Metastable Voltage States of Coupled Josephson Junctions

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We investigate a chain of capacitively coupled Josephson junctions in the regime where the charging energy dominates over the Josephson coupling, exploiting the analogy between this system and a multi-dimensional crystal. We find that the current-voltage characteristic of the current-driven chain has a staircase shape, beginning with an (insulating) non-zero voltage plateau at small currents. This behavior differs qualitatively from that of a single junction, which should show Bloch oscillations with vanishing dc voltage. The simplest system where this effect can be observed consists of three grains connected by two junctions. The theory explains the results of recent experiments on Josephson junction arrays.

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Recent experiments [1] carried out on two-dimensional Josephson junction arrays in the quantum regime – i.e. where the charging energy exceeding the Josephson coupling introduces quantum dynamics – showed remarkable steps of the current-voltage characteristics. Two voltage steps could be clearly seen, with voltage value given by the superconducting gap $2\Delta/e$. Further structure at higher voltages is washed out. The purpose of the present article is to show that these steps can be naturally explained in the framework of the Bloch oscillation description of quantum mechanical Josephson junctions. Moreover, we argue that the physics of a chain differs qualitatively from that known from single Josephson junctions [2,3]. In contrast to a current-biased single-junction, a chain of 1D Josephson junctions has no zero-voltage state, i.e. the average voltage is greater than $2\Delta/e$ for any current. The only stable state of the system has a voltage equal to $2N\Delta/e$ where N is the number of junctions. However, for low currents this stable state is reached only on astronomical time scales,

$$t_Z = 2eI^{-1}\exp(I_Z/I),$$

while the time required for the voltage $2\Delta/e$ to establish is

$$t_s \sim 2e\Delta/IE_C \ll t_Z.$$

Here E_C is the charging energy, and I_Z is defined below. Thus, in realistic experiments metastable I–V characteristic are observed, with voltage steps starting from the value $2\Delta/e$. The time required to achieve the stable voltage depends strongly on the external current. These results are a consequence of a lowered symmetry of the system, and the effect shows up already in the system shown on Fig. 1. We conjecture that the physics of 2D Josephson junction arrays is similar, which provides an explanation of the experiments [1].

First we recollect the results of a single Josephson junction with a capacitance C and Josephson coupling energy E_J. In the absence of dissipation and for energies below the superconducting energy gap Δ it can be described by the Hamiltonian [2,3]

$$\hat{H} = \frac{\hat{Q}^2}{2C} - E_J\cos\phi, \quad \hat{Q} = -2ie\frac{\partial}{\partial\phi}, \tag{1}$$

where $\hat{Q}$ and $\hat{\phi}$ are the charge and the superconducting phase difference operators, respectively. The Hamiltonian (1) is equivalent to that of a quantum particle with a coordinate ϕ in a periodic potential. If the junction is driven by an external current I (which is an analogue of an external force for a quantum particle), the junction charge increases with time, $Q = It$, until it reaches the value $Q = e$. Then the system has two options: either a Cooper pair tunnels across the junction and the junction charge is reset from $Q = e$ to $Q = -e$, or if this does not happen the charge increases further with time. The former option is analogue to a Bragg reflection when a quantum particle reaches the edge of the Brillouin zone and is reflected, thus staying in the lowest energy band. The latter option is the equivalent of Zener tunneling across the band gap to a higher band [4]. In the limit $E_J \ll E_C = e^2/2C$ (which will be discussed in this paper) the probability of such Zener tunneling event is [4]

$$\lambda = \exp(-I_Z/I), \quad I_Z = \pi e E_J^2/8E_C. \tag{2}$$

The question is which of the two options are realized and how they manifest themselves in an experiment. The answer depends on the current:

Low currents: In this regime the average time t_Z required for a Zener tunneling event to occur exceeds the time of the experiment, $t_Z = 2e(I\lambda)^{-1} \gg t_e$. This means that during the whole experiment the system stays in the lowest energy band, the charge (and the voltage) across the junction oscillates with the period $2e/I$ around zero. This is the metastable regime of Bloch oscillations [2,3,5].

High currents: In this regime, $t_e \gg t_Z$, Zener tunneling event [6] takes place during the experiment. In the absence of dissipation the system – after it jumps to the second energy band – will continue to rise in energy [7] until its energy reaches the value $2\Delta + E_C$ (the average voltage across the junction at this point is $V_0 \approx 2\Delta/e$ [3]). A further increase is not possible because of dissipation due to single electron tunneling that sets in as soon as the voltage exceeds the value $2\Delta/e$. Then the voltage will oscillate around the stable value $2\Delta/e$ [3].

Thus the difference between the two above regimes is quite clear: the first one corresponds to metastable zero-voltage state of the junction whereas in the second regime a finite stable voltage $V \approx 2\Delta/e$ is measured. The crossover between these two regimes occurs at $I \approx I_Z/\ln(It_e/2e)$.

Experimentally it is more convenient to study charging effects in systems containing several (or many) Josephson junctions. Hence one can ask whether the simple physical picture discussed above remains valid also in the case of two or more coupled Josephson junctions. Intuitively one could expect that for sufficiently low currents flowing through the chain of Josephson junctions the system will stay in the zero voltage state $\bar{V} \approx 0$ similarly to the case of a single junction.

In this paper we argue that the behavior of already two capacitively coupled Josephson junctions is *qualitatively different*: a finite voltage drop $V \simeq V_0 \equiv 2\Delta/e$ across the system will be measured *no matter how low the external current is*. This conclusion remains valid for an arbitrary number of junctions in the chain $N > 1$. To understand this result we use the analogy between our system and an electron in a multidimensional crystal. If the self-capacitance C_0 of the superconducting islands is non-zero the symmetry of the corresponding Brillouin zone is lowered in a way which prevents the system from staying in the lowest energy band for *any* non-zero current I. Zener tunneling in at least one of the junctions occurs immediately after the current is applied and the system switches to a finite voltage state. The resulting current-voltage characteristics of the Josephson junction chain has a form of a staircase. Our results can explain the low voltage behavior of two-dimensional Josephson junction arrays recently reported in Ref. [1].

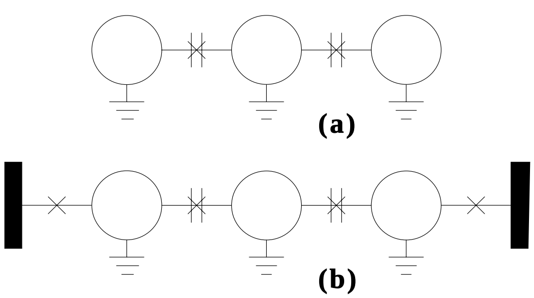

FIG. 1. (a) Two capacitively coupled Josephson junctions; (b) An experimental realization of the same system.

For the sake of simplicity we first consider the system of three superconducting grains with phases ϕ_i, $i = 1, 2, 3$, connected by two Josephson junctions (Fig. 1a). The relevant variables for description of this system are linear combinations of the phases (cf. e.g. [8])

$$\theta_1 = \phi_1 - \phi_2; \quad \theta_2 = \phi_2 - \phi_3; \quad \theta_3 = (\phi_1 + \phi_2 + \phi_3)/3.$$

We can also introduce the junction charge operators $\hat{Q}_i = -2ie\partial/\partial\theta_i$, $i = 1, 2$, and the "center-of-mass charge" $\hat{Q}_3 = -2ie\partial/\partial\theta_3$. Then the Hamiltonian reads as

$$\hat{H} = \frac{\hat{Q}_3^2}{6C_0} + \frac{1}{2}\sum_{ij} \hat{Q}_i(\hat{C}^{-1})_{ij}\hat{Q}_j - E_J \sum_i \cos\theta_i, \qquad (3)$$

and the effective 2×2 capacitance matrix $\hat{C}$ has elements $C_{11} = C_{22} = C + 2C_0/3$; $C_{12} = C_{21} = C_0/3$. The center-of-mass dynamics is independent of that of both junctions. The latter is essentially the motion of an electron in a 2D periodic potential. In the limiting case $E_J \ll e^2/\max\{C_0, C\}$ (nearly-free-electron model) the Josephson term in the Hamiltonian (3) can be considered as a perturbation. We assume furthermore that $2\Delta > e^2/\max\{C_0, C\}$. The eigenfunctions of the unperturbed Hamiltonian are "plain waves",

$$\psi_0(\boldsymbol{\theta}) = \exp(i\boldsymbol{Q\theta}/2e + iQ_3\theta_3/2e).$$

Here we have introduced two-dimensional vectors $\boldsymbol{\theta} = (\theta_1, \theta_2)$ and $\boldsymbol{Q} = (Q_1, Q_2)$.

Now we can clarify the meaning of the charges Q_1 and Q_2. Introducing the grain charge operators $\hat{q}_i = -2ie\partial/\partial\phi_i$, $i = 1, 2, 3$, with the eigenvalues q_i, we obtain

$$q_1 = Q_1 + Q_3/3; \; q_2 = -Q_1 + Q_2 + Q_3/3; \; q_3 = -Q_2 + Q_3/3.$$

The charge Q_3 is just a total charge of the chain. It is conserved, and from now on we put it equal to zero [9]. In the current-bias regime the charge q_1 grows linearly with time, except for the jumps by $2e$ due to Cooper pair tunneling (CPT) through junction 1 (see below). The charge q_3 decreases linearly with time, except for CPT through junction 2, while the charge q_2 keeps constant except for CPT through the either junction. In other words, the system can move only along lines $Q_1 - Q_2 = 2en$, with n being an integer, and CPT processes are responsible for the jumps of the system between these lines. The time evolution of the system starts from the line $Q_1 = Q_2 = It$.

The eigenvalues of the unperturbed Hamiltonian are

$$E_0(\mathbf{Q}) = \frac{1}{2}\sum_{ij} Q_i(\hat{C}^{-1})_{ij}Q_j. \qquad (4)$$

The Josephson coupling plays a role only in the vicinity of "critical surfaces", $E_0(\mathbf{Q}) = E_0(\mathbf{Q} - \mathbf{K})$, with $\mathbf{K} = (2pe, 2eq)$ (p and q are integer numbers) being the reciprocal lattice vectors. In lowest (second) order in E_J only vectors $\mathbf{K}$ with $p = \pm 1, q = 0$ and $p = 0, q = \pm 1$ are essential, and the Josephson coupling creates gaps equal to E_J on the critical surfaces. For the spectrum (4) these are given by

$$1)\ Q_1 - \alpha Q_2 = \pm e; \quad 2)\ Q_2 - \alpha Q_1 = \pm e,$$

with $\alpha = (C_0/3)(C + 2C_0/3)^{-1}$. Note that $0 \le \alpha \le 1/2$; the limits $\alpha = 0$ and $\alpha = 1/2$ correspond to the cases $C_0 = 0$ and $C = 0$, respectively. Thus, the critical surfaces are just two pairs of parallel straight lines. The romb formed by these lines is the first Brillouin zone. The Bragg reflection processes are possible on the critical surfaces, i.e. the system can jump by a vector $\mathbf{K}$. The energy is periodic in $\mathbf{Q}$-plane with two orthogonal periods equal to $2e$. It will be convenient for us to use the extended band picture. The voltage across the junction i is given by $V_i = \partial E/\partial Q_i$.

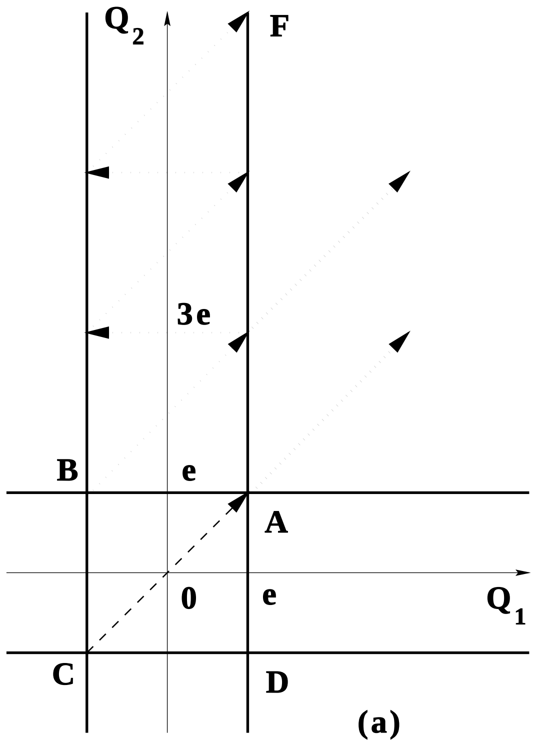

FIG. 2. The motion of the current-biased system in the plane (Q_1, Q_2); $C_0 = 0$. The critical surfaces are shown by boldface lines; the first Brillouin zone is a square ABCD. In the weak current regime the system moves along the line AC. Possible trajectories for the cases of strong currents, $\lambda \sim 1$, are shown by dotted lines.

For $C_0 = 0$ the Hamiltonian (3) is just a sum of two Hamiltonians (1) of isolated junctions, and the behavior of the chain is trivial. The critical surfaces are $Q_i = \pm e$, $i = 1, 2$ (Fig. 2). For low currents a possible time evolution (with large probability $(1-\lambda)^2$) is simultaneous Bloch oscillations in two junctions [10], i.e. Bragg reflection from point A= (e, e) to point C= $(-e, -e)$. The mean voltage is zero for this process. Another possibility (with probability $\lambda(1-\lambda)$) is the reflection to the point B= $(-e, e)$, with subsequent motion along the line $(Q, Q{+}2e)$, $Q > -e$. This means Zener tunneling in junction 2, and Bloch oscillations in junction 1. Eventually the system reaches the critical surface in the point $(e, 3e)$, where it is reflected to the point $(-e, 3e)$ and so on. The voltage across junction 1 $V_1 = Q_1/C$ oscillates with frequency $I/2e$, while the voltage across junction 2 grows linearly in time. The third possibility (with equal probability $\lambda(1-\lambda)$) is an equivalent process, where the system is initially reflected to the point $(e, -e)$. Finally, the last possibility (with probability λ^2) is to proceed along the line (Q, Q) without any reflection. This corresponds to Zener tunneling in two junctions simultaneously, i.e. the voltage grows across both junctions.

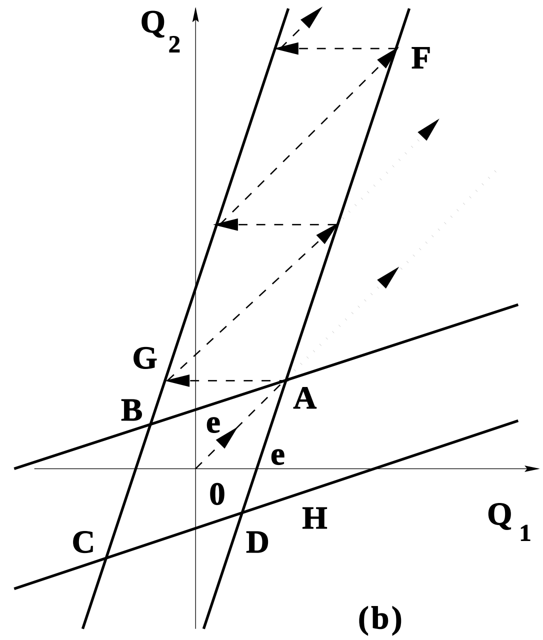

FIG. 3. The same as Fig. 2, $C_0 \neq 0$. The first Brillouin zone is a romb ABCD. In the weak current regime the system moves along the dashed line BF.

The behavior is quite different in the presence of a finite self-capacitance C_0. The system again starts from the point $(0,0)$ and reaches the critical surface in the point A= $(e\xi, e\xi)$, $\xi = (1-\alpha)^{-1}$, $1 < \xi \le 2$ (Fig. 3), *beyond* the point (e, e). Consequently, it has only three possibilities of further motion. The first one (with probability $(1-\lambda^2)/2$, where λ is given by the same expression (2) with $E_C = e^2/2(C + C_0)$) is to be reflected to the point G= $(e(\xi-2), e\xi)$. It corresponds to Zener tunneling in junction 2. The subsequent evolution of the system is voltage growth across junction 2 and Bloch oscillations in junction 1. The equivalent possibility is to be reflected to the point H(= $e\xi, e(\xi-2)$). Finally, the last allowed process (with probability λ^2) is to continue the motion along the line (It, It) without any reflection, i.e. Zener tunneling takes place in both junctions. Note that the simultaneous occurrence of Bloch oscillations in both junctions is impossible. This is a consequence of a lowered symmetry: while the case $C_0 = 0$ has the same symmetry group as a square, the introduction of C_0 lowers the latter to that of a romb.

Thus, the situation differs qualitatively from that of a single junction. Still two scenarios exist, low and high currents. The second one yields the mean voltage $2V_0$, which one could expect. However, the low current scenario leads now to a finite mean voltage V_0, in contrast to the single-junction case. The time $t_s \sim 2e\Delta/IE_C \ll t_Z$, required for this metastable value to establish, is very short. Hence at the beginning of experiment the mean value V_0 is observed. The time of order t_Z, which can be extremely long, is required to obtain the stable value $2V_0$.

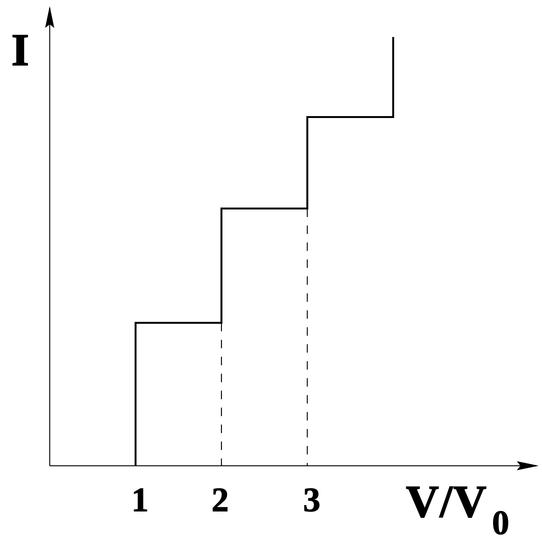

FIG. 4. A sketch of voltage-current characteristic of the system of coupled junctions. All fine structure of oscillations is eliminated, and only the mean value $\bar{V}$ in units of V_0 is shown.

The above approach can be generalized to the case of the Josephson chain consisting of $N+1$ grains and N junctions. The relevant variables are phases of junctions $\theta_i = \phi_i - \phi_{i+1}$, $1 \leq i \leq N$, and the center-of-mass coordinate $\theta_{N+1} = (N+1)^{-1}(\phi_1 + \phi_2 + \ldots + \phi_{N+1})$, with ϕ_i being the phases of the grains. The problem is equivalent to the motion of an electron in an N-dimensional periodic potential; the chain is represented by a point $\mathbf{Q}$ in the space of junction charges $Q_i, i = 1, \ldots, N$. A charge-neutral system moves along the lines

$$(Q_1 + 2eN_1, Q_2 + 2en_2, \ldots, Q_N + 2en_N), n_k \in \mathcal{Z}.$$

It starts from the coordinate origin and can jump between the lines only as a result of "Bragg reflection" on the critical surfaces, given by ($l = 1, \ldots, N$)

$$\sum_i Q_i (C^{-1})_{il} = \pm e (C^{-1})_{ll}, \tag{5}$$

where the $N \times N$ capacitance matrix is given by

$$C_{kl} = C_{lk} = C\delta_{kl} + C_0 \frac{k(N+1-l)}{N+1}; \quad k < l. \tag{6}$$

An analysis for arbitrary relation between capacitances C and C_0 would require the inversion of the capacitance matrix $\hat{C}$. However, both cases $C = 0$ and $C_0 \ll C$ allow an analytical solution and show the same symmetry lowering causing a finite (metastable) voltage V_0 across the chain.

Thus, the current-voltage characteristics can be described as follows. The stable value of the mean voltage for *any* value of current is NV_0. However, the time required to achieve this stable voltage is astronomical (exponentially long) for low currents, and may exceed the duration of the experiment t_e. Hence one observes the metastable value V_0. The main effect of an increasing current is a decrease of the time t_Z. For high currents one has $t_Z \ll t_e$ and observes the stable voltage. There are some intermediate regimes between, and metastable voltages kV_0, $k = 2, 3, \ldots, N-1$ can be measured (Fig. 4). This dependence is essentially a low-voltage that observed in the experiments [1].

In conclusion, we predict that a current-biased chain of N capacitively coupled Josephson junctions always is in a finite voltage $\bar{V} \geq 2\Delta/e$ state no matter how low the external current is [11]. This conclusion differs qualitatively from that reached for a single Josephson junction. It is based on simple symmetry arguments. While for vanishing on-site capacitance $C_0 = 0$ the structure of the Brillouin zone for a Josephson "particle" corresponds to that for an N-dimensional cubic crystal, the crystal symmetry is lowered for any non-zero C_0. As a result Bloch oscillations in the lowest Brillouin zone cease to occur and the system "jumps" to higher bands due to Zener tunneling. This value of the voltage is metastable, but can be measured experimentally since the time scale required to observe the stable value NV_0 is very long for low currents. We found a staircase-like shape of the I-V curve of the system and argue that our predictions can explain the observed low-voltage behavior of Josephson junction arrays [1].

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[9] Our considerations can be generalized to the configuration with gate voltages inducing non-zero gate charges on superconducting grains. In this case fractional parts of these charges determine the initial position of the system in the first Brillouin zone. No new physical effects arise from this complication and all our main conclusions remain valid also for the case $Q_3 \neq 0$.

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[11] In order to relate our simple model to the real experimental situations, one should connect the system with two additional junction to two reservoirs. It can be easily shown that the minimal system in order to observe the predicted effect is shown on Fig. 1b.