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Non–Marginal Time Evaluation of Special Stochastic Networks by Using Renewal Processes

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Abstract

Real–world projects often demand modelling of stochastic project evolution, stochastic project activity durations, as well as feedback in order to take care of uncertainties occurring during a project execution. These conditions can be met by project modelling through EOR networks. In order to obtain exact analysis of a project evolution in time, these EOR networks have to be evaluated non–marginally in time.

In this paper, we consider non–marginal time analysis of EOR project networks with exponentially distributed project activity durations. In this case, we derive exact formulas for the activation functions. Arbitrary project activity durations can be approximated by using Edgeworth–approximation or by approximation via Cox– or Phase–Type distributions.

This new analysis method for stochastic projects is compared to the well–known Markov Renewal Process method (MRP). It shows that, the exact method needs to calculate all paths from the network's source to a considered project state's node in the project network, but clearly compensates this disadvantage through the very quick and explicite calculation of the activation function in comparison to the MRP method, which does not need to calculate each path separately but needs to approximate integrals along the project activity arcs, i.e., for real–world project dimensions with decent network structure the new method clearly outperforms the MRP method in calculation time.

Keywords

non–marginal time evaluation, stochastic project networks, Markov Renewal Process method, Edgeworth approximation, exponential distribution, hyper–exponential distribution, Cox–distribution, Phase–Type distribution

Contents

Chapter 1

Special Stochastic Networks (EOR Networks) and Coincidence with Markov Renewal Processes

In this chapter we first introduce special stochastic project networks: EOR networks. EOR networks are a subclass of GERT networks. They are activity–on–arc project networks, where stochastic project evolution as well as stochastic activity durations can be modelled. Project feedback can be expressed by cycle structures.

For literature on activity–on–arc project networks, we refer to Elmaghraby (1977). Standard literature on GERT networks is Neumann & Steinhardt (1979) and Neumann (1990). GERT networks were first introduced by Pritsker & Happ (1966).

We then show coincidence of EOR networks and Markov Renewal Processes. This coincidence can be exploited for non–marginal time evaluation of EOR project networks.

For literature on Renewal Theory, we refer to Alsmeyer (1991) or Grimmett & Stirzaker (1982).

1.1 EOR Networks

In EOR networks, stochastic project evolution can be modelled as well as stochastic activity durations:

1. Stochastic Project Evolution:

For modelling stochastic project evolution, there are two types of nodes: STEOR–nodes and DETEOR–nodes. The corresponding symbols are shown in Figure 1.1.

Figure 1.1: Node–types in EOR networks

- **STEOR–node i:**

  When a project realization activates STEOR–node i, then exactly one outgoing activity is executed. The corresponding conditioned execution probability of activity $< i, j >$ is

  $$p_{ij} := P(< i, j > \text{is executed} \,|\, i \,\text{is active}),$$

  compare Figure 1.2. We require $\sum_{j \in S(i)} p_{ij} \stackrel{!}{=} 1$, where $S(i)$ denotes the set of all successor–nodes of i

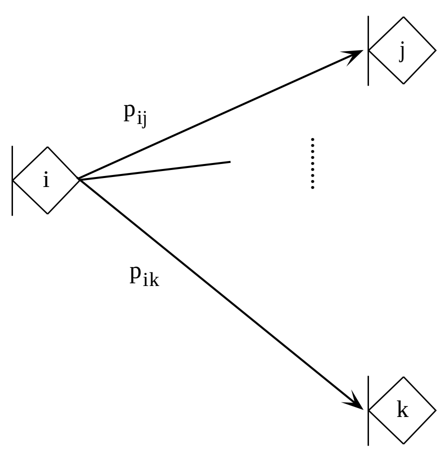

Figure 1.2: Conditioned transition probabilities of activities emanating from STEOR–node i

- **DETEOR–node i:**

  When a project realization activates DETEOR–node i, then all outgoing activities are executed. The corresponding conditioned execution probability of activity $< i, j >$ is $p_{ij} := 1 \; \forall \, j \in S(i)$. See Figure 1.3.

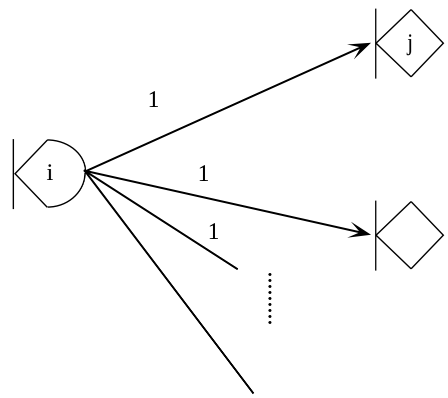

Figure 1.3: Conditioned transition probabilities of activities emanating from DETEOR–node i

2. Stochastic Activity Durations:

   Given the event "node i is active", we define:
   D_{ij}: duration of activity $< i, j >$, which is a (non–negative) random variable.
   F_{ij}: distribution function of random variable D_{ij}.
   It is required that $F_{ij}(t) = 0$ for $t < 0$ and $E(D_{ij}) < \infty$.

Convention:
If node i has at most one successor–node, we let i be STEOR.

Structural Assumptions:

- Markov–Property: A further project evolution of an EOR network only depends on the current project state and not on former project behaviour.

- Project network paths emanating from one and the same DETEOR–node are not allowed to merge again.

- Cycle Structures (feedback):
  - Only STEOR–nodes are allowed in cycle structures.
  - A cycle structure must be left with a strictly positive probability.
  - A cycle structure can be entered at most once during a project execution.
- The project possesses exactly one source[1].

Example 1.1:
Figure 1.4 shows a fictitious dentist project as a STEOR network. A project realization corresponds to a patient receiving treatment at a dentist. The activity duration distributions are shown at the corresponding arcs. Notice, that the normal distribution is not a strictly positive distribution. In order to be exact, we have to omit its negative partion.

Suppose, we are interested in the following question: What is the probability, that we have to issue an additional bill at most 20 minutes after a patient was started to get treatment?

More general: What is the probability, that a certain project state will be activated up to a certain time t? Non–marginal time analysis of projects is concerned with these questions. $\triangleleft$

1.2 Coincidence with Markov Renewal Processes

We denote an EOR network by $N =< V, E, p, F >$, where $V := \{1, 2, ..., n\}$ is the set of all nodes, E denotes the set of all arcs $< i, j >$, p the matrix $(p_{ij})_{i,j=1,...,n}$ of all conditioned transition probabilities, and F the matrix $(F_{ij})_{i,j=1,...,n}$ of all conditioned distribution functions.

We first want to consider STEOR networks, which are EOR networks only including STEOR–nodes. STEOR networks have the property that a project realization corresponds to a single "trace" through the network. Suppose, at time $t = 0$ the source is activated. We give the following:

[1] Weakening of this assumption is possible but combined with technical modelling. Compare Neumann (1990). We therefore restrict ourselves to only considering one project source.

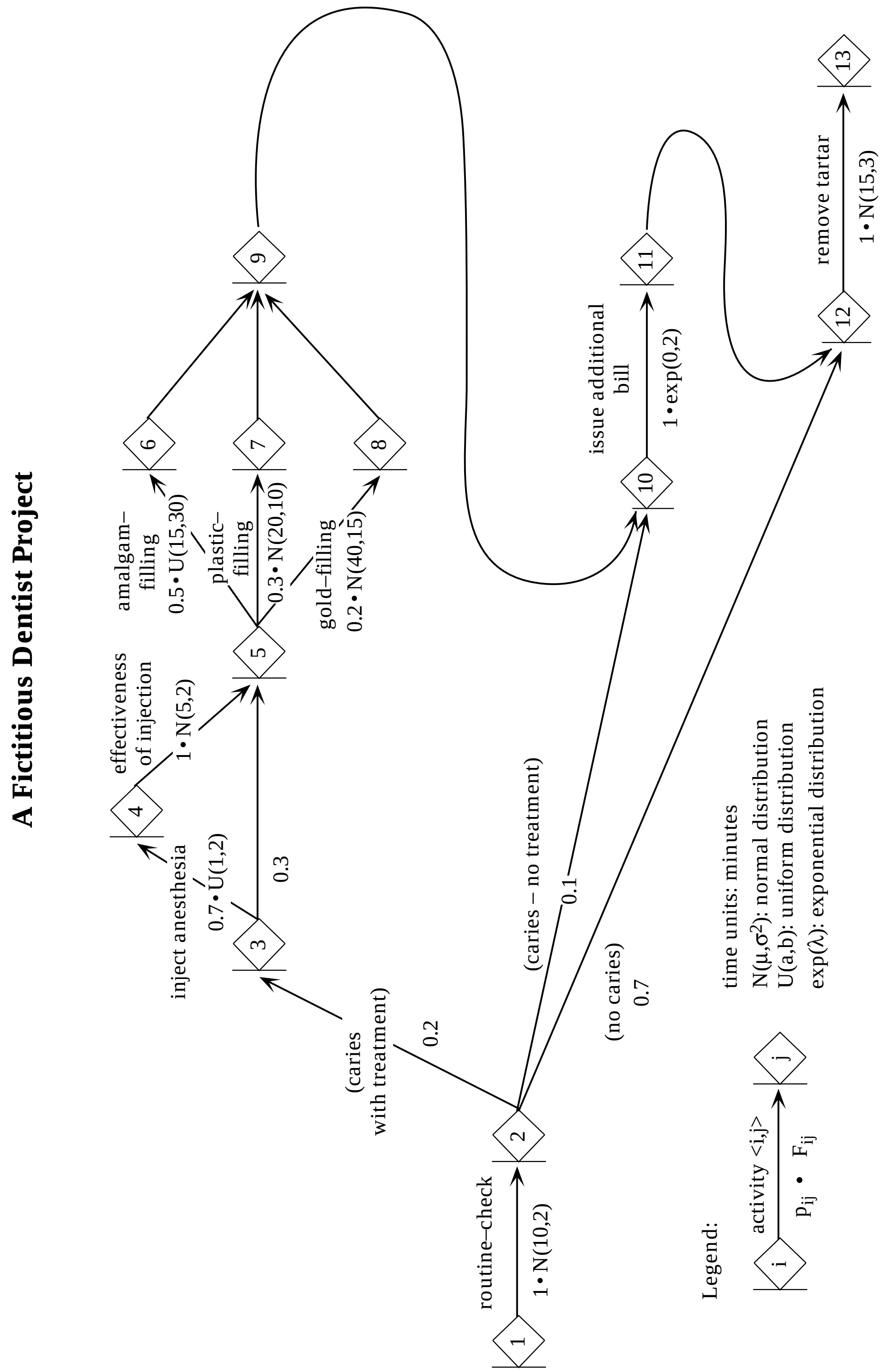

Figure 1.4: Example of STEOR project network: a fictitious dentist project

Definition 1.2:

1. We let θ_ν be that point in time, where the ν–th activation of a node occurs.
2. We let X_ν be that node which is activated at time θ_ν.

□

Theorem 1.3:
$\{X_\nu, \theta_\nu\}_{\nu \in \mathbb{N}_0}$ is a Markov Renewal Process with embedded homogeneous Markov Chain $\{X_\nu\}$ with transition matrix $(p_{ij})_{i,j=1,\ldots,n}$. ◁

Remarks to the proof:

- The theorem takes advantage of direct coincidence of a STEOR network with the above defined Markov Renewal Process, since there is a single "trace" through the network for every project realization. The transition matrix $(p_{ij})_{i,j=1,\ldots,n}$ is stochastic, since $\sum_{j=1}^{n} p_{ij} = 1$ for all STEOR–nodes $i = 1, \ldots, n$.
- For exact formulations of a proof, see Neumann & Steinhardt (1979).

We now consider the more general case, where at time $t = 0$ node i is active. I.e.: $X_0 = i$ and $\theta_0 = 0$.

Definition 1.4: (transition function, one–step transition function)

$$Q_{ij}(t) := \begin{cases} P\left(X_1 = j; \theta_1 \le t \mid X_0 = i\right) & \text{for} \quad t \ge 0 \\ 0 & \text{for} \quad t < 0 \end{cases}$$

□

Remark 1.5: $Q_{ij}(t)$ describes the transition from node i to node j in exactly one step: $Q_{ij}(t) = p_{ij} F_{ij}(t)$. ◁

Definition 1.6: (ν–step transition function)

$$Q_{ij}^{(\nu)}(t) := \begin{cases} P\left(X_\nu = j; \theta_\nu \le t \mid X_0 = i\right) & \text{for} \quad t \ge 0 \\ 0 & \text{for} \quad t < 0 \end{cases}$$

□

Definition 1.7: (activation function of node j given $X_0 = i$)

The activation function $Y_{ij}(t)$ is defined to be the expected number of activations of node j up to time t given that node i is active at time $t = 0$. □

Theorem 1.8: (argument of coincidence)

$Y_{ij}(t)$ coincides with the expected number of renewals of the renewal process $\{X_\nu, \theta_\nu\}_{\nu \in \mathbb{N}_0}$ up to time t with $X_0 = i$ and $\theta_0 = 0$.
Especially: $Y_{ij}(t)$ is solution of the renewal equation, i.e.:

$$Y_{ij}(t) = \sum_{\nu=0}^{\infty} Q_{ij}^{(\nu)}(t).$$

◁

We next will develop the renewal equation, applying a standard "one–step–backtracking" argument. In the following, δ_{ij} denotes the Kronecker–Delta, i.e.

$$\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{else} \end{cases} ,$$

$\chi_{[0,\infty)}(t)$ denotes the indicator–function on $[0, \infty)$, i.e.

$$\chi_{[0,\infty)}(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} ,$$

and "$*$" is the operator for the convolution of two (conditioned, positive) distribution functions, i.e., for two functions $F(t)$ and $G(t)$,

$$(F * G)(t) = \int_{-\infty}^{t} F(t-s)dG(s) \ .$$

We devolop the renewal equation:

$$\begin{aligned}
Y_{ij}(t) &= \sum_{\nu=0}^{\infty} Q_{ij}^{(\nu)}(t) \\
\Longleftrightarrow \quad Y_{ij}(t) &= \underbrace{d_{ij}(t)}_{:=\delta_{ij}\chi_{[0,\infty)}(t)} + \sum_{\nu=1}^{\infty} Q_{ij}^{(\nu)}(t) \\
&= d_{ij}(t) + \sum_{\nu=1}^{\infty} \sum_{k=1}^{n} \left(Q_{ik}^{(\nu-1)} * Q_{kj} \right)(t) \\
&= d_{ij}(t) + \sum_{k=1}^{n} (\underbrace{\sum_{\nu=1}^{\infty} Q_{ik}^{(\nu-1)}(t)}_{=Y_{ik}(t)}) * Q_{kj}(t) \\
&= d_{ij}(t) + \sum_{k=1}^{n} Y_{ik}(t) * Q_{kj}(t) \qquad (*)
\end{aligned}$$

The marginal case ($t \to \infty$) of ($*$):

Definition 1.9: (activation number)

$z_{ij} := \lim_{t\to\infty} Y_{ij}(t)$ is the activation number of node j when the project is started in node i. □

Remark 1.10: z_{ij} corresponds to the expected number of activations of j with start in i "in the long run" of a project. As we will see, z_{ij} is independent of F $\forall\, i, j = 1, \ldots, n$. ◁

From ($*$) we get:

$$z_{ij} = \underbrace{\lim_{t\to\infty} d_{ij}(t)}_{=\begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise}\end{cases}} + \sum_{k=1}^{n} \lim_{t\to\infty} \underbrace{Y_{ik}(t) * \underbrace{Q_{kj}(t)}_{\to p_{kj}\cdot 1}}_{\to p_{kj} z_{ik}}$$

Summarized, in the marginal case we obtain the following linear system of equations:

$$\begin{cases} z_{ii} = 1 \\ z_{ij} = \sum_{k=1}^{n} p_{kj} z_{ik} \ , \quad j = 1, \ldots, n \ , \ i \neq j \end{cases}$$

Remark 1.11: From the above linear system of equations it is obvious, that z_{ij} depends on p and not on F $\forall\, i, j = 1, \ldots, n$. ◁

Theorem 1.12:

For EOR networks, there is coincidence of several independently overlapping Markov Renewal Processes and ($*$) holds as well. ◁

Remarks to the proof:

- Here, we need the prerequisite, that paths emanating from one and the same DET–node do not merge again. I.e.: at DET–nodes, a single trace is "splitting" into several traces. For each trace, we define a coinciding Markov Renewal Process.
- An exact formulation of a proof can be found in Neumann (1990).

Chapter 2

Non–Marginal Time Analysis

We show, how coincidence of EOR networks with renewal processes can be applied to calculate the activation functions Y_{ij}.

First, we consider the well–known Markov Renewal Process Method (MRP), where the equation $(*)$ from Chapter 1 is solved by approximating Riemann–Stieltjes integrals.

Second, we develop a new exact analysis formula. This formula can be resolved to explicit representation of the $Y_{ij}(t)$ in the case where $F_{ij} \equiv \exp(\lambda)$[1] $\forall < i,j > \in E$ and in the case where $F_{ij} = \exp(\lambda_{ij}) \; \forall < i,j > \in E$ with $\lambda_{ij} \neq \lambda_{kl} \, \forall \; ij \neq kl$.

2.1 The Markov Renewal Process Method *(MRP)*

We now derive the Markov Renewal Process Method (MRP) out of equation $(*)$.

$$
\begin{array}{rrcl}
(*) & Y_{ij}(t) & = & d_{ij}(t) + \sum\limits_{k=1}^{n} (Y_{ik} * Q_{kj})(t) \\
\Leftrightarrow & Y_{ij}(t) & = & d_{ij}(t) + p_{ij}F_{ij}(t) + \sum\limits_{\substack{k=1 \\ k\neq i}}^{n} (Y_{ik} * Q_{kj})(t) \\
 & & = & d_{ij}(t) + p_{ij}F_{ij}(t) + \sum\limits_{\substack{k=1 \\ k\neq i}}^{n} \int\limits_{0}^{t} p_{kj}F_{kj}(t-s)\, dY_{ik}(s) \qquad (**)
\end{array}
$$

[1] $\exp(\lambda)$ denotes the distribution function of the exponential distribution:
$\exp(\lambda) = \begin{cases} 1 - e^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$.

Remark 2.1:

- $(**)$ can be approximated numerically, e.g., through rectangular calculus of Riemann–Stieltjes integrals: when $[a,b]$ is a small intervall, it holds:
$\int_a^b f(s)\,dg(s) \approx f(a)\,(g(a)-g(a^-)) + f\left(\frac{a+b}{2}\right)(g(b)-g(a))$

- With the integral approximation, $(**)$ becomes a linear system of equations in Y_{ij}, i fixed, $j=1,...,n\;\forall\, t=0$.

- For acyclic networks, topological ordering of nodes is possible and the above system of equations provides a triangular coefficient matrix which makes calculus easier.

- $\sum_{\substack{k=1\\k\neq i}}^{n}\ldots$ can be replaced by $\sum_{k\in P(j)\cap\dot{R}(i)}\ldots$ (where $P(j)$ denotes the set of all predecessors of node j and $\dot{R}(i)$ denotes the set of all nodes which can be reached from i minus node i), since transition probabilities $p_{kj}=0$ for $k\notin P(j)\cap\dot{R}(i)$.

- Evaluation of $(**)$ via integral approximation is referred to as *Markov Renewal Process Method* (MRP), compare Neumann (1990).

- Evaluating $(**)$ via MRP is combined with a considerable calculatorial expense: Obtaining values for $Y_{ij}(t_0)$, we need to calculate $Y_{ij}(t)$ at many intermediate values $t\in[0,t_0]$, since the integral approximation is based on former time values of Y_{ij} (which correspond to $g(a^-)$ in the above formula for integral approximation).

  Furthermore: The smoother the partition of $[0,t_0]$ is, the better is the approximation.

- We want to stress that MRP works in cycle structures as well.

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2.2 An Exact Analysis Method

Another approach to solve the renewal equation is the following:

$$\begin{aligned}
Y_{ij}(t) &= \sum_{\nu=0}^{\infty} Q_{ij}^{(\nu)}(t) \\
&= d_{ij}(t) + \sum_{\nu=1}^{\infty} Q_{ij}^{(\nu)}(t) \\
&= d_{ij}(t) + \sum_{\nu=1}^{\infty} \sum_{i_1=1}^{n} Q_{ij}^{(\nu-1)}(t) * Q_{i_1 j}(t) \\
&= d_{ij}(t) + \sum_{\nu=1}^{\infty} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} Q_{ii_2}^{(\nu-2)}(t) * Q_{i_2 i_1}(t) * Q_{i_1 j}(t) \\
&= d_{ij}(t) + \sum_{\nu=1}^{\infty} \sum_{i_1} \sum_{i_2} \dots \sum_{i_{\nu-1}} Q_{ii_{\nu-1}} * \dots * Q_{i_2 i_1} * Q_{i_1 j}(t) \\
&= d_{ij}(t) + \sum_{\nu=1}^{\infty} \sum_{\substack{(i_1,\dots,i_{\nu-1}) \\ \in\{1,\dots,n\}^{\nu-1}}} Q_{ii_1} * Q_{i_1 i_2} * \dots * Q_{i_{\nu-1} j}(t) \\
(***) \quad &= d_{ij}(t) + \sum_{\nu=1}^{\infty} \sum_{\substack{(i_1,\dots,i_{\nu-1}) \\ \in\{1,\dots,n\}^{\nu-1}}} \left(\prod_{l=0}^{\nu-1} p_{i_l i_{l+1}} \right) F_{ii_1} * F_{i_1 i_2} * \dots * F_{i_{\nu-1} j}(t)
\end{aligned}$$

where $i_0 := i$ and $i_\nu := j$.

Remark 2.2:

- $\sum_{\nu=1}^{\infty} \dots$ actually consists of finitely many terms in the case of acyclic networks.
- $|\{1,\dots,n\}^{\nu-1}| = n^{\nu-1}$; but: only "admissible" paths from i to j have to be considered, since $p_{ij} = 0$ for $\langle i,j \rangle \notin E$.

$\triangleleft$

2.2.1 The Case $F_{ij} \equiv \exp(\lambda) \quad \forall\, ij$

Theorem 2.3:
In the case where $F_{ij} \equiv \exp(\lambda) \quad \forall\, ij$ it holds:

$$(***) \Longleftrightarrow Y_{ij}(t) = d_{ij}(t) + \sum_{\nu=1}^{\infty} \sum_{\substack{(i_1,\dots,i_{\nu-1}) \\ \in\{1,\dots,n\}^{\nu-1}}} \left(\prod_{l=0}^{\nu-1} p_{i_l i_{l+1}} \right) \cdot \left(1 - e^{-\lambda t} \left(\sum_{k=0}^{\nu-1} \frac{(\lambda t)^k}{k!} \right) \right)$$

$\triangleleft$

Remark 2.4:

- The above theorem holds due to the fact, that $*_1^\nu \exp(\lambda) = \text{Erlang}(\nu, \lambda)$ (Erlang–distribution).
- $(***)$ in this case is an explicit representation of $Y_{ij}(t)$.

$\triangleleft$

2.2.2 The Case $F_{ij} = \exp(\lambda_{ij})\,;\ \lambda_{ij} \neq \lambda_{kl}\ \ \forall\, ij \neq kl$

Theorem 2.5:
In the case where $F_{ij} = \exp(\lambda_{ij})\,;\ \lambda_{ij} \neq \lambda_{kl}\ \ \forall\, ij \neq kl$ it holds:

$$(***) \Longleftrightarrow Y_{ij}(t) = d_{ij}(t) + \sum_{\nu=1}^{\infty} \sum_{\substack{(i_1,\dots,i_{\nu-1}) \\ \in \{1,\dots,n\}^{\nu-1}}} \left(\prod_{l=0}^{\nu-1} p_{i_l i_{l+1}} \right) \cdot \left(1 + \sum_{k=0}^{\nu-1} c_{k,\nu} e^{-\lambda_{i_k i_{k+1}} t} \right)$$

where $c_{k,\nu} := (-1)^\nu \prod_{\substack{l=0 \\ l \neq k}}^{\nu-1} \frac{\lambda_{i_l i_{l+1}}}{\lambda_{i_l i_{l+1}} - \lambda_{i_k i_{k+1}}}$ $\triangleleft$

Remark 2.6:

- The above holds, since the sum of exponentially distributed random variables with distinct parameters is *hypo–exponentially* or *hyper–exponentially* distributed[2]. For a proof, which is quite technical, see Ströhler (1995) or Dehon & Latouche (1982).
- $(***)$ in this case is an explicit representation of $Y_{ij}(t)$, too.

$\triangleleft$

[2]Expressions in literature differ.

Chapter 3

Extensions of the Exact Analysis Method

In this chapter, we show generalizations of the new exact and explicit representation formula of the activation functions Y_{ij} ($(***)$ in Chapter 2).

First, we show an approximation of the distribution of a sum of independent and identically distributed *iid* random variables. This approximation is directly applied for calculating the convolution of distribution functions in $(***)$.

Second, we show how Cox–distributed activity durations can be replaced by a STEOR network structure. In this network structure, all activities are exponentially distributed. Thus, the exact analysis formula in the case, where $F_{ij} = \exp(\lambda_{ij})\,;\ \lambda_{ij} \neq \lambda_{kl}\ \ \forall\, ij \neq kl$ can be applied. Since the class of Cox–distributed random variables is dense in the class of all positive distributions, arbitrary positive random durations can be approximated with this technique.

3.1 Edgeworth–Approximation of the Distribution of a Sum of *iid* Random Variables

In the following, we show how the distribution of a sum of independent and identically distributed random variables can be approximated by the Edgeworth–approximation. The Edgeworth–approximation is based on the Central Limit Theorem of probability theory. This approach can be found in Barndorff–Nielsen & Cox (1989).

For sketching the Edgeworth–approximation, we briefly introduce the following notations:

Definition 3.1:

Let $Y, Y_1, Y_2, \ldots$ be *iid* random variables. Then:

1. $\mu := E(Y)$ is the expectation of Y.
2. $\mu'_k := E\left(Y^k\right) \quad k = 1, 2, \ldots$ is the k–th moment of Y.
3. $\mu_k := E\left((Y-\mu)^k\right) \quad k = 1, 2, \ldots$ is defined to be the k–th centered moment of Y.
4. $S_n := Y_1 + \ldots + Y_n$ denotes the sum of the Y_i.
5. $S_n^* := \frac{S_n - n\mu}{\sigma\sqrt{n}}$ denotes the normed or standardized sum of the Y_i.
6. $M_{Y(t)} := E\left(e^{tY}\right)$, $t \geq 0$ is the moment–generating function of Y.
7. $K_{Y(t)} := \log M_{y(t)}$ is the cumulant–generating function of Y.

□

Remark 3.2: The moment–generating function and the cumulant–generating function of a random variable Y have the following power series representation:

- $M_{Y(t)} = 1 + \sum_{k=1}^{\infty} \mu'_k \frac{t^k}{k!}$, if convergent.
- $K_{Y(t)} = \sum_{k=1}^{\infty} \kappa_k \frac{t^k}{k!}$, if convergent, with power series coefficients $\frac{\kappa_k}{k!}$.

◁

Definition 3.3:

1. κ_k are called cumulants of Y.
2. $\varrho_k := \frac{\kappa_k}{\sigma^k} \quad k = 3, 4, \ldots$ are called standardized cumulants, where $\sigma := \sqrt{\mu_2}$.

□

Remark 3.4:

$\kappa_1 = \mu,$

$\kappa_2 = \mu_2 = \sigma^2,$

$\kappa_3 = \mu_3,$

$\kappa_4 = \mu_4 - 3\mu_2^2,$

$\kappa_5 = \mu_3 - 10\mu_3\mu_2,$ and

$\kappa_6 = \mu_6 - 15\mu_4\mu_2 - 10\mu_3^2 + 30\mu_2^3$

◁

Definition 3.5: (Hermite Polynoms)

With the density of the standard normal–distribution $\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$, the *Hermite Poynoms* H_k are defined by

$$\varphi_{(t)} H_{k(t)} = (-1)^k \left(\frac{d}{dt}\right)^k \varphi(t)$$

□

Hermite polynoms possess the following property:

$$\int_{\mathbb{R}} e^{tX} H_{k(X)} \varphi(x)\, dx = E\left(e^{tX} H_{k(X)}\right) = t^k e^{\frac{t^2}{2}}$$

The first six Hermite polynoms have the following representation:

$H_0(t) = 1$

$H_1(t) = t$

$H_2(t) = t^2 - 1$

$H_3(t) = t^3 - 3t$

$H_4(t) = t^4 - 6t^2 + 3$

$H_5(t) = t^5 - 10t^3 + 15t$

$H_6(t) = t^6 - 15t^4 + 45t^2 - 15$

Theorem 3.6: (Edgeworth–development)

For $Y_1, Y_2, \ldots$ *iid* random variables, it holds

$$\begin{aligned} F_{S_n^*(t)} &= \Phi(t) \\ &= \varphi(t)\left(\tfrac{\varrho_3}{3!\sqrt{n}} H_2(t) + \tfrac{\varrho_4}{4!n} H_3(t) + \tfrac{\varrho_3^2}{5!n} H_5(t) + \tfrac{\varrho_3\varrho_4}{6!\sqrt{n}^3} H_6(t)\right) + \mathcal{O}\left(\tfrac{1}{n^2}\right) \end{aligned}$$

◁

Remark 3.7:

- The proof is via the cumulant generating function, the moment generating function, and the Hermite Polynoms. An exact proof can be found in Barndorff–Nielsen & Cox (1989).
- The Edgeworth–approximation of $F_{S_n^*(t)}$ is obtained in the above formula, when $\mathcal{O}\left(\frac{1}{n^2}\right)$ is omitted.
- The Edgeworth–approximation is close to the central limit theorem for big n. It uses the standardized normal distribution $\varphi(t)$.
- The longer the paths in an EOR network with identically distributed random variables, the better the approximation through the Edgeworth formula.

$\triangleleft$

3.2 Representation of the Cox–Distribution as a STEOR Network

The Cox–distribution is a quite general class of distributions.
It is dense in the class of all positive distributions. It is modelled as follows: If $X \sim Cox_{m,(p_1,...,p_m),(\lambda_1,...,\lambda_m)}$, then X corresponds to the time of absorption of the Markov process $\{Y_\nu, \theta_\nu\}$ with state space of the embedded Markov chain $S = \{0, ..., m+1\}$, inter arrival times $\theta_\nu \sim exp(\lambda_\nu)$, and transition matrix[1]

$$P = \begin{pmatrix} 0 & p_1 & 0 & \dots & \dots & 0 & 1-p_1 \\ 0 & 0 & p_2 & 0 & \dots & 0 & 1-p_2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & \dots & \dots & 0 & p_m & 1-p_m \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 \end{pmatrix}$$

If an activity $< i,j >$ has a $Cox_{m,(p_1,...,p_m),(\lambda_1,...,\lambda_m)}$–distributed duration D_{ij}, then the arc $< i,j >$ in the EOR network can be replaced by a STEOR network structure, where all occurring distributions are exponential. The network structure is shown in Figure 3.1. The network structure exactly corresponds to the Markov process defining the Cox–distribution, where state 0 is represented by node i and state $m+1$ by node j.

[1] Clearly, the Markov process will be absorbed in state $m+1$.

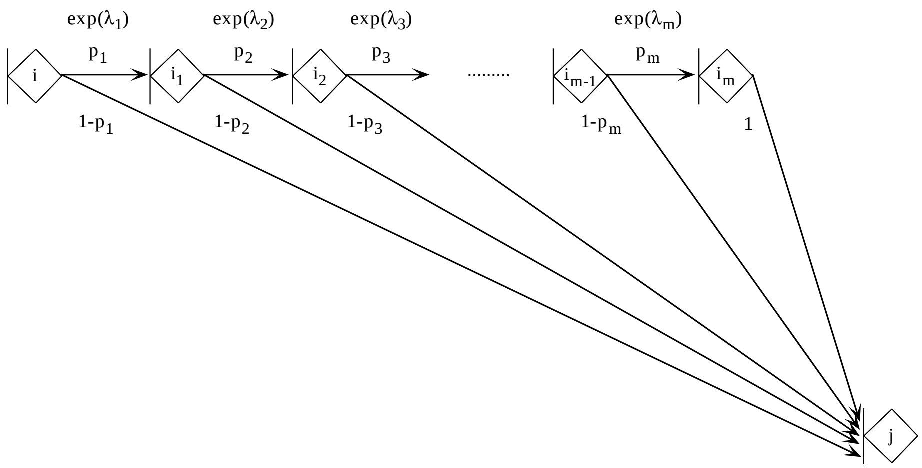

Figure 3.1: Cox–distribution as a STEOR network

Remark 3.8:

- In the more general Triangular Phase–type distribution (TPH), forward–moves not only to the next state in the Markov state space but also to following, higher–numbered states are allowed. TPH–distributed activity durations can be modelled analogously by STEOR network structures, where corresponding additional "forward arcs" are contained with their respective transition probabilities.

  TPH is dense in the class of positive distributions, too. For details, see Barlow & Proschan (1975).

- With the Cox–distribution or the TPH, any positive distribution can be approximated. One way to do this is the following: Set the first k moments of the random variable, whose distribution is to be approximated, equal to the corresponding moments of the Cox–distribution[2]. This leads to a non–linear system of equations. Solve this system approximately and obtain parameters for the Cox–distribution. The higher k, the better the approximation, but also, the more complex the system of non–linear equations.

- Replacing arbitrarily distributed activity durations by the above described network structures, we obtain a network with exclusively exponentially distributed parameters. We are able to apply the exact analysis formula in the case $F_{ij} = \exp(\lambda_{ij})$; $\lambda_{ij} \neq \lambda_{kl} \;\; \forall\, ij \neq kl$. The number of network nodes is hereby increased. Since the network structures are not dense networks,

[2]For the moments of TPH, see Barlow & Proschan (1975)

calculus of paths should still be able in an acceptable amount of time, cf. Chapter 4.

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3.3 Cycle Structures

Whereas the MRP works for cycle structures as well, the exact analysis formula has to calculate infinite sums of convolutions, since paths may contain infinitely many arcs in cycle structures. There are two possibilities to obtain approximate results for cycle structures:

1. Calculate paths through cycle structures until the probability for staying another turn in the cycle structure is sufficiently small. This can be done since the probability for leaving a cycle structure is assumed to be strictly positive:

   Let C be a cycle structure, where the probability to stay in cycle structure C is $p_C < 1$ and $1 - p_C$ is the probability for leaving C. Then, a project realization which activates C stays exactly n turns in C with probability ${p_C}^n \cdot (1 - p_C)$. This probability limits to 0 when $n \to \infty$.

2. Calculate the expected time of a single "cycle execution", $E(T_C)$. Then, calculate the expected number of cycle executions $E(N)$, which can be considered as a stopping time. Corresponding to Wald's equation, the expected time consumption within cycle structure C is $E(T_C) \cdot E(N)$. Finally, cycle structure C can be replaced by a single activity with deterministic duration $E(T_C) \cdot E(N)$. Clearly, this approach is a heuristic approximation.

Chapter 4

Computing Test Results

In this chapter, we analyze and compare the two methods in exactness and computing time. The results are extracted from Ströhler (1995).

The methods were implemented in VisualWorks\Smalltalk V.2.0a and run on a "80 MHz–Macintosh Power PC". The MRP was implemented with equidistant interval–splitting partitions and different approximation–accuracy, depending on the time–distributions of the activities and on the network–depth. In the following, approximation–accuracy parameters are denoted by "approx" and a following number. The number itself is a value for the smoothness of the partition of the considered interval[1]. E.g., approx=40 is twice as smooth as approx=20. This relation of the smoothness–values is sufficient to qualitatively interpret the following analysis.

Of course, the MRP could be implemented with higher technical standards, as professional mathematical linear equation solver software and adapted integral approximation methods. This would definitely lead to less absolute computing times for the MRP. However, the qualitative trends which can be extracted from the diagrams stay the same.

4.1 Exactness and Computing Time in Comparison

We first consider an acyclic EOR network with $n = 30$ nodes and $m = 60$ arcs, $D_{ij} \sim \exp(1)$ *iid*. This project network was evaluated in a single network sink at time $t = 5$. The exact analysis formula is referred to as the exact value, since it could be adapted in the exact and explicit form. Figure 4.1 shows the relative deviation of the MRP with different approximation accuracies and the consumed absolute computing time (in milliseconds) compared to the exact method. Figure 4.2 depicts the same situation with the relative computing time difference to

[1] Nevertheless, the corresponding absolute smoothness–value depends on the given problem.

the exact method.

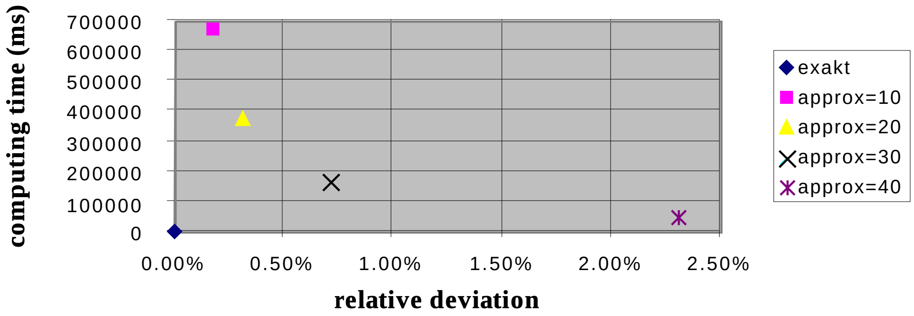

Figure 4.1: Computing time comparison: exact analysis formula versus MRP

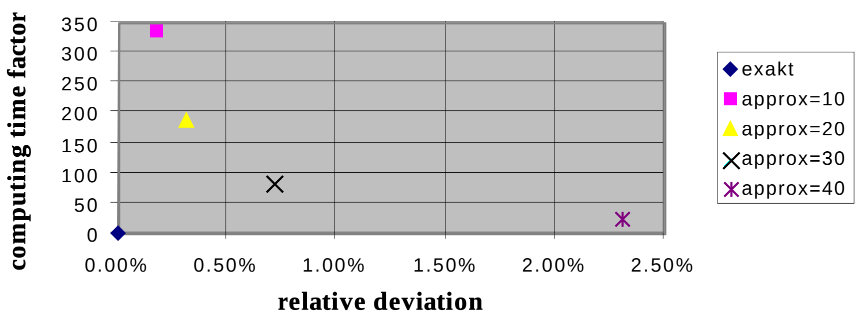

Figure 4.2: Relative computing time comparison: exact analysis formula versus MRP

Remark 4.1:

- We notice, that in this case, where the exact analysis formula can be adapted directly, we have an enormous computing time advantage in comparison to MRP.
- Computing times of the MRP increase exponentially in the smoothness of the integral partition. The reason therefore is, that the MRP needs to calculate all intermediate values up to $t_0 = 5$ in the same smoothness steps (Riemann–Stieltjes integral approximation needs former values). $\triangleleft$

Next, we investigate the behaviour of the methods in the course of time. Therefore, we consider an acyclic EOR network with again $n = 30$ nodes and $m = 60$ arcs. This time, $D_{ij} \sim \exp(5)$ *iid.* We consider non–marginal time evaluation of a sink at times $t = 2$, $t = 4$, and $t = 6$. Figure 4.3 shows the relative deviation of the MRP with different approximation parameters to the exact formula. Figure 4.4 shows the relative computing time differences of the MRP to the exact method in this case.

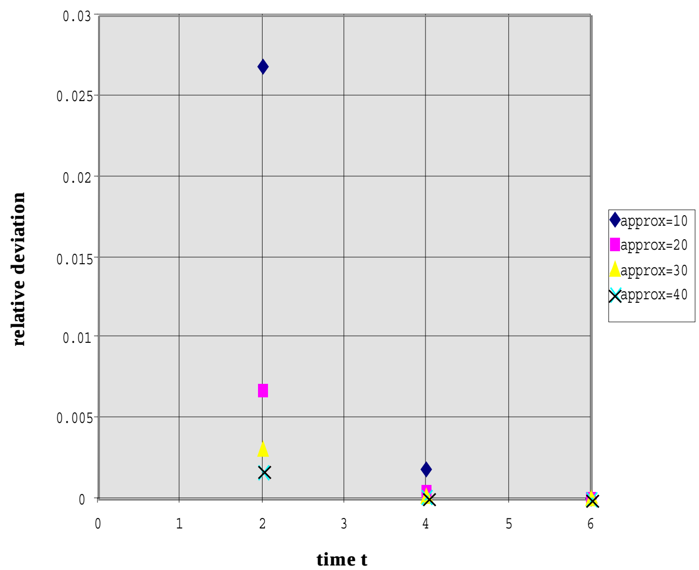

Figure 4.3: Relative deviation: exact analysis formula versus MRP

Remark 4.2:

- We see, that the MRP is getting stable, i.e., the longer (with respect to evaluation time) the MRP is applied for approximation of the activation functions, the more exact it becomes. The limit–values of the activation functions are the marginal values, to which the MRP tends with appropriate approximation accuracy.
- The computing times of the MRP increase, for any approximation accuracy, linear in time, since integral approximation is calculated in equidistant partitions.

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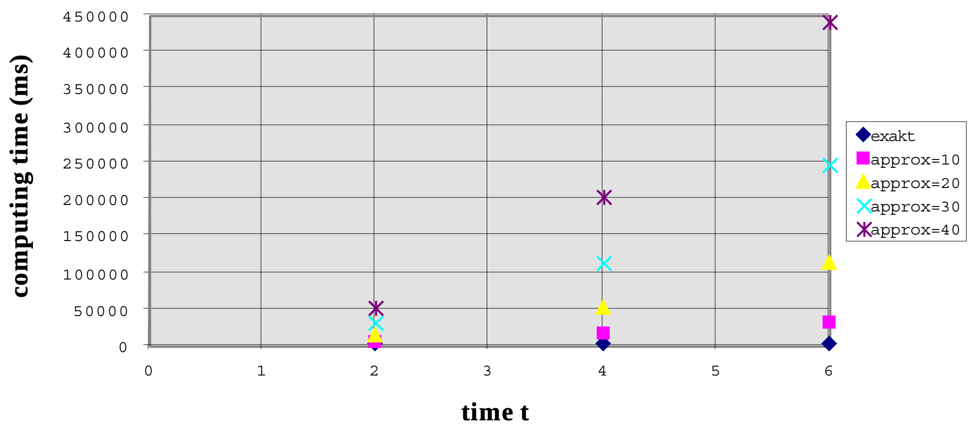

Figure 4.4: Computing time at several points in time: exact analysis formula versus MRP

4.2 Examination of the Exactness in Detail

In order to verify the exactness of the MRP, we examine a STEOR network with $n = 20$ nodes and $m = 19$ activities, all in series. Activity durations are distributed *iid* exp(1). The activation function of the sink is the distribution function of the 19–times convolution of exp(1)–random variables, i.e.: Erlang $(19, 1)$. This distribution function is "already" close to the normal distribution, due to the central limit theorem. This exact activity distribution is shown in Figure 4.5. An approximation with the MRP and a smooth approximation accuracy (approx=40) is shown in Figure 4.6.

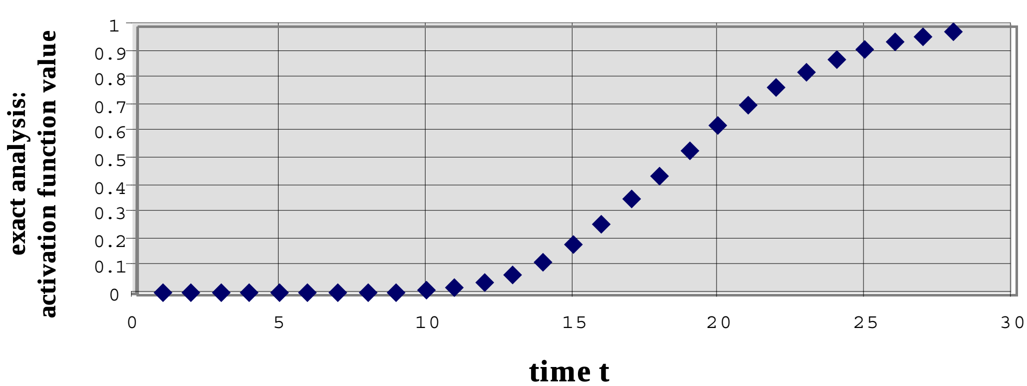

Figure 4.5: Activation function of sink for exp(1)–distributed activity durations

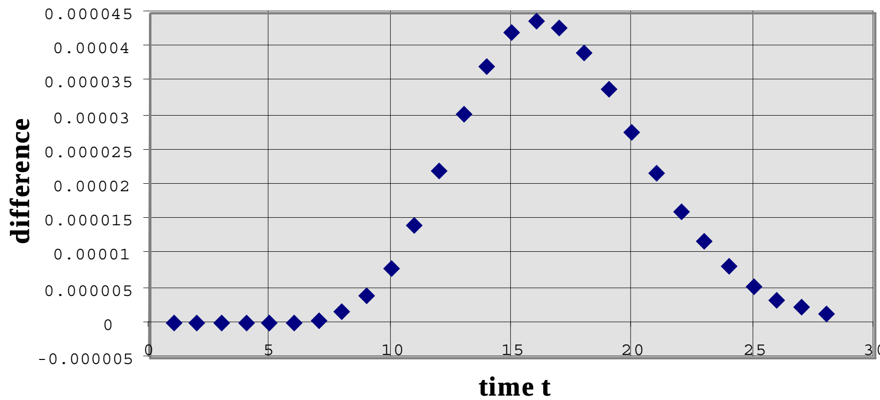

Figure 4.6: Difference of MRP and exact formula for exp(1)–distributed activity durations

Remark 4.3:

- The absolute difference of MRP with appropriate approximation accuracy to the exact analysis is not much.
- The difference is the higher, the greater the derivation of the exact distribution function is. This is reasoned by the equidistant partitioning of the integral approximation.
- We again notice, that the difference of MRP to the exact analysis converges to zero if $t \to \infty$. I.e.: "the MRP stabilizes towards the marginal case". $\triangleleft$

Finally, we want to compare the MRP with the "exact analysis formula" in the case, where activity durations are distributed uniformly. We consider the same network as above, where 19 activities are executed in series. Activity durations this time are distributed *iid* $U[0,1]$, i.e., uniform on the interval $[0,1]$. We apply the Edgeworth–approximation for the "exact formula". Figure 4.7 shows the activation function for the sink obtained with the Edgeworth–approximation. Since $n = 19$, the result is "already" close to a normal distribution, due to the central limit theorem. Figure 4.8 shows the difference of the MRP to the "exact formula" with Edgeworth–approximation.

Remark 4.4:

- The absolute difference of MRP with appropriate approximation accuracy to the "exact analysis" is not much. Here, we need to be careful since two not–exact values are compared.
- The difference again is the higher, the greater the gradient of the exact distribution function is. $\triangleleft$

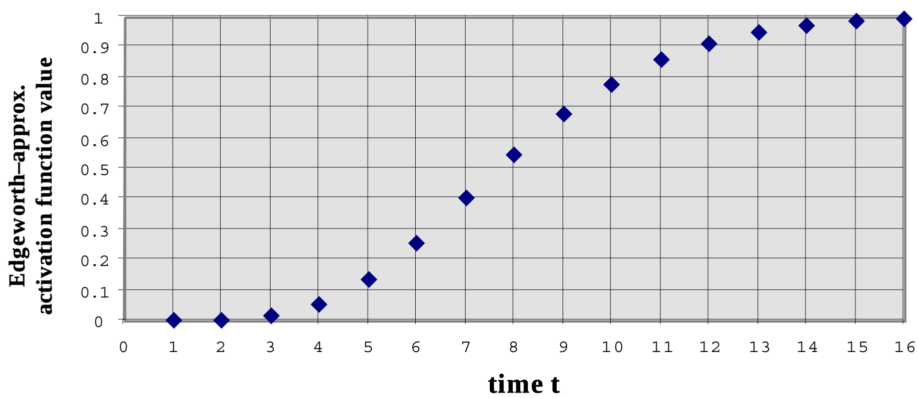

Figure 4.7: Activation function of sink for $U(0,1)$–distributed activity durations

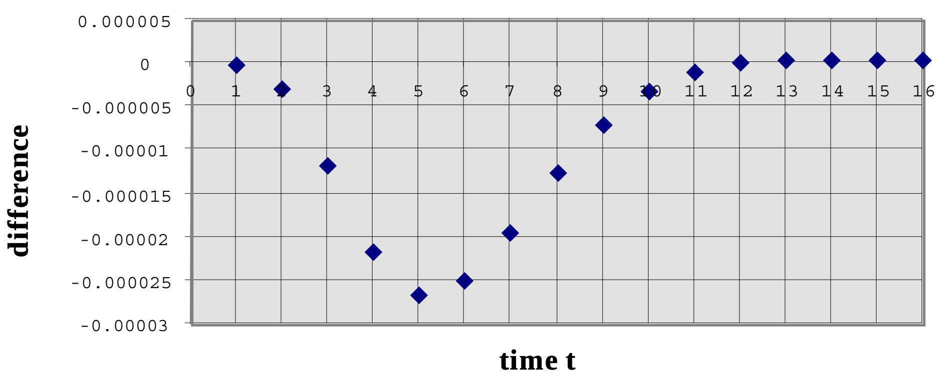

Figure 4.8: Difference of MRP and "exact formula" for $U(0,1)$–distributed activity durations

Chapter 5

Summary and Outlook

The Markov Renewal Process Method (MRP) with appropriate accuracy for the integral approximation gives good and stable results. It is a very flexible approximation method for arbitrary positively distributed activity durations. It is as good within cycle structures.

Disadvantages, however, are the large computing times to obtain good results. The qualitative trends, that computing time increases exponentially with increasing smoothness of the interval partition for integral approximation and, that computing time increases linear in evaluation time, cannot be compensated with high technical computing standards.

The exact analysis formula derived from the renewal equation gives explicit representation of the activity functions in the case where activity durations are exponentially distributed. Non–marginal time evaluation becomes exact. For projects, where the acyclic network is not too dense, the computing time for all occurring paths is acceptable. The comparison to MRP gives astonishingly souvereign results.

It was shown, that arbitrary project durations can be approximated by Edgeworth–approximation or by the Cox–distribution and using the "exact analysis formula". Cycle–structures can heuristically be applied, too.

Thus, for real–world projects, this new method should be as well taken into consideration for non–marginal time analysis due to its very fast evaluation.

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