\ell+j-1} \cdot \delta) \cdot \gamma_{\ell+j..\infty} \\
&\quad \text{by Lemma 2.15} \\
&> [\![Q^{\ell+j-1}u']\!]_e\gamma_{0..\ell+j-2} \cdot (\gamma_{\ell+j-1}) \cdot \gamma_{\ell+j..\infty} = [\![Q^{\ell+j-1}u']\!]_e\gamma \\
&\quad \text{by Lemmas 2.10 and 2.11}
\end{aligned}
$$

while:

$$
\begin{aligned}
& [\![q_\mathcal{C}(Q^\ell(q_\ell[\![u]\!]_q))]\!]_e\gamma \\
&= [\![Q^{\ell+j-1}((q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j} \circ \ldots \circ q_{i_p} \circ q_\ell)[\![u]\!]_q)]\!]_e\gamma \\
&= [\![Q^{\ell+j-1}u']\!]_e\gamma
\end{aligned}
$$

Therefore, as before, $s \succ_{eq} t$. □

We shall now assume an extra property, namely:

(P5) for every $\ell \geq 0$, $C_\ell \geq E_\ell$.

which is verified by our proposal of Section 2.3.

Lemma 2.22 *If s rewrites to t by rule $(\mathtt{ev}\circ^\ell)$, then $s \succ_{eq} t$.*

Proof: Again, basically $\_\circ^\ell\_$ is similar to $\mathtt{ev}^{\ell+1}(Q^\ell\_)\_$, with a few changes (replacing C_ℓ functions by E_ℓ, in particular). Then $\mathtt{ev}^\ell(u \circ^\ell v)w$ is similar to $\mathtt{ev}^\ell(\mathtt{ev}^{\ell+1}(Q^\ell u)v)w$, which rewrites to $\mathtt{ev}^\ell(\mathtt{ev}^\ell(Q^\ell u)w)(\mathtt{ev}^\ell vw)$ by rule $(\mathtt{ev}^\ell\mathtt{ev}^{\ell+1})$, then to $\mathtt{ev}^\ell u(\mathtt{ev}^\ell vw)$ by rule $(\mathtt{ev}Q^\ell)$.

Formally, let s be $\mathcal{C}[\mathtt{ev}^\ell(u \circ^\ell v)w]$, t be $\mathcal{C}[\mathtt{ev}^\ell u(\mathtt{ev}^\ell vw)]$. $[\![s]\!]_q = [\![\mathcal{C}]\!]_q[q_\mathcal{C}(\mathtt{ev}^\ell(q_\ell(u) \circ^\ell v)w)]$, and $[\![t]\!]_q = [\![\mathcal{C}]\!]_q[q_\mathcal{C}(\mathtt{ev}^\ell u(\mathtt{ev}^\ell vw))]$. Let's write $q_\mathcal{C}$ as $q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j} \circ \ldots \circ q_{i_p}$, where j is the greatest index such that $i_1 \leq \ldots \leq i_{j-1} \leq \ell - 1 \leq i_j \leq \ldots \leq i_p$. Then:

$$
\begin{aligned}
& [\![q_\mathcal{C}(\mathtt{ev}^\ell(q_\ell[\![u]\!]_q \circ^\ell [\![v]\!]_q)[\![w]\!]_q)]\!]_e\gamma \\
&= [\![\mathtt{ev}^{\ell+j-1}((q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j+1} \circ \ldots \circ q_{i_p+1})(q_\ell[\![u]\!]_q \circ^\ell [\![v]\!]_q))w']\!]_e\gamma \\
&\quad \text{where } w' = (q_{i_1} \circ \ldots \circ q_{i_{j-1}})[\![w]\!]_q \\
&= [\![\mathtt{ev}^{\ell+j-1}((q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j+1} \circ \ldots \circ q_{i_p+1})(q_\ell[\![u]\!]_q \circ^\ell [\![v]\!]_q))w']\!]_e\gamma \\
&= [\![\mathtt{ev}^{\ell+j-1}((q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j+2} \circ \ldots \circ q_{i_p+2} \circ q_\ell)[\![u]\!]_q \circ^{\ell+j-1} v')w']\!]_e\gamma \\
&\quad \text{where } v' = (q_{i_1} \circ \ldots \circ q_{i_{j-1}})[\![v]\!]_q \\
&= [\![\mathtt{ev}^{\ell+j-1}((q_{\ell+j-1} \circ q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j+1} \circ \ldots \circ q_{i_p+1})[\![u]\!]_q \circ^{\ell+j-1} v')w']\!]_e\gamma \\
&\quad \text{using Lemma 2.4 } p \text{ times} \\
&= [\![\mathtt{ev}^{\ell+j-1}(q_{\ell+j-1}(u') \circ^{\ell+j-1} v')w']\!]_e\gamma \\
&\quad \text{where } u' = (q_{i_1} \circ \ldots \circ q_{i_{j-1}} \circ q_{i_j+1} \circ \ldots \circ q_{i_p+1})[\![u]\!]_q \\
&= [\![q_{\ell+j-1}(u') \circ^{\ell+j-1} v']\!]_e\gamma_{0..\ell+j-2} \cdot (E_{\ell+j-1}[\![w']\!]_e\gamma_{0..\ell+j-2} \cdot \delta) \cdot \gamma_{\ell+j-1..\infty} \\
&= [\![q_{\ell+j-1}(u')]\!]_e\gamma_{0..\ell+j-2} \cdot (E_{\ell+j-1}[\![w']\!]_e\gamma_{0..\ell+j-2} \cdot \delta) \cdot \\
&\quad (C_{\ell+j-1}[\![v']\!]_e\gamma_{0..\ell+j-2} \cdot (E_{\ell+j-1}[\![w']\!]_e\gamma_{0..\ell+j-2} \cdot \delta) \cdot \delta) \cdot \gamma_{\ell+j-1..\infty} \\
&> [\![u']\!]_e\gamma_{0..\ell+j-2} \cdot (C_{\ell+j-1}[\![v']\!]_e\gamma_{0..\ell+j-2} \cdot (E_{\ell+j-1}[\![w']\!]_e\gamma_{0..\ell+j-2} \cdot \delta) \cdot \delta) \cdot \gamma_{\ell+j-1..\infty} \\
&\quad \text{by Lemma 2.15}
\end{aligned}
$$

$\geq [\![u']\!]_e\gamma_{0..\ell+j-2} \,.\, (E_{\ell+j-1}[\![v']\!]_e\gamma_{0..\ell+j-2} \,.\, (E_{\ell+j-1}[\![w']\!]_e\gamma_{0..\ell+j-2} \,.\, \delta) \,.\, \delta) \,.\, \gamma_{\ell+j-1..\infty}$
by (P5) and Lemma 2.11
while:

$$
\begin{aligned}
&[\![q_{\mathcal{C}}(\mathtt{ev}^\ell[\![u]\!]_q(\mathtt{ev}^\ell[\![v]\!]_q[\![w]\!]_q))]\!]_e\gamma \\
&= [\![\mathtt{ev}^{\ell+j-1}((q_{i_1}\circ\ldots\circ q_{i_{j-1}}\circ q_{i_j+1}\circ\ldots\circ q_{i_p+1})[\![u]\!]_q)((q_{i_1}\circ\ldots\circ q_{i_{j-1}})(\mathtt{ev}^\ell[\![v]\!]_q[\![w]\!]_q))]\!]_e\gamma \\
&= [\![\mathtt{ev}^{\ell+j-1}u'((q_{i_1}\circ\ldots\circ q_{i_{j-1}})(\mathtt{ev}^\ell[\![v]\!]_q[\![w]\!]_q))]\!]_e\gamma \\
&= [\![\mathtt{ev}^{\ell+j-1}u'(\mathtt{ev}^{\ell+j-1}v'w')]\!]_e\gamma \\
&= [\![u']\!]_e\gamma_{0..\ell+j-2} \,.\, (E_{\ell+j-1}[\![\mathtt{ev}^{\ell+j-1}v'w']\!]_e\gamma_{0..\ell+j-2} \,.\, \delta) \,.\, \gamma_{\ell+j-1..\infty} \\
&= [\![u']\!]_e\gamma_{0..\ell+j-2} \,.\, (E_{\ell+j-1}[\![v']\!]_e\gamma_{0..\ell+j-2} \,.\, (E_{\ell+j-1}[\![w']\!]_e\gamma_{0..\ell+j-2} \,.\, \delta) \,.\, \delta) \,.\, \gamma_{\ell+j-1..\infty}
\end{aligned}
$$

It just remains to apply Lemma 2.13 to get $[\![s]\!]_{eq}\gamma > [\![t]\!]_{eq}\gamma$, i.e. $s \succ_{eq} t$. □

Definition 2.6 *Let $(\mathcal{S}ort)$ be the set of rules in groups (D), (E), (F), plus the rules $(\circ^\ell)$, $(Q\circ^\ell)$, $(\mathtt{ev}\circ^\ell)$, $(\mathtt{ev}Q^\ell)$, $(\mathtt{ev}1^\ell)$, and $(\mathtt{ev}\uparrow^\ell)$.*

Let $(\mathcal{S}ort)_{H1}$ be $(\mathcal{S}ort)$ plus rule $(\eta\Uparrow^\ell)$, and $(\mathcal{S}ort)_H$ be $(\mathcal{S}ort)_{H1}$ plus rule $(\eta\mathtt{ev}^\ell)$.

Finally, let $(\mathcal{S}ort)^\bullet$, $(\mathcal{S}ort)^\bullet_{H1}$ and $(\mathcal{S}ort)^\bullet_H$ be these systems respectively plus the rules $(\bullet^\ell)$, $(\mathtt{ev}\bullet^\ell)$, $(\star^\ell)$ and $(\mathtt{ev}\star^\ell)$.

Recall that a *convergent* rewrite system is a terminating and confluent one. In particular, every term has a unique normal form, and every reduction eventually leads to it in a convergent rewrite system. The following lemma is not used in the sequel, but is interesting in its own right.

Lemma 2.23 *$(\mathcal{S}ort)$, $(\mathcal{S}ort)_{H1}$, $(\mathcal{S}ort)_H$, $(\mathcal{S}ort)^\bullet$, $(\mathcal{S}ort)^\bullet_{H1}$ and $(\mathcal{S}ort)^\bullet_H$ terminate. $(\mathcal{S}ort)$, $(\mathcal{S}ort)_{H1}$, $(\mathcal{S}ort)^\bullet$ and $(\mathcal{S}ort)^\bullet_{H1}$ are convergent rewrite systems.*

Proof: That $(\mathcal{S}ort)^\bullet_H$ (hence the other systems) terminates is a consequence of Lemma 2.14, and Lemmas 2.16, 2.17, 2.18, 2.19, 2.20, 2.21 and 2.22.

Moreover, $(\mathcal{S}ort)$, $(\mathcal{S}ort)_{H1}$, $(\mathcal{S}ort)^\bullet$ and $(\mathcal{S}ort)^\bullet_{H1}$ are locally confluent, as shown by a Knuth-Bendix-style completion procedure (see Sections 4, 5, 6 and 7 of [GL95]). Since they are terminating, they are therefore confluent, hence convergent. □

2.5 The $[\![\_]\!]_\_$ Interpretation

So $(\mathcal{S}ort)^\bullet_{H1}$ is terminating. But just adding $(\mathtt{ev}id^1)$ turns it into a non-terminating system (see Lemma 2.2). So it is here that types start to play a role.

We first make the following observation, which will allow us to cut down on the number of rules that we have to examine:

Definition 2.7 *Let the $\lambda\mathtt{ev}Q^+$-terms be those $\lambda\mathtt{ev}Q$-terms in the language defined by the following grammar:*

$$
\begin{aligned}
T^+ &::= \lambda^\ell T^+ \mid T^+ \star^\ell T^+ \mid T^+ \circ^\ell_T S^+ \mid 1^\ell \mid \mathtt{ev}^{\ell+1}_T T^+ S^+ \mid Q^{\ell+1}_T T^+ \\
S^+ &::= S^+ \circ^\ell_S S^+ \mid id^\ell \mid T^+ \bullet^\ell S^+ \mid \uparrow^\ell \mid \Uparrow^\ell S^+ \mid \mathtt{ev}^{\ell+1}_S S^+ S^+ \mid Q^{\ell+1}_S S^+
\end{aligned}
$$

where $\ell \geq 1$.

Let Σ^+ (resp. Σ^+_H) denote the subset of rules in Σ (resp. Σ_H) defined as follows: all rules in group (B) at levels $\ell \geq 1$ except (β^ℓ), all rules in group (C) at levels $\ell \geq 2$, all rules in group (D) at levels $1 \leq \ell < \mathcal{L}$, all rules in (E) and (F) at levels $2 \leq \ell < \mathcal{L}$ (resp. and $(\eta\mathtt{ev}^\ell)$ for $\ell \geq 2$, plus $(\eta\Uparrow^\ell)$, $(\eta\bullet^\ell)$ and $(\eta\bullet\circ^\ell)$ for $\ell \geq 1$).

Observe that Σ^+ and Σ^+_H define rewrite rules on $\lambda\mathtt{ev}Q^+$-terms, yielding $\lambda\mathtt{ev}Q^+$-terms. In fact:

Lemma 2.24 *Σ (resp. Σ_H) terminates on the set of typed $\lambda\mathtt{ev}Q$-terms if and only if Σ^+ (resp. Σ^+_H) terminates on the set of typed $\lambda\mathtt{ev}Q^+$-terms.*

Proof: The only if direction is obvious. For the if direction, any infinite derivation in Σ (resp. Σ_H) translates by $u \mapsto u`\rho$, where ρ is an environment $[x_1 \mapsto 0, \ldots, x_n \mapsto n-1]$, where $x_1, \ldots, x_n$ contain all the free variables in the derivation, into an infinite derivation in Σ^+ (resp. Σ_H^+) by Theorem 3.18 (resp. Lemma 4.10), part II. (Check that the resulting quoted rule or sequence of rules is indeed in Σ^+, resp. Σ_H^+.) By Theorem 3.3, part II, the quoted terms are also well-typed. Hence the claim. □

We now interpret typed $\lambda\mathtt{ev}Q^+$-terms into another typed calculus, the typed $\lambda\oplus$-calculus.

Definition 2.8 *The* positive θ-types θ^+ *and the* negative θ-types θ^- *are defined by the following grammar:*

$$\begin{array}{ll} \theta^+ & ::= o \mid \theta^- \to \theta^+ \\ \theta^- & ::= \top \mid \theta^+ \times \theta^- \end{array}$$

where o is a distinguished base type.

We go from types to θ-types by forgetting type arrows $\Rightarrow$, while converting $\stackrel{\Box}{\Rightarrow}$ to $\to$. This is summarized as follows:

Definition 2.9 *Call a* signature Σ *any expression of the form $\theta_1^-, \ldots, \theta_n^- \leadsto \theta$, where $n \geq 0$ and θ is either a negative θ-type or o. Its* arity *is n. It is a* term signature *if θ is o, and a* stack signature *otherwise.*

Given a term signature $\Sigma = \theta_1^-, \ldots, \theta_n^- \leadsto o$, let $\Sigma^\bullet$ be the positive type $\theta_1^- \to \ldots \to \theta_n^- \to o$. Also, write $\theta^- \leadsto \Sigma$ for $\theta^-, \theta_1^-, \ldots, \theta_n^- \leadsto o$.

Define the following translation from $\lambda\mathtt{ev}Q$ types to signatures:

$$\begin{array}{ll} [\![b]\!] = (\leadsto o) \text{ for any base type } b & \\ [\![\tau_1 \Rightarrow \tau_2]\!] = [\![\tau_2]\!] & [\![\varsigma]\!] = (\leadsto [\![\varsigma]\!]_1) \\ [\![\top]\!]_1 = \top & [\![\tau \times \varsigma]\!]_1 = [\![\tau]\!]^\bullet \times [\![\varsigma]\!]_1 \\ [\![\varsigma \stackrel{\Box}{\Rightarrow} \tau]\!] = [\![\varsigma]\!]_1 \leadsto [\![\tau]\!] & \end{array}$$

Lemma 2.25 *The $[\![\_]\!]$ translation on types is well-defined. Moreover, if τ is a term type, then $[\![\tau]\!]$ is a term signature; if ς is a stack type, then $[\![\varsigma]\!]_1$ is a negative type; and if μ is a metastack type, then $[\![\mu]\!]$ is a stack signature.*

Proof: By structural induction on the argument Φ of the translation. If Φ is a base type b, then it is a term type and a positive type, so $[\![\Phi]\!]$ is a term signature. If Φ is a function type $\tau_1 \Rightarrow \tau_2$, then it is a term type; moreover, τ_2 is a term type, so by induction hypothesis $[\![\tau_2]\!]$, hence $[\![\Phi]\!]$ is a term signature. If Φ is $\top$, then it is a stack type and a negative type; so $[\![\Phi]\!]_1$ is a negative type and $[\![\Phi]\!]$ is a stack signature. If Φ is $\tau \times \varsigma$, then it is a stack type, and by induction hypothesis $[\![\tau]\!]$ is a term signature and $[\![\varsigma]\!]_1$ is a negative type; so $[\![\tau]\!]^\bullet$ is a positive type, and $[\![\Phi]\!]_1$ is indeed a negative type; it also follows that $[\![\Phi]\!]$ is a stack signature.

If Φ is $\varsigma \stackrel{\Box}{\Rightarrow} \tau$, then it is a term type; by induction hypothesis $[\![\varsigma]\!]_1$ is a negative type, $[\![\tau]\!]$ is a term signature, so $[\![\Phi]\!]$ is a term signature. And if Φ is $\varsigma \stackrel{\Box}{\Rightarrow} \mu$, then it is a metastack type and by a similar argument $[\![\Phi]\!]$ is a stack signature. □

Definition 2.10 *For every $\lambda\mathtt{ev}Q$ type Φ, define its* arity $a(\Phi)$ *as the arity of $[\![\Phi]\!]$.*

For every typed $\lambda\mathtt{ev}Q$-term u, define its arity $a(u)$ *as the arity of its type.*

Observe that the arity of a term type $\overline{\varsigma^\ell \stackrel{\Box}{\Rightarrow}} \tau$ is at least ℓ. It may be greater than ℓ: for example, the arity of $\varsigma_0 \stackrel{\Box}{\Rightarrow} \varsigma_1 \stackrel{\Box}{\Rightarrow} b \Rightarrow \varsigma_2 \stackrel{\Box}{\Rightarrow} b'$ is 3, not 2.

Definition 2.11 *The* $\lambda\oplus$-calculus *is the extension of the λ-calculus with pairs $\langle \_, \_ \rangle$, projections π_1 and π_2, the unary operators ϵ and ι, the binary operators $\oplus$ and A, and the $n+1$-ary operator L, for each $n \geq 1$.*

The application of L to $n+1$ arguments $t_1, \ldots, t_n, t$ is written $L(t_1, \ldots, t_n; t)$.

The typing rules are given in Figure 3, and the reduction rules are in Figure 5.

We omit type indices on variables when they should be obvious. For example, in rule (L), the variables $x_1, \ldots, x_n$ have the types of $t_1, \ldots, t_n$ respectively as indices. Moreover, to make the notation lighter, we assume that $\oplus$ is right-associative, that is, $s \oplus t \oplus r$ denotes $s \oplus (t \oplus r)$.

$$\frac{}{\vdash x_{\theta^-} : \theta^-}$$

$$\frac{\vdash s : \theta^- \to \theta^+ \quad \vdash t : \theta^-}{\vdash st : \theta^+} \qquad \frac{\vdash s : \theta^+}{\vdash \lambda x_{\theta^-} \cdot s : \theta^- \to \theta^+}$$

$$\frac{\vdash s : \theta^+ \times \theta^-}{\vdash \pi_1 s : \theta^+} \quad \frac{\vdash s : \theta^+ \times \theta^-}{\vdash \pi_2 s : \theta^-} \qquad \frac{\vdash s : \theta^+ \quad \vdash t : \theta^-}{\vdash \langle s, t\rangle : \theta^+ \times \theta^-}$$

$$\frac{\vdash s : \theta^-}{\vdash \epsilon s : \theta^-} \quad \frac{\vdash s : \theta^-}{\vdash \iota s : \theta^-} \qquad \frac{\vdash s : \theta_1^- \quad \vdash t : \theta_2^-}{\vdash s \oplus t : \theta_2^-}$$

$$\frac{\vdash s : \theta_1^+ \quad \vdash t : \theta_2^+}{\vdash A(s,t) : \theta_1^+} \qquad \frac{\begin{array}{c}\vdash s_1 : \theta_1^- \quad \ldots \vdash s_{n-1} : \theta_{n-1}^- \quad \vdash s_n : \theta_n^- \\ \vdash s : \theta_1^- \to \ldots \to \theta_{n-1}^- \to \theta_1^+ \times \theta_n^- \to \theta_2^+\end{array}}{\vdash L(s_1, \ldots, s_{n-1}, s_n; s) : \theta_2^+}$$

Figure 3: Typing the $\lambda\oplus$-terms

$$\begin{array}{l}
[\![\mathtt{ev}^{\ell+1} uv]\!] s_1 \ldots s_n = [\![u]\!] s_1 \ldots s_\ell (\epsilon [\![v]\!] s_1 \ldots s_\ell) s_{\ell+1} \ldots s_n \\
[\![Q^\ell u]\!] s_1 \ldots s_n = [\![u]\!] s_1 \ldots s_{\ell-1} (s_\ell \oplus s_{\ell+1}) s_{\ell+2} \ldots s_n \\
[\![u \circ^\ell v]\!] s_1 \ldots s_n = [\![u]\!] s_1 \ldots s_{\ell-1} (s_\ell \oplus \epsilon [\![v]\!] s_1 \ldots s_\ell) s_{\ell+1} \ldots s_n \\
[\![\lambda^\ell u]\!] s_1 \ldots s_n = L(s_1, \ldots, s_\ell; \lambda x_1 \ldots x_\ell \cdot [\![u]\!] x_1 \ldots x_\ell s_{\ell+1} \ldots s_n) \\
[\![u \star^\ell v]\!] s_1 \ldots s_n = A([\![u]\!] s_1 \ldots s_n, \lambda y_{\ell+1\, {\theta'}^-_{\ell+1}} \cdot \ldots \cdot \lambda y_{m\, {\theta'}^-_m} \cdot [\![v]\!] s_1 \ldots s_\ell y_{\ell+1} \ldots y_m) \\
\qquad \text{where } v : \tau',\ [\![\tau']\!] = \theta_1^-, \ldots, \theta_\ell^-, {\theta'}^-_{\ell+1}, \ldots, {\theta'}^-_m \leadsto o \\
\qquad \text{with } m \geq \ell \\
[\![u \bullet^\ell v]\!] s_1 \ldots s_\ell = \iota \langle \lambda y_{\ell+1\, {\theta'}^-_{\ell+1}} \cdot \ldots \cdot \lambda y_{m\, {\theta'}^-_m} \cdot [\![u]\!] s_1 \ldots s_\ell y_{\ell+1} \ldots y_m, [\![v]\!] s_1 \ldots s_\ell \rangle \\
\qquad \text{where } u : \tau',\ [\![\tau']\!] = \theta_1^-, \ldots, \theta_\ell^-, {\theta'}^-_{\ell+1}, \ldots, {\theta'}^-_m \leadsto o \\
\qquad \text{with } m \geq \ell \\
[\![id^\ell]\!] s_1 \ldots s_\ell = \iota(s_1 \oplus \ldots \oplus s_{\ell-1} \oplus s_\ell) \\
[\![1^\ell]\!] s_1 \ldots s_n = \pi_1(s_1 \oplus \ldots \oplus s_{\ell-1} \oplus s_\ell) s_{\ell+1} \ldots s_n \\
[\![\uparrow^\ell]\!] s_1 \ldots s_\ell = \pi_2(s_1 \oplus \ldots \oplus s_{\ell-1} \oplus s_\ell) \\
[\![\Uparrow^\ell u]\!] s_1 \ldots s_\ell = \iota \langle \pi_1(s_1 \oplus \ldots \oplus s_\ell), [\![u]\!] s_1 \ldots s_{\ell-1}(s_\ell \oplus \epsilon \pi_2(s_1 \oplus \ldots \oplus s_\ell)) \rangle
\end{array}$$

Figure 4: The $[\![\_]\!]\_$ interpretation ($\ell \geq 1$).

(β)	$(\lambda x \cdot s)t \to s[t/x]$		
(π_1)	$\pi_1\langle s, t\rangle \to s$	$(\oplus)$	$(s \oplus t) \oplus r \to s \oplus (t \oplus r)$
(π_2)	$\pi_2\langle s, t\rangle \to t$		
$(\eta\pi)$	$\langle \pi_1 s, \pi_2 s\rangle \to s$	$(\iota\pi_1)$	$\iota\langle \pi_1 s, t\rangle \oplus s_1 \to (s \oplus \epsilon\pi_2 s) \oplus s_1$
$(\oplus-)$	$s \oplus t \to t$	$(\iota\pi_2)$	$(s_1 \oplus \epsilon\iota\langle \pi_1 s, t\rangle) \oplus s_2 \to (s_1 \oplus \epsilon\pi_2 s) \oplus s_2$
(ϵ)	$\epsilon s \to s$		
(ι)	$\iota s \to s$		

(L) $L(t_1, \ldots, t_k, t[t'_1/x'_1, \ldots, t'_m/x'_m], t_{k+1}, \ldots, t_n; s)$
$\to L(t_1, \ldots, t_k, t'_1, \ldots, t'_m, t_{k+1}, \ldots, t_n;$
$\lambda x_1 \ldots x_k x'_1 \ldots x'_m x_{k+1} \ldots x_n \cdot$
$s x_1 \ldots x_k t x_{k+1} \ldots x_n)$
for any term t where $x'_{1\,\theta'^{-}_1}, \ldots, x'_{m\,\theta'^{-}_m}$ occur free,
the x'_j are pairwise distinct and different from t,
and $0 \leq k < n$, $0 \leq m$
and $\vdash t'_1 : \theta'^{-}_1, \ldots, \vdash t'_m : \theta'^{-}_m$

$(L\oplus)$ $L(t_1, \ldots, t_{n-1}, t \oplus t'; s)$
$\to L(t_1, \ldots, t_{n-1}, t, t'; \lambda x_1 \ldots x_{n-1} x x' \cdot s x_1 \ldots x_{n-1}(x \oplus x'))$

(LC) $L(t_1, \ldots, t_i, \ldots, t_{j-1}, t_j, t_{j+1}, \ldots, t_n; s)$
$\to L(t_1, \ldots, t_i, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n;$
$\lambda x_1 \ldots x_i \ldots x_{j-1} x_{j+1} \ldots x_n \cdot$
$s x_1 \ldots x_i \ldots x_{j-1} x_i x_{j+1} \ldots x_n)$
if $t_i = t_j$, $1 \leq i < j < n$

$(L\epsilon)$ $L(t_1, \ldots, t_n, \epsilon t[t_1/x'_1, \ldots, t_n/x'_n]; s)$
$\to L(t_1, \ldots, t_n;$
$\lambda x_1 \ldots x_n \cdot s x_1 \ldots x_{n-1}(x_n \oplus \epsilon\pi_2(x_1 \oplus \ldots \oplus x_n))$
$(\epsilon\iota\langle \pi_1(x_1 \oplus \ldots \oplus x_n),$
$(\lambda x'_1 \ldots x'_n \cdot t) x_1 \ldots x_{n-1}(x_n \oplus \epsilon\pi_2(x_1 \oplus \ldots \oplus x_n))\rangle))$
where for each i, $1 \leq i \leq n$, $\vdash t'_i : \theta^{-}_i$, where $x'_i = x'_{i\,\theta^{-}_i}$ are pairwise distinct

Figure 5: Reduction rules of the $\lambda\oplus$-calculus

Definition 2.12 *We define the $\lambda\oplus$-term $[\![t]\!]s_1 s_2 \ldots s_n$, for every typed λev$Q$-term t of type Φ of arity n, and for every sequence of n $\lambda\oplus$-terms s_1 of type θ_1^-, ..., s_n of type θ_n^-, where $[\![\Phi]\!] = \theta_1^-, \ldots, \theta_n^- \rightsquigarrow \theta$, as shown in Figure 4, where all λ-bound variables are assumed to be fresh.*

Check that the definition is well-formed, i.e. that all the type constraints are verified. A side-effect of this definition is that $[\![t]\!]s_1 s_2 \ldots s_n$ has θ-type o if t is of sort T, and that it has a negative type if t is a stack.

It just remains to show that the typed $\lambda\oplus$-calculus is terminating and that every rewrite in the typed version of Σ_H is interpreted as some sequence of rewrite steps in the typed $\lambda\oplus$-calculus.

Lemma 2.26 *The typed $\lambda\oplus$-calculus has the subject reduction property, that is, whenever $\vdash s : \theta$ and $s \longrightarrow^* t$, then $\vdash t : \theta$.*

Proof: This is standard for rules (β) , (π_1), (π_2) and $(\eta\pi)$. This is obvious for rules $(\oplus-)$, $(\oplus)$, (ϵ) and (ι).

In the case of rule $(\iota\pi_1)$, to type the left-hand side we must have derived $s : \theta_1^+ \times \theta_1^-$, $t : \theta_2^-$ and $s_1 : \theta_3^-$, and the left-hand side then has type θ_3^-; then $s \oplus \epsilon\pi_2 s$ has type θ_1^-, and the right-hand side has type θ_3^- again. Notice that we could not have simplified this rule to:

$$\iota\langle \pi_1 s, t\rangle \to (s \oplus \epsilon\pi_2 s)$$

because then the left-hand side would have type $\theta_1^+ \times \theta_2^-$, not θ_3^-.

The argument and the remark are similar for rule $(\iota\pi_2)$.

Consider rule (L). To type the left-hand side, we must have derived $\vdash t_i : \theta_i^-$ for every i, $1 \leq i \leq n$, and $\vdash t[t'_1/x'_1, \ldots, t'_m/x'_m] : \theta^-$ for some type θ^-. Moreover, we must have derived:

$$\vdash s : \theta_1^- \to \ldots \to \theta_k^- \to \theta^- \to \theta_{k+1}^- \to \ldots \to \theta_{n-1}^- \to \theta_1^+ \times \theta_n^- \to \theta_2^+$$

and then the type of the left-hand side is θ_2^+. Because the types of x'_j match those of t'_j for every j, $1 \leq j \leq m$, we can also derive $\vdash t : \theta^-$. Then we have:

$$\begin{aligned}\vdash\ & (\lambda x_1 \ldots x_k x'_1 \ldots x'_m x_{k+1} \ldots x_n \cdot \\ & \quad s x_1 \ldots x_k t x_{k+1} \ldots x_n) \\ & : \theta_1^- \to \ldots \to \theta_k^- \to \theta'^-_1 \to \ldots \to \theta'^-_m \to \theta_{k+1}^- \to \ldots \to \theta_{n-1}^- \to \theta_1^+ \times \theta_n^- \to \theta_2^+\end{aligned}$$

So the right-hand side also has type θ_2^+. Observe that this would not work if we allowed $k = n$, hence the more restricted condition $k < n$.

Consider now rule $(L\oplus)$. We must have derived $\vdash t_i : \theta_i^-$ for every i, $1 \leq i \leq n-1$, and also $\vdash t \oplus t' : \theta_n^-$ for some θ_n^-. Therefore we must have derived $\vdash t : \theta^-$ and $\vdash t' : \theta'^-_n$ for some θ^-. Moreover, we must have derived:

$$\vdash s : \theta_1^- \to \ldots \to \theta_{n-1}^- \to \theta_1^+ \times \theta_n^- \to \theta_2^+$$

and the left-hand side then has type θ_2^+. Letting x_1, ..., x_{n-1}, x and x' have types θ_1^-, ..., θ_{n-1}^-, θ^- and $\theta_1^+ \times \theta_n^-$ respectively, it follows that:

$$\vdash (\lambda x_1 \ldots x_{n-1} x x' \cdot s x_1 \ldots x_{n-1}(x \oplus x')) : \theta_1^- \to \ldots \to \theta_{n-1}^- \to \theta^- \to \theta_1^+ \times \theta_n^- \to \theta_2^+$$

so the right-hand side has type θ_2^+ as well.

The case of rule (LC) follows from similar considerations.

Consider finally rule $(L\epsilon)$. To type the left-hand side, we must have derived $\vdash t[t_1/x'_1, \ldots, t_n/x'_n] : \theta^-$ for some type θ^-, so we can also derive $\vdash t : \theta^-$. Moreover, we must have derived:

$$\vdash s : \theta_1^- \to \ldots \to \theta_n^- \to \theta_1^+ \times \theta^- \to \theta_2^+$$

Let x_1, ..., x_{n-1}, x_n have type θ_1^-, ..., θ_{n-1}^- and $\theta_1^+ \times \theta_n^-$ respectively. Then we have:

$$\vdash (x_n \oplus \epsilon\pi_2(x_1 \oplus \ldots \oplus x_n)) : \theta_n^-$$

and:

$$\vdash (\epsilon\iota\langle \pi_1(x_1 \oplus \ldots \oplus x_n), (\lambda x'_1 \ldots x'_n \cdot t)x_1 \ldots x_{n-1}(x_n \oplus \epsilon\pi_2(x_1 \oplus \ldots \oplus x_n))\rangle) : \theta_1^+ \times \theta^-$$

so the expression $\lambda x_1 \ldots x_n \cdot \ldots$ on the right-hand side has type $\theta_1^- \to \ldots \to \theta_{n-1}^- \to \theta_1^+ \times \theta_n^- \to \theta_2^+$, and the right-hand side has type θ_2^+. □

To prove the termination of the typed $\lambda\oplus$-calculus, we shall use Jouannaud and Rubio's higher-order recursive path ordering $\succ_{horpo}$ [JR96]. This ordering uses a well-founded quasi-ordering on types, which we shall simply take to be the identity. In this case, the definition of $\succ_{horpo}$ is exactly the same as for $\succ_{rpo}$, based on a precedence $\succeq$ (with strict part $\succ$ and associated equivalence $\approx$), with the following provisos:

- for every bound variable x, $\lambda x\cdot$ is viewed as a unary function symbol λ, which is strictly less than any other function symbol in the precedence $\succ$;
- every bound variable x is viewed as a constant (i.e., a zero-ary function symbol); any two bound variables are equivalent under $\approx$, and are strictly less than any other constant;

The definition of the rpo can then be enriched by letting some operators having multiset status (as we did before) or, say, lexicographic status. We let @ denote the (invisible) application of the λ-calculus, and take it to have lexicographic status. Two equivalent function symbols must have the same status. Moreover, two equivalent function symbols of lexicographic status must have the same arity.

Then the definition of $\succeq_{horpo}$, $\succ_{horpo}$ and $\approx_{horpo}$ is as follows. Given $s = f(s_1, \ldots, s_m)$ and $t = g(t_1, \ldots, t_n)$, we have $s \succ_{horpo} t$ if and only if:

1. $s_i \succeq_{horpo} t$ for some i, $1 \leq i \leq m$,
2. or $f \succ g$ and $s \succ_{horpo} t_j$ for all j, $1 \leq j \leq n$,
3. or $f \approx g$ has multiset status, and $\{\!|s_1, \ldots, s_m|\!\} \succ_{horpo}^{mul} \{\!|t_1, \ldots, t_n|\!\}$,
4. or $f \approx g$ has lexicographic status (then $m = n$), and for some i, $1 \leq i \leq n$, $s_1 \approx_{horpo} t_1$, $\ldots$, $s_{i-1} \approx_{horpo} t_{i-1}$, $s_i \succ_{horpo} t_i$, $s \succ_{horpo} t_{i+1}$, $\ldots$, $s \succ_{horpo} t_n$.

The main theorem of [JR96] is that, whenever $\succ$ is well-founded, $\succ_{horpo} \cup \xrightarrow{\beta\eta}$ is well-founded on typed terms. Technically speaking, their types only include types built from some set of base types with the function arrow only. This is not a problem: just take the set of all negative types as set of base types.

Recall also that $\succ_{horpo}$ is *monotonic*: $s \succ_{horpo} t$ implies $\mathcal{C}[s] \succ_{horpo} \mathcal{C}[t]$ for every context $\mathcal{C}$ such that all terms are well-typed. And that $\succ_{horpo}$ has the *subterm property*: $\mathcal{C}[s] \succ_{horpo} s$ for any context $\mathcal{C}$ other than $[]$. And finally that $\succ_{horpo}$ is *stable*: if $s \succ_{horpo} t$, then $s\sigma \succ_{horpo} t\sigma$ for any substitution σ.

Lemma 2.27 *The typed $\lambda\oplus$-calculus terminates.*

Proof: Choose the following precedence:

$$L \succ \left\{ \begin{array}{c} \iota \succ \left\{ \begin{array}{c} \oplus \\ \epsilon \\ \pi_2 \end{array} \right. \\ @ \\ \pi_1 \\ \langle \_, \_ \rangle \end{array} \right\} \succ \lambda$$

whereas all variables (viewed as constants) are considered equivalent and incomparable with any other function symbol.

We let @ and $\oplus$ have lexicographic status. L has a status that is a combination of the multiset and lexicographic status; we let:

- $L(s_1, \ldots, s_m; s) \succ_{horpo} L(t_1, \ldots, t_n; t)$ if and only if:

$$\{\!| s_1, \ldots, s_m |\!\} \succ_{horpo}^{mul} \{\!| t_1, \ldots, t_n |\!\} \text{ and } L(s_1, \ldots, s_m; s) \succ_{horpo} t,$$
$$\text{or}$$
$$\{\!| s_1, \ldots, s_m |\!\} \approx_{horpo}^{mul} \{\!| t_1, \ldots, t_n |\!\} \text{ and } s \succ_{horpo} t.$$

This is no real extension of the horpo: let $L(s_1, \ldots, s_m; s)$ be an abbreviation for $L_2(L_1(s_1, \ldots, s_m), s)$, where L_1 has multiset status, L_2 is a binary operator with lexicographic status, and L_1 and L_2 take the place of L in the precedence.

To prove the Lemma, it is enough to prove that the left-hand side of every rule but (β) is greater than the right-hand side under $\succ_{horpo}$.

This is clear for rules (π_1), (π_2), $(\eta\pi)$, $(\oplus-)$, (ϵ) and (ι), by clause 1 of the definition of $\succ_{horpo}$.

Consider rule $(\oplus)$: $s \oplus t \succ_{horpo} s$ by clause 1, and $(s \oplus t) \oplus r \succ_{horpo} t \oplus r$ by clauses 1 and monotonicity; so by clause 4 (with $i = 1$), $(s \oplus t) \oplus r \succ_{horpo} s \oplus (t \oplus r)$.

Consider rule $(\iota\pi_1)$. We have: $\iota\langle \pi_1 s, t\rangle \succ_{horpo} s$ by clause 1. Because $\iota \succ \pi_2$, $\iota \succ \epsilon$ and $\iota \succ \oplus$, and using clause 2, it follows that $\iota\langle \pi_1 s, t\rangle \succ_{horpo} s \oplus \epsilon\pi_2 s$. By monotonicity, $(\iota\langle \pi_1 s, t\rangle) \oplus s_1 \succ_{horpo} (s \oplus \epsilon\pi_2 s) \oplus s_1$.

Similarly for rule $(\iota\pi_2)$: $\iota\langle \pi_1 s, t\rangle \succ_{horpo} \pi_2 s$, so by monotonicity $(s_1 \oplus \iota\langle \pi_1 s, t\rangle) \oplus s_2 \succ_{horpo} (s_1 \oplus \epsilon\pi_2 s) \oplus s_2$.

We now deal with rules (L), $(L\oplus)$, (LC) and $(L\epsilon)$. We first claim that: (1) for every term t, for every variable x, $t \succeq_{horpo} x$. Indeed, because the language of $\lambda\oplus$ does not contain any constants, t must contain some variable y (free or bound) as a subterm. Therefore $t \succeq_{horpo} y \approx_{horpo} x$.

Rule (L). Because $x'_1, \ldots, x'_m$ are all free in t, and because t is neither of these variables we have $t \succeq_{horpo} x'_j$ for every j, $1 \leq j \leq m$, by the subterm property. By stability, $t[t'_1/x'_1, \ldots, t'_m/x'_m] \succ_{horpo} t'_j$. It follows that:

$$\{\!| t_1, \ldots, t_k, t[t'_1/x'_1, \ldots, t'_m/x'_m], t_{k+1}, \ldots, t_n |\!\} \succ_{horpo}^{mul} \{\!| t_1, \ldots, t_k, t'_1, \ldots, t'_m, t_{k+1}, \ldots, t_n |\!\} \quad (2)$$

(Observe that this holds even when $m = 0$.) Let l denote the right-hand side of the rule. We have: (3) $l \succ_{horpo} s$ by the subterm property; (4) $l \succ_{horpo} x_i$ for every i, $1 \leq i \leq n$ (indeed, $l \succ_{horpo} t_i$ by the subterm property, and $t_i \succeq_{horpo} x_i$ by (1)); (5) $l \succ_{horpo} t$. To prove (5), observe that by (1), $t'_j \succeq_{horpo} x'_j$ for every j, $1 \leq j \leq m$; by stability, it follows that $t[t'_1/x'_1, \ldots, t'_m/x'_m] \succeq_{horpo} t$, so by clause 1, $l \succ_{horpo} t$. By clause 2, since $L \succ @$ and using (3), (4) and (5):

$$l \succ_{horpo} s x_1 \ldots x_k t x_{k+1} \ldots x_n$$

Since $L \succ \lambda$, and using clause 3:

$$l \succ_{horpo} \lambda x_1 \ldots x_k x'_1 \ldots x'_m x_{k+1} \ldots x_n \cdot s x_1 \ldots x_k t x_{k+1} \ldots x_n \quad (6)$$

By (2) and (6), using clause 4, it follows that l is greater than the right-hand side of rule (L) in $\succ_{horpo}$.

The argument is the same for rule $(L\oplus)$. For rule (LC), letting l be the left-hand side, we have:

$$l \succ_{horpo} \lambda x_1 \ldots x_i \ldots x_{j-1} x_{j+1} \ldots x_n \cdot s x_1 \ldots x_i \ldots x_{j-1} x_i x_{j+1} \ldots x_n$$

by similar arguments, using the facts that $L \succ @$, $L \succ \lambda$ and $t_k \succeq_{horpo} x_k$ for every k, $1 \leq k \leq n$ by (1). Since the first group of arguments to L decreases in the multiset ordering, l is greater than the right-hand side by clause 4.

Finally, consider rule $(L\epsilon)$. Trivially, we have:

$$\{\!| t_1, \ldots, t_n, \epsilon t[t_1/x'_1, \ldots, t_n/x'_n] |\!\} \succ_{horpo}^{mul} \{\!| t_1, \ldots, t_n |\!\} \quad (7)$$

By the subterm property: (8) $l \succ_{horpo} s$. By (1), $t_i \succeq_{horpo} x_i$ for every i, $1 \leq i \leq n$. By clause 1: (9) $l \succ_{horpo} x_i$ for every i. Since $L \succ \oplus$, $L \succ \epsilon$ and $L \succ \pi_2$, and using clause 2:

$$l \succ_{horpo} x_n \oplus \epsilon\pi_2(x_1 \oplus \ldots \oplus x_n) \quad (10)$$

Moreover, $t_i \succeq_{horpo} x'_i$ for every i by (1) again, so by monotonicity $t[t_1/x'_1, \ldots, t_n/x'_n] \succeq_{horpo} t$. By clause 1, $l \succ_{horpo} t$. Since $L \succ \epsilon$, $L \succ \iota$, $L \succ \langle \_, \_ \rangle$, $L \succ \pi_1$, $L \succ \oplus$, $L \succ \lambda$, $L \succ @$ and $L \succ \pi_2$, by clause 2:

$$l \succ_{horpo} \epsilon\iota\langle \begin{array}{l} \pi_1(x_1 \oplus \ldots \oplus x_n), \\ (\lambda x'_1 \ldots x'_n \cdot t)x_1 \ldots x_{n-1}(x_n \oplus \epsilon\pi_2(x_1 \oplus \ldots \oplus x_n))\rangle \end{array} \quad (11)$$

By (8), (9), (10), (11) and since $L \succ \lambda$, $L \succ @$, l is greater than the big $\lambda x_1 \ldots x_n \cdot \ldots$ on the right-hand side. By (7) and clause 4, it follows that l is greater than the right-hand side. □

Lemma 2.28 *For every typed λ*`ev`*Q-term u, for every $\lambda\oplus$-terms $s_1, \ldots, s_n$ of the correct types, $s_1, \ldots, s_n$ are proper subterms of $[\![u]\!]s_1 \ldots s_n$. Moreover:*

$$[\![u]\!]s_1 \ldots s_n = ([\![u]\!]x_1 \ldots x_n)[s_1/x_1, \ldots, s_n/x_n]$$

for every n distinct variables $x_1, \ldots, x_n$.

Proof: Easy induction on the definition of $[\![u]\!]s_1 \ldots s_n$. The only difficulty lies in checking that $s_1, \ldots, s_n$ indeed occur as subterms of $[\![u]\!]s_1 \ldots s_n$: it is precisely the purpose of terms like $s \oplus t$ to represent t while keeping s around. □

It follows:

Lemma 2.29 *If $s_i \longrightarrow^* s'_i$ (resp. $s_i \longrightarrow^+ s'_i$) for some i, $1 \leq i \leq n$, then:*

$$[\![u]\!]s_1 \ldots s_{i-1}s_is_{i+1} \ldots s_n \longrightarrow^* [\![u]\!]s_1 \ldots s_{i-1}s'_is_{i+1} \ldots s_n$$

(resp. $\longrightarrow^+$)

Another monotonicity property is the following:

Lemma 2.30 *Let $\succ_\lambda$ be defined by $u \succ_\lambda v$ if and only u and v have the same θ-type of arity n, and for every $s_1, \ldots, s_n$ of the right θ-types, $[\![u]\!]s_1 \ldots s_n \longrightarrow^+ [\![v]\!]s_1 \ldots s_n$ in the typed $\lambda\oplus$-calculus.*

For every context $\mathcal{C}$ respecting the θ-types, if $u \succ_\lambda v$, then $\mathcal{C}[u] \succ_\lambda \mathcal{C}[v]$.

Proof: An easy induction on the context $\mathcal{C}$. □

We can now proceed to examine how each rule in Σ_H^+ translates by the $[\![\_]\!]$ translation.

We say that a rule $l \to r$ is *decreasing* if and only if $[\![l]\!]s_1 \ldots s_n \longrightarrow^+ [\![r]\!]s_1 \ldots s_n$ for every $s_1, \ldots, s_n$ of the right type. We say that it is *non-increasing* if $[\![l]\!]s_1 \ldots s_n \longrightarrow^* [\![r]\!]s_1 \ldots s_n$ for every $s_1, \ldots, s_n$ of the right type.

Lemma 2.31

$$[\![\Uparrow^\ell u]\!]s_1 \ldots s_\ell = [\![1^\ell \bullet^\ell (u\circ^\ell \uparrow^\ell)]\!]s_1 \ldots s_\ell$$

Proof:

$$\begin{array}{l} [\![1^\ell \bullet^\ell u\circ^\ell \uparrow^\ell]\!]s_1 \ldots s_\ell \\ = \langle [\![1^\ell]\!]s_1 \ldots s_\ell, [\![u\circ^\ell \uparrow^\ell]\!]s_1 \ldots s_\ell \rangle \\ = \iota\langle \pi_1(s_1 \oplus \ldots \oplus s_\ell), [\![u]\!]s_1 \ldots s_{\ell-1}(s_\ell \oplus \epsilon[\![\uparrow^\ell]\!]s_1 \ldots s_\ell)\rangle \\ = \iota\langle \pi_1(s_1 \oplus \ldots \oplus s_\ell), [\![u]\!]s_1 \ldots s_{\ell-1}(s_\ell \oplus \epsilon\pi_2(s_1 \oplus \ldots \oplus s_\ell))\rangle \\ = [\![\Uparrow^\ell u]\!]s_1 \ldots s_\ell \end{array}$$

□

Lemma 2.32 *Rule* (`ev`Q^ℓ) *is decreasing for every $\ell \geq 2$.*

Proof: Let $\ell \geq 1$, and consider the rule (`ev`$Q^{\ell+1}$):

$$\begin{array}{ll} [\![\mathtt{ev}^{\ell+1}(Q^{\ell+1}u)v]\!]s_1 \ldots s_n & \\ = [\![Q^{\ell+1}u]\!]s_1 \ldots s_\ell(\epsilon[\![v]\!]s_1 \ldots s_\ell)s_{\ell+1} \ldots s_n & \\ = [\![u]\!]s_1 \ldots s_\ell(\epsilon([\![v]\!]s_1 \ldots s_\ell) \oplus s_{\ell+1})s_{\ell+2} \ldots s_n & \\ \longrightarrow [\![u]\!]s_1 \ldots s_n & \text{by } (\oplus-) \end{array}$$

□

Lemma 2.33 *Rule ($\eta\mathtt{ev}^\ell$) is non-increasing, for every $\ell \geq 1$.*

Proof:

$$\begin{aligned}
&[\![\mathtt{ev}^{\ell+1}(Q^\ell u)v]\!] s_1 \ldots s_n \\
&= [\![Q^\ell u]\!] s_1 \ldots s_\ell(\epsilon[\![v]\!] s_1 \ldots s_\ell) s_{\ell+1} \ldots s_n \\
&= [\![u]\!] s_1 \ldots s_{\ell-1}(s_\ell \oplus \epsilon[\![v]\!] s_1 \ldots s_\ell) s_{\ell+1} \ldots s_n \\
&= [\![u \circ^\ell v]\!] s_1 \ldots s_n
\end{aligned}$$

□

Lemma 2.34 *Rule ($Q\circ^\ell$) is decreasing, for every $\ell \geq 1$.*

Proof:

$$\begin{aligned}
&[\![Q^\ell u \circ^\ell v]\!] s_1 \ldots s_n \\
&= [\![Q^\ell u]\!] s_1 \ldots s_{\ell-1}(s_\ell \oplus \epsilon[\![v]\!] s_1 \ldots s_\ell) s_{\ell+1} \ldots s_n \\
&= [\![u]\!] s_1 \ldots s_{\ell-1}((s_\ell \oplus \epsilon[\![v]\!] s_1 \ldots s_\ell) \oplus s_{\ell+1}) s_{\ell+2} \ldots s_n \\
&\longrightarrow [\![u]\!] s_1 \ldots s_{\ell-1}(s_\ell \oplus (\epsilon([\![v]\!] s_1 \ldots s_\ell) \oplus s_{\ell+1})) s_{\ell+2} \ldots s_n \quad \text{by } (\oplus) \\
&\longrightarrow [\![u]\!] s_1 \ldots s_{\ell-1}(s_\ell \oplus s_{\ell+1}) s_{\ell+2} \ldots s_n \quad \text{by } (\oplus -) \\
&= [\![Q^\ell u]\!] s_1 \ldots s_n
\end{aligned}$$

□

To deal with the rules involving λ^ℓ, which are the most difficult, first prove a few auxiliary lemmas.

Lemma 2.35 *For every term t where $x'_1, \ldots, x'_m$ occur free, are pairwise distinct and none is t itself, with $0 \leq k < n$, $0 \leq m$:*

$$\begin{aligned}
&L(t_1, \ldots, t_k, t[t'_1/x'_1, \ldots, t'_m/x'_m], t_{k+1}, \ldots, t_n; \lambda x_1 \ldots x_k x x_{k+1} \ldots x_n \cdot s) \\
&\longrightarrow^+ L(t_1, \ldots, t_k, t'_1, \ldots, t'_m, t_{k+1}, \ldots, t_n; \lambda x_1 \ldots x_k x'_1 \ldots x'_m x_{k+1} \ldots x_n \cdot s[t/x])
\end{aligned}$$

Proof: By rule (L) and $n+1$ applications of rule (β). □

Lemma 2.36 *For every $0 \leq k \leq n$:*

$$\begin{aligned}
&L(t_1, \ldots, t_k, t \oplus t', t_{k+1}, \ldots, t_n; \lambda x_1 \ldots x_k x x_{k+1} \ldots x_n \cdot s) \\
&\longrightarrow^+ L(t_1, \ldots, t_k, t, t', t_{k+1}, \ldots, t_n; \lambda x_1 \ldots x_k x'_1 x'_2 x_{k+1} \ldots x_n \cdot s[x'_1 \oplus x'_2/x])
\end{aligned}$$

Proof: By Lemma 2.35 if $k < n$, otherwise by rule ($L\oplus$) and n applications of rule (β). □

Lemma 2.37 *For every $1 \leq i < j < n$, if $t_i = t_j$:*

$$\begin{aligned}
&L(t_1, \ldots, t_i, \ldots, t_{j-1}, t_j, t_{j+1}, \ldots, t_n; \lambda x_1 \ldots x_n \cdot s) \\
&\longrightarrow^+ L(t_1, \ldots, t_i, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n; \\
&\qquad \lambda x_1 \ldots x_{j-1} x_{j+1} \ldots x_n \cdot s[x_i/x_j])
\end{aligned}$$

Proof: By rule (LC) and n applications of rule (β). □

Lemma 2.38

$$\begin{aligned}
&L(t_1, \ldots, t_n, \epsilon t[t_1/x'_1, \ldots, t_n/x'_n]; \lambda x_1 \ldots x_{n+1} \cdot s) \\
&\longrightarrow^+ L(t_1, \ldots, t_n; \\
&\qquad \lambda x_1 \ldots x_n \cdot s[x_n \oplus \epsilon\pi_2(x_1 \oplus \ldots \oplus x_n)/x_n, \\
&\qquad\quad \epsilon\iota\langle \pi_1(x_1 \oplus \ldots \oplus x_n), \\
&\qquad\qquad t[x_1/x'_1, \ldots, x_{n-1}/x'_{n-1}, (x_n \oplus \epsilon\pi_2(x_1 \oplus \ldots \oplus x_n))/x'_n]\rangle/x_{n+1}])
\end{aligned}$$

Proof: By rule ($L\epsilon$) and $2n+1$ applications of rule (β). □

Lemma 2.39 *Rule ($Q^\ell\lambda^{\mathcal{L}}$) is decreasing, for every $1 \leq \ell < \mathcal{L}$.*

Proof:

$$
\begin{aligned}
&[\![Q^\ell(\lambda^{\mathcal{L}-1}u)]\!]s_1 \dots s_n \\
&= [\![\lambda^{\mathcal{L}-1}u]\!]s_1 \dots s_{\ell-1}(s_\ell \oplus s_{\ell+1})s_{\ell+2} \dots s_n \\
&= L(s_1, \dots, s_{\ell-1}, s_\ell \oplus s_{\ell+1}, s_{\ell+2}, \dots, s_{\mathcal{L}}; \lambda x_1 \dots x_{\mathcal{L}-1} \cdot [\![u]\!]x_1 \dots x_{\mathcal{L}-1}s_{\mathcal{L}+1} \dots s_n) \\
&\longrightarrow^+ L(s_1, \dots, s_{\ell-1}, s_\ell, s_{\ell+1}, s_{\ell+2}, \dots, s_{\mathcal{L}}; \\
&\qquad \lambda x_1 \dots x_{\ell-1}x'_1x'_2x_{\ell+1} \dots x_{\mathcal{L}-1} \cdot \\
&\qquad\quad [\![u]\!]x_1 \dots x_{\ell-1}(x'_1 \oplus x'_2)x_{\ell+1} \dots x_{\mathcal{L}-1}s_{\mathcal{L}+1} \dots s_n) \qquad \text{by Lemma 2.36}
\end{aligned}
$$

while:

$$
\begin{aligned}
&[\![\lambda^{\mathcal{L}}(Q^\ell u)]\!]s_1 \dots s_n \\
&= L(s_1, \dots, s_{\mathcal{L}}; \lambda x_1 \dots x_{\mathcal{L}} \cdot [\![Q^\ell u]\!]x_1 \dots x_{\mathcal{L}}s_{\mathcal{L}+1} \dots s_n) \\
&= L(s_1, \dots, s_{\mathcal{L}}; \lambda x_1 \dots x_{\mathcal{L}} \cdot [\![u]\!]x_1 \dots x_{\ell-1}(x_\ell \oplus x_{\ell+1})x_{\ell+2} \dots x_{\mathcal{L}}s_{\mathcal{L}+1} \dots s_n)
\end{aligned}
$$

These two terms are α-equivalent. □

Lemma 2.40 *Rule* $(\mathtt{ev}^\ell\lambda^{\mathcal{L}})$ *is decreasing, for every* $1 \leq \ell < \mathcal{L}$.

Proof:

$$
\begin{aligned}
&[\![\mathtt{ev}^\ell(\lambda^{\mathcal{L}}u)v]\!]s_1 \dots s_n \\
&= [\![\lambda^{\mathcal{L}}u]\!]s_1 \dots s_{\ell-1}(\epsilon[\![v]\!]s_1 \dots s_{\ell-1})s_\ell \dots s_n \\
&= L(s_1, \dots, s_{\ell-1}, (\epsilon[\![v]\!]s_1 \dots s_{\ell-1}), s_\ell, \dots, s_{\mathcal{L}-1}; \\
&\qquad \lambda x_1 \dots x_{\mathcal{L}} \cdot [\![u]\!]x_1 \dots x_{\mathcal{L}}s_{\mathcal{L}} \dots s_n) \\
&\longrightarrow^+ L(s_1, \dots, s_{\ell-1}, s_1, \dots, s_{\ell-1}, s_\ell, \dots, s_{\mathcal{L}-1}; \\
&\qquad \lambda x_1 \dots x_{\ell-1}x'_1 \dots x'_{\ell-1}x_{\ell+1} \dots x_{\mathcal{L}} \cdot \\
&\qquad\quad [\![u]\!]x_1 \dots x_{\ell-1}(\epsilon[\![v]\!]x'_1 \dots x'_{\ell-1})x_{\ell+1} \dots x_{\mathcal{L}}s_{\mathcal{L}} \dots s_n) \qquad \text{by Lemma 2.35, which is applicable because of Lemma 2.28} \\
&\longrightarrow^+ L(s_1, \dots, s_{\ell-1}, s_\ell, \dots, s_{\mathcal{L}-1}; \\
&\qquad \lambda x_1 \dots x_{\ell-1}x_{\ell+1} \dots x_{\mathcal{L}} \cdot \\
&\qquad\quad [\![u]\!]x_1 \dots x_{\ell-1}(\epsilon[\![v]\!]x_1 \dots x_{\ell-1})x_{\ell+1} \dots x_{\mathcal{L}}s_{\mathcal{L}} \dots s_n) \qquad \text{by Lemma 2.37 } \ell-1 \text{ times}
\end{aligned}
$$

while:

$$
\begin{aligned}
&[\![\lambda^{\mathcal{L}-1}(\mathtt{ev}^\ell uv)]\!]s_1 \dots s_n \\
&= L(s_1, \dots, s_{\mathcal{L}-1}; \lambda x_1 \dots x_{\mathcal{L}-1} \cdot [\![\mathtt{ev}^\ell uv]\!]x_1 \dots x_{\mathcal{L}-1}s_{\mathcal{L}} \dots s_n) \\
&= L(s_1, \dots, s_{\mathcal{L}-1}; \lambda x_1 \dots x_{\mathcal{L}-1} \cdot [\![u]\!]x_1 \dots x_{\ell-1}(\epsilon[\![v]\!]x_1 \dots x_{\ell-1})x_\ell \dots x_{\mathcal{L}-1}s_{\mathcal{L}} \dots s_n)
\end{aligned}
$$

which is equal to the latter. □

Lemma 2.41 *Rule* $(\lambda^{\mathcal{L}}\circ^\ell)$ *is decreasing, for every* $1 \leq \ell < \mathcal{L}$.

Proof:

$$
\begin{aligned}
&[\![(\lambda^{\mathcal{L}}u) \circ^\ell w]\!]s_1 \dots s_n \\
&= [\![\mathtt{ev}^{\ell+1}(Q^\ell(\lambda^{\mathcal{L}}u))w]\!]s_1 \dots s_n && \text{by Lemma 2.33} \\
&\longrightarrow^+ [\![\mathtt{ev}^{\ell+1}(\lambda^{\mathcal{L}+1}(Q^\ell u))w]\!]s_1 \dots s_n && \text{by Lemma 2.39} \\
&\longrightarrow^+ [\![\lambda^{\mathcal{L}}(\mathtt{ev}^{\ell+1}(Q^\ell u)w)]\!]s_1 \dots s_n && \text{by Lemma 2.40} \\
&= [\![\lambda^{\mathcal{L}}(u \circ^\ell w)]\!]s_1 \dots s_n && \text{by Lemma 2.33}
\end{aligned}
$$

□

Lemma 2.42 *Rule* $(\mathtt{ev}\lambda^{\ell+1})$ *is decreasing, for every* $\ell \geq 1$.

Proof:

$$
\begin{aligned}
&[\![\mathtt{ev}^{\ell+1}(\lambda^{\ell+1}u)v]\!]s_1 \dots s_n \\
&= [\![\lambda^{\ell+1}u]\!]s_1 \dots s_\ell(\epsilon[\![v]\!]s_1 \dots s_\ell)s_{\ell+1} \dots s_n \\
&= L(s_1, \dots, s_\ell, \epsilon[\![v]\!]s_1 \dots s_\ell; \lambda x_1 \dots x_{\ell+1} \cdot [\![u]\!]x_1 \dots x_{\ell+1}s_{\ell+1} \dots s_n) \\
&\longrightarrow^+ L(s_1, \dots, s_\ell; \\
&\qquad \lambda x_1 \dots x_\ell \cdot [\![u]\!]x_1 \dots x_{\ell-1}(x_\ell \oplus \epsilon\pi_2(x_1 \oplus \dots \oplus x_\ell)) \\
&\qquad\quad (\epsilon\iota\langle\pi_1(x_1 \oplus \dots \oplus x_\ell), \\
&\qquad\qquad [\![v]\!]x_1 \dots x_{\ell-1}(x_\ell \oplus \epsilon\pi_2(x_1 \oplus \dots \oplus x_\ell))\rangle) \\
&\qquad\quad s_{\ell+1} \dots s_n) \qquad \text{by Lemma 2.38}
\end{aligned}
$$

while:

$$
\begin{aligned}
&[\![\lambda^\ell(\mathtt{ev}^{\ell+1}(u\circ^\ell\uparrow^\ell)(1^\ell\bullet^\ell v\circ^\ell\uparrow^\ell))]\!]s_1\ldots s_n\\
&=[\![\lambda^\ell(\mathtt{ev}^{\ell+1}(u\circ^\ell\uparrow^\ell)(\Uparrow^\ell v))]\!]s_1\ldots s_n && \text{by definition}\\
&=L(s_1,\ldots,s_\ell;\lambda x_1\ldots x_\ell\cdot[\![\mathtt{ev}^{\ell+1}(u\circ^\ell\uparrow^\ell)(\Uparrow^\ell v)]\!]x_1\ldots x_\ell s_{\ell+1}\ldots s_n)\\
&=L(s_1,\ldots,s_\ell;\lambda x_1\ldots x_\ell\cdot[\![u\circ^\ell\uparrow^\ell]\!]x_1\ldots x_\ell(\epsilon[\![\Uparrow^\ell v]\!]x_1\ldots x_\ell)s_{\ell+1}\ldots s_n)\\
&=L(s_1,\ldots,s_\ell;\\
&\quad\lambda x_1\ldots x_\ell\cdot\\
&\qquad[\![u]\!]x_1\ldots x_{\ell-1}(x_\ell\oplus\epsilon\pi_2(x_1\oplus\ldots\oplus x_\ell))(\epsilon[\![\Uparrow^\ell v]\!]x_1\ldots x_\ell)s_{\ell+1}\ldots s_n)\\
&=L(s_1,\ldots,s_\ell;\\
&\quad\lambda x_1\ldots x_\ell\cdot\\
&\qquad[\![u]\!]x_1\ldots x_{\ell-1}(x_\ell\oplus\epsilon\pi_2(x_1\oplus\ldots\oplus x_\ell))\\
&\qquad\quad(\epsilon\iota\langle\pi_1(x_1\oplus\ldots\oplus x_\ell),\\
&\qquad\qquad[\![v]\!]x_1\ldots x_{\ell-1}(x_\ell\oplus\epsilon\pi_2(x_1\oplus\ldots\oplus x_\ell))\rangle)\\
&\qquad\quad s_{\ell+1}\ldots s_n) && \text{by Lemma 2.31}
\end{aligned}
$$

□

Lemma 2.43 *Rule* $(\lambda\circ^\ell)$ *is decreasing, for every* $\ell\geq 1$.

Proof:

$$
\begin{aligned}
&[\![\lambda^\ell u\circ^\ell v]\!]s_1\ldots s_n\\
&=[\![\mathtt{ev}^{\ell+1}(Q^\ell(\lambda^\ell u))v]\!]s_1\ldots s_n && \text{by Lemma 2.33}\\
&=[\![Q^\ell(\lambda^\ell u)]\!]s_1\ldots s_\ell(\epsilon[\![v]\!]s_1\ldots s_\ell)s_{\ell+1}\ldots s_n\\
&\longrightarrow^+[\![\lambda^{\ell+1}(Q^\ell u)]\!]s_1\ldots s_\ell(\epsilon[\![v]\!]s_1\ldots s_\ell)s_{\ell+1}\ldots s_n && \text{by Lemma 2.39}\\
&=[\![\mathtt{ev}^{\ell+1}(\lambda^{\ell+1}(Q^\ell u))v]\!]s_1\ldots s_n\\
&\longrightarrow^+[\![\lambda^\ell(\mathtt{ev}^{\ell+1}(Q^\ell u\circ^\ell\uparrow^\ell)(\Uparrow^\ell v))]\!]s_1\ldots s_n && \text{by Lemma 2.42 and Lemma 2.31}\\
&\longrightarrow^+[\![\lambda^\ell(\mathtt{ev}^{\ell+1}(Q^\ell u)(\Uparrow^\ell v))]\!]s_1\ldots s_n && \text{by Lemma 2.34}\\
&=[\![\lambda^\ell(u\circ^\ell(\Uparrow^\ell v))]\!]s_1\ldots s_n && \text{by Lemma 2.33}
\end{aligned}
$$

□

This ends the difficult cases involving λ^ℓ. We now turn to the other, simpler cases.

Lemma 2.44 *For every* $1\leq\ell<\mathcal{L}$,

$$[\![\mathtt{ev}^\ell(\mathtt{ev}^{\mathcal{L}}uv)w]\!]s_1\ldots s_n=[\![\mathtt{ev}^{\mathcal{L}-1}(\mathtt{ev}^\ell uw)(\mathtt{ev}^\ell vw)]\!]s_1\ldots s_n$$

Proof:

$$
\begin{aligned}
&[\![\mathtt{ev}^\ell(\mathtt{ev}^{\mathcal{L}}uv)w]\!]s_1\ldots s_n\\
&=[\![\mathtt{ev}^{\mathcal{L}}uv]\!]s_1\ldots s_{\ell-1}(\epsilon[\![w]\!]s_1\ldots s_{\ell-1})s_\ell\ldots s_n\\
&=[\![u]\!]s_1\ldots s_{\ell-1}(\epsilon[\![w]\!]s_1\ldots s_{\ell-1})s_\ell\ldots s_{\mathcal{L}-2}(\epsilon[\![v]\!]s_1\ldots s_{\ell-1}(\epsilon[\![w]\!]s_1\ldots s_{\ell-1})s_\ell\ldots s_{\mathcal{L}-2})s_{\mathcal{L}-1}\ldots s_n
\end{aligned}
$$

while:

$$
\begin{aligned}
&[\![\mathtt{ev}^{\mathcal{L}-1}(\mathtt{ev}^\ell uw)(\mathtt{ev}^\ell vw)]\!]s_1\ldots s_n\\
&=[\![\mathtt{ev}^\ell uw]\!]s_1\ldots s_{\mathcal{L}-2}(\epsilon[\![\mathtt{ev}^\ell vw]\!]s_1\ldots s_{\mathcal{L}-2})s_{\mathcal{L}-1}\ldots s_n\\
&=[\![u]\!]s_1\ldots s_{\ell-1}(\epsilon[\![w]\!]s_1\ldots s_{\ell-1})s_\ell\ldots s_{\mathcal{L}-2}(\epsilon[\![\mathtt{ev}^\ell vw]\!]s_1\ldots s_{\mathcal{L}-2})s_{\mathcal{L}-1}\ldots s_n\\
&=[\![u]\!]s_1\ldots s_{\ell-1}(\epsilon[\![w]\!]s_1\ldots s_{\ell-1})s_\ell\ldots s_{\mathcal{L}-2}(\epsilon[\![v]\!]s_1\ldots s_{\ell-1}(\epsilon[\![w]\!]s_1\ldots s_{\ell-1})s_\ell\ldots s_{\mathcal{L}-2})s_{\mathcal{L}-1}\ldots s_n
\end{aligned}
$$

□

Lemma 2.45 *For every* $1\leq\ell<\mathcal{L}$,

$$[\![Q^\ell(Q^{\mathcal{L}-1}u)]\!]s_1\ldots s_n\longrightarrow^*[\![Q^{\mathcal{L}}(Q^\ell u)]\!]s_1\ldots s_n$$

Proof:

$$\begin{array}{l}
[\![Q^\ell(Q^{\mathcal{L}-1}u)]\!]s_1\dots s_n \\
= [\![Q^{\mathcal{L}-1}u]\!]s_1\dots s_{\ell-1}(s_\ell \oplus s_{\ell+1})s_{\ell+2}\dots s_n \\
= \left\{\begin{array}{ll} [\![u]\!]s_1\dots s_{\ell-1}((s_\ell \oplus s_{\ell+1})\oplus s_{\ell+2})s_{\ell+3}\dots s_n & \text{if } \mathcal{L}=\ell+1 \\ [\![u]\!]s_1\dots s_{\ell-1}(s_\ell \oplus s_{\ell+1})s_{\ell+2}\dots s_{\mathcal{L}-1}(s_{\mathcal{L}} \oplus s_{\mathcal{L}+1})s_{\mathcal{L}+2}\dots s_n & \text{if } \mathcal{L}\geq\ell+2 \end{array}\right.
\end{array}$$

while:

$$\begin{array}{l}
[\![Q^{\mathcal{L}}(Q^\ell u)]\!]s_1\dots s_n \\
= [\![Q^\ell u]\!]s_1\dots s_{\mathcal{L}-1}(s_{\mathcal{L}} \oplus s_{\mathcal{L}+1})s_{\mathcal{L}+2}\dots s_n \\
= \left\{\begin{array}{ll} [\![u]\!]s_1\dots s_{\ell-1}(s_\ell \oplus (s_{\ell+1}\oplus s_{\ell+2}))s_{\ell+3}\dots s_n & \text{if } \mathcal{L}=\ell+1 \\ [\![u]\!]s_1\dots s_{\ell-1}(s_\ell \oplus s_{\ell+1})s_{\ell+2}\dots s_{\mathcal{L}-1}(s_{\mathcal{L}} \oplus s_{\mathcal{L}+1})s_{\mathcal{L}+2}\dots s_n & \text{if } \mathcal{L}\geq\ell+2 \end{array}\right.
\end{array}$$

These quantities are equal if $\mathcal{L}\geq\ell+2$, and the former reduces to the latter by $(\oplus)$ if $\mathcal{L}=\ell+1$. □

Lemma 2.46 *For every* $1\leq\ell<\mathcal{L}$,

$$[\![\mathtt{ev}^\ell(Q^{\mathcal{L}}u)v]\!]s_1\dots s_n = [\![Q^{\mathcal{L}-1}(\mathtt{ev}^\ell uv)]\!]s_1\dots s_n$$

Proof:

$$\begin{array}{l}
[\![\mathtt{ev}^\ell(Q^{\mathcal{L}}u)v]\!]s_1\dots s_n \\
= [\![Q^{\mathcal{L}}u]\!]s_1\dots s_{\ell-1}(\epsilon[\![v]\!]s_1\dots s_{\ell-1})s_\ell\dots s_n \\
= [\![u]\!]s_1\dots s_{\ell-1}(\epsilon[\![v]\!]s_1\dots s_{\ell-1})s_\ell\dots s_{\mathcal{L}-2}(s_{\mathcal{L}-1}\oplus s_{\mathcal{L}})s_{\mathcal{L}+1}\dots s_n
\end{array}$$

while:

$$\begin{array}{l}
[\![Q^{\mathcal{L}-1}(\mathtt{ev}^\ell uv)]\!]s_1\dots s_n \\
= [\![\mathtt{ev}^\ell uv]\!]s_1\dots s_{\mathcal{L}-2}(s_{\mathcal{L}-1}\oplus s_{\mathcal{L}})s_{\mathcal{L}+1}\dots s_n \\
= [\![u]\!]s_1\dots s_{\ell-1}(\epsilon[\![v]\!]s_1\dots s_{\ell-1})s_\ell\dots s_{\mathcal{L}-2}(s_{\mathcal{L}-1}\oplus s_{\mathcal{L}})s_{\mathcal{L}+1}\dots s_n
\end{array}$$

□

Lemma 2.47 *For every* $1\leq\ell<\mathcal{L}$,

$$[\![Q^\ell(\mathtt{ev}^{\mathcal{L}}uv)]\!]s_1\dots s_n = [\![\mathtt{ev}^{\mathcal{L}+1}(Q^\ell u)(Q^\ell v)]\!]s_1\dots s_n$$

Proof:

$$\begin{array}{l}
[\![Q^\ell(\mathtt{ev}^{\mathcal{L}}uv)]\!]s_1\dots s_n \\
= [\![\mathtt{ev}^{\mathcal{L}}uv]\!]s_1\dots s_{\ell-1}(s_\ell\oplus s_{\ell+1})s_{\ell+2}\dots s_n \\
= [\![u]\!]s_1\dots s_{\ell-1}(s_\ell\oplus s_{\ell+1})s_{\ell+2}\dots s_{\mathcal{L}}(\epsilon[\![v]\!]s_1\dots s_{\ell-1}(s_\ell\oplus s_{\ell+1})s_{\ell+2}\dots s_{\mathcal{L}})s_{\mathcal{L}+1}\dots s_n
\end{array}$$

while:

$$\begin{array}{l}
[\![\mathtt{ev}^{\mathcal{L}+1}(Q^\ell u)(Q^\ell v)]\!]s_1\dots s_n \\
= [\![Q^\ell u]\!]s_1\dots s_{\mathcal{L}}(\epsilon[\![Q^\ell v]\!]s_1\dots s_{\mathcal{L}})s_{\mathcal{L}+1}\dots s_n \\
= [\![u]\!]s_1\dots s_{\ell-1}(s_\ell\oplus s_{\ell+1})s_{\ell+2}\dots s_{\mathcal{L}}(\epsilon[\![Q^\ell v]\!]s_1\dots s_{\mathcal{L}})s_{\mathcal{L}+1}\dots s_n \\
= [\![u]\!]s_1\dots s_{\ell-1}(s_\ell\oplus s_{\ell+1})s_{\ell+2}\dots s_{\mathcal{L}}(\epsilon[\![v]\!]s_1\dots s_{\ell-1}(s_\ell\oplus s_{\ell+1})s_{\ell+2}\dots s_{\mathcal{L}})s_{\mathcal{L}+1}\dots s_n
\end{array}$$

□

Lemma 2.48 *For every* $1\leq\ell<\mathcal{L}$,

$$\begin{array}{lll}
[\![\mathtt{ev}^\ell(u\circ^{\mathcal{L}}v)w]\!]s_1\dots s_n & = & [\![(\mathtt{ev}^\ell uw)\circ^{\mathcal{L}-1}(\mathtt{ev}^\ell vw)]\!]s_1\dots s_n \\
[\![Q^\ell(u\circ^{\mathcal{L}-1}v)]\!]s_1\dots s_n & \longrightarrow^* & [\![(Q^\ell u)\circ^{\mathcal{L}}(Q^\ell v)]\!]s_1\dots s_n
\end{array}$$

Proof:

$$\begin{array}{ll}
[\![\mathtt{ev}^\ell(u\circ^{\mathcal{L}}v)w]\!]s_1\dots s_n & \\
= [\![\mathtt{ev}^\ell(\mathtt{ev}^{\mathcal{L}+1}(Q^{\mathcal{L}}u)v)w]\!]s_1\dots s_n & \text{by Lemma 2.33} \\
= [\![\mathtt{ev}^{\mathcal{L}}(\mathtt{ev}^\ell(Q^{\mathcal{L}}u)w)(\mathtt{ev}^\ell vw)]\!]s_1\dots s_n & \text{by Lemma 2.44} \\
= [\![\mathtt{ev}^{\mathcal{L}}(Q^{\mathcal{L}-1}(\mathtt{ev}^\ell uw))(\mathtt{ev}^\ell vw)]\!]s_1\dots s_n & \text{by Lemma 2.47} \\
= [\![(\mathtt{ev}^\ell uw)\circ^{\mathcal{L}-1}(\mathtt{ev}^\ell vw)]\!]s_1\dots s_n & \text{by Lemma 2.33}
\end{array}$$

$$
\begin{array}{ll}
[\![Q^\ell(u \circ^{\mathcal{L}-1} v)]\!] s_1 \dots s_n & \\
= [\![Q^\ell(\mathtt{ev}^{\mathcal{L}}(Q^{\mathcal{L}-1}u)v)]\!] s_1 \dots s_n & \text{by Lemma 2.33} \\
= [\![\mathtt{ev}^{\mathcal{L}+1}(Q^\ell(Q^{\mathcal{L}-1}u))(Q^\ell v)]\!] s_1 \dots s_n & \text{by Lemma 2.46} \\
\longrightarrow^* [\![\mathtt{ev}^{\mathcal{L}+1}(Q^{\mathcal{L}}(Q^\ell u))(Q^\ell v)]\!] s_1 \dots s_n & \text{by Lemma 2.45} \\
= [\![(Q^\ell u) \circ^{\mathcal{L}} (Q^\ell v)]\!] s_1 \dots s_n & \text{by Lemma 2.33}
\end{array}
$$

□

Lemma 2.49 *For every $1 \leq \ell < \mathcal{L}$,*

$$
\begin{array}{ll}
[\![(\mathtt{ev}^{\mathcal{L}} uv) \circ^\ell w]\!] s_1 \dots s_n & = [\![\mathtt{ev}^{\mathcal{L}}(u \circ^\ell v)(v \circ^\ell w)]\!] s_1 \dots s_n \\
[\![(Q^{\mathcal{L}} u) \circ^\ell w]\!] s_1 \dots s_n & \longrightarrow^* [\![Q^{\mathcal{L}}(u \circ^\ell w)]\!] \\
[\![(u \circ^{\mathcal{L}} v) \circ^\ell w]\!] s_1 \dots s_n & \longrightarrow^* [\![(u \circ^\ell w) \circ^{\mathcal{L}} (v \circ^\ell w)]\!] s_1 \dots s_n
\end{array}
$$

Proof:

$$
\begin{array}{ll}
[\![(\mathtt{ev}^{\mathcal{L}} uv) \circ^\ell w]\!] s_1 \dots s_n & \\
= [\![\mathtt{ev}^{\ell+1}(Q^\ell(\mathtt{ev}^{\mathcal{L}} uv))w]\!] s_1 \dots s_n & \text{by Lemma 2.33} \\
= [\![\mathtt{ev}^{\ell+1}(\mathtt{ev}^{\mathcal{L}+1}(Q^\ell u)(Q^\ell v))w]\!] s_1 \dots s_n & \text{by Lemma 2.47} \\
= [\![\mathtt{ev}^{\mathcal{L}}(\mathtt{ev}^{\ell+1}(Q^\ell u)w)(\mathtt{ev}^{\ell+1}(Q^\ell v)w)]\!] s_1 \dots s_n & \text{by Lemma 2.44} \\
= [\![\mathtt{ev}^{\mathcal{L}}(u \circ^\ell v)(v \circ^\ell w)]\!] s_1 \dots s_n & \text{by Lemma 2.33}
\end{array}
$$

$$
\begin{array}{ll}
[\![(Q^{\mathcal{L}} u) \circ^\ell w]\!] s_1 \dots s_n & \\
= [\![\mathtt{ev}^{\ell+1}(Q^\ell(Q^{\mathcal{L}} u))w]\!] s_1 \dots s_n & \text{by Lemma 2.33} \\
\longrightarrow^* [\![\mathtt{ev}^{\ell+1}(Q^{\mathcal{L}+1}(Q^\ell u))w]\!] s_1 \dots s_n & \text{by Lemma 2.45} \\
= [\![Q^{\mathcal{L}}(\mathtt{ev}^{\ell+1}(Q^\ell u)w)]\!] s_1 \dots s_n & \text{by Lemma 2.46} \\
= [\![Q^{\mathcal{L}}(u \circ^\ell w)]\!] s_1 \dots s_n & \text{by Lemma 2.33}
\end{array}
$$

$$
\begin{array}{ll}
[\![(u \circ^{\mathcal{L}} v) \circ^\ell w]\!] s_1 \dots s_n & \\
[\![\mathtt{ev}^{\ell+1}(Q^\ell(u \circ^{\mathcal{L}} v))w]\!] s_1 \dots s_n & \text{by Lemma 2.33} \\
\longrightarrow^* [\![\mathtt{ev}^{\ell+1}(Q^\ell u \circ^{\mathcal{L}+1} Q^\ell v)w]\!] s_1 \dots s_n & \text{by Lemma 2.48} \\
= [\![(\mathtt{ev}^{\ell+1}(Q^\ell u)w) \circ^{\mathcal{L}} (\mathtt{ev}^{\ell+1}(Q^\ell v)w)]\!] s_1 \dots s_n & \text{by Lemma 2.48} \\
= [\![(u \circ^\ell w) \circ^{\mathcal{L}} (v \circ^\ell w)]\!] s_1 \dots s_n & \text{by Lemma 2.33}
\end{array}
$$

□

Lemma 2.50 *For every $\ell \geq 1$,*

$$[\![\mathtt{ev}^\ell id^\ell w]\!] s_1 \dots s_{\ell-1} \longrightarrow^+ [\![w]\!] s_1 \dots s_{\ell-1}$$

Proof: We have $[\![\mathtt{ev}^\ell id^\ell w]\!] s_1 \dots s_{\ell-1} = [\![id^\ell]\!] s_1 \dots s_{\ell-1}(\epsilon [\![w]\!] s_1 \dots s_{\ell-1}) = \iota(s_1 \oplus \dots \oplus s_{\ell-1} \oplus \epsilon [\![w]\!] s_1 \dots s_{\ell-1})$, which rewrites in $\ell - 1$ applications of $(\oplus-)$ to $\iota\epsilon [\![w]\!] s_1 \dots s_{\ell-1}$, then to $[\![w]\!] s_1 \dots s_{\ell-1}$ by rules (ϵ) and (ι). □

Lemma 2.51 *For every $\ell \geq 1$,*

$$[\![u \circ^\ell id^\ell]\!] s_1 \dots s_n \longrightarrow^+ [\![u]\!] s_1 \dots s_n$$

Proof:

$$
\begin{array}{ll}
[\![u \circ^\ell id^\ell]\!] s_1 \dots s_n & \\
= [\![u]\!] s_1 \dots s_{\ell-1}(s_\ell \oplus \epsilon [\![id^\ell]\!] s_1 \dots s_\ell) s_{\ell+1} \dots s_n & \\
= [\![u]\!] s_1 \dots s_{\ell-1}(s_\ell \oplus \epsilon\iota(s_1 \oplus \dots \oplus s_\ell)) s_{\ell+1} \dots s_n & \\
\longrightarrow^+ [\![u]\!] s_1 \dots s_{\ell-1}(\epsilon\iota(s_1 \oplus \dots \oplus s_\ell)) s_{\ell+1} \dots s_n & \text{by } (\oplus-) \\
\longrightarrow^* [\![u]\!] s_1 \dots s_{\ell-1}(\epsilon\iota s_\ell) s_{\ell+1} \dots s_n & \text{by } (\oplus-)\ \ell - 1 \text{ times} \\
\longrightarrow^+ [\![u]\!] s_1 \dots s_n & \text{by } (\epsilon),\ (\iota)
\end{array}
$$

and where we have implicitly used Lemma 2.29 all along. □

Lemma 2.52 *For every* $\ell \geq 1$,

$$[\![id^\ell \circ^\ell u]\!]s_1 \ldots s_\ell \longrightarrow^+ [\![u]\!]s_1 \ldots s_\ell$$

Proof: $[\![id^\ell \circ^\ell u]\!]s_1 \ldots s_\ell = [\![id^\ell]\!]s_1 \ldots s_{\ell-1}(s_\ell \oplus \epsilon[\![u]\!]s_1 \ldots s_\ell) = \iota(s_1 \oplus \ldots \oplus s_{\ell-1} \oplus s_\ell \oplus \epsilon[\![u]\!]s_1 \ldots s_\ell)$ rewrites to $[\![u]\!]s_1 \ldots s_\ell$ by (ι) and ℓ applications of $(\oplus-)$. □

Lemma 2.53 *For every* $\ell \geq 1$,

$$\begin{array}{ll} [\![1^\ell \circ^\ell (u \bullet^\ell v)]\!]s_1 \ldots s_n & \longrightarrow^+ [\![u]\!]s_1 \ldots s_n \\ [\![\uparrow^\ell \circ^\ell (u \bullet^\ell v)]\!]s_1 \ldots s_\ell & \longrightarrow^+ [\![v]\!]s_1 \ldots s_\ell \end{array}$$

Proof:

$$\begin{array}{ll} [\![1^\ell \circ^\ell (u \bullet^\ell v)]\!]s_1 \ldots s_n & \\ = [\![1^\ell]\!]s_1 \ldots s_{\ell-1}(s_\ell \oplus \epsilon[\![u \bullet^\ell v]\!]s_1 \ldots s_\ell)s_{\ell+1} \ldots s_n & \\ = \pi_1(s_1 \oplus \ldots \oplus s_{\ell-1} \oplus s_\ell \oplus \epsilon[\![u \bullet^\ell v]\!]s_1 \ldots s_\ell)s_{\ell+1} \ldots s_n & \\ = \pi_1(s_1 \oplus \ldots \oplus s_{\ell-1} \oplus s_\ell \oplus \epsilon\iota\langle \lambda\overline{y} \cdot [\![u]\!]s_1 \ldots s_\ell\overline{y}, [\![v]\!]s_1 \ldots s_\ell\rangle)s_{\ell+1} \ldots s_n & \\ \longrightarrow^+ \pi_1(\epsilon\iota\langle \lambda\overline{y} \cdot [\![u]\!]s_1 \ldots s_\ell\overline{y}, [\![v]\!]s_1 \ldots s_\ell\rangle)s_{\ell+1} \ldots s_n & \text{by } (\oplus-)\ \ell \text{ times} \\ \longrightarrow^+ \pi_1(\langle \lambda\overline{y} \cdot [\![u]\!]s_1 \ldots s_\ell\overline{y}, [\![v]\!]s_1 \ldots s_\ell\rangle)s_{\ell+1} \ldots s_n & \text{by } (\epsilon),\ (\iota) \\ \longrightarrow (\lambda\overline{y} \cdot [\![u]\!]s_1 \ldots s_\ell\overline{y})s_{\ell+1} \ldots s_n & \text{by } (\pi_1) \\ \longrightarrow [\![u]\!]s_1 \ldots s_\ell s_{\ell+1} \ldots s_n & \text{by } (\beta) \end{array}$$

where $\overline{y}$ abbreviates the appropriate sequence $y_{\ell+1}, \ldots, y_n$ of fresh variables of the right θ-types.

For the second reduction, involving $\uparrow^\ell$ instead of 1^ℓ, the argument is similar, using (π_2) instead of (π_1), and noticing that $n = \ell$. □

Lemma 2.54 *For every* $\ell \geq 1$,

$$\begin{array}{ll} [\![\mathsf{ev}^{\ell+1} 1^{\ell+1} w]\!]s_1 \ldots s_n & = [\![1^\ell \circ^\ell w]\!]s_1 \ldots s_n \\ [\![\mathsf{ev}^{\ell+1} \uparrow^{\ell+1} w]\!]s_1 \ldots s_\ell & = [\![\uparrow^\ell \circ^\ell w]\!]s_1 \ldots s_\ell \end{array}$$

Proof:

$$\begin{array}{l} [\![\mathsf{ev}^{\ell+1} 1^{\ell+1} w]\!]s_1 \ldots s_n \\ = [\![1^{\ell+1}]\!]s_1 \ldots s_\ell(\epsilon[\![w]\!]s_1 \ldots s_\ell)s_{\ell+1} \ldots s_n \\ = \pi_1(s_1 \oplus \ldots s_\ell \oplus \epsilon[\![w]\!]s_1 \ldots s_\ell)s_{\ell+1} \ldots s_n \\ = \pi_1(s_1 \oplus \ldots s_{\ell-1} \oplus (s_\ell \oplus \epsilon[\![w]\!]s_1 \ldots s_\ell))s_{\ell+1} \ldots s_n \\ = [\![1^\ell]\!]s_1 \ldots s_{\ell-1}(s_\ell \oplus \epsilon[\![w]\!]s_1 \ldots s_\ell)s_{\ell+1} \ldots s_n \\ = [\![1^\ell \circ^\ell w]\!]s_1 \ldots s_n \end{array}$$

and similarly for the other equation. □

Lemma 2.55 *For every* $1 \leq \ell \leq \mathcal{L}$,

$$\begin{array}{ll} [\![\mathsf{ev}^{\ell+1}(u \bullet^{\mathcal{L}+1} v)w]\!]s_1 \ldots s_{\mathcal{L}} & = [\![(\mathsf{ev}^{\ell+1} uw) \bullet^{\mathcal{L}} (\mathsf{ev}^{\ell+1} vw)]\!]s_1 \ldots s_{\mathcal{L}} \\ [\![\mathsf{ev}^{\ell+1}(u \star^{\mathcal{L}+1} v)w]\!]s_1 \ldots s_n & = [\![(\mathsf{ev}^{\ell+1} uw) \star^{\mathcal{L}} (\mathsf{ev}^{\ell+1} vw)]\!]s_1 \ldots s_n \end{array}$$

Proof:

$$\begin{array}{l} [\![\mathsf{ev}^{\ell+1}(u \bullet^{\mathcal{L}+1} v)w]\!]s_1 \ldots s_{\mathcal{L}} \\ = [\![u \bullet^{\mathcal{L}+1} v]\!]s_1 \ldots s_\ell(\epsilon[\![w]\!]s_1 \ldots s_\ell)s_{\ell+1} \ldots s_{\mathcal{L}} \\ = \iota\langle \lambda\overline{y} \cdot [\![u]\!]s_1 \ldots s_\ell(\epsilon[\![w]\!]s_1 \ldots s_\ell)s_{\ell+1} \ldots s_{\mathcal{L}}\overline{y}, [\![v]\!]s_1 \ldots s_\ell(\epsilon[\![w]\!]s_1 \ldots s_\ell)s_{\ell+1} \ldots s_{\mathcal{L}}\rangle \\ = \iota\langle \lambda\overline{y} \cdot [\![\mathsf{ev}^{\ell+1} uw]\!]s_1 \ldots s_{\mathcal{L}}\overline{y}, [\![\mathsf{ev}^{\ell+1} vw]\!]s_1 \ldots s_{\mathcal{L}}\rangle \\ = [\![(\mathsf{ev}^{\ell+1} uw) \bullet^{\mathcal{L}} (\mathsf{ev}^{\ell+1} vw)]\!]s_1 \ldots s_{\mathcal{L}} \end{array}$$

where $\overline{y}$ denotes an appropriate sequence of variables.

$$\begin{array}{l} [\![\mathsf{ev}^{\ell+1}(u \star^{\mathcal{L}+1} v)w]\!]s_1 \ldots s_n \\ = [\![u \star^{\ell+1} v]\!]s_1 \ldots s_\ell(\epsilon[\![w]\!]s_1 \ldots s_\ell)s_{\ell+1} \ldots s_n \\ = A([\![u]\!]s_1 \ldots s_\ell(\epsilon[\![w]\!]s_1 \ldots s_\ell)s_{\ell+1} \ldots s_n, \lambda\overline{y} \cdot [\![v]\!]s_1 \ldots s_\ell(\epsilon[\![w]\!]s_1 \ldots s_\ell)s_{\ell+1} \ldots s_{\mathcal{L}}\overline{y}) \\ = A([\![\mathsf{ev}^{\ell+1} uw]\!]s_1 \ldots s_n, \lambda\overline{y} \cdot [\![\mathsf{ev}^{\ell+1} vw]\!]s_1 \ldots s_{\mathcal{L}}\overline{y}) \\ = [\![(\mathsf{ev}^{\ell+1} uw) \star^\ell (\mathsf{ev}^{\ell+1} vw)]\!]s_1 \ldots s_n \end{array}$$

□

Lemma 2.56 *For every* $1 \leq \ell < \mathcal{L}$,

$$\begin{array}{ll} [\![Q^\ell(u \bullet^{\mathcal{L}-1} v)]\!]s_1 \dots s_{\mathcal{L}} & = [\![Q^\ell u \bullet^{\mathcal{L}} Q^\ell v]\!]s_1 \dots s_{\mathcal{L}} \\ [\![Q^\ell(u \star^{\mathcal{L}-1} v)]\!]s_1 \dots s_n & = [\![Q^\ell u \star^{\mathcal{L}} Q^\ell v]\!]s_1 \dots s_n \end{array}$$

Proof:

$$\begin{array}{l} [\![Q^\ell(u \bullet^{\mathcal{L}-1} v)]\!]s_1 \dots s_{\mathcal{L}} \\ = [\![u \bullet^{\mathcal{L}-1} v]\!]s_1 \dots s_{\ell-1}(s_\ell \oplus s_{\ell+1})s_{\ell+2} \dots s_{\mathcal{L}} \\ = \iota\langle \lambda\overline{y} \cdot [\![u]\!]s_1 \dots s_{\ell-1}(s_\ell \oplus s_{\ell+1})s_{\ell+2} \dots s_{\mathcal{L}}\overline{y}, [\![v]\!]s_1 \dots s_{\ell-1}(s_\ell \oplus s_{\ell+1})s_{\ell+2} \dots s_{\mathcal{L}}\rangle \\ = \iota\langle \lambda\overline{y} \cdot [\![Q^\ell u]\!]s_1 \dots s_{\mathcal{L}}\overline{y}, [\![Q^\ell v]\!]s_1 \dots s_{\mathcal{L}}\rangle \\ = [\![Q^\ell u \bullet^{\mathcal{L}} Q^\ell v]\!]s_1 \dots s_{\mathcal{L}} \end{array}$$

and similarly for the second equation. □

Lemma 2.57 *For every* $1 \leq \ell \leq \mathcal{L}$,

$$\begin{array}{ll} [\![(u \bullet^{\mathcal{L}} v) \circ^\ell w]\!]s_1 \dots s_{\mathcal{L}} & = [\![(u \circ^\ell w) \bullet^{\mathcal{L}} (v \circ^\ell w)]\!]s_1 \dots s_{\mathcal{L}} \\ [\![(u \star^{\mathcal{L}} v) \circ^\ell w]\!]s_1 \dots s_n & = [\![(u \circ^\ell w) \star^{\mathcal{L}} (v \circ^\ell w)]\!]s_1 \dots s_n \end{array}$$

Proof:

$$\begin{array}{ll} [\![(u \bullet^{\mathcal{L}} v) \circ^\ell w]\!]s_1 \dots s_{\mathcal{L}} & \\ = [\![\mathtt{ev}^{\ell+1}(Q^\ell(u \bullet^{\mathcal{L}} v))w]\!]s_1 \dots s_{\mathcal{L}} & \text{by Lemma 2.33} \\ = [\![\mathtt{ev}^{\ell+1}(Q^\ell u \bullet^{\mathcal{L}+1} Q^\ell v)w]\!]s_1 \dots s_{\mathcal{L}} & \text{by Lemma 2.56} \\ = [\![(\mathtt{ev}^{\ell+1}(Q^\ell u)w) \bullet^{\mathcal{L}} (\mathtt{ev}^{\ell+1}(Q^\ell v)w)]\!]s_1 \dots s_{\mathcal{L}} & \text{by Lemma 2.55} \\ = [\![(u \circ^\ell w) \bullet^{\mathcal{L}} (v \circ^\ell w)]\!]s_1 \dots s_{\mathcal{L}} & \text{by Lemma 2.33} \end{array}$$

and similarly for the second equation. □

Lemma 2.58 *For every* $1 \leq \ell < \mathcal{L}$,

$$\begin{array}{ll} [\![Q^\ell 1^{\mathcal{L}-1}]\!]s_1 \dots s_n & \longrightarrow^* [\![1^{\mathcal{L}}]\!]s_1 \dots s_n \\ [\![Q^\ell \uparrow^{\mathcal{L}-1}]\!]s_1 \dots s_{\mathcal{L}} & \longrightarrow^* [\![\uparrow^{\mathcal{L}}]\!]s_1 \dots s_{\mathcal{L}} \\ [\![Q^\ell id^{\mathcal{L}-1}]\!]s_1 \dots s_{\mathcal{L}} & \longrightarrow^* [\![id^{\mathcal{L}}]\!]s_1 \dots s_{\mathcal{L}} \end{array}$$

Proof:

$$\begin{array}{ll} [\![Q^\ell 1^{\mathcal{L}-1}]\!]s_1 \dots s_n & \\ = [\![1^{\mathcal{L}-1}]\!]s_1 \dots s_{\ell-1}(s_\ell \oplus s_{\ell+1})s_{\ell+2} \dots s_n & \\ = \pi_1(s_1 \oplus \dots \oplus s_{\ell-1} \oplus (s_\ell \oplus s_{\ell+1}) \oplus s_{\ell+2} \oplus \dots \oplus s_{\mathcal{L}})s_{\mathcal{L}+1} \dots s_n & \\ \longrightarrow^* \pi_1(s_1 \oplus \dots \oplus s_{\ell-1} \oplus s_\ell \oplus s_{\ell+1} \oplus s_{\ell+2} \oplus \dots \oplus s_{\mathcal{L}})s_{\mathcal{L}+1} \dots s_n & \text{by } (\oplus) \\ \text{(in one step if } \mathcal{L} > \ell + 1\text{, in no step otherwise)} & \\ = [\![1^{\mathcal{L}}]\!]s_1 \dots s_n & \end{array}$$

The other reductions are proved similarly. □

Lemma 2.59 *For every* $1 \leq \ell < \mathcal{L}$,

$$\begin{array}{ll} [\![\mathtt{ev}^\ell 1^{\mathcal{L}} w]\!]s_1 \dots s_n & \longrightarrow [\![1^{\mathcal{L}-1}]\!]s_1 \dots s_n \\ [\![\mathtt{ev}^\ell \uparrow^{\mathcal{L}} w]\!]s_1 \dots s_{\mathcal{L}-1} & \longrightarrow [\![\uparrow^{\mathcal{L}-1}]\!]s_1 \dots s_{\mathcal{L}-1} \\ [\![\mathtt{ev}^\ell id^{\mathcal{L}} w]\!]s_1 \dots s_{\mathcal{L}-1} & \longrightarrow [\![id^{\mathcal{L}-1}]\!]s_1 \dots s_{\mathcal{L}-1} \end{array}$$

Proof:

$$\begin{array}{ll} [\![\mathtt{ev}^\ell 1^{\mathcal{L}} w]\!]s_1 \dots s_n & \\ = [\![1^{\mathcal{L}}]\!]s_1 \dots s_{\ell-1}(\epsilon[\![w]\!]s_1 \dots s_{\ell-1})s_\ell \dots s_n & \\ = \pi_1(s_1 \oplus \dots \oplus s_{\ell-1} \oplus (\epsilon[\![w]\!]s_1 \dots s_{\ell-1}) \oplus s_\ell \oplus \dots \oplus s_{\mathcal{L}-1})s_{\mathcal{L}} \dots s_n & \\ \longrightarrow \pi_1(s_1 \oplus \dots \oplus s_{\ell-1} \oplus s_\ell \oplus \dots \oplus s_{\mathcal{L}-1})s_{\mathcal{L}} \dots s_n & \text{by } (\oplus-) \\ = [\![1^{\mathcal{L}-1}]\!]s_1 \dots s_n & \end{array}$$

and similarly for the other reductions. □

Lemma 2.60 *For every $1 \leq \ell < \mathcal{L}$,*

$$\begin{array}{ll} [\![1^{\mathcal{L}} \circ^{\ell} w]\!] s_1 \ldots s_n & \longrightarrow^{+} [\![1^{\mathcal{L}}]\!] s_1 \ldots s_n \\ [\![\uparrow^{\mathcal{L}} \circ^{\ell} w]\!] s_1 \ldots s_{\mathcal{L}} & \longrightarrow^{+} [\![\uparrow^{\mathcal{L}}]\!] s_1 \ldots s_{\mathcal{L}} \\ [\![id^{\mathcal{L}} \circ^{\ell} w]\!] s_1 \ldots s_{\mathcal{L}} & \longrightarrow^{+} [\![id^{\mathcal{L}}]\!] s_1 \ldots s_{\mathcal{L}} \end{array}$$

Proof:

$$\begin{array}{ll} [\![1^{\mathcal{L}} \circ^{\ell} w]\!] s_1 \ldots s_n & \\ = [\![\mathtt{ev}^{\ell+1}(Q^{\ell} 1^{\mathcal{L}}) w]\!] s_1 \ldots s_n & \text{by Lemma 2.33} \\ \longrightarrow^{*} [\![\mathtt{ev}^{\ell+1} 1^{\mathcal{L}+1} w]\!] s_1 \ldots s_n & \text{by Lemma 2.58} \\ \longrightarrow [\![1^{\mathcal{L}}]\!] s_1 \ldots s_n & \text{by Lemma 2.59} \end{array}$$

and similarly for the other rules. □

Lemma 2.61 *For every ell ≥ 1,*

$$[\![1^{\ell} \bullet^{\ell} \uparrow^{\ell}]\!] s_1 \ldots s_{\ell} \longrightarrow [\![id^{\ell}]\!] s_1 \ldots s_{\ell}$$

Proof:

$$\begin{array}{ll} [\![1^{\ell} \bullet^{\ell} \uparrow^{\ell}]\!] s_1 \ldots s_{\ell} & \\ = \iota \langle [\![1^{\ell}]\!] s_1 \ldots s_{\ell}, [\![\uparrow^{\ell}]\!] s_1 \ldots s_{\ell} \rangle & \\ = \iota \langle \pi_1(s_1 \oplus \ldots \oplus s_{\ell}), \pi_2(s_1 \oplus \ldots \oplus s_{\ell}) \rangle & \\ \longrightarrow \iota(s_1 \oplus \ldots \oplus s_{\ell}) & \text{by } (\eta\pi) \\ = [\![id^{\ell}]\!] s_1 \ldots s_{\ell} & \end{array}$$

□

Lemma 2.62 *For every $\ell \geq 1$,*

$$[\![(1^{\ell} \circ^{\ell} u) \bullet^{\ell} (\uparrow^{\ell} \circ^{\ell} u)]\!] s_1 \ldots s_{\ell} \longrightarrow^{+} [\![u]\!] s_1 \ldots s_{\ell}$$

Proof:

$$\begin{array}{ll} [\![(1^{\ell} \circ^{\ell} u) \bullet^{\ell} (\uparrow^{\ell} \circ^{\ell} u)]\!] s_1 \ldots s_{\ell} & \\ = \iota \langle [\![1^{\ell} \circ^{\ell} u]\!] s_1 \ldots s_{\ell}, [\![\uparrow^{\ell} \circ^{\ell} u]\!] s_1 \ldots s_{\ell} \rangle & \\ = \iota \langle [\![1^{\ell}]\!] s_1 \ldots s_{\ell-1}(s_{\ell} \oplus \epsilon [\![u]\!] s_1 \ldots s_{\ell}), & \\ \quad [\![\uparrow^{\ell}]\!] s_1 \ldots s_{\ell-1}(s_{\ell} \oplus \epsilon [\![u]\!] s_1 \ldots s_{\ell}) \rangle & \\ = \iota \langle \pi_1(s_1 \oplus \ldots \oplus s_{\ell-1} \oplus s_{\ell} \oplus \epsilon [\![u]\!] s_1 \ldots s_{\ell}), & \\ \quad \pi_2(s_1 \oplus \ldots \oplus s_{\ell-1} \oplus s_{\ell} \oplus \epsilon [\![u]\!] s_1 \ldots s_{\ell}) \rangle & \\ \longrightarrow \iota(s_1 \oplus \ldots \oplus s_{\ell-1} \oplus s_{\ell} \oplus \epsilon [\![u]\!] s_1 \ldots s_{\ell}) & \text{by } (\eta\pi) \\ \longrightarrow^{+} \iota\epsilon [\![u]\!] s_1 \ldots s_{\ell} & \text{by } (\oplus-)\ \ell \text{ times} \\ \longrightarrow^{+} [\![u]\!] s_1 \ldots s_{\ell} & \text{by } (\epsilon),\ (\iota) \end{array}$$

□

Lemma 2.63 *For every $\ell \geq 1$,*

$$[\![(u \circ^{\ell} v) \circ^{\ell} w]\!] s_1 \ldots s_n \longrightarrow^{+} [\![u \circ^{\ell} (v \circ^{\ell} w)]\!] s_1 \ldots s_n$$

Proof:

$$\begin{array}{ll} [\![(u \circ^{\ell} v) \circ^{\ell} w]\!] s_1 \ldots s_n & \\ = [\![u \circ^{\ell} v]\!] s_1 \ldots s_{\ell-1}(s_{\ell} \oplus \epsilon [\![w]\!] s_1 \ldots s_{\ell}) s_{\ell+1} \ldots s_n & \\ = [\![u]\!] s_1 \ldots s_{\ell-1} & \\ \quad ((s_{\ell} \oplus \epsilon [\![w]\!] s_1 \ldots s_{\ell}) \oplus \epsilon [\![v]\!] s_1 \ldots s_{\ell-1}(s_{\ell} \oplus \epsilon [\![w]\!] s_1 \ldots s_{\ell})) s_{\ell+1} \ldots s_n & \\ \longrightarrow^{+} [\![u]\!] s_1 \ldots s_{\ell-1} & \\ \qquad (s_{\ell} \oplus (\epsilon [\![w]\!] s_1 \ldots s_{\ell}) \oplus \epsilon [\![v]\!] s_1 \ldots s_{\ell-1}(s_{\ell} \oplus \epsilon [\![w]\!] s_1 \ldots s_{\ell})) s_{\ell+1} \ldots s_n & \text{by } (\oplus) \\ \longrightarrow^{+} [\![u]\!] s_1 \ldots s_{\ell-1} & \\ \qquad (s_{\ell} \oplus \epsilon [\![v]\!] s_1 \ldots s_{\ell-1}(s_{\ell} \oplus \epsilon [\![w]\!] s_1 \ldots s_{\ell})) s_{\ell+1} \ldots s_n & \text{by } (\oplus-) \end{array}$$

where we have used Lemma 2.29 implicitly. On the other hand:

$$\begin{array}{l}
[\![u \circ^\ell (v \circ^\ell w)]\!] s_1 \dots s_n \\
= [\![u]\!] s_1 \dots s_{\ell-1} (s_\ell \oplus \epsilon [\![v \circ^\ell w]\!] s_1 \dots s_\ell) s_{\ell+1} \dots s_n \\
= [\![u]\!] s_1 \dots s_{\ell-1} (s_\ell \oplus \epsilon [\![v]\!] s_1 \dots s_{\ell-1} (s_\ell \oplus [\![w]\!] s_1 \dots s_\ell)) s_{\ell+1} \dots s_n
\end{array}$$

□

Lemma 2.64 *For every $\ell \geq 2$,*

$$[\![\mathtt{ev}^\ell (u \circ^\ell v) w]\!] s_1 \dots s_n \longrightarrow^+ [\![\mathtt{ev}^\ell u (\mathtt{ev}^\ell v w)]\!] s_1 \dots s_n$$

Proof:

$$\begin{array}{ll}
[\![\mathtt{ev}^\ell (u \circ^\ell v) w]\!] s_1 \dots s_n & \\
= [\![u \circ^\ell v]\!] s_1 \dots s_{\ell-1} (\epsilon [\![w]\!] s_1 \dots s_{\ell-1}) s_\ell \dots s_n & \\
= [\![u]\!] s_1 \dots s_{\ell-1} ((\epsilon [\![w]\!] s_1 \dots s_{\ell-1}) \oplus \epsilon [\![v]\!] s_1 \dots s_{\ell-1} (\epsilon [\![w]\!] s_1 \dots s_{\ell-1})) s_\ell \dots s_n & \\
\longrightarrow^+ [\![u]\!] s_1 \dots s_{\ell-1} (\epsilon [\![v]\!] s_1 \dots s_{\ell-1} (\epsilon [\![w]\!] s_1 \dots s_{\ell-1})) s_\ell \dots s_n & \text{by } (\oplus -) \\
= [\![u]\!] s_1 \dots s_{\ell-1} (\epsilon [\![\mathtt{ev}^\ell v w]\!] s_1 \dots s_\ell) s_\ell \dots s_n & \\
= [\![\mathtt{ev}^\ell u (\mathtt{ev}^\ell v w)]\!] s_1 \dots s_n &
\end{array}$$

where we have again used Lemma 2.29 implicitly. □

Lemma 2.65 *Rules $(1 \Uparrow^\ell)$, $(1 \Uparrow \circ^\ell)$, $(\uparrow\Uparrow^\ell)$, $(\uparrow\Uparrow \circ^\ell)$, $(\Uparrow\bullet^\ell)$, and $(\Uparrow id^\ell)$ are decreasing, for every $\ell \geq 1$.*

Proof:

$$\begin{array}{ll}
[\![1^\ell \circ^\ell \Uparrow^\ell u]\!] s_1 \dots s_n & \\
= [\![1^\ell \circ^\ell (1^\ell \bullet^\ell u \circ^\ell \uparrow^\ell)]\!] s_1 \dots s_n & \text{by Lemma 2.31} \\
\longrightarrow^+ [\![1^\ell]\!] s_1 \dots s_n & \text{by Lemma 2.53}
\end{array}$$

$$\begin{array}{ll}
[\![1^\ell \circ^\ell (\Uparrow^\ell u \circ^\ell v)]\!] s_1 \dots s_n & \\
= [\![1^\ell \circ^\ell ((1^\ell \bullet^\ell u \circ^\ell \uparrow^\ell) \circ^\ell v)]\!] s_1 \dots s_n & \text{by Lemma 2.31} \\
= [\![1^\ell \circ^\ell ((1^\ell \circ^\ell v) \bullet^\ell ((u \circ^\ell \uparrow^\ell) \circ^\ell v))]\!] s_1 \dots s_n & \text{by Lemma 2.57} \\
\longrightarrow^+ = [\![1^\ell \circ^\ell v]\!] s_1 \dots s_n & \text{by Lemma 2.53}
\end{array}$$

$$\begin{array}{ll}
[\![\uparrow^\ell \circ^\ell \Uparrow^\ell u]\!] s_1 \dots s_n & \\
= [\![\uparrow^\ell \circ^\ell (1^\ell \bullet^\ell u \circ^\ell \uparrow^\ell)]\!] s_1 \dots s_n & \text{by Lemma 2.31} \\
\longrightarrow^+ [\![u \circ^\ell \uparrow^\ell]\!] s_1 \dots s_n & \text{by Lemma 2.53}
\end{array}$$

$$\begin{array}{ll}
[\![\uparrow^\ell \circ^\ell (\Uparrow^\ell u \circ^\ell v)]\!] s_1 \dots s_n & \\
= [\![\uparrow^\ell \circ^\ell ((1^\ell \bullet^\ell u \circ^\ell \uparrow^\ell) \circ^\ell v)]\!] s_1 \dots s_n & \text{by Lemma 2.31} \\
= [\![\uparrow^\ell \circ^\ell ((1^\ell \circ^\ell v) \bullet^\ell ((u \circ^\ell \uparrow^\ell) \circ^\ell v))]\!] s_1 \dots s_n & \text{by Lemma 2.57} \\
\longrightarrow^+ = [\![(u \circ^\ell \uparrow^\ell) \circ^\ell v]\!] s_1 \dots s_n & \text{by Lemma 2.53} \\
\longrightarrow^+ = [\![u \circ^\ell (\uparrow^\ell \circ^\ell v)]\!] s_1 \dots s_n & \text{by Lemma 2.63}
\end{array}$$

$$\begin{array}{ll}
[\![\Uparrow^\ell u \circ^\ell (v \bullet^\ell w)]\!] s_1 \dots s_\ell & \\
= [\![(1^\ell \bullet^\ell u \circ^\ell \uparrow^\ell) \circ^\ell (v \bullet^\ell w)]\!] s_1 \dots s_\ell & \text{by Lemma 2.31} \\
= [\![(1^\ell \circ^\ell (v \bullet^\ell w)) \bullet^\ell ((u \circ^\ell \uparrow^\ell) \circ^\ell (v \bullet^\ell w))]\!] s_1 \dots s_\ell & \text{by Lemma 2.57} \\
\longrightarrow^+ [\![v \bullet^\ell ((u \circ^\ell \uparrow^\ell) \circ^\ell (v \bullet^\ell w))]\!] s_1 \dots s_\ell & \text{by Lemma 2.53} \\
\longrightarrow^+ [\![v \bullet^\ell (u \circ^\ell (\uparrow^\ell \circ^\ell (v \bullet^\ell w)))]\!] s_1 \dots s_\ell & \text{by Lemma 2.63} \\
\longrightarrow^+ [\![v \bullet^\ell (u \circ^\ell w)]\!] s_1 \dots s_\ell & \text{by Lemma 2.53}
\end{array}$$

$$\begin{array}{ll}
[\![\Uparrow^\ell id^\ell]\!] s_1 \dots s_\ell & \\
= [\![1^\ell \bullet^\ell id^\ell \circ^\ell \uparrow^\ell]\!] s_1 \dots s_\ell & \text{by Lemma 2.31} \\
\longrightarrow^+ [\![1^\ell \bullet^\ell \uparrow^\ell]\!] s_1 \dots s_\ell & \text{by Lemma 2.52} \\
\longrightarrow^+ [\![id^\ell]\!] s_1 \dots s_\ell & \text{by Lemma 2.61}
\end{array}$$

□

Lemma 2.66 *Rules* $(\mathtt{ev}\Uparrow^{\ell+1})$, $(1\mathtt{ev}\Uparrow^{\ell+1})$ *are decreasing, for every* $\ell \geq 1$.

Proof: First observe that:

$$\begin{array}{ll}
[\![\mathtt{ev}^{\ell+1}(\Uparrow^{\ell+1} u)w]\!]s_1\dots s_\ell & \\
= [\![\mathtt{ev}^{\ell+1}(1^{\ell+1} \bullet^{\ell+1} (u\circ^{\ell+1}\uparrow^{\ell+1}))w]\!]s_1\dots s_\ell & \text{by Lemma 2.31}\\
= [\![(\mathtt{ev}^{\ell+1}1^{\ell+1}(w_1 \bullet^\ell w_2)) \bullet^\ell (\mathtt{ev}^{\ell+1}(u\circ^{\ell+1}\uparrow^{\ell+1})w)]\!]s_1\dots s_\ell & \text{by Lemma 2.55}\\
= [\![(1^\ell \circ^\ell w) \bullet^\ell (\mathtt{ev}^{\ell+1}(u\circ^{\ell+1}\uparrow^{\ell+1})w)]\!]s_1\dots s_\ell & \text{by Lemma 2.54}\\
\longrightarrow^+ [\![(1^\ell \circ^\ell w) \bullet^\ell (\mathtt{ev}^{\ell+1}u(\mathtt{ev}^{\ell+1}\uparrow^{\ell+1} w))]\!]s_1\dots s_\ell & \text{by Lemma 2.64}\\
\longrightarrow^+ [\![(1^\ell \circ^\ell w) \bullet^\ell (\mathtt{ev}^{\ell+1}u(\uparrow^\ell \circ^\ell w))]\!]s_1\dots s_\ell & \text{by Lemma 2.54}
\end{array}$$

$$\begin{array}{ll}
[\![\mathtt{ev}^{\ell+1}(\Uparrow^{\ell+1} u)(w_1 \bullet^\ell w_2)]\!]s_1\dots s_\ell & \\
\longrightarrow^+ [\![(1^\ell \circ^\ell (w_1 \bullet^\ell w_2)) \bullet^\ell (\mathtt{ev}^{\ell+1}u(\uparrow^\ell \circ^\ell(w_1 \bullet^\ell w_2)))]\!]s_1\dots s_\ell & \text{by the remark above}\\
\longrightarrow^+ [\![w_1 \bullet^\ell (\mathtt{ev}^{\ell+1}u(\uparrow^\ell \circ^\ell(w_1 \bullet^\ell w_2)))]\!]s_1\dots s_\ell & \text{by Lemma 2.53}\\
\longrightarrow^+ [\![w_1 \bullet^\ell (\mathtt{ev}^{\ell+1}uw_2)]\!]s_1\dots s_\ell & \text{by Lemma 2.53}
\end{array}$$

$$\begin{array}{ll}
[\![1^\ell \circ^\ell \mathtt{ev}^{\ell+1}(\Uparrow^{\ell+1} u)w]\!]s_1\dots s_n & \\
\longrightarrow^+ [\![1^\ell \circ^\ell ((1^\ell \circ^\ell w) \bullet^\ell (\mathtt{ev}^{\ell+1}u(\uparrow^\ell \circ^\ell w)))]\!]s_1\dots s_n & \text{by the remark above}\\
\longrightarrow^+ [\![1^\ell \circ^\ell w]\!]s_1\dots s_n & \text{by Lemma 2.53}
\end{array}$$

$$\begin{array}{ll}
[\![\uparrow^\ell \circ^\ell(\mathtt{ev}^{\ell+1}(\Uparrow^{\ell+1} u)w)]\!]s_1\dots s_\ell & \\
\longrightarrow^+ [\![\uparrow^\ell \circ^\ell((1^\ell \circ^\ell w) \bullet^\ell (\mathtt{ev}^{\ell+1}u(\uparrow^\ell \circ^\ell w)))]\!]s_1\dots s_\ell & \text{by the remark}\\
\longrightarrow^+ [\![\mathtt{ev}^{\ell+1}u(\uparrow^\ell \circ^\ell w)]\!]s_1\dots s_\ell & \text{by Lemma 2.53}
\end{array}$$

□

Lemma 2.67 *The rules* $(\mathtt{ev}^\ell \Uparrow^{\mathcal{L}})$ *and* $(\Uparrow^{\mathcal{L}} \circ^\ell)$ *are decreasing. Rule* $(Q^\ell \Uparrow^{\mathcal{L}})$ *is non-increasing.*

Proof:

$$\begin{array}{ll}
[\![\mathtt{ev}^\ell(\Uparrow^{\mathcal{L}} u)w]\!]s_1\dots s_{\mathcal{L}-1} & \\
= [\![\mathtt{ev}^\ell(1^{\mathcal{L}} \bullet^{\mathcal{L}} (u\circ^{\mathcal{L}}\uparrow^{\mathcal{L}}))w]\!]s_1\dots s_{\mathcal{L}-1} & \text{by Lemma 2.31}\\
= [\![(\mathtt{ev}^\ell 1^{\mathcal{L}} w) \bullet^{\mathcal{L}-1} (\mathtt{ev}^\ell(u\circ^{\mathcal{L}}\uparrow^{\mathcal{L}})w)]\!]s_1\dots s_{\mathcal{L}-1} & \text{by Lemma 2.55}\\
\longrightarrow^+ [\![1^{\mathcal{L}-1} \bullet^{\mathcal{L}-1} (\mathtt{ev}^\ell(u\circ^{\mathcal{L}}\uparrow^{\mathcal{L}})w)]\!]s_1\dots s_{\mathcal{L}-1} & \text{by Lemma 2.59}\\
= [\![1^{\mathcal{L}-1} \bullet^{\mathcal{L}-1} ((\mathtt{ev}^\ell uw) \circ^{\mathcal{L}-1} (\mathtt{ev}^\ell \uparrow^{\mathcal{L}} w))]\!]s_1\dots s_{\mathcal{L}-1} & \text{by Lemma 2.48}\\
= [\![1^{\mathcal{L}-1} \bullet^{\mathcal{L}-1} ((\mathtt{ev}^\ell uw)\circ^{\mathcal{L}-1}\uparrow^{\mathcal{L}-1})]\!]s_1\dots s_{\mathcal{L}-1} & \text{by Lemma 2.59}\\
= [\![\Uparrow^{\mathcal{L}-1} (\mathtt{ev}^\ell uw)]\!]s_1\dots s_{\mathcal{L}-1} & \text{by Lemma 2.31}
\end{array}$$

$$\begin{array}{ll}
[\![Q^\ell(\Uparrow^{\mathcal{L}-1} u)]\!]s_1\dots s_{\mathcal{L}} & \\
= [\![Q^\ell(1^{\mathcal{L}-1} \bullet^{\mathcal{L}-1} (u\circ^{\mathcal{L}-1}\uparrow^{\mathcal{L}-1}))]\!]s_1\dots s_{\mathcal{L}} & \text{by Lemma 2.31}\\
= [\![(Q^\ell 1^{\mathcal{L}-1}) \bullet^{\mathcal{L}} (Q^\ell(u\circ^{\mathcal{L}-1}\uparrow^{\mathcal{L}-1}))]\!]s_1\dots s_{\mathcal{L}} & \text{by Lemma 2.56}\\
\longrightarrow^* [\![1^{\mathcal{L}} \bullet^{\mathcal{L}} (Q^\ell(u\circ^{\mathcal{L}-1}\uparrow^{\mathcal{L}-1}))]\!]s_1\dots s_{\mathcal{L}} & \text{by Lemma 2.58}\\
\longrightarrow^* [\![1^{\mathcal{L}} \bullet^{\mathcal{L}} (Q^\ell u \circ^{\mathcal{L}} Q^\ell \uparrow^{\mathcal{L}-1})]\!]s_1\dots s_{\mathcal{L}} & \text{by Lemma 2.48}\\
\longrightarrow^* [\![1^{\mathcal{L}} \bullet^{\mathcal{L}} (Q^\ell u\circ^{\mathcal{L}}\uparrow^{\mathcal{L}})]\!]s_1\dots s_{\mathcal{L}} & \text{by Lemma 2.58}\\
= [\![\Uparrow^{\mathcal{L}} (Q^\ell u)]\!]s_1\dots s_{\mathcal{L}} & \text{by Lemma 2.31}
\end{array}$$

$$\begin{array}{ll}
[\![(\Uparrow^{\mathcal{L}} u) \circ^\ell w]\!]s_1\dots s_{\mathcal{L}} & \\
= [\![\mathtt{ev}^{\ell+1}(Q^\ell(\Uparrow^{\mathcal{L}} u))w]\!]s_1\dots s_{\mathcal{L}} & \text{by Lemma 2.33}\\
\longrightarrow^* [\![\mathtt{ev}^{\ell+1}(\Uparrow^{\mathcal{L}+1} (Q^\ell u))w]\!]s_1\dots s_{\mathcal{L}} & \text{by the above}\\
\longrightarrow^+ [\![\Uparrow^{\mathcal{L}} (\mathtt{ev}^{\ell+1}(Q^\ell u)w)]\!]s_1\dots s_{\mathcal{L}} & \text{by the above}\\
= [\![\Uparrow^{\mathcal{L}} (u \circ^\ell w)]\!]s_1\dots s_{\mathcal{L}} & \text{by Lemma 2.33}
\end{array}$$

□

The remaining rules that involve $\Uparrow^\ell$ tend to involve rather heavy calculations. It is also here that the strange rules $(\iota\pi_1)$ and $(\iota\pi_2)$ are needed.

Lemma 2.68 *For every $\ell \geq 1$,*

$$[\![\Uparrow^\ell u\circ^\ell \Uparrow^\ell v]\!]s_1\ldots s_\ell \longrightarrow^+ [\![\Uparrow^\ell (u\circ^\ell v)]\!]s_1\ldots s_\ell$$

Proof:

$$\begin{array}{l}
[\![\Uparrow^\ell u\circ^\ell \Uparrow^\ell v]\!]s_1\ldots s_\ell \\
= [\![\Uparrow^\ell u]\!]s_1\ldots s_{\ell-1}(s_\ell \oplus \epsilon[\![\Uparrow^\ell v]\!]s_1\ldots s_\ell) \\
= [\![\Uparrow^\ell u]\!]s_1\ldots s_{\ell-1}(s_\ell \oplus \epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle) \\
= \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle), \\
\quad [\![u]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle) \\
\qquad \oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle))\rangle \\
\longrightarrow^+ \iota\langle\pi_1(\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle), \\
\quad [\![u]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle) \\
\qquad \oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle))\rangle \\
\quad \text{by } (\oplus-)\ \ell \text{ times} \\
\longrightarrow^+ \iota\langle\pi_1\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle, \\
\quad [\![u]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle) \\
\qquad \oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle))\rangle \quad \text{by } (\epsilon),\ (\iota) \\
\longrightarrow \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), \\
\quad [\![u]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle) \\
\qquad \oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle))\rangle \quad \text{by } (\pi_1) \\
\longrightarrow^+ \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), \\
\quad [\![u]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle) \\
\qquad \oplus\epsilon\pi_2(\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle))\rangle \\
\quad \text{by } (\oplus-)\ \ell \text{ times} \\
\longrightarrow^+ \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), \\
\quad [\![u]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle) \\
\qquad \oplus\epsilon\pi_2\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle)\rangle \quad \text{by } (\epsilon),\ (\iota) \\
\longrightarrow^+ \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), \\
\quad [\![u]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle) \\
\qquad \oplus\epsilon[\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell)))\rangle \quad \text{by } (\pi_2) \\
\longrightarrow^+ \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), \\
\quad [\![u]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell)) \\
\qquad \oplus\epsilon[\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell)))\rangle \quad \text{by } (\iota\pi_2)
\end{array}$$

while:

$$\begin{array}{l}
[\![\Uparrow^\ell (u\circ^\ell v)]\!]s_1\ldots s_\ell \\
= \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), [\![u\circ^\ell v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell))\rangle \\
= \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell), \\
\quad [\![u]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell)) \\
\qquad \oplus\epsilon[\![v]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell)))\rangle
\end{array}$$

□

Lemma 2.69 *For every $\ell \geq 1$,*

$$[\![\mathtt{ev}^{\ell+1}(\Uparrow^{\ell+1} u)(\mathtt{ev}^{\ell+1}(\Uparrow^{\ell+1} v)w)]\!]s_1\ldots s_\ell \longrightarrow^+ [\![\mathtt{ev}^{\ell+1}(\Uparrow^{\ell+1} (u\circ^{\ell+1} v))w]\!]s_1\ldots s_\ell$$

Proof:

$$
\begin{array}{l}
\llbracket \mathtt{ev}^{\ell+1}(\Uparrow^{\ell+1}\ u)(\mathtt{ev}^{\ell+1}(\Uparrow^{\ell+1}\ v)w)\rrbracket s_1\ldots s_\ell \\
= \llbracket \Uparrow^{\ell+1}\ u\rrbracket s_1\ldots s_\ell(\epsilon\llbracket \mathtt{ev}^{\ell+1}(\Uparrow^{\ell+1}\ v)w\rrbracket s_1\ldots s_\ell) \\
= \llbracket \Uparrow^{\ell+1}\ u\rrbracket s_1\ldots s_\ell(\epsilon\llbracket \Uparrow^{\ell+1}\ v\rrbracket s_1\ldots s_\ell(\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)) \\
= \llbracket \Uparrow^{\ell+1}\ u\rrbracket s_1\ldots s_\ell(\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle) \\
= \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad\qquad\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle), \\
\qquad \llbracket u\rrbracket s_1\ldots s_\ell((\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad\qquad\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle) \\
\qquad\qquad \oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad\qquad\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle))\rangle \\
\longrightarrow^+ \iota\langle\pi_1(\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad\qquad\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle), \\
\qquad \llbracket u\rrbracket s_1\ldots s_\ell((\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad\qquad\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle) \\
\qquad\qquad \oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad\qquad\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle))\rangle \quad \text{by } (\oplus-)\ \ell \text{ times} \\
\longrightarrow^+ \iota\langle\pi_1\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad\qquad\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle, \\
\qquad \llbracket u\rrbracket s_1\ldots s_\ell((\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad\qquad\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle) \\
\qquad\qquad \oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad\qquad\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle))\rangle \quad \text{by } (\epsilon),\ (\iota) \\
\longrightarrow \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad \llbracket u\rrbracket s_1\ldots s_\ell((\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad\qquad\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle) \\
\qquad\qquad \oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad\qquad\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle))\rangle \quad \text{by } (\pi_1) \\
\longrightarrow^+ \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad \llbracket u\rrbracket s_1\ldots s_\ell((\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad\qquad\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle) \\
\qquad\qquad \oplus\epsilon\pi_2(\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad\qquad\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle))\rangle \quad \text{by } (\oplus-)\ \ell \text{ times} \\
\longrightarrow^+ \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad \llbracket u\rrbracket s_1\ldots s_\ell((\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad\qquad\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle) \\
\qquad\qquad \oplus\epsilon\pi_2\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad\qquad\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle)\rangle \quad \text{by } (\epsilon),\ (\iota) \\
\longrightarrow^+ \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad \llbracket u\rrbracket s_1\ldots s_\ell((\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad\qquad\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle) \\
\qquad\qquad \oplus\epsilon\llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)))\rangle \quad \text{by } (\pi_2) \\
\longrightarrow^+ \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad \llbracket u\rrbracket s_1\ldots s_\ell((\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad\qquad\qquad \llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell))\rangle) \\
\qquad\qquad \oplus\epsilon\llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)))\rangle \quad \text{by } (\epsilon) \\
\longrightarrow^+ \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad \llbracket u\rrbracket s_1\ldots s_\ell(((s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)) \\
\qquad\qquad \oplus\epsilon\llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)))\rangle \quad \text{by } (\iota\pi_1) \\
\longrightarrow^+ \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell), \\
\qquad \llbracket u\rrbracket s_1\ldots s_\ell(((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)) \\
\qquad\qquad \oplus\epsilon\llbracket v\rrbracket s_1\ldots s_\ell((\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon\llbracket w\rrbracket s_1\ldots s_\ell)))\rangle \quad \text{by } (\oplus-)\ \ell \text{ times}
\end{array}
$$

while:
$$
\begin{aligned}
&[\![\mathtt{ev}^{\ell+1}(\Uparrow^{\ell+1}(u\circ^{\ell+1}v))w]\!]s_1\dots s_\ell\\
&=[\![\Uparrow^{\ell+1}(u\circ^{\ell+1}v)]\!]s_1\dots s_\ell(\epsilon[\![w]\!]s_1\dots s_\ell)\\
&=\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad[\![u\circ^{\ell+1}v]\!]s_1\dots s_\ell((\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\rangle\\
&=\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad[\![u]\!]s_1\dots s_\ell(((\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\\
&\qquad\qquad\oplus\epsilon[\![v]\!]s_1\dots s_\ell((\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell)))\rangle
\end{aligned}
$$

□

Lemma 2.70 *For every $\ell\geq 1$,*

$$[\![\Uparrow^\ell u\circ^\ell(\Uparrow^\ell v\circ^\ell w)]\!]s_1\dots s_\ell\longrightarrow^+[\![\Uparrow^\ell(u\circ^\ell v)\circ^\ell w]\!]s_1\dots s_\ell$$

Proof:
$$
\begin{aligned}
&[\![\Uparrow^\ell u\circ^\ell(\Uparrow^\ell v\circ^\ell w)]\!]s_1\dots s_\ell\\
&=[\![\Uparrow^\ell u]\!]s_1\dots s_{\ell-1}(s_\ell\oplus\epsilon[\![\Uparrow^\ell v\circ^\ell w]\!]s_1\dots s_\ell)\\
&=[\![\Uparrow^\ell u]\!]s_1\dots s_{\ell-1}(s_\ell\oplus\epsilon[\![\Uparrow^\ell v]\!]s_1\dots s_{\ell-1}(s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\\
&=[\![\Uparrow^\ell u]\!]s_1\dots s_{\ell-1}(s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad[\![v]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\rangle)\\
&=\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad\qquad[\![v]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\rangle),\\
&\qquad[\![u]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad\qquad[\![v]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\rangle)\\
&\qquad\quad\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad\qquad[\![v]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\rangle))\rangle\\
&\longrightarrow^+\iota\langle\pi_1(\epsilon\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad\qquad[\![v]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\rangle),\\
&\qquad[\![u]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad\qquad[\![v]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\rangle)\\
&\qquad\quad\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad\qquad[\![v]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\rangle))\rangle\\
&\qquad\text{by }(\oplus-)\ \ell\text{ times}\\
&\longrightarrow^+\iota\langle\pi_1\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad\qquad[\![v]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\rangle,\\
&\qquad[\![u]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad\qquad[\![v]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\rangle)\\
&\qquad\quad\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad\qquad[\![v]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\rangle))\rangle\quad\text{by }(\epsilon),\ (\iota)\\
&\longrightarrow\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad[\![u]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad\qquad[\![v]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\rangle)\\
&\qquad\quad\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad\qquad[\![v]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\rangle))\rangle\quad\text{by }(\pi_1)\\
&\longrightarrow^+\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad[\![u]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad\qquad[\![v]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\rangle)\\
&\qquad\quad\oplus\epsilon\pi_2(\epsilon\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad\qquad[\![v]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\rangle))\rangle\\
&\qquad\text{by }(\oplus-)\ \ell\text{ times}\\
&\longrightarrow^+\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad[\![u]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad\qquad[\![v]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\rangle)\\
&\qquad\quad\oplus\epsilon\pi_2\langle\pi_1(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell),\\
&\qquad\qquad[\![v]\!]s_1\dots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\dots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\dots s_\ell))\rangle)\rangle\quad\text{by }(\epsilon),\ (\iota)
\end{aligned}
$$

$$
\begin{aligned}
\longrightarrow^+ & \ \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell),\\
& [\![u]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell),\\
& \qquad [\![v]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell))\rangle)\\
& \quad \oplus\epsilon[\![v]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)))\rangle \quad \text{by } (\pi_2)\\
\longrightarrow^+ & \ \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell),\\
& [\![u]\!]s_1\ldots s_{\ell-1}((\epsilon\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell),\\
& \qquad [\![v]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell))\rangle)\\
& \quad \oplus\epsilon[\![v]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)))\rangle \quad \text{by } (\oplus-)\\
\longrightarrow^+ & \ \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell),\\
& [\![u]\!]s_1\ldots s_{\ell-1}((\iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell),\\
& \qquad [\![v]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell))\rangle)\\
& \quad \oplus\epsilon[\![v]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)))\rangle \quad \text{by } (\epsilon)\\
\longrightarrow^+ & \ \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell),\\
& [\![u]\!]s_1\ldots s_{\ell-1}(((s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell))\\
& \quad \oplus\epsilon[\![v]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)))\rangle \quad \text{by } (\iota\pi_1)\\
\longrightarrow^* & \ \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell),\\
& [\![u]\!]s_1\ldots s_{\ell-1}(((s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell))\\
& \quad \oplus\epsilon[\![v]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)))\rangle
\end{aligned}
$$

by $(\oplus-)$ $\ell-1$ times

while:

$$
\begin{aligned}
& [\![\Uparrow^\ell (u\circ^\ell v)\circ^\ell w]\!]s_1\ldots s_\ell\\
& = [\![\Uparrow^\ell (u\circ^\ell v)]\!]s_1\ldots s_{\ell-1}(s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)\\
& = \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell),\\
& \qquad [\![u\circ^\ell v]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell))\rangle\\
& = \iota\langle\pi_1(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell),\\
& \qquad [\![u]\!]s_1\ldots s_{\ell-1}(((s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell))\\
& \qquad\quad \oplus\epsilon[\![v]\!]s_1\ldots s_{\ell-1}((s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)\oplus\epsilon\pi_2(s_1\oplus\ldots\oplus s_\ell\oplus\epsilon[\![w]\!]s_1\ldots s_\ell)))\rangle
\end{aligned}
$$

□

Figure 6 and Figure 7 sum up the results. Read them as a short justification, for every rule R, of the fact that whenever u rewrites to v by R, then u is greater than v in the lexicographic product of $\succ_\lambda$ and $\succ_{eq}$. For instance, if u rewrites to v by rule $(\bullet^\ell)$, then $u =_\lambda v$ by Lemma 2.57, and $u \succ_{eq} v$ by Lemma 2.20. The $\succ_{eq}$ signs in Figure 7 come from Lemma 2.14. Observe that some $\succ_{eq}$ signs are in fact not needed. It follows:

Theorem 2.71 *Σ and Σ_H terminate.*

Proof: $\succ_\lambda$ is well-founded, because it is the non-empty intersection of orderings induced by Jouannaud and Rubio's higher-order recursive path ordering, and all these orderings are well-founded. Therefore the lexicographic product of $\succ_\lambda$ and $\succ_{eq}$ is also well-founded. By the results summed up in Figures 6 and 7, all rules in Σ_H^+ are decreasing in this ordering. By Lemma 2.24, Σ_H terminates, hence also Σ, which is a subsystem of Σ_H. □

2.6 Comments

Can we relax the well-typedness condition on $\lambda\mathtt{ev}Q$-terms while still keeping Σ_H terminating? We don't know yet, but here is an idea. Observe that we didn't really use the whole power of $\lambda\mathtt{ev}Q$ types: we only used θ-types, or skeletons of the real types, where every function type $\tau_1 \Rightarrow \tau_2$ has been abstracted away as τ_2 (recursively).

This suggests endowing the $\lambda\mathtt{ev}Q$-terms with a new type system based on θ-types instead of real types. The result is shown in Figure 8, where we abuse the notation by identifying signatures and θ-types.

It is immediate that every typed $\lambda\mathtt{ev}Q$-term also has a θ-type, namely the $[\![\_]\!]$-translation of its type. Let's call *stratified* any $\lambda\mathtt{ev}Q$-term that is typable in the system of Figure 8.

The whole proof of termination transfers to the stratified case, with the proviso that whenever u rewrites to v in $\lambda\mathtt{ev}Q$, we can use the θ-type of u to compute the $[\![\_]\!]$-translation of v; or, in other words, provided

Group (B) (level ℓ, $\ell \geq 1$):

$(\circ id^{\ell})$	$\succ_{\lambda}$	(Lemma 2.51)			$(1 \Uparrow^{\ell})$	$\succ_{\lambda}$	(Lemma 2.65)
$(id\circ^{\ell})$	$\succ_{\lambda}$	(Lemma 2.52)			$(1 \Uparrow \circ^{\ell})$	$\succ_{\lambda}$	(Lemma 2.65)
$(\circ^{\ell})$	$\succ_{\lambda}$	(Lemma 2.63)	$\succ_{eq}$	(Lemma 2.21)	$(\uparrow\Uparrow^{\ell})$	$\succ_{\lambda}$	(Lemma 2.65)
$(\uparrow^{\ell})$	$\succ_{\lambda}$	(Lemma 2.53)			$(\uparrow\Uparrow \circ^{\ell})$	$\succ_{\lambda}$	(Lemma 2.65)
(1^{ℓ})	$\succ_{\lambda}$	(Lemma 2.53)			$(\Uparrow\Uparrow^{\ell})$	$\succ_{\lambda}$	(Lemma 2.68)
$(\bullet^{\ell})$	$=_{\lambda}$	(Lemma 2.57)	$\succ_{eq}$	(Lemma 2.20)	$(\Uparrow\Uparrow \circ^{\ell})$	$\succ_{\lambda}$	(Lemma 2.70)
(λ^{ℓ})	$\succ_{\lambda}$	(Lemma 2.43)			$(\Uparrow\bullet^{\ell})$	$\succ_{\lambda}$	(Lemma 2.65)
$(\star^{\ell})$	$=_{\lambda}$	(Lemma 2.57)	$\succ_{eq}$	(Lemma 2.20)	$(\Uparrow id^{\ell})$	$\succ_{\lambda}$	(Lemma 2.65)
$(Q\circ^{\ell})$	$\succ_{\lambda}$	(Lemma 2.34)	$\succ_{eq}$	(Lemma 2.21)			

Group (C) ($\ell \geq 1$):

$(\mathtt{ev}\lambda^{\ell+1})$	$\succ_{\lambda}$	(Lemma 2.42)		
$(\mathtt{ev}\star^{\ell+1})$	$=_{\lambda}$	(Lemma 2.55)	$\succ_{eq}$	(Lemma 2.20)
$(\mathtt{ev}id^{\ell+1})$	$\succ_{\lambda}$	(Lemma 2.50)		
$(\mathtt{ev}\circ^{\ell+1})$	$\succ_{\lambda}$	(Lemma 2.64)	$\succ_{eq}$	(Lemma 2.22)
$(\mathtt{ev} \uparrow^{\ell+1})$	$=_{\lambda}$	(Lemma 2.54)	$\succ_{eq}$	(Lemma 2.19)
$(\mathtt{ev}1^{\ell+1})$	$=_{\lambda}$	(Lemma 2.54)	$\succ_{eq}$	(Lemma 2.19)
$(\mathtt{ev} \bullet^{\ell+1})$	$=_{\lambda}$	(Lemma 2.55)	$\succ_{eq}$	(Lemma 2.20)
$(\mathtt{ev} \Uparrow^{\ell+1})$	$\succ_{\lambda}$	(Lemma 2.66)		
$(1\mathtt{ev} \Uparrow^{\ell+1})$	$\succ_{\lambda}$	(Lemma 2.66)		
$(\uparrow \mathtt{ev} \Uparrow^{\ell+1})$	$\succ_{\lambda}$	(Lemma 2.66)		
$(\mathtt{ev} \Uparrow\Uparrow^{\ell+1})$	$\succ_{\lambda}$	(Lemma 2.69)		
$(\mathtt{ev}Q^{\ell+1})$	$\succ_{\lambda}$	(Lemma 2.32)	$\succ_{eq}$	(Lemma 2.16)

Figure 6: Termination of Σ_H^+, part 1

Group (D) $(1 \leq \ell < \mathcal{L})$:

$(\lambda^{\mathcal{L}} \circ^{\ell})$	$\succ_\lambda$	(Lemma 2.41)	$\succ_{eq}$
$(\star^{\mathcal{L}} \circ^{\ell})$	$=_\lambda$	(Lemma 2.57)	$\succ_{eq}$
$(id^{\mathcal{L}} \circ^{\ell})$	$\succ_\lambda$	(Lemma 2.60)	$\succ_{eq}$
$(\circ^{\mathcal{L}} \circ^{\ell})$	$\succeq_\lambda$	(Lemma 2.49)	$\succ_{eq}$
$(\uparrow^{\mathcal{L}} \circ^{\ell})$	$\succ_\lambda$	(Lemma 2.60)	$\succ_{eq}$
$(1^{\mathcal{L}} \circ^{\ell})$	$\succ_\lambda$	(Lemma 2.60)	$\succ_{eq}$
$(\bullet^{\mathcal{L}} \circ^{\ell})$	$=_\lambda$	(Lemma 2.57)	$\succ_{eq}$
$(\Uparrow^{\mathcal{L}} \circ^{\ell})$	$\succ_\lambda$	(Lemma 2.67)	$\succ_{eq}$
$(Q^{\mathcal{L}} \circ^{\ell})$	$\succeq_\lambda$	(Lemma 2.49)	$\succ_{eq}$
$(\mathtt{ev}^{\mathcal{L}} \circ^{\ell})$	$=_\lambda$	(Lemma 2.49)	$\succ_{eq}$

Group (E) $(2 \leq \ell < \mathcal{L})$:

$(\mathtt{ev}^{\ell} \lambda^{\mathcal{L}})$	$\succ_\lambda$	(Lemma 2.40)	$\succ_{eq}$
$(\mathtt{ev}^{\ell} \star^{\mathcal{L}})$	$=_\lambda$	(Lemma 2.55)	$\succ_{eq}$
$(\mathtt{ev}^{\ell} id^{\mathcal{L}})$	$\succ_\lambda$	(Lemma 2.59)	$\succ_{eq}$
$(\mathtt{ev}^{\ell} \circ^{\mathcal{L}})$	$=_\lambda$	(Lemma 2.48)	$\succ_{eq}$
$(\mathtt{ev}^{\ell} \uparrow^{\mathcal{L}})$	$\succ_\lambda$	(Lemma 2.59)	$\succ_{eq}$
$(\mathtt{ev}^{\ell} 1^{\mathcal{L}})$	$\succ_\lambda$	(Lemma 2.59)	$\succ_{eq}$
$(\mathtt{ev}^{\ell} \bullet^{\mathcal{L}})$	$=_\lambda$	(Lemma 2.55)	$\succ_{eq}$
$(\mathtt{ev}^{\ell} \Uparrow^{\mathcal{L}})$	$\succ_\lambda$	(Lemma 2.67)	$\succ_{eq}$
$(\mathtt{ev}^{\ell} Q^{\mathcal{L}})$	$=_\lambda$	(Lemma 2.46)	$\succ_{eq}$
$(\mathtt{ev}^{\ell} \mathtt{ev}^{\mathcal{L}})$	$=_\lambda$	(Lemma 2.44)	$\succ_{eq}$

(F) Quoting $(2 \leq \ell < \mathcal{L})$:

$(Q^{\ell} \lambda^{\mathcal{L}})$	$\succ_\lambda$	(Lemma 2.39)	$\succ_{eq}$
$(Q^{\ell} \star^{\mathcal{L}})$	$=_\lambda$	(Lemma 2.56)	$\succ_{eq}$
$(Q^{\ell} id^{\mathcal{L}})$	$\succeq_\lambda$	(Lemma 2.58)	$\succ_{eq}$
$(Q^{\ell} \circ^{\mathcal{L}})$	$\succeq_\lambda$	(Lemma 2.48)	$\succ_{eq}$
$(Q^{\ell} \uparrow^{\mathcal{L}})$	$\succeq_\lambda$	(Lemma 2.58)	$\succ_{eq}$
$(Q^{\ell} 1^{\mathcal{L}})$	$\succeq_\lambda$	(Lemma 2.58)	$\succ_{eq}$
$(Q^{\ell} \bullet^{\mathcal{L}})$	$=_\lambda$	(Lemma 2.56)	$\succ_{eq}$
$(Q^{\ell} \Uparrow^{\mathcal{L}})$	$\succeq_\lambda$	(Lemma 2.67)	$\succ_{eq}$
$(Q^{\ell} Q^{\mathcal{L}})$	$\succeq_\lambda$	(Lemma 2.45)	$\succ_{eq}$
$(Q^{\ell} \mathtt{ev}^{\mathcal{L}})$	$=_\lambda$	(Lemma 2.47)	$\succ_{eq}$

Group (H) $(1 \leq \ell)$:

$(\eta \mathtt{ev}^{\ell})$	$\succeq_\lambda$	(Lemma 2.33)	$\succ_{eq}$	(Lemma 2.17)
$(\eta \Uparrow^{\ell})$	$=_\lambda$	(Lemma 2.31)	$\succ_{eq}$	(Lemma 2.18)
$(\eta \bullet^{\ell})$	$\succ_\lambda$	(Lemma 2.61)		
$(\eta \bullet \circ^{\ell})$	$\succ_\lambda$	(Lemma 2.62)		

Figure 7: Termination of Σ_H^+, part 2

Level 0:

$$\frac{}{,\,, x : \theta^+ \vdash x : \theta^+}$$

$$\frac{\begin{array}{c}, \vdash u : \theta_1^+ \\ , \vdash v : \theta_2^+\end{array}}{, \vdash uv : \theta_1^+} \qquad \frac{,\,, x : \theta_1^+ \vdash u : \theta_2^+}{, \vdash \lambda x \cdot u : \theta_2^+}$$

$$\frac{, \vdash u : \theta^+ \times \theta^-}{, \vdash 1u : \theta^+} \qquad \frac{\begin{array}{c}, \vdash u : \theta^+ \\ , \vdash v : \theta^-\end{array}}{, \vdash u \bullet v : \theta^+ \times \theta^-}$$

(resp. $, \vdash\uparrow u : \theta^-$)

$$\frac{}{, \vdash () : \top}$$

Level $\ell \geq 1$:

$$\frac{}{, \vdash 1^\ell : \theta_1^-, \ldots, \theta_{\ell-1}^-, \theta^+ \times \theta^- \rightsquigarrow \theta^+} \qquad \frac{}{, \vdash id^\ell : \theta_1^-, \ldots, \theta_{\ell-1}^-, \theta \rightsquigarrow \theta}$$

$$\frac{}{, \vdash\uparrow^\ell : \theta_1^-, \ldots, \theta_{\ell-1}^-, \theta^+ \times \theta^- \rightsquigarrow \theta^-}$$

$$\frac{\begin{array}{c}, \vdash u : \theta_1^-, \ldots, \theta_\ell^- \rightsquigarrow \theta_1^+ \\ , \vdash v : \theta_1^-, \ldots, \theta_\ell^- \rightsquigarrow \theta_2^+\end{array}}{, \vdash u \star^\ell v : \theta_1^-, \ldots, \theta_\ell^- \rightsquigarrow \theta_1^+} \qquad \frac{, \vdash u : \theta_1^-, \ldots, \theta_{\ell-1}^-, \theta_1^+ \times \theta^- \rightsquigarrow \theta_2^+}{, \vdash \lambda^\ell u : \theta_1^-, \ldots, \theta_{\ell-1}^-, \theta^- \rightsquigarrow \theta_2^+}$$

$$\frac{, \vdash u : \theta_1^-, \ldots, \theta_{\ell-1}^-, \theta_\ell^- \rightsquigarrow \theta^-}{, \vdash\Uparrow^\ell u : \theta_1^-, \ldots, \theta_{\ell-1}^-, \theta^+ \times \theta_\ell^- \rightsquigarrow \theta^+ \times \theta^-} \qquad \frac{\begin{array}{c}, \vdash u : \theta_1^-, \ldots, \theta_\ell^- \rightsquigarrow \theta^+ \\ , \vdash v : \theta_1^-, \ldots, \theta_\ell^- \rightsquigarrow \theta^-\end{array}}{, \vdash u \bullet^\ell v : \theta_1^-, \ldots, \theta_\ell^- \rightsquigarrow \theta^+ \times \theta^-}$$

$$\frac{\begin{array}{c}, \vdash u : \theta_1^-, \ldots, \theta_{\ell-1}^-, \theta_\ell^- \rightsquigarrow \theta \\ , \vdash w : \theta_1^-, \ldots, \theta_{\ell-1}^- \rightsquigarrow \theta_\ell^-\end{array}}{, \vdash \mathtt{ev}^\ell uw : \theta_1^-, \ldots, \theta_{\ell-1}^- \rightsquigarrow \theta} \qquad \frac{, \vdash u : \theta_1^-, \ldots, \theta_{\ell-1}^-, \theta_\ell^- \rightsquigarrow \theta}{\begin{array}{r}, \vdash Q^\ell u : \theta_1^-, \ldots, \theta_{\ell-1}^-, \\ \theta^-, \theta_\ell^- \rightsquigarrow \theta\end{array}}$$

$$\frac{, \vdash u : \theta_1^-, \ldots, \theta_{\ell-1}^-, \theta_\ell^- \rightsquigarrow \theta \quad , \vdash v : \theta_1^-, \ldots, \theta_{\ell-1}^-, {\theta'}_\ell^- \rightsquigarrow \theta_\ell^-}{, \vdash u \circ^\ell v : \theta_1^-, \ldots, \theta_{\ell-1}^-, {\theta'}_\ell^- \rightsquigarrow \theta}$$

Figure 8: Stratifying by θ-types

that subject reduction holds in the stratified calculus. Unfortunately, it does not, as the following derivation shows:

$$\dfrac{\dfrac{\dfrac{}{x:\theta^+ \vdash x:\theta^+}}{\vdash \lambda x\cdot x:\theta^+} \qquad \dfrac{\vdots}{\vdash u:\theta_2^+}}{\vdash (\lambda x\cdot x)u:\theta^+}$$

which rewrites to u, of θ-type θ_2^+, not θ^+.

For want of an intermediate type system which would allow us to interpret all untyped λ_{S4}-terms via G, we shall therefore stick to the full type system of $\lambda\mathtt{ev}Q$, which only allows us to interpret the typed λ_{S4}-terms.

3 Confluence

3.1 Confluence

The results of this section are the following: the $\lambda\mathtt{ev}Q$-calculus and the $\lambda\mathtt{ev}Q_H$-calculus are locally confluent, whether untyped, semi-stratified or typed. In the typed case, the $\lambda\mathtt{ev}Q$-calculus is also confluent. The untyped and semi-stratified $\lambda\mathtt{ev}Q_H$-calculi are not confluent. We conjecture that the typed $\lambda\mathtt{ev}Q_H$-calculus is confluent: this will be dealt with in part IIIb.

Lemma 3.1 *The $\lambda\mathtt{ev}Q$-calculus, the $\lambda\mathtt{ev}Q_H$-calculus, Σ and Σ_H are locally confluent.*

Proof: The proof is easy but tedious: consider all critical pairs between all rules, and show that they are joinable. As this job can be mechanized, we have built a computer program to check this automatically. (Notice, however, that a standard Knuth-Bendix completion program won't work, as all terms are indexed by integer expressions subject to linear constraints of the form $\alpha\ell \leq \alpha'\ell' + \beta$, where α, α' are either 0 or 1, and β is a relative integer.) The results are shown in a separate appendix [GL95]. □

To prove that $\lambda\mathtt{ev}Q$ is confluent, we mimic the proof of [HL89]. The latter was inpired by [Yok89], and is in the spirit of the Tait-Martin-Löf method of parallel reductions:

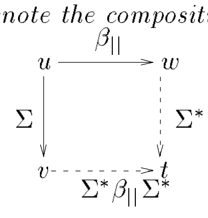
Definition 3.1 *Let $\stackrel{\beta_{||}}{\longrightarrow}$ be the relation on $\lambda\mathtt{ev}Q$-terms defined as follows:*

$$u \stackrel{\beta_{||}}{\longrightarrow} u \qquad \dfrac{u \stackrel{\beta_{||}}{\longrightarrow} u' \qquad v \stackrel{\beta_{||}}{\longrightarrow} v'}{(\lambda^\ell u) \star^\ell v \stackrel{\beta_{||}}{\longrightarrow} u' \circ^\ell (v' \bullet^\ell id^\ell)} \qquad \dfrac{u \stackrel{\beta_{||}}{\longrightarrow} u' \qquad v \stackrel{\beta_{||}}{\longrightarrow} v'}{(\lambda x\cdot u)v \stackrel{\beta_{||}}{\longrightarrow} u[v/x]} \qquad \dfrac{u_1 \stackrel{\beta_{||}}{\longrightarrow} u_1' \quad \ldots \quad u_n \stackrel{\beta_{||}}{\longrightarrow} u_n'}{f(u_1,\ldots,u_n) \stackrel{\beta_{||}}{\longrightarrow} f(u_1',\ldots,u_n')}$$

for every $\ell \geq 1$ and every n-ary operator f, $n \geq 0$.

In the sequel, we shall use diagrams represent reductions. These diagrams are read as follows: for all reductions represented as solid lines in the diagram, there are reductions represented as dashed lines such that the diagram commutes.

Lemma 3.2 *Let Σ denote the reduction relation $\stackrel{\Sigma}{\longrightarrow}$, Σ^* denote its reflexive transitive closure, and $\Sigma^*\beta_{||}\Sigma^*$ denote the composition of Σ^*, $\stackrel{\beta_{||}}{\longrightarrow}$ and Σ^*. Then:*



Proof: The proof is as in [HL89], proposition 3.2. Because all rules are left-linear, we only have to consider the critical pairs between Σ and $\stackrel{\beta_{||}}{\longrightarrow}$. There are five interesting cases, which parallel the five critical pairs between (β^ℓ) and the rules of Σ is Section 14 of [GL95]; there are no critical pairs with (β).

Case 1: $u = ((\lambda^\ell u_1) \star^\ell u_2) \circ^\ell u_3$, $v = (\lambda^\ell u_1 \circ^\ell u_3) \star^\ell (u_2 \circ^\ell u_3)$ is obtained by rule $(\star^\ell)$ and $w = (u'_1 \circ^\ell (u'_2 \bullet^\ell id^\ell)) \circ^\ell u'_3$, where $u_1 \xrightarrow{\beta_{||}} u'_1$, $u_2 \xrightarrow{\beta_{||}} u'_2$ and $u_3 \xrightarrow{\beta_{||}} u'_3$. Then:

$$\begin{array}{lll} w & = (u'_1 \circ^\ell (u'_2 \bullet^\ell id^\ell)) \circ^\ell u'_3 & \\ & \longrightarrow u'_1 \circ^\ell ((u'_2 \bullet^\ell id^\ell) \circ^\ell u'_3) & \text{by } (\circ^\ell) \\ & \longrightarrow u'_1 \circ^\ell (u'_2 \circ^\ell u'_3 \bullet^\ell id^\ell \circ^\ell u'_3) & \text{by } (\bullet^\ell) \\ & \longrightarrow u'_1 \circ^\ell (u'_2 \circ^\ell u'_3 \bullet^\ell u'_3) & \text{by } (id\circ^\ell) \end{array}$$

while:

$$\begin{array}{lll} v & = (\lambda^\ell u_1 \circ^\ell u_3) \star^\ell (u_2 \circ^\ell u_3) & \\ & \longrightarrow \lambda^\ell (u_1 \circ^\ell \Uparrow^\ell u_3) \star^\ell (u_2 \circ^\ell u_3) & \text{by } (\lambda^\ell) \\ & \xrightarrow{\beta_{||}} (u'_1 \circ^\ell \Uparrow^\ell u'_3) \circ^\ell (u'_2 \circ^\ell u'_3 \bullet^\ell id^\ell) & \\ & \longrightarrow u'_1 \circ^\ell (\Uparrow^\ell u'_3 \circ^\ell (u'_2 \circ^\ell u'_3 \bullet^\ell id^\ell)) & \text{by } (\circ^\ell) \\ & \longrightarrow u'_1 \circ^\ell (u'_2 \circ^\ell u'_3 \bullet^\ell u'_3 \circ^\ell id^\ell) & \text{by } (\Uparrow\bullet^\ell) \\ & \longrightarrow u'_1 \circ^\ell (u'_2 \circ^\ell u'_3 \bullet^\ell u'_3) & \text{by } (\circ id^\ell) \end{array}$$

Case 2: $u = \mathtt{ev}^\ell((\lambda^\ell u_1) \star^\ell u_2) u_3$, $v = \mathtt{ev}^\ell(\lambda^\ell u_1) u_3 \star^{\ell-1} \mathtt{ev}^\ell u_2 u_3$ is obtained by rule $(\mathtt{ev}\star^\ell)$ and $w = \mathtt{ev}^\ell(u'_1 \circ^\ell (u'_2 \bullet^\ell id^\ell)) u'_3$, where $u_1 \xrightarrow{\beta_{||}} u'_1$, $u_2 \xrightarrow{\beta_{||}} u'_2$ and $u_3 \xrightarrow{\beta_{||}} u'_3$. Then:

$$\begin{array}{lll} w & = \mathtt{ev}^\ell(u'_1 \circ^\ell (u'_2 \bullet^\ell id^\ell)) u'_3 & \\ & \longrightarrow \mathtt{ev}^\ell u'_1 (\mathtt{ev}^\ell (u'_2 \bullet^\ell id^\ell) u'_3) & \text{by } (\mathtt{ev}\circ^\ell) \\ & \longrightarrow \mathtt{ev}^\ell u'_1 (\mathtt{ev}^\ell u'_2 u'_3 \bullet^{\ell-1} \mathtt{ev}^\ell id^\ell u'_3) & \text{by } (\mathtt{ev}\bullet^\ell) \\ & \longrightarrow \mathtt{ev}^\ell u'_1 (\mathtt{ev}^\ell u'_2 u'_3 \bullet^{\ell-1} u'_3) & \text{by } (\mathtt{ev}id^\ell) \end{array}$$

(let the last term be called t), while if $\ell = 1$:

$$\begin{array}{lll} v & = (\mathtt{ev}^1(\lambda^1 u_1) u_3)(\mathtt{ev}^1 u_2 u_3) & \\ & \longrightarrow (\lambda x \cdot \mathtt{ev}^1 u_1 (x \bullet u_3))(\mathtt{ev}^1 u_2 u_3) & \text{by } (\mathtt{ev}\lambda^1) \\ & \xrightarrow{\beta_{||}} \mathtt{ev}^1 u'_1 (\mathtt{ev}^1 u'_2 u'_3 \bullet u'_3) = t & \end{array}$$

and if $\ell > 1$:

$$\begin{array}{lll} v & = \mathtt{ev}^\ell(\lambda^\ell u_1) u_3 \star^{\ell-1} \mathtt{ev}^\ell u_2 u_3 & \\ & \longrightarrow \lambda^{\ell-1}(\mathtt{ev}^\ell(u_1 \circ^{\ell-1} \uparrow^{\ell-1})(1^{\ell-1} \bullet^{\ell-1} u_3 \circ^{\ell-1} \uparrow^{\ell-1})) \star^{\ell-1} \mathtt{ev}^\ell u_2 u_3 & \text{by } (\mathtt{ev}\lambda^\ell) \\ & \xrightarrow{\beta_{||}} (\mathtt{ev}^\ell(u'_1 \circ^{\ell-1} \uparrow^{\ell-1})(1^{\ell-1} \bullet^{\ell-1} u'_3 \circ^{\ell-1} \uparrow^{\ell-1})) \circ^{\ell-1} (\mathtt{ev}^\ell u'_2 u'_3 \bullet^{\ell-1} id^{\ell-1}) & \\ & \longrightarrow \mathtt{ev}^\ell((u'_1 \circ^{\ell-1} \uparrow^{\ell-1}) \circ^{\ell-1} (\mathtt{ev}^\ell u'_2 u'_3 \bullet^{\ell-1} id^{\ell-1})) & \\ & \quad ((1^{\ell-1} \bullet^{\ell-1} u'_3 \circ^{\ell-1} \uparrow^{\ell-1}) \circ^{\ell-1} (\mathtt{ev}^\ell u'_2 u'_3 \bullet^{\ell-1} id^{\ell-1})) & \text{by } (\mathtt{ev}^\ell \circ^{\ell-1}) \\ & \longrightarrow \mathtt{ev}^\ell(u'_1 \circ^{\ell-1} (\uparrow^{\ell-1} \circ^{\ell-1} (\mathtt{ev}^\ell u'_2 u'_3 \bullet^{\ell-1} id^{\ell-1}))) & \\ & \quad ((1^{\ell-1} \bullet^{\ell-1} u'_3 \circ^{\ell-1} \uparrow^{\ell-1}) \circ^{\ell-1} (\mathtt{ev}^\ell u'_2 u'_3 \bullet^{\ell-1} id^{\ell-1})) & \text{by } (\circ^{\ell-1}) \\ & \longrightarrow \mathtt{ev}^\ell(u'_1 \circ^{\ell-1} id^{\ell-1})((1^{\ell-1} \bullet^{\ell-1} u'_3 \circ^{\ell-1} \uparrow^{\ell-1}) \circ^{\ell-1} (\mathtt{ev}^\ell u'_2 u'_3 \bullet^{\ell-1} id^{\ell-1})) & \text{by } (\uparrow^{\ell-1}) \\ & \longrightarrow \mathtt{ev}^\ell u'_1 ((1^{\ell-1} \bullet^{\ell-1} u'_3 \circ^{\ell-1} \uparrow^{\ell-1}) \circ^{\ell-1} (\mathtt{ev}^\ell u'_2 u'_3 \bullet^{\ell-1} id^{\ell-1})) & \text{by } (\circ id^{\ell-1}) \\ & \longrightarrow \mathtt{ev}^\ell u'_1 (1^{\ell-1} \circ^{\ell-1} (\mathtt{ev}^\ell u'_2 u'_3 \bullet^{\ell-1} id^{\ell-1}) \bullet^{\ell-1} (u'_3 \circ^{\ell-1} \uparrow^{\ell-1}) \circ^{\ell-1} (\mathtt{ev}^\ell u'_2 u'_3 \bullet^{\ell-1} id^{\ell-1})) & \text{by } (\bullet^{\ell-1}) \\ & \longrightarrow \mathtt{ev}^\ell u'_1 (\mathtt{ev}^\ell u'_2 u'_3 \bullet^{\ell-1} (u'_3 \circ^{\ell-1} \uparrow^{\ell-1}) \circ^{\ell-1} (\mathtt{ev}^\ell u'_2 u'_3 \bullet^{\ell-1} id^{\ell-1})) & \text{by } (1^{\ell-1}) \\ & \longrightarrow \mathtt{ev}^\ell u'_1 (\mathtt{ev}^\ell u'_2 u'_3 \bullet^{\ell-1} u'_3 \circ^{\ell-1} (\uparrow^{\ell-1} \circ^{\ell-1} (\mathtt{ev}^\ell u'_2 u'_3 \bullet^{\ell-1} id^{\ell-1}))) & \text{by } (\circ^{\ell-1}) \\ & \longrightarrow \mathtt{ev}^\ell u'_1 (\mathtt{ev}^\ell u'_2 u'_3 \bullet^{\ell-1} u'_3 \circ^{\ell-1} id^{\ell-1}) & \text{by } (\uparrow^{\ell-1}) \\ & \longrightarrow \mathtt{ev}^\ell u'_1 (\mathtt{ev}^\ell u'_2 u'_3 \bullet^{\ell-1} u'_3) = t & \text{by } (\circ id^{\ell-1}) \end{array}$$

Case 3: $u = (\lambda^{\mathcal{L}} u_1 \star^{\mathcal{L}} u_2) \circ^\ell u_3$, with $1 \leq \ell < \mathcal{L}$, $v = (\lambda^{\mathcal{L}} u_1 \circ^\ell u_3) \star^{\mathcal{L}} (u_2 \circ^\ell u_3)$ is obtained by rule $(\star^{\mathcal{L}} \circ^\ell)$, and $w = (u'_1 \circ^{\mathcal{L}} (u'_2 \bullet^{\mathcal{L}} id^{\mathcal{L}})) \circ^\ell u'_3$, where $u_1 \xrightarrow{\beta_{||}} u'_1$, $u_2 \xrightarrow{\beta_{||}} u'_2$ and $u_3 \xrightarrow{\beta_{||}} u'_3$. Then:

$$\begin{array}{lll} w & = (u'_1 \circ^{\mathcal{L}} (u'_2 \bullet^{\mathcal{L}} id^{\mathcal{L}})) \circ^\ell u'_3 & \\ & \longrightarrow (u'_1 \circ^\ell u'_3) \circ^{\mathcal{L}} ((u'_2 \bullet^{\mathcal{L}} id^{\mathcal{L}}) \circ^\ell u'_3) & \text{by } (\circ^{\mathcal{L}} \circ^\ell) \\ & \longrightarrow (u'_1 \circ^\ell u'_3) \circ^{\mathcal{L}} (u'_2 \circ^\ell u'_3 \bullet^{\mathcal{L}} id^{\mathcal{L}} \circ^\ell u'_3) & \text{by } (\bullet^{\mathcal{L}} \circ^\ell) \\ & \longrightarrow (u'_1 \circ^\ell u'_3) \circ^{\mathcal{L}} (u'_2 \circ^\ell u'_3 \bullet^{\mathcal{L}} id^{\mathcal{L}}) & \text{by } (id^{\mathcal{L}} \circ^\ell) \end{array}$$

while

$$\begin{aligned} v &= (\lambda^{\mathcal{L}} u_1 \circ^{\ell} u_3) \star^{\mathcal{L}} (u_2 \circ^{\ell} u_3) \\ &\longrightarrow \lambda^{\mathcal{L}}(u_1 \circ^{\ell} u_3) \star^{\mathcal{L}} (u_2 \circ^{\ell} u_3) && \text{by } (\lambda^{\mathcal{L}} \circ^{\ell}) \\ &\xrightarrow{\beta_{||}} (u_1' \circ^{\ell} u_3') \circ^{\mathcal{L}} (u_2' \circ^{\ell} u_3' \bullet^{\mathcal{L}} id^{\mathcal{L}}) \end{aligned}$$

Case 4: $u = \mathtt{ev}^{\ell}(\lambda^{\mathcal{L}} u_1 \star^{\mathcal{L}} u_2)u_3$, with $1 \leq \ell < \mathcal{L}$, $v = (\mathtt{ev}^{\ell}(\lambda^{\mathcal{L}} u_1)u_3) \star^{\mathcal{L}-1} (\mathtt{ev}^{\ell} u_2 u_3)$ is obtained by rule $(\mathtt{ev}^{\ell}\star^{\mathcal{L}})$, and $w = \mathtt{ev}^{\ell}(u_1' \circ^{\mathcal{L}} (u_2' \bullet^{\mathcal{L}} id^{\mathcal{L}}))u_3'$, where $u_1 \xrightarrow{\beta_{||}} u_1'$, $u_2 \xrightarrow{\beta_{||}} u_2'$ and $u_3 \xrightarrow{\beta_{||}} u_3'$. Then:

$$\begin{aligned} w &= \mathtt{ev}^{\ell}(u_1' \circ^{\mathcal{L}} (u_2' \bullet^{\mathcal{L}} id^{\mathcal{L}}))u_3' \\ &\longrightarrow (\mathtt{ev}^{\ell} u_1' u_3') \circ^{\mathcal{L}-1} (\mathtt{ev}^{\ell}(u_2' \bullet^{\mathcal{L}} id^{\mathcal{L}})u_3') && \text{by } (\mathtt{ev}^{\ell} \circ^{\mathcal{L}}) \\ &\longrightarrow (\mathtt{ev}^{\ell} u_1' u_3') \circ^{\mathcal{L}-1} (\mathtt{ev}^{\ell} u_2' u_3' \bullet^{\mathcal{L}-1} \mathtt{ev}^{\ell} id^{\mathcal{L}} u_3') && \text{by } (\mathtt{ev}^{\ell} \bullet^{\mathcal{L}}) \\ &\longrightarrow (\mathtt{ev}^{\ell} u_1' u_3') \circ^{\mathcal{L}-1} (\mathtt{ev}^{\ell} u_2' u_3' \bullet^{\mathcal{L}-1} id^{\mathcal{L}-1}) && \text{by } (\mathtt{ev}^{\ell} id^{\mathcal{L}}) \end{aligned}$$

while

$$\begin{aligned} v &= (\mathtt{ev}^{\ell}(\lambda^{\mathcal{L}} u_1)u_3) \star^{\mathcal{L}-1} (\mathtt{ev}^{\ell} u_2 u_3) \\ &\longrightarrow \lambda^{\mathcal{L}-1}(\mathtt{ev}^{\ell} u_1 u_3) \star^{\mathcal{L}-1} (\mathtt{ev}^{\ell} u_2 u_3) && \text{by } (\mathtt{ev}^{\ell} \lambda^{\mathcal{L}}) \\ &\xrightarrow{\beta_{||}} (\mathtt{ev}^{\ell} u_1' u_3') \circ^{\mathcal{L}-1} (\mathtt{ev}^{\ell} u_2' u_3' \bullet^{\mathcal{L}-1} id^{\mathcal{L}-1}) \end{aligned}$$

Case 5: $u = Q^{\ell}(\lambda^{\mathcal{L}-1} u_1 \star^{\mathcal{L}-1} u_2)$, with $1 \leq \ell < \mathcal{L}$, $v = Q^{\ell}(\lambda^{\mathcal{L}-1} u_1) \star^{\mathcal{L}} (Q^{\ell} u_2)$ is obtained by rule $(Q^{\ell}\star^{\mathcal{L}})$, and $w = Q^{\ell}(u_1' \circ^{\mathcal{L}-1} (u_2' \bullet^{\mathcal{L}-1} id^{\mathcal{L}-1}))$, where $u_1 \xrightarrow{\beta_{||}} u_1'$ and $u_2 \xrightarrow{\beta_{||}} u_2'$. Then:

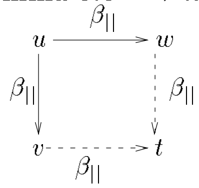
$$\begin{aligned} w &= Q^{\ell}(u_1' \circ^{\mathcal{L}-1} (u_2' \bullet^{\mathcal{L}-1} id^{\mathcal{L}-1})) \\ &\longrightarrow Q^{\ell} u_1' \circ^{\mathcal{L}} Q^{\ell}(u_2' \bullet^{\mathcal{L}-1} id^{\mathcal{L}-1}) && \text{by } (Q^{\ell} \circ^{\mathcal{L}}) \\ &\longrightarrow Q^{\ell} u_1' \circ^{\mathcal{L}} (Q^{\ell} u_2' \bullet^{\mathcal{L}} Q^{\ell} id^{\mathcal{L}-1}) && \text{by } (Q^{\ell} \bullet^{\mathcal{L}}) \\ &\longrightarrow Q^{\ell} u_1' \circ^{\mathcal{L}} (Q^{\ell} u_2' \bullet^{\mathcal{L}} id^{\mathcal{L}}) && \text{by } (Q^{\ell} id^{\mathcal{L}}) \end{aligned}$$

while

$$\begin{aligned} v &= Q^{\ell}(\lambda^{\mathcal{L}-1} u_1) \star^{\mathcal{L}} (Q^{\ell} u_2) \\ &\longrightarrow \lambda^{\mathcal{L}}(Q^{\ell} u_1) \star^{\mathcal{L}} (Q^{\ell} u_2) && \text{by } (Q^{\ell} \lambda^{\mathcal{L}}) \\ &\xrightarrow{\beta_{||}} Q^{\ell} u_1' \circ^{\mathcal{L}} (Q^{\ell} u_2' \bullet^{\mathcal{L}} id^{\mathcal{L}}) \end{aligned}$$

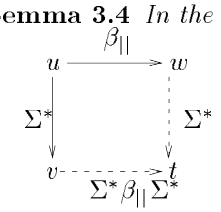
In any other case, it is readily verified that $v \xrightarrow{\beta_{||}} t$, where w rewrites in one step to t by the same rule that was used from u to v. □

Lemma 3.3 $\xrightarrow{\beta_{||}}$ *is strongly confluent. More precisely, the following holds:*



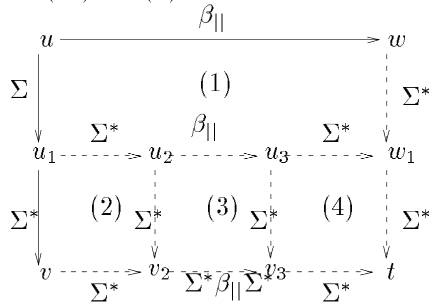
Proof: $\xrightarrow{\beta_{||}}$ is defined as a left linear system, and has no critical pairs. □

Lemma 3.4 *In the typed case, we have:*



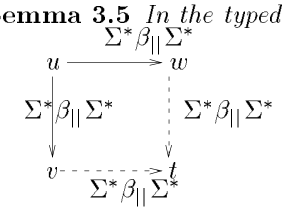
Proof: By induction on $\nu(u)$, the length of the longest Σ derivation starting from u. Observe that by Lemma 3.1 Σ is confluent.

If $\nu(u) = 0$, then the result is clear. Otherwise, let the first reduction step from u to v rewrite u to u_1, with $\nu(u_1) < \nu(u)$. We have:



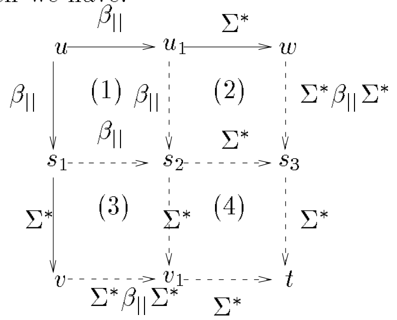
where (1) follows from Lemma 3.2, (2) follows from the fact that Σ is confluent, (3) follows by induction hypothesis, noticing that $\nu(u_2) \leq \nu(u_1) < \nu(u)$ and (4) follows from the confluence of Σ. $\Box$

Lemma 3.5 *In the typed case, we have:*



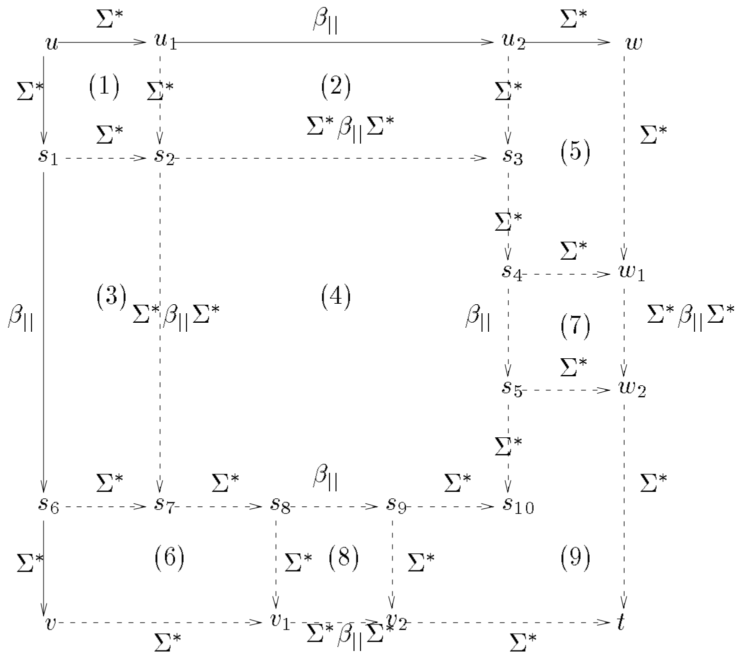
that is, $\Sigma^*\beta_{||}\Sigma^*$ *is strongly confluent.*

Proof: By induction on $\nu(u)$ again. If the reductions from u to v and to w both begin by the $\beta_{||}$ reduction, then we have:



where (1) follows from Lemma 3.3, (2) and (3) follow from Lemma 3.4, and (4) follows from the confluence of Σ.

Otherwise, we have:



where (1) follows from the confluence of Σ, (2) and (3) come from Lemma 3.4, (4) comes from the induction hypothesis, since $\nu(s_2) \leq \min(\nu(u_1), \nu(s_1)) < \nu(u)$, (5) and (6) follow from the confluence of Σ, (7) and (8) follow from Lemma 3.4, and (9) comes from the fact that Σ is Church-Rosser. □

Theorem 3.6 *The typed $\lambda\mathtt{ev}Q$-calculus is confluent.*

Proof: By Lemma 3.5, $\Sigma^*\beta_{||}\Sigma^*$ is confluent. Furthermore, its reflexive transitive closure is exactly the reduction relation for $\lambda\mathtt{ev}Q$, hence the result. □

We now examine the question whether the $\lambda\mathtt{ev}Q_H$-calculus is confluent. Although we have taken some precautions (namely, separating the terms into two sorts, and allowing variables only of sort T), the untyped $\lambda\mathtt{ev}Q_H$-calculus is *not* confluent. Indeed, we may embed variables of sort T in a stack by using, for example $\bullet^1$, and replay Klop or Hardin's counterexample to confluence:

Theorem 3.7 *The untyped $\lambda\mathtt{ev}Q_H$-calculus is not confluent.*

Proof: We replay, almost unchanged, Hardin's proof [Har89]. The only difference is the introduction of the operators 1^1, $\bullet^1$ and id^1 below.

Let $P = \lambda x \cdot \lambda y \cdot y((xx)y)$, and $\Theta = PP$ be Turing's fixed point combinator; it is such that $\Theta u \longrightarrow^* u(\Theta u)$ for every u. Let also:

$$\begin{aligned} I &= \lambda x \cdot x \\ U &= \lambda x \cdot \lambda y \cdot \left(1^1 \circ^1_T \left(1\left((\lambda z \cdot z(xy)) \bullet^1 id^1\right) \bullet\!\uparrow \left((\lambda z \cdot zy) \bullet^1 id^1\right)\right)\right)\left(\lambda z \cdot I\right) \\ C &= \Theta U \\ B &= \Theta C \end{aligned}$$

Check that I, U, C and B are of sort T (in the untyped calculus; U is not a semi-stratified term).

Now, by definition of Θ, (1) $C \longrightarrow^* UC$, and (2) $B \longrightarrow^* CB$.

So for every term u, Cu rewrites to UCu by (1), hence by two applications of (β) to $X(u)$, where:

$$X(u) = \left(1^1 \circ^1_T \left(1\left((\lambda z \cdot z(Cu)) \bullet^1 id^1\right) \bullet\!\uparrow \left((\lambda z \cdot zu) \bullet^1 id^1\right)\right)\right)\left(\lambda z \cdot I\right)$$

by definition. To sum up, (3) $Cu \longrightarrow^* X(u)$.

Then we have:

$$
\begin{array}{ll}
B \longrightarrow^* CB & \text{by } (2) \\
\longrightarrow^* X(B) & \text{by } (3) \\
= \left(1^1 \circ^1_T \left(1\left((\lambda z \cdot z(CB)) \bullet^1 id^1\right) \bullet\!\uparrow \left((\lambda z \cdot zB) \bullet^1 id^1\right)\right)\right)\left(\lambda z \cdot I\right) & \\
\longrightarrow^* \left(1^1 \circ^1_T \left(1\left((\lambda z \cdot z(CB)) \bullet^1 id^1\right) \bullet\!\uparrow \left((\lambda z \cdot z(CB)) \bullet^1 id^1\right)\right)\right)\left(\lambda z \cdot I\right) & \text{by } (2) \\
\longrightarrow \left(1^1 \circ^1_T \left((\lambda z \cdot z(CB)) \bullet^1 id^1\right)\right)(\lambda z \cdot I) & \text{by } (\eta\, \bullet) \\
\longrightarrow (\lambda z \cdot z(CB))(\lambda z \cdot I) & \text{by } (1^1) \\
\longrightarrow (\lambda z \cdot I)(CB) & \text{by } (\beta) \\
\longrightarrow I & \text{by } (\beta)
\end{array}
$$

So: (4) $B \longrightarrow^* I$.

It follows: (5) $B \longrightarrow^* CI$, by (2) and (4).

We now claim: (6) If $\lambda\mathtt{ev}Q_H$ has the unique normal form property (i.e., any two normal forms of the same term are equal) and u has a normal form u_0 different from I, then Cu and u have no common reduct. Indeed, by (3) $Cu \longrightarrow^* X(u)$, and if Cu and u had a common reduct v, then:

$$
\begin{array}{ll}
X(u) \longrightarrow^* \left(1^1 \circ^1_T \left(1\left((\lambda z \cdot zv) \bullet^1 id^1\right) \bullet\!\uparrow \left((\lambda z \cdot zv) \bullet^1 id^1\right)\right)\right)\left(\lambda z \cdot I\right) & \text{by rewriting } Cu \text{ and } u \\
\longrightarrow \left(1^1 \circ^1_T \left((\lambda z \cdot zv) \bullet^1 id^1\right)\right)(\lambda z \cdot I) & \text{by } (\eta\, \bullet) \\
\longrightarrow (\lambda z \cdot zv)(\lambda z \cdot I) & \text{by } (1^1) \\
\longrightarrow (\lambda z \cdot I)v & \text{by } (\beta) \\
\longrightarrow I & \text{by } (\beta)
\end{array}
$$

but since $\lambda\mathtt{ev}Q_H$ is assumed to have the unique normal form property, then $u_0 = I$, which contradicts the assumption $u_0 \neq I$.

We now claim that (8) if $\lambda\mathtt{ev}Q_H$ has the unique normal form property, then CI does not reduce to I. Indeed, assume that $\lambda\mathtt{ev}Q_H$ has the unique normal form property, and let R be a derivation from CI to I using rule $(\eta\, \bullet)$ the least many times. Now CI has only one redex, namely the one in $\Theta = (\lambda x \cdot \lambda y \cdot y((xx)y))P$. So the first step in R must rewrite CI into A_1, where:

$$A_1 = (\lambda y \cdot y((PP)y))UI = (\lambda y \cdot y(\Theta y))UI$$

Let R_1 be the the subsequence of R leading from A_1 to I. Since U and I are normal, the only possible reductions in A_1 are to rewrite under $\lambda y \cdot$ in A_1 (in fact to rewrite Θy) or to contract the outermost redex $(\lambda y \cdot y(\Theta y))U$. Note that the outermost redex must eventually be contracted, because there is no such redex in the end-term of R_1, namely I. So R_1 decomposes into, first, a reduction R_1' from Θy to some term that we denote by $A(y)$, and second a sequence R_2 of rewriting steps from:

$$A_2 = (\lambda y \cdot y\, A(y))UI$$

to I. Then if we choose R_1' to be of maximal length, R_2 is:

$$A_2 \longrightarrow U\, A(U)\, \underbrace{I \longrightarrow^* I}_{R_3}$$

Observe that $A(U)$ can be obtained from ΘU, i.e. from C by a sequence of rewriting steps, which we shall again call R_1'. R_3 may rewrite $A(U)$, but by the same argument it must eventually contract the redex $U\, A(U)$. Without loss of generality, assume that R_3 begins by contracting the latter. Then $U\, A(U)\, I$ contracts to A_3, where:

$$A_3 = \left(\lambda y \cdot \left(1^1 \circ^1_T \left(1\left((\lambda z \cdot z(A(U)y)) \bullet^1 id^1\right) \bullet\!\uparrow \left((\lambda z \cdot zy) \bullet^1 id^1\right)\right)\right)\left(\lambda z \cdot I\right)\right) I$$

Let R_4 be the rest of the derivation. R_4 may first rewrite $A(U)y$, so in general it has the form:

$$A_3 \longrightarrow^* \left(\lambda y \cdot \left(1^1 \circ^1_T \left(1\left((\lambda z \cdot zD) \bullet^1 id^1\right) \bullet\!\uparrow \left((\lambda z \cdot zy) \bullet^1 id^1\right)\right)\right)\left(\lambda z \cdot I\right)\right) I \underbrace{\longrightarrow^* I}_{R_5}$$

where R_5 does not start by reducing D, and where $A(U)y \longrightarrow^* D$ by some subsequence R'_4 of rewriting steps in R_4. Then (7) $Cy \longrightarrow^* D$ by R'_1 followed by R'_4. Consider the first step of R_5: it may either contract the outermost (β) redex or the inner $(\eta\ \bullet)$ redex. In the latter case, we must have $D = y$, therefore by (7) $Cy \longrightarrow^* y$, which is impossible by (6), since y is a normal form different from I. So the first step of R_5 contracts the outermost (β) redex, leading to A_4, where:

$$A_4 = \left(1^1 \circ^1_T \left(1\left((\lambda z \cdot zD) \bullet^1 id^1\right) \bullet\!\uparrow \left((\lambda z \cdot zI) \bullet^1 id^1\right)\right)\right)\left(\lambda z \cdot I\right)$$

The only way that A_4 can reduce to I involves making the part on the left of $\lambda z \cdot I$ an $(\eta\ \bullet)$ redex. So R_5 must eventually reduce D to I, then apply $(\eta\ \bullet)$. Consider the subderivation R'_5 of R_5 reducing D to I. By (7), the concatenation of R'_1, R'_4 and R'_5 then reduces CI to I. This concatenation is a subderivation of R, and uses at least one less instance of $(\eta\ \bullet)$, contradicting the minimality of R.

So, if $\lambda\mathsf{ev}Q_H$ was confluent, by (4) and (5) CI and I would have a common reduct, that is CI would reduce to I, since I is normal. Then, $\lambda\mathsf{ev}Q_H$ would also have the unique normal form property, so by (8) CI cannot reduce to I: this is a contradiction. □

The problem in the untyped $\lambda\mathsf{ev}Q_H$-calculus is that we may mix operators from levels that have nothing to do with each other. As already announced, we leave the question of the confluence of the typed $\lambda\mathsf{ev}Q_H$-calculus open until part IIIb.

4 From $\lambda\mathsf{ev}Q$ To $\lambda^{\approx}_{\mathrm{S4}}$

Although reduction in $\lambda^{\approx}_{\mathrm{S4}}$ (resp. $\lambda^{\approx}_{\mathrm{S4}H}$) can be simulated by reduction in $\lambda\mathsf{ev}Q$ (resp. $\lambda\mathsf{ev}Q_H$), it is not obvious that the converse holds. Ideally, we would like to show that the $\lambda\mathsf{ev}Q$-calculus (resp. $\lambda\mathsf{ev}Q_H$) is a conservative m-extension of the $\lambda^{\approx}_{\mathrm{S4}}$-calculus (resp. $\lambda^{\approx}_{\mathrm{S4}H}$). It is an m-extension [Har89] if and only if:

(1) G is injective from $\lambda^{\approx}_{\mathrm{S4}}$ to $\lambda\mathsf{ev}Q$,

(2) for every $\lambda^{\approx}_{\mathrm{S4}}$-terms u and v, u reduces to v in λ_{S4} (resp. $\lambda_{\mathrm{S4}H}$) if and only if $G(u)$ reduces to $G(v)$ in $\lambda\mathsf{ev}Q$ (resp. $\lambda\mathsf{ev}Q_H$),

(3) and for every $\lambda^{\approx}_{\mathrm{S4}}$-term u, if $G(u)$ reduces to some term t in $\lambda\mathsf{ev}Q$ (resp. $\lambda\mathsf{ev}Q_H$), then t reduces to some term of the form $G(v)$, with v a $\lambda^{\approx}_{\mathrm{S4}}$-term.

And it is conservative if and only if:

(4) for every $\lambda^{\approx}_{\mathrm{S4}}$-terms u and v, u and v are λ_{S4}-equivalent (resp. $\lambda_{\mathrm{S4}H}$-equivalent) if and only if $G(u)$ and $G(v)$ are $\lambda\mathsf{ev}Q$-equivalent (resp. $\lambda\mathsf{ev}Q_H$-equivalent).

But G does not obey property (2). Whenever u reduces to v, then $G(u)$ reduces to $G(v)$, but the converse fails: consider indeed $u = \mathsf{unbox}\ (xy)^{\text{‘}}$, where $x : \Phi_1 \Rightarrow \Box\Phi_2$ and $y : \Phi_1$, and $v = (\mathsf{unbox}\ x^{\text{‘}})(\mathsf{unbox}\ y^{\text{‘}})$. We have $G(u) = \mathsf{ev}^1(Q^1 x \star^1 Q^1 y)()$, which rewrites by $(\mathsf{ev}\star^1)$ to $(\mathsf{ev}^1(Q^1 x)())(\mathsf{ev}^1(Q^1 y)()) = G(v)$; but u does not rewrite to v in λ_{S4} or $\lambda_{\mathrm{S4}H}$: indeed, the only term to which u can rewrite is xy.

So we shall actually only prove that the $\lambda\mathsf{ev}Q$-calculus (resp. $\lambda\mathsf{ev}Q_H$) is a conservative extension of the $\lambda^{\approx}_{\mathrm{S4}}$-calculus (resp. $\lambda^{\approx}_{\mathrm{S4}H}$), i.e. property (4).

We first prove property (1). Observe that we have chosen to see G as a function from $\lambda^{\approx}_{\mathrm{S4}}$-terms, not λ_{S4}-terms, to $\lambda\mathsf{ev}Q$-terms. This is the only reasonable definition, because of Theorem 3.9 and Lemmas 3.10 and 3.11 in Part II: we must interpret λ_{S4}-terms modulo (gc) and (ctract).

Lemma 4.1 *For all $n \in \mathbb{N}$, for every environment ρ of cardinality at least n, for every substitution σ, if $(u^{\text{‘}}\rho)\sigma = \mathsf{pop}^1_n$, then $u = ()$.*

Proof: If $n = 0$, then $\mathsf{pop}_n^1 = id^1$. Since $u`\rho$ cannot be a variable (see Figure 2, Part II), we must have $u`\rho = id^1$. But this can only happen when $u = ()$.

If $n \geq 1$, we prove the result by induction on n. We use the fact that $\mathsf{pop}_1^1 = \uparrow^1$, $\mathsf{pop}_{n+1}^1 = \uparrow^1 \circ^1 \mathsf{pop}_n^1$, $n \geq 1$.

When $n = 1$, if $(u`\rho)\sigma = \uparrow^1$, then by the same argument as above $u`\rho = \uparrow^1$, and the only applicable quotation rule entails that ρ has cardinality 1 and $u = ()$.

Assume that the claim holds for $n \geq 1$, and prove it for $n+1$: let ρ have cardinality at least $n+1$, and $\mathsf{pop}_{n+1}^1 = (u`\rho)\sigma$. Since $(u`\rho)\sigma$ has the form $\uparrow^1 \circ^1 \ldots$, inspection of the quotation rules shows that u must be either $()$ or of the form $\uparrow u'$ for some term u' such that (*) $\mathsf{pop}_n^1 = (u'`\rho)\sigma$. In the latter case, we apply the induction hypothesis, since the cardinality of $\rho$ is greater than $n$, so $u' = ()$: then $(u'`\rho)\sigma$ would be pop_{n+1}^1 by definition of $\_`$, contradicting (*). The only possible case is therefore the former, $u = ()$. ☐

Before we continue, we introduce a family of variables ξ_u for each term u. More formally, let W be a given set of variables, such that there are infinitely many variables outside of W. We build a family of variables ξ_u for every term u whose free variables are in W, in such a way that: ξ_u is not in W, and $u = v$ if and only if $\xi_u = \xi_v$.

We say that a term is a *ξ-term* if and only if all its free variables are ξ-variables. A *regular term* is any term whose free variables are all in W. We shall consider that W is so large that any $\lambda\mathsf{ev}Q$-term that we ordinarily use is regular.

We denote by ζ the (infinite) substitution mapping ξ_u to u. It maps ξ-variables to regular terms.

Lemma 4.2 *For any environment ρ, for every term u, there is at most one ξ-term s such that $u = (s`\rho)\zeta$.*

Proof: By structural induction on u.

If u is of the form $Q^1 v$, then the only quoting rule that applies is that for variables, so the only possible ξ-term s is ξ_v (and u is, more precisely, $Q_T^1 v$).

If u is of the form, say, $u_1 \bullet^1 u_2$, then if $u = (s`\rho)\sigma$, then $s$ must be of the form $s_1 \bullet s_2$, with $u_1 = (s_1`\rho)\zeta$ and $u_2 = (s_2`\rho)\zeta$. By induction hypothesis, there is at most one $\xi$-term $s_1$ and at most one $\xi$-term $s_2$ such that $u_1 = (s_1`\rho)\zeta$ and $u_2 = (s_2`\rho)\zeta$, so s is unique.

All other cases are similar, except when u is of the form $\lambda^1 v$ or $v \circ^1 w$. In the first case, we have to apply the induction hypothesis with $\rho[x \mapsto n]$ instead of ρ, where x is some new variable (in W) and n is the cardinality of ρ.

In the last case, where $u = v \circ^1 w$, u may be the translation of a variable in the domain of ρ, or of $()$, or of a projection $1u'$ or $\uparrow u'$. In any case, let n be the cardinality of ρ.

If $v = \uparrow^1$, then we have two possibilities, namely $s = ()$ or $s = \uparrow u'$. But these possibilities are exclusive: if $s = ()$, then $u = \mathsf{pop}_n^1$; and if $s = \uparrow u'$, then by Lemma 4.1 u cannot be pop_n^1. So, either $u = \mathsf{pop}_n^1$ and the only possible s is $()$, or $u \neq \mathsf{pop}_n^1$. In this latter case, s must be $\uparrow u'$, and we must have $w = (u'`\rho)\zeta$: by induction hypothesis, u' is unique, hence also $s = \uparrow u'$.

If $v = 1^1$, then u may be the quotation of a variable in the domain of ρ, or of a 1 projection.

If $w = \mathsf{pop}_k^1$, with $0 \leq k \leq n-1$, we claim that s cannot be a projection: indeed, if $s = 1u'$, then $u = ((1u')`\rho)\zeta$, so $w = (u'`\rho)\zeta$, and since $w = \mathsf{pop}_k^1$ by assumption, by Lemma 4.1 using the fact that $n \geq k$, we must have $u' = ()$. But then $w = \mathsf{pop}_k^1 = \mathsf{pop}_n^1$, which is impossible since $k \neq n$. So s can only be a variable, namely that which ρ maps to $n-1-k$. So s is unique.

And if $w \neq \mathsf{pop}_k^1$ for every $0 \leq k \leq n-1$, then s must be of the form $1u'$, so $w = (u'`\rho)\zeta$. But then u', hence s, is unique by induction hypothesis. ☐

Lemma 4.3 *G, as a function from $\lambda_{\mathrm{S4}}^{\approx}$ to $\lambda\mathsf{ev}Q$, is injective.*

Proof: We have to prove that every $\lambda\mathsf{ev}Q$-term u is the image of at most one term by G up to $\approx$, and we prove it by structural induction on u.

If u is a variable, observe that u cannot be of the form $(s`\rho)\sigma$ for any s, ρ and σ, so the only $\lambda_{\mathrm{S4}}^{\approx}$-term v such that $G(v) = u$ is u itself.

If u is of the form $u_1 u_2$, similarly u cannot be a quotation. So, if $u = G(v)$ for some v, then v has the form $v_1 v_2$, where $u_1 = G(v_1)$ and $u_2 = G(v_2)$, and we apply the induction hypothesis. The cases of the λ-abstractions and of ev^1-terms is similar.

In all other cases, if $u = G(v)$, then v must be of the form $\mathsf{box}\ w\ \mathsf{with}\ \sigma$. Then u must equal $((G(w))^{\text{`}}[])G(\sigma)$, where $G(\sigma)$ is defined as the substitution mapping x to $G(x\sigma)$. Without loss of generality, we may assume v to be in $(\mathsf{gc}), (\mathsf{ctract})$-normal form. In particular, the domain of σ is exactly the set of free variables of v, and σ is one-to-one. Build the renaming substitution r mapping each free variable of v to $\xi_{G(x\sigma)}$. Because σ is one-to-one, r is also one-to-one. So, u must equal $((G(w))^{\text{`}})rr^{-1}G(\sigma)$.

By Lemma 3.6 of Part II, property (ii), $((G(w))^{\text{`}}[])r = (G(w)r)^{\text{`}}[]$. By Lemma 3.6 again, property (i), the free variables of $G(w)r$ and of $(G(w)r)^{\text{`}}[]$ are the same, namely those in the domain of $r^{-1}G(\sigma)$. Since $r^{-1}G(\sigma)$ agrees with $\zeta$ on this set, it follows that $u$ must equal $((G(w)r)^{\text{`}}[])\zeta$. Notice also that $G(w)r$ is a ξ-term.

By Lemma 4.2, there is a unique ξ-term s such that $G(v) = (s^{\text{`}}[])\zeta$, so $G(w)r$ must equal s. Hence, $G(w)$ must equal sr^{-1}, and by induction hypothesis w is unique. Now, for every free variable x of w, $\xi_{G(x\sigma)}$ is also determined uniquely as the variable xr. So $G(x\sigma)$ is determined uniquely for each x. By induction hypothesis, $x\sigma$ is itself determined uniquely. Since σ is (gc)-normal, σ itself is determined uniquely.

To sum up, w and σ are determined uniquely up to a renaming substitution r, i.e. up to α-equivalence. Thus the claim is proved. □

We also observe that G transforms normal forms into normal forms. This is Lemma 4.5 below.

Lemma 4.4 *For every $\lambda\mathsf{ev}Q$-terms u, v_1, ..., v_n, if u, v_1, ..., v_n are $\lambda\mathsf{ev}Q$-normal (resp. $\lambda\mathsf{ev}Q_H$-normal with u not of the form ev^1xw where x is some variable), and v_1, ..., v_n are at level 0, then $(u^{\text{`}}\rho)[v_1/x_1, \ldots, v_n/x_n]$ is $\lambda\mathsf{ev}Q$-normal (resp. $\lambda\mathsf{ev}Q_H$-normal) for any environment ρ.*

Proof: By structural induction on u. Let σ be the substitution $[v_1/x_1, \ldots, v_n/x_n]$. If u is a variable x outside the domain of ρ, then $(u^{\text{`}}\rho)\sigma = Q^1(x\sigma)$. Since $x\sigma$ is at level 0, $Q^1(x\sigma)$ is not a redex. Since moreover $x\sigma$ is normal, $Q^1(x\sigma)$ is normal. If $u$ is a variable $x$ inside the domain of $\rho$, then $(u^{\text{`}}\rho)\sigma = \mathsf{get}_i^1$ for some some $i \geq 0$, which is normal.

If u is an application vw, with v and w normal and v not a λ-abstraction, then $(u^{\text{`}}\rho)\sigma = (v^{\text{`}}\rho)\sigma \star^1 (w^{\text{`}}\rho)\sigma$, where by induction hypothesis $(v^{\text{`}}\rho)\sigma$ and $(w^{\text{`}}\rho)\sigma$ are normal. If $(u^{\text{`}}\rho)\sigma$ was not normal, then it would itself be a redex. The only possibility is that it is a (β^1) redex. Then $(v^{\text{`}}\rho)\sigma$ would have the form $\lambda^1 v'$, and the only possibility for this to happen is for v to be a λ-abstraction, which is impossible.

The argument is similar when u is $1v$ or $\uparrow v$.

If u is a λ-abstraction $\lambda x \cdot v$, with v normal, then $(u^{\text{`}}\rho)\sigma = \lambda^1((v^{\text{`}}\rho[x \mapsto n])\sigma)$, where n is the cardinality of ρ, and by induction hypothesis $(v^{\text{`}}\rho[x \mapsto n])\sigma$ is normal. No rule can apply at the top of $u^{\text{`}}\rho$, so $u^{\text{`}}\rho$ is again normal.

If u has the form $v \bullet w$, the argument is similar.

If u has the form ev^1vw, then $(u^{\text{`}}\rho)\sigma = \mathsf{ev}^2(v^{\text{`}}\rho)\sigma(w^{\text{`}}\rho)\sigma$, where by induction hypothesis $(v^{\text{`}}\rho)\sigma$ and $(w^{\text{`}}\rho)\sigma$ are normal. So if $(u^{\text{`}}\rho)\sigma$ is not normal, it is itself the redex. In $\lambda\mathsf{ev}Q$, this means that $(v^{\text{`}}\rho)\sigma$ is at level at least 2, hence that $v$ is at level 1, but then $u$ would be a redex as well, which is impossible. In $\lambda\mathsf{ev}Q_H$, if $(u^{\text{`}}\rho)\sigma$ is not normal, there is the other possibility that it is an $(\eta\mathsf{ev}^1)$ redex, namely that $(v^{\text{`}}\rho)\sigma$ is of the form Q^1v_1. By inspection of the rules of Figure 2, Part II, the only possibility is that v be some variable x outside the domain of ρ and $v_1 = x\sigma$; but then u would be ev^1xw, which was precisely excluded in the assumptions.

In all other cases, u if of the form $f^\ell(v_1, \ldots, v_n)$, where $\ell \geq 1$ ($\ell \geq 2$ if $f = \mathsf{ev}$) and $n \geq 0$, with v_1, ..., v_n normal. By induction hypothesis $(v_1^{\text{`}}\rho)\sigma$, ..., $(v_n^{\text{`}}\rho)\sigma$ are also normal, so if $(u^{\text{`}}\rho)\sigma = f^{\ell+1}((v_1^{\text{`}}\rho)\sigma, \ldots, (v_n^{\text{`}}\rho)\sigma)$ was not normal, some rule in groups (B) through (F) (resp. through (H)) would apply at the top. Then the same rule taken at levels decreased by one would also apply at the top of u, which is impossible since u is normal. □

Lemma 4.5 *For every λ_{S4}-term u, if u is λ_{S4}-normal (resp. λ_{S4H}-normal), then $G(u)$ is $\lambda\mathsf{ev}Q$-normal (resp. $\lambda\mathsf{ev}Q_H$-normal).*

Proof: By structural induction on u. If u is a variable, an application or a λ-abstraction, then this is clear.

If $u = \mathsf{unbox}\ v$, where v is normal and not a box-term, then $G(u) = \mathsf{ev}^1G(v)()$. By induction hypothesis, $G(v)$ is normal. Moreover since v is not a box-term, $G(v)$ must be of the form x, v_1v_2, $\lambda x \cdot v_1$ or $\mathsf{ev}^1v_1()$: in any case $G(v)$ is at level 0. But no rule of $\lambda\mathsf{ev}Q_H$ applies in these cases, so $G(u)$ is normal.

In the final case, $u = \mathsf{box}\ v\ \mathsf{with}\ w_1, \ldots, w_n\ \mathsf{for}\ x_1, \ldots, x_n$, where v is normal (resp. and not of the form $\mathsf{unbox}\ x_i$ for any $1 \leq i \leq n$), $x_1, \ldots, x_n$ are exactly the free variables of u, $w_1, \ldots, w_n$ are normal, not box terms and are pairwise distinct. By induction hypothesis, $G(v)$ is normal, and $G(w_1), \ldots, G(w_n)$ are normal. Furthermore, since $w_1, \ldots, w_n$ are not box terms, $G(w_1), \ldots, G(w_n)$ are at level 0.

Then, in the $\lambda\mathsf{ev}Q$ case, by Lemma 4.4 $((G(v))^{\text{`}}[])[G(w_1)/x_1, \ldots, G(w_n)/x_n]$ is normal, i.e. $G(u)$ is normal.

In the $\lambda\mathsf{ev}Q_H$ case, in addition we know that v is not of the form $\mathsf{unbox}\ x_i$ for any $1 \leq i \leq n$. If $G(v)$ was of the form $\mathsf{ev}^1 xw$ for some variable x and some term w, then v would be of the form $\mathsf{unbox}\ v'$ by inspection of Figure 2, Part II, where $v' = x$. But $u = \mathsf{box}\ \mathsf{unbox}\ x\ \mathsf{with}\ w_1, \ldots, w_n\ \mathsf{for}\ x_1, \ldots, x_n$ is only well-formed if x is some x_i, $1 \leq i \leq n$, and this is impossible by assumption. So again Lemma 4.4 applies, showing that $G(u)$ is normal. □

We have the following property, which is stronger than property (3), but would be equivalent to it if (2) held.

Theorem 4.6 *For every typed $\lambda^{\approx}_{S4}$-term u, if $G(u)$ reduces to some term t in $\lambda\mathsf{ev}Q$, then t reduces to some term of the form $G(v)$, for some $\lambda^{\approx}_{S4}$-term v such that u reduces to v in λ_{S4}.*

Similarly, under the conjecture that the typed $\lambda\mathsf{ev}Q_H$-calculus is confluent, if $G(u)$ reduces to some term t in $\lambda\mathsf{ev}Q_H$, then t reduces to some term of the form $G(v)$, for some $\lambda^{\approx}_{S4}$-term v such that u reduces to v in λ_{S4H}.

Proof: By Theorem 5.1, in Part I (resp. 4.1, in Part II), u has a unique normal form v in λ_{S4} (resp. λ_{S4H}). By Theorem 3.29, Part II (resp. 4.11), $G(u)$ reduces to $G(v)$ as well. By confluence, $G(v)$ and t then have a common reduct. By Lemma 4.5, however, $G(v)$ is normal, so t must reduce to $G(v)$. □

Finally:

Theorem 4.7 (Conservativity) *The typed $\lambda\mathsf{ev}Q$-calculus is a conservative extension of the typed $\lambda^{\approx}_{S4}$-calculus, i.e. for every typed λ_{S4}-terms u and v, u and v are interconvertible modulo the rules of λ_{S4} if and only if $G(u)$ and $G(v)$ are interconvertible modulo the rules of $\lambda\mathsf{ev}Q$.*

Similarly, under the conjecture that the typed $\lambda\mathsf{ev}Q_H$-calculus is confluent, it is a conservative extension of the typed $\lambda^{\approx}_{S4H}$-calculus.

Proof: The only if direction comes from Theorems 3.29 and 4.11, Part II. As for the if direction, assume that $G(u)$ and $G(v)$ are interconvertible. Let u' and v' be the respective unique normal forms of u and v in λ_{S4} (resp. λ_{S4H}). Then $G(u')$ and $G(v')$ are interconvertible. By confluence, there is a $\lambda\mathsf{ev}Q$-term t such that $G(u')$ and $G(v')$ both reduce to t. By Lemma 4.5, both $G(u')$ and $G(v')$ are $\lambda\mathsf{ev}Q$-normal (resp. $\lambda\mathsf{ev}Q_H$-normal), so $G(u') = G(v')$. By Lemma 4.3, $u' \approx v'$. In particular, u and v are interconvertible modulo the rules of λ_{S4} (resp. λ_{S4H}). □

Theorem 4.6 and Theorem 4.7 were only stated for the typed version of the calculus. In both, we use the strong normalization property of the typed λ_{S4} (resp. λ_{S4H}) calculus. The proof techniques that we have used generalize to different type systems, for example in the spirit of System F [Gir71, GLT89], provided that only term types, and not metastack types, are quantified over. However, the same results in the untyped case are still open. In particular, we don't know whether $\lambda\mathsf{ev}Q$-equivalence is conservative over λ_{S4}-equivalence in the untyped case.

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