A New Approach to Tchebycheffian B-Splines

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Abstract. Originally, Tchebycheffian B-splines have been defined by generalized divided differences. In this paper, we define Tchebycheffian B-splines by integration. Based upon this definition, all basic algorithms for Tchebycheffian splines can be derived in a straightforward manner. As an example, a knot insertion algorithm for Tchebycheffian splines is constructed.

§1. Introduction

The class of Tchebycheffian splines contains many different kinds of splines: for example B-splines, exponential splines, and hyberbolic splines, see [11]. Algorithms for Tchebycheffian splines have been constructed by generalized divided differences, see e.g. [5], by generalized polar forms [8,6], and by generalized de-Boor-Fix dual functionals [1]. A fourth possibility based upon a new construction method for Tchebycheffian B-splines is presented in this paper. This construction method, which can be considered as a generalized convolution having its origin in the derivative formula for B-splines, makes it possible to derive all basic algorithms for Tchebycheffian splines in a straightforward elementary manner [2]. In this paper, we will present a knot insertion algorithm for Tchebycheffian splines to illustrate the method.

§2. Definition of Basis Splines

One can introduce ordinary B-splines by their derivative formula and derive all further properties from this definition [9]. It is also possible to construct exponential B-splines of arbitray order by this method [4]. We will use this approach with a simple modification and get a much more general class of splines. In the following section, we will show that this class contains Tchebycheffian splines.

First, let us recall some basic concepts from analysis. A function $f\colon \mathbb{R} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\}$ is called `locally integrable`, abbreviated $f \in L_{loc}$, if f is Lebesgue integrable over every compact interval J with $J \subset \mathbb{R}$, see [12]. The

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space L_{loc} is a function algebra: with $f, g \in L_{loc}$ and $\lambda \in \mathbb{R}$, the functions $f+g$, λf, and $f \cdot g$ belong to L_{loc}. A locally integrable function is called `integral-positive` if the integral

$$\int_J f(x)\,dx$$

is positive for every compact interval J with positive length. Further, a `knot sequence` is a non-decreasing sequence of numbers. With these concepts we are able to construct certain basis splines:

Definition 1. *Let $\mathbf{t} = (t_i)_{i\in\mathbb{Z}}$ be a knot sequence and $\mathbf{w} = (w_0, \ldots, w_n)$ be a sequence of integral-positive functions. Then the* `basis splines` *$A_i^n(x) = A_i^n(x; \mathbf{t}, \mathbf{w})$ of order $n+1$ over $\mathbf{t}$ with respect to $\mathbf{w}$ are recursively defined by*

(0)

$$A_i^0(x) := \begin{cases} w_0(x), & \text{if } t_i \le x < t_{i+1}, \\ 0, & \text{otherwise,} \end{cases}$$

(1)

$$A_i^n(x) := w_n(x) \int_{-\infty}^{x} \left(A_i^{n-1}(y)/\alpha_i^{n-1} - A_{i+1}^{n-1}(y)/\alpha_{i+1}^{n-1} \right) dy$$

where $\alpha_j^{n-1} := \int_{-\infty}^{\infty} A_j^{n-1}(y)\,dy$ is the area of A_j^{n-1} and the following rule is used if $t_j = t_{j+n}$:

$$\int_{-\infty}^{x} A_j^{n-1}(y)/\alpha_j^{n-1}\,dy := \begin{cases} 0, & \text{if } x < t_j, \\ 1, & \text{if } x \ge t_j. \end{cases}$$

Example 2. *If we choose $w_0(x) = w_1(x) = \cdots = w_n(x) = 1$, we obtain B-splines, see [9].*

Next, we state some properties of the basis splines A_i^n which can be verified by straightforward induction. The details and further properties are given in [2].

Positivity. For $t_i < t_{i+n+1}$ the area α_i^n of the basis spline A_i^n is positive. Hence, the basis splines in Definition 1 are well-defined. Moreover, the integral

$$\int_J A_i^n(x)\,dx$$

is positive for every interval J with positive length and with $J \subseteq [t_i, t_{i+n+1}]$. For $t_i = t_{i+n+1}$ the basis spline A_i^n is zero.

Local Support. If $t_i < t_{i+n+1}$, the support of A_i^n is the interval $[t_i, t_{i+n+1}]$.

Basis Property. The basis splines $A_i^n, A_{i+1}^n, \ldots, A_{i+n}^n$ are linearly independent over any non-empty interval (t_{i+n}, t_{i+n+1}).

Remark 3. *It is possible to replace the Lebesgue integral in Definition 1 by a Lebesgue-Stieltjes integral*

$$A_i^n(x) := w_n(x) \int_{-\infty}^{x} (\cdots)\,d\sigma_n(y),$$

where σ_n is a locally bounded, strictly increasing, and right-continuous function.

§3. Tchebycheffian Splines

Let us now examine what kinds of basis splines can be constructed by Definition 1. To do this, we repeat the definition of Tchebycheffian splines given in the book of Schumaker [11].

Let $I = [a, b]$ be a compact subinterval of $\mathbb{R}$ and let $(u_0, \ldots, u_n)$ be a sequence of functions in $C^n(I)$. Then $(u_0, \ldots, u_n)$ is called an Extended Complete Tchebycheff system on I, short ECT-system, if for all $k = 0, \ldots, n$ and and each non-decreasing sequence $(t_0, \ldots, t_k)$ of numbers in I the determinant

$$\det\left(\left[D^{d_i} u_j(t_i)\right]_{i,j=0}^{k}\right)$$

is positive, where

$$d_i := \max\{\, r \mid t_i = \cdots = t_{i-r} \,\}.$$

A linear space is called an **ECT-space** on I if it has a basis forming an ECT-system on I.

Definition 4. *Let $I = [a, b]$ be a compact interval, let U be an $(n+1)$-dimensional ECT-space on I, and let $\mathbf{t} = (t_0, \ldots, t_{m+n+1})$ be a knot sequence. Suppose $t_0 = t_n = a$, $t_{m+1} = t_{m+n+1} = b$, and $\ell_i \le n+1$ for $n < i \le m$, where ℓ_i denotes the multiplicity of the knot t_i in $\mathbf{t}$. Then a function $s\colon [a, b) \to \mathbb{R}$ is called a* `Tchebycheffian spline`*, abbreviated $s \in \mathcal{S}(U, \mathbf{t})$ if s agrees on every non-empty knot interval $[t_i, t_{i+1})$ with a function in U and if $s \in C^{n-\ell_i}(t_i)$ for any knot t_i, where $n < i \le m$.*

Theorem 5. *Every space $\mathcal{S}(U, \mathbf{t})$ of Tchebycheffian splines has a basis of basis splines $A_0^n, \ldots, A_m^n$ where the A_i^n are constructed by Definition 1.*

Proof: Let $(u_0, \ldots, u_n)$ be an ECT-system for U. A theorem in [3, p. 379] says that every ECT-system $(u_0, \ldots, u_n)$ can be written as iterated integrals of positive weight functions $w_i \in C^i$:

$$\begin{aligned}
u_0(x) &= w_n(x) \\
u_1(x) &= w_n(x) \int_a^x w_{n-1}(s_{n-1})\, ds_{n-1} \\
&\ \vdots \\
u_n(x) &= w_n(x) \int_a^x w_{n-1}(s_{n-1}) \cdots \int_a^{s_1} w_0(s_0)\, ds_{n-1} \cdots ds_0.
\end{aligned}$$

Hence, the basis splines $A_0^n, \ldots, A_m^n$ over $\mathbf{t}$ with respect to $\mathbf{w} = (w_0, \ldots, w_n)$ belong to $\mathcal{S}(U, \mathbf{t})$. Since the dimension of $\mathcal{S}(U, \mathbf{t})$ is $m + 1$, see [11, p. 378], the assertion follows form the basis properties of the basis splines A_i^n. ■

It is also possible to produce non-Tchebycheffian B-splines with Definition 1. Consider the functions $u_0(x) = 1$, $u_1(x) = 2x^{1/2}$, $u_2(x) = \frac{2}{3}x^{3/2}$, and $u_3(x) = \frac{1}{5}x^{5/2}$ with $x \in [0, 1]$ as examined in [10]. They do not span an

ECT-system since u_1 is not differentiable at $x=0$. However, the corresponding weight functions $w_3(x) = w_1(x) = w_0(x) = 1$ and $w_2(x) = x^{-1/2}$ are Lebesgue integrable and positive on $[0,1]$, so the construction of basis splines with Definition 1 is feasible.

§4. Knot Insertion

Let $\mathbf{t}$ be a knot sequence, and let $\mathbf{w}$ be a sequence of integral-positive functions. A spline $\mathbf{s}$ over $\mathbf{t}$ with respect to $\mathbf{w}$ is defined as a linear combination of the basis splines $A_i^n(x) = A_i^n(x;\mathbf{t},\mathbf{w})$, i.e.,

$$\mathbf{s}(x) = \sum_i \mathbf{c}_i A_i^n(x), \quad \text{where } \mathbf{c}_i \in \mathbb{R}^d\,.$$

The points $\mathbf{c}_i$ are called `control points`. They form the `control polygon` of $\mathbf{s}$.

We want to construct a knot insertion algorithm for these splines. Let $\hat{t} \in \mathbb{R}$ be a number occuring with multiplicity ℓ in $\mathbf{t} = (t_i)_{i\in\mathbb{Z}}$. If $\hat{t}$ is not contained in $\mathbf{t}$, we set $\ell := 0$. Let r be the number with $t_r < \hat{t} \le t_{r+1}$. If $\hat{t}$ is inserted in $\mathbf{t}$, we obtain the refined knot sequence $\hat{\mathbf{t}} = (\hat{t}_i)_{i\in\mathbb{Z}}$ where

$$\hat{t}_i := \begin{cases} t_i & \text{if } i < r+1, \\ \hat{t} & \text{if } i = r+1, \\ t_{i-1} & \text{if } i > r+1. \end{cases}$$

We write $\hat{\mathbf{t}} = \mathbf{t}[\hat{t}]$ to indicate that $\hat{\mathbf{t}}$ is obtained by inserting $\hat{t}$ into $\mathbf{t}$.

Theorem 6. *Let A_i^n be the basis splines over $\mathbf{t}$ with respect to $\mathbf{w}$, and let $B_i^n(\cdot) = A_i^n(\cdot;\hat{\mathbf{t}},\mathbf{w})$ be the basis splines over the refined knot sequence $\hat{\mathbf{t}} = \mathbf{t}[\hat{t}]$ with respect to $\mathbf{w}$. Then there exist numbers $\lambda_i^n \in \mathbb{R}$ and $\mu_i^n \in \mathbb{R}$ with*

$$A_i^n = \lambda_i^n B_i^n + \mu_i^n B_{i+1}^n. \tag{1}$$

Proof: We show the theorem by induction. Let r be such that $t_r < \hat{t} \le t_{r+1}$, and let ℓ be the multiplicity of $\hat{t}$ in $\mathbf{t}$. For $n \le \ell$, we obtain from Definition 1

$$A_i^n = \begin{cases} B_i^n & \text{if } i < r, \\ B_i^n + B_{i+1}^n & \text{if } i = r, \\ B_{i+1}^n & \text{if } i > r. \end{cases}$$

Thus equation (1) holds for

$$\lambda_i^n := \begin{cases} 1 & \text{if } i \le r, \\ 0 & \text{if } i > r \end{cases} \quad \text{and} \quad \mu_i^n := \begin{cases} 0 & \text{if } i < r, \\ 1 & \text{if } i \ge r. \end{cases}$$

Suppose $n > \ell$. Let α_j^{n-1} and β_j^{n-1} be the areas of A_j^{n-1} and B_j^{n-1} respectively, and assume $t_i < t_{i+n}$ and $t_{i+1} < t_{i+n+1}$ so that α_i^{n-1} and α_{i+1}^{n-1} do not vanish. Suppose for the induction that there are numbers λ_j^{n-1} and μ_j^{n-1} such that

$$A_j^{n-1} = \lambda_j^{n-1} B_j^{n-1} + \mu_j^{n-1} B_{j+1}^{n-1}. \tag{2}$$

Using this expression (2) for a substitution in the definition of A_i^n gives

$$A_i^n(x) = w_n(x) \int_{-\infty}^{x} \Big[\left(\lambda_i^{n-1} B_i^{n-1}(y) + \mu_i^{n-1} B_{i+1}^{n-1}(y)\right) / \alpha_i^{n-1} \\ - \left(\lambda_{i+1}^{n-1} B_{i+1}^{n-1}(y) + \mu_{i+1}^{n-1} B_{i+2}^{n-1}(y)\right) / \alpha_{i+1}^{n-1} \Big] \, dy.$$

Applying Definition 1 to B_i^n and B_{i+1}^n, we obtain

$$A_i^n(x) = \lambda_i^{n-1} \frac{\beta_i^{n-1}}{\alpha_i^{n-1}} B_i^n(x) + \mu_{i+1}^{n-1} \frac{\beta_{i+2}^{n-1}}{\alpha_{i+1}^{n-1}} B_{i+1}^n(x) \\ + \left(\frac{\mu_i^{n-1}}{\alpha_i^{n-1}} + \frac{\lambda_i^{n-1} \beta_i^{n-1}}{\alpha_i^{n-1} \beta_{i+1}^{n-1}} - \frac{\lambda_{i+1}^{n-1}}{\alpha_{i+1}^{n-1}} - \frac{\mu_{i+1}^{n-1} \beta_{i+2}^{n-1}}{\alpha_{i+1}^{n-1} \beta_{i+1}^{n-1}} \right) w_n(x) \int_{-\infty}^{x} B_{i+1}^{n-1}(y) \, dy.$$

The last term in this equation vanishes since integrating equation (2) gives

$$\alpha_j^{n-1} = \lambda_j^{n-1} \beta_j^{n-1} + \mu_j^{n-1} \beta_{j+1}^{n-1}.$$

Hence equation (1) is valid for

$$\lambda_i^n := \lambda_i^{n-1} \frac{\beta_i^{n-1}}{\alpha_i^{n-1}} \quad \text{and} \quad \mu_i^n := \mu_{i+1}^{n-1} \frac{\beta_{i+2}^{n-1}}{\alpha_{i+1}^{n-1}}.$$

A similar computation gives

$$\lambda_i^n := 1 \quad \text{and} \quad \mu_i^n := \mu_{i+1}^{n-1} \frac{\beta_{i+2}^{n-1}}{\alpha_{i+1}^{n-1}} \qquad \text{for } t_i = t_{i+n} < t_{i+n+1}$$

and

$$\lambda_i^n := \lambda_i^{n-1} \frac{\beta_i^{n-1}}{\alpha_i^{n-1}} \quad \text{and} \quad \mu_i^n := 1 \qquad \text{for } t_i < t_{i+1} = t_{i+n+1}.$$

The case $t_i = t_{i+n+1}$ is trivial. ■

The proof of Theorem 6 shows that the numbers λ_i^n and μ_i^n are as follows:

Corollary 7. *Let α_i^m and β_i^m be the areas of A_i^m and B_i^m, respectively. Then the numbers λ_i^n and μ_i^n in Theorem 6 can be computed by*

$$\lambda_i^n = \begin{cases} 1 & \text{if } i \le r - n + \ell, \\ \prod_{m=\ell+r-i}^{n-1} (\beta_i^m / \alpha_i^m) & \text{if } r - n + \ell < i \le r, \\ 0 & \text{if } i > r, \end{cases}$$

$$\mu_i^n = 1 - \lambda_{i+1}^n$$

where r is such that $t_r < \hat{t} \le t_{r+1}$ and ℓ is the multiplicity of $\hat{t}$ in $\mathbf{t}$.

If we apply Theorem 6 to linear combinations of the basis splines A_i^n, we obtain the following knot insertion algorithm:

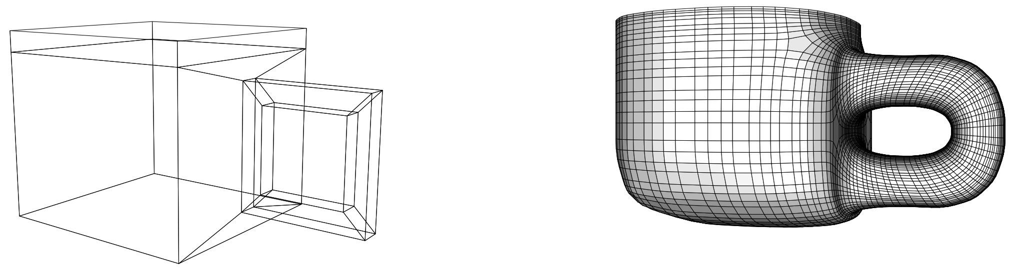

Fig. 1. Generating a cup by subdivision.

Algorithm 8 (Knot Insertion). *Every spline* $\mathbf{s} = \sum_i \mathbf{c}_i A_i^n$ *over* $\mathbf{t}$ *can be written as a spline* $\mathbf{s} = \sum_i \mathbf{d}_i B_i^n$ *over* $\mathbf{t}[\hat{t}]$ *where the control points* $\mathbf{d}_i$ *are given by*

$$\mathbf{d}_i = (1 - \lambda_i^n)\,\mathbf{c}_{i-1} + \lambda_i^n \mathbf{c}_i, \quad \lambda_i^n \text{ as in Corollary 7.}$$

Algorithm 8 implies that knot insertion is a corner cutting algorithm, see [7] for a detailed description of corner cutting algorithms. With a knot insertion algorithm it is easy to derive subdivision algorithms. For example, if the functions $w_0(x) = x(1-x)$ and $w_1(x) = w_2(x) = 1$ defined on the interval $[0,1]$ are periodically continued to $\mathbb{R}$, we can construct a local corner cutting algorithm by repeated knot insertion which produces C^2-curves with flat points, see [2] for full details. Also, by forming tensor products and introducing special rules for non-quadrilateral meshes, we can extend this local corner cutting algorithm to control nets of arbitrary topology, see Fig. 1 for an illustration.

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