Limit Sets of Discrete Groups acting on Symmetric Spaces

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Vorwort

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Contents

Introduction

The study of discrete isometry groups acting on Hadamard manifolds has been a subject of continuing interest since group invariants allow to draw conclusions about the topology and geometry of their quotient manifolds. An important example are fundamental groups of compact manifolds. A remarkable theorem of Mostow says that closed hyperbolic manifolds of dimension greater or equal than three are determined up to isometry by their fundamental group.

An important tool to prove such rigidity theorems is the extension of the group action to an appropriate compactification of the Hadamard manifold. The structure of the so obtained limit set and properties of certain equivariant measures on it are also intimately related with the geodesic flow on the quotient manifold.

In the case of nonelementary discrete groups acting on real hyperbolic spaces, the theory is rather well developped. A good survey is given in the book of P. J. Nicholls ([N]). In 1976, S. J. Patterson ([P]) constructed a family of equivariant measures for fuchsian groups in order to determine the Hausdorff dimension of their limit sets. Subsequently, D. Sullivan ([S]) extended his results to real hyperbolic spaces of arbitrary dimension. He showed that the Hausdorff dimension of the limit set of a geometrically finite group Γ is given by its critical exponent $\delta(\Gamma)$.

More recently, part of the theory has been generalized to discrete isometry groups of Hadamard manifolds with pinched variable negative curvature (see for example [Y]), and to Gromov hyperbolic spaces ([Co]). K. Corlette ([C]) generalized Sullivan's results on the Hausdorff dimension to symmetric spaces of strictly negative curvature, using a family of conformally equivalent subriemannian metrics on the sphere at infinity.

For general Hadamard manifolds with tangent planes of zero sectional curvature, results about the limit set of discrete isometry groups are sparse. A good introduction to the subject as well as a description of individual isometries is given in the books of P. Eberlein ([E]) and of W. Ballmann, M. Gromov, V. Schroeder ([Ba], [BGS]). Beautiful examples of such manifolds are provided by higher rank symmetric spaces of noncompact type, which, due to their rich algebraic structure, allow more precise results concerning the limit sets of discrete isometry groups.

The goal of this thesis is to give more insight into the dynamics of individual isometries acting on symmetric spaces of higher rank, to describe geometrically the structure of the limit sets of discrete isometry groups, and finally to estimate their size in terms of certain equivariant measures. The main difficulties we face in this context compared to the situation in Hadamard manifolds with pinched negative curvature arise from the fact, that the structure of the geometric boundary is much more complicated and the isometry group does not act transitively on it. This seems to make impossible the definition of an appropriate metric on the geometric boundary in the sense that the size of the shadow of a ball in the space viewed from different points is conformally the same.

Moreover, the incomplete picture we have about the dynamics of parabolic isometries leads to difficulties in the investigation of the structure of the limit set of discrete isometry groups, which in general contain parabolics. Nevertheless, we are able to obtain precise results using an approximation argument, which works for a large class of groups. We call these groups nonelementary, since our definition generalizes the familiar notion of nonelementary groups in the context of real hyperbolic spaces in a natural geometric way.

The thesis is organized as follows. The first two chapters are of introductory nature and provide the basics about the algebraic structure of symmetric spaces, as well as a precise description of the sphere at infinity and the Furstenberg boundary endowed with their natural topologies. We describe the local product structure of the geometric boundary composed of a transversal factor and a direction unchanged under the action of the isotropy group of some chosen base point. The third chapter introduces Buseman functions and a family of (possibly nonsymmetric) pseudo distances, which will play an important role in the sequel. It further contains some elementary geometric estimates.

In the second part of the work, results concerning the dynamics and structure of the limit set are developped from a purely geometric point of view. Chapter 4 introduces the limit set L_Γ, characterizes the radial limit set L_Γ^{rad} and supplies a precise description of the different kinds of individual isometries.

In chapter 5, we investigate the dynamics of axial isometries in order to extend the well-known results in Hadamard manifolds of pinched negative curvature to symmetric spaces of higher rank. Since the isometry group does not act transitively on the geometric boundary, it is not possible that a sequence of axial isometries maps all of the geometric boundary to the limit of its attractive fixed points. Nonetheless, we are able to prove similar dynamics for certain sequences of axial isometries and certain boundary subsets, which will provide the key to Theorem 5.6, a natural construction of free groups far more general than the Schottky group construction proposed by Y. Benoist in ([Be]). The following theorem gives an impression of the more extensive statement of Theorem 5.4.

THEOREM **1** *If (γ_j) is a suitably nondegenerate sequence of axial isometries such that $(\gamma_j x_0)$ converges to a point ξ in the regular boundary, then a dense open subset of the geometric boundary is mapped by the sequence (γ_j) to a neighborhood of a Weyl chamber at infinity which contains ξ.*

We will see that for every sequence of axial isometries in a nonelementary group, there exists a suitably nondegenerate sequence of axial isometries with the same attractive fixed points. In fact, Proposition 5.11 states that every limit point of a discrete nonelementary isometry group can be approximated by a sequence of axial isometries, and leads directly to

THEOREM **2** *If $\Gamma \subset Isom^o(X)$ is a nonelementary discrete isometry group of a symmetric space X of noncompact type, then either the regular limit set $L_\Gamma \cap \partial X^{reg}$ is empty or the set of fixed points of axial isometries is a dense subset of the limit set L_Γ.*

It also provides a means to overcome the above mentioned difficulties concerning parabolic isometries and allows to deduce the important Theorems 5.14, 5.15 and 5.16 about the structure of the limit set. The simplified statements read as follows:

THEOREM **3** *If $\Gamma \subset Isom^o(X)$ is a nonelementary discrete group acting on a globally symmetric space X of noncompact type with nonempty regular limit set, then the limit set K_Γ, considered as a subset of the Furstenberg boundary, is a minimal closed set under the action of Γ, the geometric limit set is a product $L_\Gamma \cong K_\Gamma \times P_\Gamma$, where P_Γ is the set of directions of the limit points, and P_Γ equals the closure of the set of translation directions of axial isometries $\mathcal{L}_\Gamma = \{L(\gamma) \mid \gamma \in \Gamma\,,\ \gamma \text{ axial}\}$.*

For Zariski dense isometry groups, Theorem 3 has been proved by Y. Benoist ([Be]) using algebraic methods. Similar results have also been obtained by Y. Guivarc'h ([G]). The advantage of our proofs is their geometric nature which allows to easily adapt the methods to products of pinched Hadamard manifolds ([DaK]). Furthermore, our notion of nonelementary groups is more natural from a dynamical point of view and less restrictive than Zariski density.

The final part of this thesis is dedicated to the construction and study of equivariant measures supported on the limit set. According to Theorem A in ([Al]), for Zariski dense discrete groups Γ acting on a globally symmetric space G/K of higher rank, the support of any $\delta(\Gamma)$-dimensional conformal density either lies in the singular boundary or is contained in a unique G-invariant subset of the regular boundary. Since we are interested in the size of the whole geometric limit set, the use of conformal densities, which serves well in the case of Hadamard manifolds of pinched negative curvature, does not seem appropriate. Introducing more degrees of freedom and replacing the critical exponent $\delta(\Gamma)$ by $\delta_{G\cdot\xi}(\Gamma)$, the exponent of growth of Γ in direction $G\cdot\xi \subseteq \partial X$, we are able to construct for any discrete group Γ and for every $\xi \in \partial X$ families of Γ-equivariant, absolutely continuous orbital measures supported on the limit set. With minor restrictions on the behaviour of the exponent of growth in a neighborhood of the considered direction which are satisfied at least for Zariski dense discrete groups Γ by a result due to J. F. Quint ([Q]), there exist parameters such that our Γ-equivariant, absolutely continuous orbital measures are supported on the geometric limit set intersected with the given subset $G\cdot\xi \subseteq \partial X$. Depending on the parameters occuring in the Radon-Nikodym derivative, we call such a family of measures a $(b, \Gamma\cdot\xi)$-density.

In chapter 6, we define the exponent of growth of Γ in every direction and work out the details of the above construction using generalized Poincaré series. For the sake of illustration, we give precise parameters in the case of a few typical examples of discrete isometry groups. We remark that independently, J. F. Quint ([Q]) constructed a similar class of generalized Patterson Sullivan measures by different methods. His measures, however, do not seem appropriate to estimate the Hausdorff dimension of the geometric limit set, because they are all supported on the Furstenberg boundary and therefore lack an essential piece of information concerning the geometry of Γ-orbits.

In Chapter 7 we derive properties of $(b, \Gamma{\cdot}\xi)$-densities invariant by nonelementary discrete groups $\Gamma \subset \mathrm{Isom}^o(X)$. Following P. Albuquerque ([Al]), we prove our main tool, the shadow lemma Theorem 7.6, in our more general situation. A first application gives an upper bound for the exponent of growth $\delta_{G\cdot\xi}(\Gamma)$ in direction $G{\cdot}\xi \subseteq \partial X$. If r denotes the rank of X, and $H_1, H_2, \ldots, H_r$ are certain linearly independent vectors in the tangent space of a base point $x_0 \in X$, this upper bound is determined by the parameters $b^1, b^2, \ldots, b^r$ and the direction H_ξ of a $(b, \Gamma{\cdot}\xi)$-density.

Theorem 4 *If a $(b, \Gamma{\cdot}\xi)$-density exists, then $\delta_{G\cdot\xi}(\Gamma) \le \sum_{i=1}^r b^i \langle H_i, H_\xi\rangle$.*

In certain cases, we also obtain a lower bound for the exponent of growth of Γ in direction $G{\cdot}\xi \subseteq \partial X$, and state

Theorem 5 *If a $(b, \Gamma{\cdot}\xi)$-density μ with $\mu_{x_0}(L_\Gamma^{rad}) > 0$ exists, then*

$$\delta_{G\cdot\xi}(\Gamma) = \sum_{i=1}^r b^i \langle H_i, H_\xi\rangle\,.$$

We further examine the atomic part of $(b, \Gamma{\cdot}\xi)$-densities and prove

Theorem 6 *A regular radial limit point of a nonelementary discrete isometry group Γ is not a point mass for any $(b, \Gamma{\cdot}\xi)$-density.*

Next we address the question of ergodicity of a $(b, \Gamma{\cdot}\xi)$-density μ. If Γ is strongly nonelementary and if every Γ-invariant subset of the radial limit set possesses a suitable density point, then an application of Theorem 7.6, the shadow lemma, allows to conclude that Γ acts ergodically on L_Γ^{rad} with respect to the measure class defined by μ. It is not clear, however, if such density points exist in the general case, because it seems difficult to construct a Vitali cover from shadows.

The final section introduces Hausdorff measure using a conformal structure on the geometric boundary as proposed by G. Knieper ([Kn]). A first result is

Theorem 7 *Let $\Gamma \subset Isom^o(X)$ be a discrete nonelementary subgroup and $\xi \in \partial X^{reg}$. Then*

$$\dim_{\mathrm{Hd}}(L_\Gamma^{rad} \cap G{\cdot}\xi) \le \delta_{G\cdot\xi}(\Gamma)\,.$$

For a certain class of groups which we call radially cocompact, we even have equality.

Theorem 8 *If $\Gamma \subset Isom^o(X)$ is a nonelementary discrete radially cocompact subgroup, then the Hausdorff dimension of the radial limit set intersected with a given G-invariant subset $G{\cdot}\xi \subseteq \partial X$ is equal to the exponent of growth $\delta_{G\cdot\xi}(\Gamma)$ in that direction.*

The class of radially cocompact groups represents a natural generalization of the class of cocompact lattices in G and includes for example the classical convex cocompact and geometrically finite groups acting on rank one symmetric spaces, as well as products of convex cocompact or geometrically finite groups acting on the corresponding product manifold.

Chapter 1

Symmetric spaces

In this chapter, we will give a short review of the basic properties and the geometry of symmetric spaces. We will also describe decompositions of semisimple Lie groups and Lie algebras, which are intimitely related to the structure of symmetric spaces. The main reference will be [H], chapters III, IV, V and IX. A more geometric description is given in chapter 2 of [E].

1.1 The Riemannian structure

Let M be a Riemannian manifold, $p \in M$ and N_0 a symmetric neighborhood of the origin in T_pM such that the Riemannian exponential map $\exp_p$ is a diffeomorphism of N_0 onto $N_p := \exp_p(N_0) \subset M$. Then the mapping

$$\begin{array}{rrcl} s_p : & N_p & \to & N_p \\ & q & \mapsto & \exp_p(-\exp_p^{-1} q) \end{array}$$

is called the geodesic symmetry with respect to $p \in M$.

Definition 1.1 *A* symmetric space *X is a complete, connected Riemannian manifold such that for any point $x \in X$ the geodesic symmetry s_x belongs to the isometry group $Isom(X)$ of X. If every geodesic symmetry in X is only a local isometry, X is called* locally symmetric*, otherwise X is called* globally symmetric.

It is well known that simply connected symmetric spaces are globally symmetric. Since every locally symmetric space can be realized as a quotient of a globally symmetric space X by a discrete subgroup of $\mathrm{Isom}(X)$, we will restrict our attention to globally symmetric spaces.

The connected component $G := \mathrm{Isom}^o(X)$ of the isometry group of X which contains the identity acts transitively on X and can be equipped with a Lie group structure. The geodesic symmetry at some point x_0 in X defines an involutive automorphism $\sigma : G \to G$, $g \mapsto s_{x_0} g s_{x_0}$, which descends to an involutive isomorphism $\theta : \mathfrak{g} \to \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$

of G. This Cartan involution gives rise to a direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ in its $+1$ and -1 eigenspaces, $\theta(\mathfrak{k}) = \mathfrak{k}$, $\theta(\mathfrak{p}) = -\mathfrak{p}$.

The stabilizer $K := \mathrm{Stab}_G(x_0)$ of x_0 in G is a maximal compact subgroup with Lie algebra $\mathfrak{k}$. The natural projection $g \mapsto g \cdot x_0$ induces an isomorphism of $\mathfrak{p}$ to the tangent space $T_{x_0}X$. Via G-translation we may identify every tangent space in X with $\mathfrak{p}$.

The restriction of the Killing form of G

$$\begin{array}{rccl} B: & \mathfrak{g} \times \mathfrak{g} & \to & \mathbb{R} \\ & (X_1, X_2) & \mapsto & \mathrm{Tr}(\mathrm{ad}X_1 \circ \mathrm{ad}X_2) \end{array}$$

to K is a negative definite bilinear form, since K is compact. The Cartan relations

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}\,, \qquad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}\,, \qquad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$$

imply $B(\mathfrak{k}, \mathfrak{p}) = 0$. Thus we may endow $\mathfrak{g}$ with an $\mathrm{Ad}(K)$-invariant bilinear form

$$\langle X_1, X_2 \rangle = -B(X_1, \theta(X_2))\,.$$

The restriction to $\mathfrak{p}$ of this bilinear form is positive definite and extends to a G-invariant Riemannian metric on X. This Riemannian structure is independent of the choice of $x_0 \in X$ (see [H], ch. IV, §3).

DEFINITION 1.2 *X is of* noncompact type, *if Isom(X) is noncompact.*

In this case, the sectional curvature is nonpositive (see [H], ch. V, §3), and therefore the Riemannian exponential map is a diffeomorphism from $\mathfrak{p}$ to X.

1.2 Flats, Weyl chambers and the Weyl group

From here on, X will be a globally symmetric space of noncompact type. Then the connected component $G = \mathrm{Isom}^o(X)$ of its isometry group is a semisimple Lie group without compact factor and with trivial center. Up to scaling in each factor, the metric induced by the Killing form equals the original one. We use [H], chapter V, §6, as a reference for this section.

DEFINITION 1.3 *The* rank *of X is the dimension of a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and is denoted by rank(X). We will abbreviate this number by the integer r.*

DEFINITION 1.4 *An l-*flat *in X is an isometric imbedding of $\mathbb{R}^l$ into X, where $l \leq r =$ rank(X). An r-flat is called a* maximal flat *or simply a* flat.

Maximal flats are G-translates of the set Ax_0, where $A = e^{\mathfrak{a}}$ is a maximal abelian subgroup of G, and 1-flats are geodesics. If $\mathrm{rank}(X) = 1$, flats coincide with geodesics.

DEFINITION 1.5 *A geodesic is called* regular *if it lies in exactly one flat, otherwise it is called* singular*. A vector $Y \in \mathfrak{p}$ is regular, if the geodesic $e^{tY}x_0$ is regular, and singular otherwise.*

Let $\mathfrak{a}^{sing}$ denote the subset of singular vectors in $\mathfrak{a}$. In rank one spaces, all geodesics are regular, and therefore $\mathfrak{a}^{sing} = \{0\}$.

DEFINITION 1.6 *An open* Weyl chamber $\mathfrak{a}^+$ *in $\mathfrak{a}$ is a connected component of $\mathfrak{a} \setminus \mathfrak{a}^{sing}$. An open Weyl chamber in X is a G-translate of the set $e^{\mathfrak{a}^+}x_0$.*

The Weyl group W *of the pair $(\mathfrak{g}, \mathfrak{a})$ is the finite group generated by reflections at the hyperplanes which bound a Weyl chamber $\mathfrak{a}^+$ in $\mathfrak{a}$.*

Note that the Weyl group is independent of the choice of $\mathfrak{a}^+$ and acts simply transitively on the set of Weyl chambers in $\mathfrak{a}$. It is isomorphic to the quotient M^*/M, where M^* denotes the normalizer and M the centralizer of $\mathfrak{a}$ in K.

Let w_* be the unique element in W which maps the set $\mathfrak{a}^+$ to

$$-\mathfrak{a}^+ := \{-H \in \mathfrak{a} \mid H \in \mathfrak{a}^+\}\,.$$

We have $w_* \cdot w_* = \mathrm{id}$, and put $\overline{\mathfrak{a}_1^+} = \overline{\mathfrak{a}^+} \cap \mathfrak{a}_1$, where $\overline{\mathfrak{a}^+}$ denotes the closure of $\mathfrak{a}^+$ in $\mathfrak{a}$ and $\mathfrak{a}_1 \subset \mathfrak{a}$ the subset of elements of unit length.

DEFINITION 1.7 *The* opposition involution ι *is defined as the map*

$$\begin{array}{rccc} \iota : & \overline{\mathfrak{a}_1^+} & \to & \overline{\mathfrak{a}_1^+} \\ & H & \mapsto & -\mathrm{Ad}(w_*)H\,, \end{array}$$

We remark that $\iota = \mathrm{id}$ if and only if $w_* = -\mathrm{id}$.

1.3 Root spaces and the Iwasawa decomposition

For this section, we refer the reader to [H], chapter III, §4.

DEFINITION 1.8 *A* root *of the pair $(\mathfrak{g}, \mathfrak{a})$ is a nontrivial linear form α on $\mathfrak{a}$, for which the subspace $\mathfrak{g}_\alpha = \{Z \in \mathfrak{g} \mid \mathrm{ad}(H)(Z) = \alpha(H)Z\} \neq \{0\}$. $\mathfrak{g}_\alpha$ is called a* root space.

Since for any choice of maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ the operators $\{\mathrm{ad}(H) \mid H \in \mathfrak{a}\}$ commute and are selfadjoint, we obtain the root space decomposition of $\mathfrak{g}$ in simultaneous eigenspaces (see [H], chapter III, §4)

$$\mathfrak{g} = \mathfrak{g}_0 \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha\,.$$

Here Σ denotes the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$, $\mathfrak{g}_0 = \{Z \in \mathfrak{g} \mid \mathrm{ad}(H)(Z) = 0\}$.

The choice of a Weyl chamber $\mathfrak{a}^+$ further determines subsets Σ^+ and Σ^- of Σ by

$$\alpha \in \Sigma^{\pm} \iff \pm\alpha(H) > 0 \quad \forall\, H \in \mathfrak{a}^+ .$$

An element in Σ^+ will be called a positive root.

The set of roots Σ contains a fundamental set $\Upsilon = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ with cardinality $\#\Upsilon = \mathrm{rank}(X) = r$ and the following properties. Υ is a basis of the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$, no $\alpha_i \in \Upsilon$ can be written as a sum of positive roots, and any root $\alpha \in \Sigma$ can be written in the form

$$\alpha = \sum_{i=1}^{r} n_i \alpha_i \qquad \text{or} \qquad \alpha = -\sum_{i=1}^{r} n_i \alpha_i$$

with integers $n_i \geq 0$. The roots $\alpha_1, \alpha_2, \ldots, \alpha_r$ are called simple roots of Σ.

The barycenter of a Weyl chamber $\mathfrak{a}^+$ is the unique direction $H_* \in \mathfrak{a}_1^+$ such that for any fundamental set of roots $\Upsilon = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ we have $\alpha_i(H_*) = \alpha_j(H_*)$ for all $1 \leq i, j \leq r$. If $w_* \neq -\mathrm{id}$, then H_* is the unique fixed point of ι.

From the sets $\Sigma^{\pm}$ we obtain the Lie algebras

$$\mathfrak{n}^{\pm} = \bigoplus_{\alpha \in \Sigma^{\pm}} \mathfrak{g}_\alpha ,$$

which are nilpotent, because $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ for $\alpha, \beta \in \Sigma$. We will call the corresponding nilpotent Lie groups $N^{\pm} := e^{\mathfrak{n}^{\pm}} \subset G$.

THEOREM 1.9 *([H], chapter IX, Theorem 1.3)*
Let G be a connected semisimple Lie group. Then the map

$$\begin{array}{rcl} N^+ \times A \times K & \to & G \\ (n, a, k) & \mapsto & n{\cdot}a{\cdot}k \end{array}$$

is a diffeomorphism onto G. It is called the Iwasawa decomposition *of G.*

If X is a globally symmetric space and $G = \mathrm{Isom}^o(X)$, then an Iwasawa decomposition $G = N^+AK$ gives rise to a diffeomorphism $X \cong N^+Ax_0$, where $x_0 \in X$ is the unique point stabilized by the maximal compact subgroup $K \subset G$. We therefore obtain horospherical coordinates

$$\begin{array}{rcl} X & \to & N^+ \times \mathfrak{a} \\ x = ne^H x_0 & \mapsto & (n, H) . \end{array}$$

Here $n \in N^+$ is called the horospherical projection, and $H \in \mathfrak{a}$ the Iwasawa projection of $x \in X$.

The map

$$\begin{array}{rrcl} \mu : & N^+A & \to & X \\ & n{\cdot}a & \mapsto & nax_0 \end{array}$$

is a diffeomorphism from the solvable subgroup $N^+A \subset G$ onto X. Let da^2 denote the left invariant scalar product $\langle\cdot,\cdot\rangle$ from section 1.1 on $\mathfrak{a}$, and h_α the left invariant scalar product on $N^+ = \exp(\sum_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha)$ which equals $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}_\alpha$ and is zero on $\mathfrak{g}_\beta$ for $\beta \neq \alpha$. The following formula follows directly from Proposition 4.3 in [B].

PROPOSITION 1.10 *The pullback metric $\mu^* g$ of the Riemannian metric g on X to the solvable subgroup N^+A of G is given by*

$$ds^2_{(n,a)} = \frac{1}{2}\sum_{\alpha\in\Sigma^+} e^{-2\alpha(\log a)}\, h_\alpha \oplus da^2 \,.$$

Here $\log(a)$ *denotes the unique element in* $\mathfrak{a}$ *with the property* $\exp(\log a) = a$.

An easy consequence of this formula for the pullback metric is the following

COROLLARY 1.11 *Let $a_1, a_2 \in A$. Then for any $n \in N^+ \setminus \{\mathrm{id}\}$ we have*

$$d(na_1x_0, a_2x_0) > d(a_1x_0, a_2x_0)\,.$$

We will call the orbits N^+y, $y \in X$, horocycles in X through y. If $y = n_0e^{H_0}x_0$ in horospherical coordinates, then $N^+y = N^+e^{H_0}x_0$. The metric $ds^2_{H_0}$ of the submanifold $N^+e^{H_0}x_0$ induced from the Riemannian metric of X can then be written as

$$ds^2_{H_0} = \frac{1}{2}\sum_{\alpha\in\Sigma^+} e^{-2\alpha(H_0)}\, h_\alpha \,.$$

Note that we also obtain a Riemannian distance d_N on the nilpotent Lie group N^+ from the left invariant metric

$$dn^2 := \frac{1}{2}\sum_{\alpha\in\Sigma^+} h_\alpha \,.$$

Due to the fact that $ds^2_0 = \frac{1}{2}\sum_{\alpha\in\Sigma^+} h_\alpha$, we have for all $H \in \mathfrak{a}$ and for all $n \in N^+$

$$d(ne^Hx_0, e^Hx_0) \le \max_{\alpha\in\Sigma^+} e^{-\alpha(H)} d_N(n, \mathrm{id})\,.$$

1.4 The Cartan decomposition

Let X be a globally symmetric space, $G = \mathrm{Isom}^o(X)$ and $K \subset G$ a maximal compact subgroup which stabilizes a unique point x_0 in X. We call $x_0 \in X$ the base point of $X = G/K$ corresponding to K. Further let $\mathfrak{a}^+ \subset \mathfrak{p} \cong T_{x_0}X$ be an open Weyl chamber. The following decomposition is called the Cartan decomposition of G.

THEOREM 1.12 *([H], chapter IX, Theorem 1.1)*
We have $G = Ke^{\overline{\mathfrak{a}^+}}K$, i.e. each $g \in G$ can be written $g = ke^Hk'$ where $k, k' \in K$ and $H \in \overline{\mathfrak{a}^+}$. Moreover, $H = H(g)$ is unique.

As a consequence, we obtain a surjective map

$$\begin{array}{rrcl} \Phi_0 : & K \times \overline{\mathfrak{a}^+} & \to & X \\ & (k,H) & \mapsto & ke^H x_0 \,. \end{array}$$

If $x \in X$, there exists a unique element $H \in \overline{\mathfrak{a}^+}$ such that $x = ke^H x_0 = \Phi_0(k,H)$ for some $k \in K$. We will call H the Cartan projection of x.

We remark that the map Φ_0 is not injective, because (k_1,H) and (k_2,H) have the same image in X if and only if $k_1^{-1}k_2$ belongs to the centralizer of H in K. If $H \in \overline{\mathfrak{a}^+}$ is the Cartan projection of a point $x \in X$, then every element $k \in K$ with the property $x = ke^H x_0$ will be called an angular projection of x.

Due to [H], chapter IX, Corollary 1.2, the map

$$\begin{array}{rrcl} \Phi : & K/M \times \mathfrak{a}^+ & \to & Y \\ & (kM,H) & \mapsto & ke^H x_0 \end{array}$$

is a diffeomorphism onto a dense open submanifold $Y \subset X$.

Proposition 1.13 *(see [L], section 3)*
If $\tau : G \times G/K \to G/K$ denotes the natural action of G on the space G/K, and $d\tau$ its differential, then the pullback metric $\Phi^* g$ of the Riemannian metric g on $Y \subset G/K$ to $K/M \times \mathfrak{a}^+$ is given by

$$\Phi^* g_{(k(s)M,H(s))}\left((d\tau(k(s))Z_1,H_1),(d\tau(k(s))Z_2,H_2)\right) = \langle H_1,H_2\rangle + \langle \sinh \mathrm{ad} H(s) Z_1, \sinh \mathrm{ad} H(s) Z_2\rangle \,.$$

1.5 Geometric estimates

In this section, we prove a few useful geometric estimates for $X = G/K$ in terms of a given Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$. The following lemma can be found in [H].

Lemma 1.14 *Let $H_1, H_2 \in \overline{\mathfrak{a}^+}$. Then for any $k \in K$ we have*

$$\langle \mathrm{Ad}(k)H_1, H_2\rangle \le \langle H_1, H_2\rangle \,.$$

Proof. Consider the map $f : K \to \mathbb{R}$ defined by $f(k) = \langle \mathrm{Ad}(k)H_1, H_2\rangle$ and suppose k_0 is a critical point of f. Then

$$\begin{array}{rcl} \forall Z \in \mathfrak{k} & & \left.\frac{d}{dt}\right|_{t=0} f(k_0e^{tZ}) = 0 \quad \Longleftrightarrow \quad \forall Z \in \mathfrak{k} : \\ 0 & = & \left.\frac{d}{dt}\right|_{t=0} \langle \mathrm{Ad}(k_0e^{tZ})H_1, H_2\rangle = \left.\frac{d}{dt}\right|_{t=0} B(\mathrm{Ad}(k_0)\mathrm{Ad}(e^{tZ})H_1, H_2) \\ & = & B(\mathrm{Ad}(k_0)\left.\frac{d}{dt}\right|_{t=0}(\mathrm{Ad}(e^{tZ})H_1), H_2) \\ & = & B(\mathrm{Ad}(k_0)(\mathrm{ad}Z)H_1, H_2) = B(\mathrm{Ad}(k_0)[Z,H_1], H_2) \\ & = & B(\mathrm{Ad}(k_0)Z, [\mathrm{Ad}(k_0)H_1, H_2]) \le 0 \end{array}$$

since both $\mathrm{Ad}(k_0)Z$ and $[\mathrm{Ad}(k_0)H_1, H_2]$ belong to $\mathfrak{k}$ and the restriction of the Killing form B to the compact Lie algebra $\mathfrak{k}$ is negative definite. Now

$$\frac{d}{dt}\Big|_{t=0} f(k_0 e^{tZ}) = 0 \qquad \forall\, Z \in \mathfrak{k}$$

implies $[\mathrm{Ad}(k_0)H_1, H_2] = 0$, which means that $\mathrm{Ad}(k_0)H_1$ is contained in the normalizer of H_2 in $\mathfrak{p}$. Therefore $k_0 w \in \{k \in K \mid \mathrm{Ad}(k)H_2 = H_2\}$ for some $w \in W$, and we conclude $\langle \mathrm{Ad}(k_0)H_1, H_2\rangle = \langle \mathrm{Ad}(w^{-1})H_1, \mathrm{Ad}((k_0w)^{-1})H_2\rangle = \langle \mathrm{Ad}(w^{-1})H_1, H_2\rangle$.

Since $H_1, H_2 \in \overline{\mathfrak{a}^+}$, and the Weyl group W is generated by reflections at the hyperplanes which bound $\mathfrak{a}^+$, this number is maximal for $w = \mathrm{id}$. Hence f assumes its maximum value for $k \in \{k \in K \mid \mathrm{Ad}(k)H_2 = H_2\}$. □

Let $x_0 \in X$ denote the base point of $X = G/K$ corresponding to K, and θ the Cartan involution introduced in section 1.1. From the previous lemma, we deduce

Corollary 1.15 *Let $H_1, H_2 \in \overline{\mathfrak{a}^+}$. Then for all $k \in K$ we have*

$$d(ke^{H_1}x_0, e^{H_2}x_0) \ge d(e^{H_1}x_0, e^{H_2}x_0)\,.$$

Proof. Let $k \in K$ arbitrary and put $\alpha(k) := \angle_{x_0}\left(ke^{H_1}x_0, e^{H_2}x_0\right)$. By the previous lemma we have

$$\|H_1\|\|H_2\|\cos\alpha(k) = \langle \mathrm{Ad}(k)H_1, H_2\rangle \le \langle H_1, H_2\rangle\,. \tag{1.1}$$

Comparison of the hinge $\angle_{x_0}\left(ke^{H_1}x_0, e^{H_2}x_0\right)$ with the corresponding one in the Euclidian plane of the same angle and sidelengths yields

$$\begin{aligned} d(ke^{H_1}x_0, e^{H_2}x_0)^2 &\ge d(x_0, ke^{H_1}x_0)^2 + d(x_0, e^{H_2}x_0)^2 - 2d(x_0, ke^{H_1}x_0)d(x_0, e^{H_2}x_0)\cos\alpha(k) \\ &= \|H_1\|^2 + \|H_2\|^2 - 2\|H_1\|\|H_2\|\cos\alpha(k)\,. \end{aligned}$$

Since the geodesic triangle $\triangle(x_0, e^{H_1}x_0, e^{H_2}x_0)$ is contained in a totally geodesic subspace of X isometric to the Euclidian plane, we conclude

$$\begin{aligned} d(e^{H_1}x_0, e^{H_2}x_0)^2 &= \|H_1 - H_2\|^2 = \|H_1\|^2 + \|H_2\|^2 - 2\langle H_1, H_2\rangle \\ &\overset{(1.1)}{\le} \|H_1\|^2 + \|H_2\|^2 - 2\langle \mathrm{Ad}(k)H_1, H_2\rangle = \|H_1\|^2 + \|H_2\|^2 \\ &\quad -2\|H_1\|\|H_2\|\cos\alpha(k) \le d(ke^{H_1}x_0, e^{H_2}x_0)^2\,. \end{aligned}$$ □

The following estimate will be needed in section 7.5 to measure the Hausdorff dimension.

Lemma 1.16 *Fix $c > 0$, $H_0 \in \mathfrak{a}_1^+$ and put $A_0 := \max\{\|\alpha\|/\alpha(H_0) \mid \alpha \in \Sigma^+\}$, where $\|\alpha\|$ denotes the operator norm of $\alpha \in \mathfrak{a}^*$. Then there exists $\varphi_0 = \varphi(H_0) \in (0, \pi/4)$ and $R_0 = R_0(H_0, c) > 0$ such that the following holds:*

If $\varphi \in (0, \varphi_0]$, $H \in \overline{\mathfrak{a}_1^+}$ with $\angle(H, H_0) < \varphi$, $t \ge R_0$ and $t_0 := t(\cos\varphi - A_0\sin\varphi) - 2A_0c$, then for any $k \in K$

$$d(ke^{Ht}x_0, e^{\mathfrak{a}^+}x_0) \le c \qquad \Longrightarrow \qquad d(ke^{t_0H_0}x_0, e^{t_0H_0}x_0) \le c\,.$$

Proof. Put $\varphi_0 := (2A_0)^{-1} = \frac{1}{2}\min\{\alpha(H_0)/\|\alpha\| \mid \alpha \in \Sigma^+\}$, let $\varphi \in (0, \varphi_0]$ and $H \in \overline{\mathfrak{a}_1^+}$ with $\angle(H, H_0) < \varphi \leq \varphi_0$. Then $\|H - H_0\|^2 = 2 - 2\langle H, H_0\rangle < \varphi^2$, and by the Cauchy Schwartz inequality

$$\alpha(H) = \alpha(H_0) + \alpha(H - H_0) > \alpha(H_0) - \|\alpha\|\varphi_0 \geq \alpha(H_0)/2 > 0 \tag{1.2}$$

for any $\alpha \in \Sigma^+$. We conclude $H \in \mathfrak{a}_1^+$.

Let $\varepsilon \in (0, c)$ and $t > 2c$. By the assumption, there exists a curve $\kappa : [0,1] \to X$ with $\kappa(0) \in e^{\mathfrak{a}^+}$, $\kappa(1) = ke^{Ht}x_0$, and

$$\int_0^1 \|\dot{\kappa}(s)\| ds \leq c + \varepsilon < 2c\,. \tag{1.3}$$

For $s \in [0,1]$ we write $\kappa(s) = k(s)e^{H(s)}x_0$ using the Cartan decomposition. By Corollary 1.15, we have

$$2c > d(ke^{Ht}x_0, \kappa(s)) \geq d(e^{Ht}x_0, e^{H(s)}x_0) = \|Ht - H(s)\|\,,$$

and again by the Cauchy Schwartz inequality and (1.2)

$$\begin{aligned} \alpha(H(s)) &\geq \alpha(H_0t) + \alpha(Ht - H_0t) + \alpha(H(s) - Ht) \\ &\geq t\alpha(H_0) - t\|\alpha\|\varphi - \|\alpha\|2c \geq t\alpha(H_0)/2 - 2c\|\alpha\| \end{aligned}$$

for any $\alpha \in \Sigma^+$. This shows that if $t \geq R_0 := \max\{4c\|\alpha\|/\alpha(H_0) \mid \alpha \in \Sigma^+\}$, then $H(s)$ belongs to $\mathfrak{a}^+$ for all $s \in [0,1]$.

Recall from the previous section that $\tau : G \times G/K \to G/K$ denotes the natural action of G on the space G/K, and $d\tau$ its differential. For $t \geq R_0$ and $s \in [0,1]$ let $Z(s) \in \mathfrak{k}/\mathfrak{m}$ such that

$$\frac{d}{d\sigma}\Big|_{\sigma=s} k(\sigma) = d\tau(k(s))Z(s)\,,$$

and write

$$Z(s) = \sum_{\alpha\in\Sigma^+} \left(Z_\alpha(s) + \theta Z_\alpha(s)\right)\,, \qquad Z_\alpha(s) \in \mathfrak{g}_\alpha\,.$$

Using the formula for the induced metric on $K/M \times \mathfrak{a}^+$ given in Proposition 1.13, we estimate for $s \in [0,1]$

$$2\sum_{\alpha\in\Sigma^+} \sinh^2\alpha(H(s))\langle Z_\alpha(s), Z_\alpha(s)\rangle \leq \|\dot{\kappa}(s)\|^2\,. \tag{1.4}$$

Put $t_0 := t(\cos\varphi - A_0\sin\varphi) - 2A_0c$ and let $s \in [0,1]$. We compute for all $\alpha \in \Sigma^+$

$$\begin{aligned} \alpha(H(s)) &\geq \alpha(Ht) + \alpha(H(s) - Ht) > t\alpha(H) - \|\alpha\|2c \\ &\geq t\langle H, H_0\rangle\alpha(H_0) - t\sin\varphi\|\alpha\| - \|\alpha\|2c \\ &\geq \alpha(H_0)\left(t\langle H, H_0\rangle - t\sin\varphi A_0 - A_02c\right) \\ &\geq \alpha(H_0)\left(t(\cos\varphi - A_0\sin\varphi) - 2A_0c\right) = t_0\alpha(H_0)\,. \end{aligned}$$

The curve defined by $c(s) = k(s)e^{H_0 t_0}x_0$ with $k(s)$ as above joins $e^{H_0 t_0}x_0$ to $ke^{H_0 t_0}x_0$, and for $s \in [0,1]$ we calculate using (1.4)

$$\begin{aligned} \|\dot{c}(s)\|^2 &= 2\sum_{\alpha\in\Sigma^+} \sinh^2\left(t_0\alpha(H_0)\right)\langle Z_\alpha(s), Z_\alpha(s)\rangle \\ &\le 2\sum_{\alpha\in\Sigma^+} \sinh^2\alpha(H(s))\langle Z_\alpha(s), Z_\alpha(s)\rangle \le \|\dot{\kappa}(s)\|^2 . \end{aligned}$$

Hence $d(ke^{H_0 t_0}x_0, e^{H_0 t_0}x_0) \le \int_0^1 \|\dot{c}(s)\|ds \le \int_0^1 \|\dot{\kappa}(s)\|ds < c + \varepsilon$, and the assertion follows as $\varepsilon \searrow 0$. □

1.6 The coset space $SL(n,\mathbb{R})/SO(n)$

For each integer $n \ge 2$, the Lie group $G = SL(n,\mathbb{R})$ is simple with center $\{\mathrm{id}, -\mathrm{id}\}$, and $K = SO(n)$ is a maximal compact subgroup with Lie algebra $\mathfrak{k}$. Furthermore, the involutive isomorphism θ of the Lie algebra $\mathfrak{g}$ of $SL(n,\mathbb{R})$ defined by $X \mapsto -X^t$ satisfies $\theta\mathfrak{k} = \mathfrak{k}$. The Cartan decomposition is given by the direct sum of $\mathfrak{k}$ and the vector subspace $\mathfrak{p}$ consisting of symmetric (n,n)-matrices with trace equal to zero. The coset space $SL(n,\mathbb{R})/SO(n)$ endowed with the left invariant metric determined by

$$\langle X_1, X_2\rangle := \mathrm{Tr}(X_1X_2) = -\frac{1}{2n}B(X_1, \theta(X_2)), \quad X_1, X_2 \in \mathfrak{p}$$

is a globally symmetric space of noncompact type. Furthermore, by the Imbedding Theorem 2.6.5 in [E], every globally symmetric space X of noncompact type can be imbedded isometrically up to scaling on each irreducible De Rham factor in $SL(n,\mathbb{R})/SO(n)$ as a totally geodesic submanifold, where n equals the dimension of $\mathrm{Isom}^o(X)$.

An abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is given by the diagonal matrices with trace zero, which shows that the rank of $SL(n,\mathbb{R})/SO(n)$ is $r = n-1$. The standard choice of positive Weyl chamber is

$$\mathfrak{a}^+ := \{\mathrm{Diag}(t_1, t_2, \ldots, t_n) \mid t_1 > t_2 > \ldots > t_n, \sum_{i=1}^n t_i = 0\} .$$

Identifying the dual space $\mathfrak{a}^*$ with $\mathfrak{a}$, the l-th simple root α_l is given by

$$\alpha_l = \mathrm{Diag}(d_1, d_2, \ldots, d_n), \quad \text{where} \quad d_i = \delta_{il} - \delta_{i,l+1}, \ 1 \le i \le n .$$

The remaining roots occuring in Σ^+ are given by all possible linear combinations

$$\sum_{i=j}^{j+k} \alpha_i, \quad \text{where} \quad 1 \le j \le r-1, \ 1 \le k \le r-j .$$

Let ρ denote the half sum of positive roots. In $SL(n,\mathbb{R})/SO(n)$ we may write ρ in terms of the simple roots

$$2\rho = \sum_{j=1}^{r-1}\sum_{k=0}^{r-j}\sum_{i=j}^{j+k} \alpha_i .$$

Chapter 2

The sphere at infinity

In this chapter, we discuss the structure at infinity of globally symmetric spaces of noncompact type. Such spaces are in particular Hadamard manifolds, i.e. complete, simply connected Riemannian manifolds of nonpositive sectional curvature. We first describe a standard method to compactify Hadamard manifolds following reference [Ba], chapter II, and then proceed by explaining different aspects occurring in the special case of symmetric spaces. Our main references will be [BGS], appendix 5, [H], chapter IX, and [W], chapter 1.2.

2.1 The geometric boundary

Let M be a complete Riemannian manifold. The restriction of a geodesic $\sigma : \mathbb{R} \to M$ to the intervall $[0, \infty)$ is called a geodesic ray with initial point $\sigma(0)$.

DEFINITION 2.1 *The* geometric boundary ∂X *of a globally symmetric space* X *of noncompact type is defined as the set of equivalence classes of geodesic rays under the equivalence relation*

$$\sigma_1 \sim \sigma_2 :\Longleftrightarrow \quad d(\sigma_1(t), \sigma_2(t)) \quad \textit{bounded}.$$

An equivalence class will be denoted by $\sigma(\infty)$.

In order to topologize the space $\overline{X} := X \cup \partial X$, we introduce the following sets. For $\varepsilon > 0$, $R >> 1$, $x \in X$ and $\eta \in \partial X$ let $\mathrm{C}_{x,\eta}^{R,\varepsilon} \subset \overline{X}$ be the truncated cone

$$\mathrm{C}_{x,\eta}^{R,\varepsilon} := \{y \in \overline{X} \mid d(x,y) > R\,,\ d(\sigma_{x,\eta}(R), \sigma_{x,y}(R)) < \varepsilon\}$$

in $\overline{X}$, where $\sigma_{x,y}$ denotes the unique unit speed geodesic emanating from $x \in X$ passing through $y \in \overline{X}$.

DEFINITION 2.2 *([Ba], chapter II)*
The cone topology *on* $\overline{X}$ *is the topology generated by the open sets in* X *and these truncated cones.*

If not stated otherwise, convergence in $X \cup \partial X$ will always mean convergence with respect to the cone topology. The relative topology on ∂X turns the geometric boundary into a topological space.

The action of an isometry $g \in \mathrm{Isom}(X)$ of X extends naturally to a homeomorphism on the geometric boundary, since isometries map geodesic rays to geodesic rays.

The geometric boundary is homeomorphic to the unit tangent space of an arbitrary point in X. We remark that every Hadamard manifold can be compactified in such a way.

Let $X = G/K$ be a globally symmetric space of noncompact type with base point x_0 corresponding to K. If a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ is fixed, the writing (k, H) with $k \in K$ and $H \in \overline{\mathfrak{a}_1^+}$ will denote the equivalence class of geodesic rays which contains the geodesic ray σ given by $\sigma(t) = ke^{Ht}x_0 = \Phi_0(k, Ht)$, $t \geq 0$. We remark that by the properties of the map Φ_0 defined in section 1.4, $(k_1, H) = (k_2, H)$ in ∂X if and only if $k_1^{-1}k_2$ belongs to the centralizer of H in K.

For every $\xi \in \partial X$, however, there exists exactly one element $H = H_\xi \in \overline{\mathfrak{a}_1^+}$ which we will call the Cartan projection of ξ. If $k \in K$ is an element with the property $\xi = (k, H_\xi)$, we will call k an angular projection of ξ. Note that the action of $G = \mathrm{Isom}^o(X)$ on the geometric boundary leaves invariant the Cartan projections of boundary points.

If $\mathrm{rank}(X) \geq 2$, then $\overline{\mathfrak{a}^+}$ consists of a regular and a singular part. The geometric boundary therefore decomposes into a disjoint union

$$\partial X = \partial X^{reg} \cup \partial X^{sing}\,.$$

We will give a precise description of the singular boundary in section 2.5.

2.2 The Furstenberg boundary

For the remainder of this chapter, $X = G/K$ will be a globally symmetric space of noncompact type. Note that the writing $X = G/K$ depends on a base point $x_0 \in X$, the unique point in X stabilized by K. The following statements, however, remain true for any choice of base point in X, because its stabilizer is then simply a conjugate of K in G.

Definition 2.3 *Two Weyl chambers in X are called* asymptotic *if their Hausdorff distance is bounded.*

The Furstenberg boundary *∂X^F of X is defined as the set of equivalence classes of asymptotic Weyl chambers.*

If $G = Ke^{\overline{\mathfrak{a}^+}}K$ is a Cartan decompositon for $X = G/K$, and M denotes the centralizer of $\mathfrak{a}$ in K, we may identify the Furstenberg boundary with the homogeneous space K/M using the projection

$$\begin{array}{cccc} \pi^B: & \partial X^{reg} & \to & K/M \\ & (k, H) & \mapsto & kM\,. \end{array}$$

Furthermore, we may endow K/M with the natural differentiable structure arising from the Lie group structure of K. We obtain a topology on the Furstenberg boundary by the requirement that ∂X^F is diffeomorphic to K/M.

We remark that if two boundary points $\xi, \eta \in \partial X$ possess a common angular projection $k \in K$, then ξ and η belong to the closure of every Weyl chamber asymptotic to $kM \in K/M \cong \partial X^F$. We say that ξ and η belong to the closure of a common Weyl chamber at infinity.

Note that for rank one symmetric spaces, the Furstenberg boundary coincides with the geometric boundary as sets. Lemma 2.9 will further show, that in this case the topology on K/M is equivalent to cone topology.

2.3 The Bruhat decomposition

We fix an Iwasawa decomposition $G = N^+AK$ as in section 1.3 and consider the closed subgroup $P = MAN^+ \subset G$. Since P equals the stabilizer of the unique equivalence class of asymptotic Weyl chambers which contains $e^{\mathfrak{a}^+}x_0 \subset \overline{X}$, we may identify the Furstenberg boundary $\partial X^F \cong K/M$ with the homogeneous space G/P under the bijection

$$\begin{array}{rccl} \overline{\kappa}: & G/P & \to & K/M \\ & gP & \mapsto & kM\,. \end{array}$$

Here $kM \in K/M$ is uniquely determined by any element k in the Iwasawa decomposition $g = kan = (n^{-1}a^{-1}k^{-1})^{-1}$.

Definition 2.4 *([W], chapter 1.2) A* minimal parabolic subgroup *of G is a conjugate of the closed subgroup $P = MAN^+$ in G. A* parabolic subgroup *is any subgroup of G containing a minimal parabolic subgroup.*

For a minimal parabolic subgroup P we have the Bruhat decomposition of G as a disjoint union

$$G = \bigcup_{w\in W} N^+m_wP = \bigcup_{w\in W} U_wm_wP\,,$$

where m_w is an arbitrary representative of w in M^*, and the sets U_w are the Lie group exponentials of the subspaces

$$\mathfrak{u}_w := \mathfrak{n}^+ \cap \mathrm{Ad}(m_w)\mathfrak{n}^- \subset \mathfrak{n}^+\,.$$

Note that the orbit corresponding to the element w_* is parametrized by $N^+ = U_{w_*}$, and the restriction of the above bijection $\overline{\kappa}$ to N^+w_*P defines a map

$$\begin{array}{rccl} \kappa: & N^+ & \to & K/M \\ & n & \mapsto & \overline{\kappa}(nw_*P)\,. \end{array}$$

PROPOSITION 2.5 *([H], chapter IX, Corollary 1.9)*
The map κ is a diffeomorphism onto an open submanifold of K/M whose complement consists of finitely many disjoint manifolds of strictly lower dimension.

This proposition implies that the orbit N^+w_*P is a dense and open submanifold of the Furstenberg boundary $\partial X^F \cong K/M$. We will call a G-translate of the set $N^+w_*P \subset G/P$ a big cell in $\partial X^F \cong K/M$ using the diffeomorphism $\overline{\kappa}$.

If $n \in N^+$, $\kappa(n) \in K/M$ is the unique element such that the Weyl chamber $\kappa(n)e^{\mathfrak{a}^+}x_0$ is asymptotic to the Weyl chamber $ne^{-\mathfrak{a}^+}x_0$.

The choice of a subset $\Theta \subseteq \Upsilon$ determines a standard face of type Θ

$$\mathfrak{a}^\Theta := \{H \in \overline{\mathfrak{a}^+} \mid \alpha(H) = 0 \ \ \forall\, \alpha \in \Theta\}$$

in $\mathfrak{a}$. If $\langle\Theta\rangle^\pm \subseteq \Sigma^\pm$ denotes the set of those $\alpha \in \Sigma^\pm$ which are linear combinations of the elements of Θ, then its centralizer in K is a closed subgroup M_Θ with Lie algebra

$$\mathfrak{m}_\Theta := \mathfrak{m} \oplus \sum_{\alpha\in\langle\Theta\rangle^+} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}\,.$$

We have $\mathfrak{a}^\emptyset = \overline{\mathfrak{a}^+}$, $M_\emptyset = M$, and if $\Theta \subset \tilde\Theta$, then $\mathfrak{a}^{\tilde\Theta}$ is a vector subspace of $\mathfrak{a}^\Theta$ and M_Θ is a closed subgroup of $M_{\tilde\Theta}$. If $W_\Theta = M_\Theta \cap W$, the parabolic subgroup $P_\Theta = M_\Theta A N^+$ may be written as

$$P_\Theta = \bigcup_{w\in W_\Theta} Pm_wP\,.$$

P_Θ is minimal if and only if $\Theta = \emptyset$. We remark that for any $H \in \mathfrak{a}_1^\Theta \setminus \bigcup_{\tilde\Theta\supsetneq\Theta} \mathfrak{a}_1^{\tilde\Theta}$, the parabolic subgroup P_Θ equals the stabilizer of $(\mathrm{id}, H) \in \partial X$.

The generalized Bruhat decomposition of G with respect to the parabolic subgroup P_Θ can now be written as a disjoint union

$$G = \bigcup_{w\in W/W_\Theta} N^+ m_w P_\Theta\,.$$

Again, the orbit corresponding to the class $w_*W_\Theta \in W/W_\Theta$ has maximal dimension in G/P_Θ and may be parametrized by

$$N_\Theta^+ = \exp(\mathfrak{n}_\Theta^+)\,, \qquad \text{where} \quad \mathfrak{n}_\Theta^+ := \sum_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \mathfrak{g}_\alpha\,.$$

The G-translates of this orbit are sometimes called Schubert cells in G/P_Θ. Note that for $\Theta = \Upsilon$ the decomposition is trivial because $P_\Upsilon = G$.

2.4 The generalized Iwasawa decomposition

Let X be a globally symmetric space of noncompact type, $G = \mathrm{Isom}^o(X)$ and $G = N^+AK$ an Iwasawa decomposition for G. The goal of this section is to generalize the well known decomposition result of Iwasawa described in section 1.3.

For any subset $\Theta \subset \Upsilon$, we introduce the homomorphisms

$$\begin{array}{rccl} T_\Theta : & P_\Theta & \to & G \\ & g & \mapsto & \lim_{t\to\infty} e^{-Ht} g e^{Ht}\,, \quad H \in \mathfrak{a}^\Theta \text{ arbitrary}\,, \end{array}$$

where $\Theta \subset \Upsilon$. These homomorphisms exist by the following proposition, are well defined and have kernel $N_\Theta^+ \subseteq N^+$ as may be easily checked.

Proposition 2.6 *([E], Proposition 2.17.3)*
Fix $\Theta \subset \Upsilon$, let $H \in \mathfrak{a}_\Theta$. Then $g \in P_\Theta$ if and only if T_Θ exists in G.

Using the notation $Z_\Theta := \{g \in G \mid \mathrm{Ad}(g)H = H\}$ for arbitrary $H \in \mathfrak{a}^\Theta$,
$\mathfrak{p}_\Theta := \mathfrak{a} \oplus \sum_{\alpha\in\langle\Theta\rangle^+} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{p}$, we restate

Proposition 2.7 *([E], Proposition 2.17.5)*

1. *N_Θ^+ is a connected normal subgroup of P_Θ.*
2. *T_Θ has image $Z_\Theta \subset P_\Theta$.*
3. *T_Θ fixes every element of Z_Θ. Moreover, $Z_\Theta = M_\Theta \cdot \exp(\mathfrak{p}_\Theta) = \exp(\mathfrak{p}_\Theta) \cdot M_\Theta$ and the decomposition of an element Z_Θ into a product $h \cdot m$, where $h \in \exp(\mathfrak{p}_\Theta)$ and $m \in M_\Theta$, is unique.*
4. *$P_\Theta = M_\Theta \exp(\mathfrak{p}_\Theta) N_\Theta^+ = M_\Theta N_\Theta^+ \exp(\mathfrak{p}_\Theta)$. The indicated decomposition of elements of P_Θ is unique.*
5. *We have the generalized Iwasawa decomposition $G = K \exp(\mathfrak{p}_\Theta) N_\Theta^+$. The indicated decomposition of elements of G is unique.*
6. *Z_Θ has finitely many components.*

We remark that Theorem 1.2.4.11 in [W] gives a slightly different version of the generalized Iwasawa decomposition.

2.5 The structure of the boundary

Let $X = G/K$ be a globally symmetric space of noncompact type with base point $x_0 \in X$ corresponding to K, and fix a Cartan decomposition $G = K e^{\overline{\mathfrak{a}^+}} K$. Then every subset $\Theta \subset \Upsilon$ determines a homogeneous space K/M_Θ, which, as in the case $\Theta = \emptyset$, can be endowed with the natural differentiable structure arising from the Lie group structure of K. Recall from section 2.3 that we also obtain a standard face of type Θ, $\mathfrak{a}^\Theta \subset \overline{\mathfrak{a}^+}$. Putting

$$\partial X^\Theta = \{(k, H) \mid k \in K\,,\ H \in \mathfrak{a}_1^\Theta \setminus \bigcup_{\tilde\Theta \supsetneq \Theta} \mathfrak{a}_1^{\tilde\Theta}\}\,,$$

the geometric boundary decomposes into a disjoint union

$$\partial X = \bigcup_{\Theta \subset \Upsilon} \partial X^{\Theta} .$$

Note that $\partial X^{\emptyset} = \partial X^{reg}$ and $\partial X^{\Upsilon} = \emptyset$.

This decomposition allows to extend the projection π^B from the regular boundary ∂X^{reg} to the whole geometric boundary by the requirement that for any subset $\Theta \subset \Upsilon$

$$\pi^B\Big|_{\partial X^{\Theta}} : \quad \begin{array}{rcl} \partial X^{\Theta} & \to & K/M_{\Theta} \\ (k,H) & \mapsto & kM_{\Theta} . \end{array}$$

We remark that for $k_0 M_\Theta \in K/M_\Theta$ every preimage $(\pi^B)^{-1}(k_0 M_\Theta)$ is of the form $(k,H) \in \partial X^{\Theta}$ with $k^{-1}k_0 \in M_\Theta$ and $H \in \mathfrak{a}_1^{\Theta} \setminus \bigcup_{\tilde\Theta \supsetneq \Theta} \mathfrak{a}_1^{\tilde\Theta}$. The natural G-action on K/M_Θ is given by

$$g(kM_\Theta) := \pi^B \circ g \circ (\pi^B)^{-1}(kM_\Theta) , \;\; g \in G , \;\; kM_\Theta \in K/M_\Theta . \tag{2.1}$$

Note that the action does not depend on the choice of preimage of $(\pi^B)^{-1}(kM_\Theta) \in \partial X^{\Theta}$, because $(k,H) = (k',H) \in \partial X^{\Theta}$ if and only if $k^{-1}k' \in M_\Theta$, and the Cartan projections remain unchanged by the action of G.

The following equivalence will be useful in comparing the cone topology of $\overline{X} = X \cup \partial X$ with the topology on K/M_Θ, $\Theta \subset \Upsilon$. By abuse of notation, $k_j \to k_0$ in K/M_Θ wil+l mean that the cosets $k_j M_\Theta$ converge to the coset $k_0 M_\Theta \in K/M_\Theta$.

LEMMA 2.8 *Let $\Theta \subset \Upsilon$, $H \in \mathfrak{a}_1^{\Theta}$ and (k_j) a sequence in K. Then $k_j \to \mathrm{id}$ in K/M_Θ if and only if $d(k_j e^H x_0, e^H x_0)$ tends to zero.*

Proof. Fix $\Theta \subset \Upsilon$, $H \in \mathfrak{a}_1^{\Theta}$. Suppose $k_j \to \mathrm{id}$ in K/M_Θ. By definition of the quotient topology, there exists a sequence $(m_j) \subset M_\Theta$ such that $k_j m_j$ converges to $\mathrm{id} \in K$. If

$$\mathfrak{h} := \sum_{\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{k} ,$$

we have the direct sum decomposition $\mathfrak{k} = \mathfrak{m}_\Theta \oplus \mathfrak{h}$, where $\mathfrak{m}_\Theta$ denotes the Lie algebra of M_Θ. Since the exponential map is a local diffeomorphism, there exist sequences $(Z_j) \subset \mathfrak{m}_\Theta$, $(Y_j) \subset \mathfrak{h}$ such that $k_j m_j = \exp(Z_j + Y_j)$ and $(Z_j + Y_j) \to 0 \in \mathfrak{k}$ as $j \to \infty$. We write $Y_j := \sum_{\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+} \left(Z_\alpha^{(j)} + \theta Z_\alpha^{(j)} \right)$, where $Z_\alpha^{(j)} \in \mathfrak{g}_\alpha$ and θ denotes the Cartan involution.

For the following estimates we will use the norm $\|\cdot\|$ on $\mathfrak{g}$ induced from the scalar product introduced in section 1.1. Since the spaces $\mathfrak{m}_\Theta$, $(\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}$, $\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$, are orthogonal to each other with respect to this norm, we conclude $\|Z_j\| \to 0$ and $\|Z_\alpha^{(j)}\|$, $\|\theta Z_\alpha^{(j)}\| \to 0$ for any $\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$. Now $\mathrm{ad}H(Z_j) = 0$ implies

$$X_j := e^{-\mathrm{ad}H}(Z_j + Y_j) = Z_j + e^{-\mathrm{ad}H} Y_j = Z_j + \sum_{\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+} \left(e^{-\alpha(H)} Z_\alpha^{(j)} + e^{\alpha(H)} \theta Z_\alpha^{(j)} \right)$$

and therefore

$$\|X_j\| \le \|Z_j\| + \sum_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \left(e^{-\alpha(H)}\|Z_\alpha^{(j)}\| + e^{\alpha(H)}\|\theta Z_\alpha^{(j)}\|\right) \to 0\,.$$

We conclude

$$d(k_j e^H x_0, e^H x_0) = d(k_j m_j e^H x_0, e^H x_0) = d(e^{-H} k_j m_j e^H x_0, x_0) = d(e^{X_j} x_0, x_0) \to 0\,.$$

Conversely, suppose $d(k_j e^H x_0, e^H x_0)$ tends to zero. For $j \in \mathbb{N}$, let $X_j \in \mathfrak{g}$ such that $e^{-H} k_j e^H = e^{X_j}$ and write

$$k_j = \exp\left(Z_j + \sum_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \left(Z_\alpha^{(j)} + \theta Z_\alpha^{(j)}\right)\right) \qquad \text{with } Z_j \in \mathfrak{m}_\Theta\,,\ Z_\alpha^{(j)} \in \mathfrak{g}_\alpha\,.$$

As before, we have

$$\begin{aligned} X_j &= e^{-\mathrm{ad}H}\left(Z_j + \sum_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \left(Z_\alpha^{(j)} + \theta Z_\alpha^{(j)}\right)\right) = \sum_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \cosh(\alpha(H))\left(Z_\alpha^{(j)} + \theta Z_\alpha^{(j)}\right) \\ &\quad + Z_j - \sum_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \sinh(\alpha(H))\left(Z_\alpha^{(j)} - \theta Z_\alpha^{(j)}\right) \in \mathfrak{k}\oplus\mathfrak{p}\,. \end{aligned}$$

Now $d(e^{X_j}x_0, x_0) \to 0$ implies that X_j converges to an element in $\mathfrak{k}$, i.e. the component of X_j in $\mathfrak{p}$ tends to zero. This component is given by $\sum_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \sinh(\alpha(H))\left(Z_\alpha^{(j)} - \theta Z_\alpha^{(j)}\right)$. Since the root spaces are orthogonal and $\alpha(H) > 0$ for any $\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$, we conclude that $\|Z_\alpha^{(j)}\| \to 0$ for any such α. This is equivalent to $k_j \to \mathrm{id}$ in K/M_Θ. □

We are now able to state the following equivalence between the cone topology of $\overline{X}$ and the topology of the differentiable manifold K/M_Θ.

LEMMA 2.9 *Fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ and let $\Theta \subset \Upsilon$. Then $(y_j) \subset X\cup\partial X$ converges to $\eta \in \partial X^\Theta$ in the cone topology if and only if every sequence $(k_j) \subset K$ of angular projections of (y_j) converges to $\pi^B(\eta)$ in K/M_Θ, if the sequence of unit length Cartan projections $(H_j) \subset \overline{\mathfrak{a}_1^+}$ converges to the Cartan projection H_η of η in $\mathfrak{a}$, and if $d(x_0, y_j) \to \infty$ as $j \to \infty$.*

Proof. Suppose $(y_j) \subset X$ converges to $\eta \in \partial X^\Theta$ in the cone topology. For $j \in \mathbb{N}$, let $k_j \in K$ be an angular projection, $H_j \in \overline{\mathfrak{a}_1^+}$ the unit length Cartan projection of y_j. Now $y_j \to \eta$ in the cone topology implies that for $R >> 1$ and $\varepsilon > 0$ arbitrary, there exists $N_0 \in \mathbb{N}$ such that $y_j \in \mathrm{C}_{x_0,\eta}^{R,\varepsilon}$ for $j > N_0$. In particular $d_j := d(x_0, y_j) > R$ for $j > N_0$. If H_η denotes the Cartan projection and $k_\eta \in K$ an angular projection of η, Corollary 1.15 gives

$$\begin{aligned} \varepsilon &> d(\sigma_{x_0,y_j}(R), \sigma_{x_0,\eta}(R)) = d(k_j e^{H_j R} x_0, k_\eta e^{H_\eta R} x_0) \\ &\ge d(e^{H_j R} x_0, e^{H_\eta R} x_0) = R\cdot\|H_j - H_\eta\|\,, \end{aligned}$$

which shows that H_j converges to H_η in $\mathfrak{a}$. We further conclude

$$\begin{aligned} d(k_j e^{H_\eta R}x_0, k_\eta e^{H_\eta R}x_0) &\leq d(k_j e^{H_\eta R}x_0, k_j e^{H_j R}x_0) + d(k_j e^{H_j R}x_0, k_\eta e^{H_\eta R}x_0) \\ &\leq d(e^{H_\eta R}x_0, e^{H_j R}x_0) + d(\sigma_{x_0,y_j}(R), \sigma_{x_0,\eta}(R)) < 2\varepsilon\,. \end{aligned}$$

This proves $k_j^{-1}k_\eta \to \mathrm{id} \in K/M_\Theta$ by the previous lemma.

Conversely, let $(k_j) \subset K$ be a sequence converging to k_η in K/M_Θ, $(H_j) \in \overline{\mathfrak{a}_1^+}$ a sequence converging to H_η in $\mathfrak{a}$ and (d_j) a sequence of positive numbers which tends to infinity. Let $R >> 0$, $\varepsilon > 0$ arbitrary. We have to show the existence of $N_0 \in \mathbb{N}$ such that for any $j > N_0$ we have $d(x_0, k_j e^{H_j d_j}x_0) > R$ and

$$d(k_j e^{H_j R}x_0, k_\eta e^{H_\eta R}x_0) < \varepsilon\,.$$

The first claim follows from the fact that $d_j = d(x_0, k_j e^{H_j d_j}x_0)$ tends to infinity. For the second claim, we calculate as before

$$d(k_j e^{H_j R}x_0, k_\eta e^{H_\eta R}x_0) \leq R\cdot\|H_j - H_\eta\| + d(k_j e^{H_\eta R}x_0, k_\eta e^{H_\eta R}x_0)\,.$$

Since R is fixed, the claim follows from the fact that $\|H_j - H_\eta\| \to 0$ and with the above lemma from $k_j \to k_\eta$ in K/M_Θ. □

As in the previous section for the case $\Theta = \emptyset$, we will identify the homogeneous space G/P_Θ, $\Theta \subset \Upsilon$, with the component K/M_Θ of $\pi^B(\partial X^\Theta)$ using the natural bijection

$$\begin{aligned} \overline{\kappa}_\Theta \;:\; G/P_\Theta &\to K/M_\Theta \\ gP_\Theta &\mapsto kM_\Theta \end{aligned}$$

arising from the generalized Iwasawa decomposition. Its restriction to the big cell $N^+w_*P_\Theta = N_\Theta^+ w_* P_\Theta$ induces a map

$$\begin{aligned} \kappa_\Theta \;:\; N_\Theta^+ &\to K/M_\Theta \\ n &\mapsto \overline{\kappa}_\Theta(nw_*P_\Theta)\,. \end{aligned}$$

Proposition 1.2.4.10 of [W] shows, that the orbit $N^+w_*P_\Theta = N_\Theta^+ w_*P_\Theta$ is dense and open in G/P_Θ. We will call the image of such a Schubert cell under the map $\overline{\kappa}_\Theta$ a big cell in K/M_Θ. We further remark, that κ_Θ is a diffeomorphism onto an open submanifold of K/M_Θ.

Geometrically, if $n \in N_\Theta^+$, then $\kappa_\Theta(n) \in K/M_\Theta$ is the unique element such that for any $H \in \mathfrak{a}^\Theta$ the geodesic rays $nw_*e^{Ht}x_0$ and $\kappa_\Theta(n)e^{Ht}x_0$ are asymptotic.

2.6 The visibility sets at infinity

In the case of a complete, simply connected Riemannian manifold with a negative upper bound for the sectional curvature, any two points in the geometric boundary can be joined by a geodesic. If the tangent bundle contains planes with zero sectional curvature, this is no longer true. Nevertheless, for globally symmetric spaces X of noncompact type, the set of all points in the geometric boundary which can be joined to a given point $\xi \in \partial X$ by a geodesic can be described in a simple way.

DEFINITION 2.10 *The* visibility set at infinity *viewed from $\xi \in \partial X$ is the set*

$$Vis^\infty(\xi) := \{\eta \in \partial X \mid \exists \text{ geodesic } \sigma \text{ such that } \sigma(-\infty) = \xi\,, \sigma(\infty) = \eta\}\,.$$

If $\mathrm{rank}(X) = 1$, the sectional curvature of X has a negative upper bound. Thus $\xi \in \partial X$ can be joined to any other point in ∂X by a geodesic, hence $\mathrm{Vis}^\infty(\xi) = \partial X \setminus \{\xi\}$.

If $\mathrm{rank}(X) \geq 2$, however, the sets $\mathrm{Vis}^\infty(\xi)$, $\xi \in \partial X$, are sparse in the geometric boundary. We will therefore need to consider some larger sets.

Let $X = G/K$ be a globally symmetric space of noncompact type with base point $x_0 \in X$ corresponding to K and fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$.

DEFINITION 2.11 *The* Bruhat visibility set *viewed from $\xi \in \partial X$ is the image of $Vis^\infty(\xi)$ under the projection $\pi^B :\ \partial X \to \bigcup_{\Theta\subset\Upsilon} K/M_\Theta$, i.e.*

$$Vis^B(\xi) = \pi^B(\,Vis^\infty(\xi))\,.$$

For regular points $\xi \in \partial X^{reg}$, the subset $\mathrm{Vis}^B(\xi) \subset K/M$ can be identified with the set of equivalence classes of asymptotic Weyl chambers which possess a representative $\mathcal{C} \subset X$ with the following property: There exists ageodesic $\sigma :\ \mathbb{R} \to X$ with extremity $\sigma(-\infty) = \xi$, and the geodesic ray $\sigma(t)$, $t \geq 0$, is contained in $\mathcal{C}$. In rank one symmetric spaces, the Bruhat visibility sets coincide with the visibility sets at infinity.

We remark that for ξ, $\eta \in \partial X$, $\eta \in \mathrm{Vis}^\infty(\xi)$ is equivalent to $\xi \in \mathrm{Vis}^\infty(\eta)$. This also implies the equivalence of $\pi^B(\eta) \in \mathrm{Vis}^B(\xi)$ and $\pi^B(\xi) \in \mathrm{Vis}^B(\eta)$.

The opposition involution ι from Definition 1.7 will play an important role in the sequel.

DEFINITION 2.12 *For any subset $\Theta \subset \Upsilon$, we define the* opposition set Θ^* *by the condition*

$$\alpha \in \Theta^* \quad :\Longleftrightarrow \quad \alpha(\iota(H)) = 0 \quad \forall\, H \in \mathfrak{a}^\Theta\,.$$

The standard face of type Θ^* is then given by $\mathfrak{a}^{\Theta^*} = \iota(\mathfrak{a}^\Theta)$ and we have $M_{\Theta^*} = w_* M_\Theta w_*^{-1}$. Note that $\Theta^* = \Theta$ if $\iota = \mathrm{id}$.

Furthermore, if $\xi = (\mathrm{id}, H) \in \partial X^\Theta$ we have $\sigma_{x_0,\xi}(-\infty) = (w_*, \iota(H)) \in \partial X^{\Theta^*}$, because $\sigma_{x_0,\xi}(-t) = e^{-Ht}x_0 = w_* e^{-\mathrm{Ad}(w_*)Ht}x_0 = w_* e^{\iota(H)t}x_0$. Using the natural extension of the opposition involution to the geometric boundary

$$\begin{array}{rccl} \iota: & \partial X^\Theta & \to & \partial X^{\Theta^*} \\ & (k,H) & \mapsto & (w_* k w_*^{-1}, \iota(H))\,, \end{array}$$

this implies $\mathrm{Vis}^\infty(\iota{\cdot}\xi) \subset G{\cdot}\xi \subset \partial X^\Theta$.

The following lemma will describe the visibility sets at infinity in terms of the Bruhat decomposition of $G = \mathrm{Isom}^o(X)$. Given $x \in X$ and $\xi \in \partial X$, there exists a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ such that K is the unique maximal compact subgroup which stabilizes x, and $\mathrm{id} \in K$ is an angular projection of ξ (see section 2.1). We will call this decomposition a Cartan decomposition with respect to x and ξ.

Similarly, there exists an Iwasawa decomposition $G = N^+AK$ such that K is the unique maximal compact subgroup which stabilizes x, and N^+ is the nilpotent subgroup in the stabilizer of a closed Weyl chamber at infinity which contains ξ. We will call this decomposition $G = N^+AK$ an Iwasawa decomposition with respect to x and ξ. Note that if $\xi \in \partial X^{sing}$, then N^+ depends on the chosen Weyl chamber at infinity, and we might as well choose a conjugate of N^+ by an element in the stabilizer $K_\xi \subset K$ of ξ.

Using the subsets $N^+_\Theta \subset N^+$ and the maps κ_Θ defined at the end of section 2.5, the lemma reads as follows.

LEMMA 2.13 *Let $\xi \in \partial X$, $G = Ke^{\overline{\mathfrak{a}^+}}K$ a Cartan decomposition with respect to some base point $x_0 \in X$ and $\xi \in \partial X$, and let $\Theta \subset \Upsilon$ such that $\xi \in \partial X^{\Theta^*}$. Then*

$$Vis^\infty(\xi) = \{(k, \iota(H_\xi)) \mid kM_\Theta \in \kappa_\Theta(N^+_\Theta)\}\,.$$

Proof. Let $k \in K$ such that $kM_\Theta = \kappa_\Theta(n)$ with $n \in N^+_\Theta \subset N^+$. Put

$$\sigma(t) := n\sigma_{x_0,\xi}(-t) = ne^{-H_\xi t}x_0 = nw_* e^{\iota(H_\xi)t}x_0\,.$$

Then $\sigma(-\infty) = \xi$ because N^+ stabilizes ξ, and $\sigma(t)$ is asymptotic to $ke^{\iota(H_\xi)t}x_0$ as $t \to \infty$ by the property of the map κ_Θ. Hence $\sigma(\infty) = (k, \iota(H_\xi))$ which proves that $(k, \iota(H_\xi)) \in \mathrm{Vis}^\infty(\xi)$.

Conversely, let $\eta \in \mathrm{Vis}^\infty(\xi)$ and take a geodesic σ with extremities $\sigma(\infty) = \eta$ and $\sigma(-\infty) = \xi$. Since G acts transitively on X, there exists $g \in G$ such that $\sigma(0) = gx_0$. Using the generalized Iwasawa decomposition with respect to $\Theta \subset \Upsilon$ and write $gx_0 = nbx_0$, where $n \in N^+_\Theta$ and $b \in \exp(\mathfrak{p}_\Theta)$. The geodesic $\sigma_0 := (nb)^{-1}\sigma = b^{-1}n^{-1}\sigma$ satisfies $\sigma_0(-\infty) = \sigma(-\infty) = \xi$ because n and b stabilize ξ. Since $\sigma_0(0) = x_0$ and $\xi \in \partial X^{\Theta^*}$, we conclude $\sigma_0(\infty) = \sigma_{x_0,\xi}(-\infty) = (w_*, \iota(H_\xi)) \in \partial X^\Theta$. Since b also stabilizes $\sigma_{x_0,\xi}(-\infty)$ we obtain $\eta = \sigma(\infty) = nb\sigma_0(\infty) = n\sigma_0(\infty) = (k, \iota(H_\xi))$, where $k \in K/M_\Theta$ is the unique element such that $n\sigma_0(t) = nw_*e^{\iota(H_\xi)t}x_0$ is asymptotic to $ke^{\iota(H_\xi)t}x_0$. Therefore $kM_\Theta = \kappa_\Theta(n)$ which proves the claim. □

COROLLARY 2.14 *Let $\xi \in \partial X$, $G = Ke^{\overline{\mathfrak{a}^+}}K$ a Cartan decomposition with respect to some base point $x_0 \in X$ and $\xi \in \partial X$, and let $\Theta \subset \Upsilon$ such that $\xi \in \partial X^{\Theta^*}$. Then $\eta \in Vis^\infty(\xi)$ if and only if there exists $n \in N^+_\Theta$ with the property $n\sigma_{x_0,\xi}(-\infty) = \eta$.*

COROLLARY 2.15 *Let $\xi \in \partial X$, $G = Ke^{\overline{\mathfrak{a}^+}}K$ a Cartan decomposition with respect to some base point $x_0 \in X$ and $\xi \in \partial X$, and let $\Theta \subset \Upsilon$ such that $\xi \in \partial X^{\Theta^*}$. Then the Bruhat visibility set $Vis^B(\xi)$ is the image under κ_Θ of the Schubert cell $N^+_\Theta w_* P_\Theta$ in G/P_Θ.*

The following lemma and its corollary will be used in the construction of free groups in section 5.2.

LEMMA 2.16 *For* $1 \le i \le l$ *let* $\Theta_i \subset \Upsilon$ *and* $\xi_i \in \partial X^{\Theta_i^*}$. *If* $\Theta \supseteq \bigcup_{i=1}^{l} \Theta_i$, *then*

$$\bigcap_{i=1}^{l} Vis^B(\xi_i) \text{ is a dense and open subset of } K/M_\Theta \,.$$

Proof. We fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ and let $i \in \{1, 2, \dots, l\}$. Let $k \in K$ be an angular projection, $H \in \mathfrak{a}^{\Theta_i^*}$ the Cartan projection of ξ_i. Since $\Theta_i^* \subseteq \Theta^*$, there exists a point $\eta_i \in \partial X^{\Theta^*}$ such that ξ_i and η_i define points in the closure of a common Weyl chamber at infinity. In particular, k is an angular projection of η_i, and therefore the natural projection of $\pi^B(\xi_i) \in K/M_{\Theta_i^*}$ to K/M_{Θ^*} is equal to $\pi^B(\eta_i)$. Consequently, the natural projection of the subset $\mathrm{Vis}^B(\xi_i) \subset K/M_{\Theta_i}$ to K/M_Θ is equal to $\mathrm{Vis}^B(\eta_i)$. Now the set $\mathrm{Vis}^B(\eta_i) = \kappa_\Theta(N_\Theta^+)$ is a dense and open submanifold of K/M_Θ and the claim follows, because a finite intersection of dense and open sets remains dense and open in K/M_Θ. □

COROLLARY 2.17 *If* $\xi_1, \xi_2, \dots, \xi_l \in \partial X^{reg}$, *then the intersection* $\bigcap_{i=1}^{l} Vis^B(\xi_i)$ *is a dense and open subset of the Furstenberg boundary* $\partial X^F \cong K/M$.

The following result is an easy consequence of the facts that the Bruhat visiblity sets are open and the maps κ_Θ introduced in section 2.5 are diffeomorphisms.

LEMMA 2.18 *For* $\Theta \subset \Upsilon$ *let* $\eta \in \partial X^\Theta$ *and* $\xi \in Vis^\infty(\eta)$. *Then for any sequence* $(\eta_j) \subset G\cdot\eta \subseteq \partial X^\Theta$ *which converges to* η *in the cone topology, we have* $\xi \in Vis^\infty(\eta_j)$ *for all but finitely many* $j \in \mathbb{N}$.

Proof. We choose a unit speed geodesic σ in X such that $\sigma(-\infty) = \xi$ and $\sigma(\infty) = \eta$. Fix an Iwasawa decomposition $G = N^+AK$ and a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ with respect to $x_0 := \sigma(0) \in X$ and $\xi \in \partial X$. If $H \in \mathfrak{a}^\Theta$ denotes the Cartan projection of η, we have $\xi = (\mathrm{id}, \iota(H))$ and $\eta = (w_*, H)$.

For $j \in \mathbb{N}$ let $k_j \in K$ denote an angular projection of η_j. The convergence of η_j to η implies the existence of $N_0 \in \mathbb{N}$ such that for $j > N_0$, $k_j M_\Theta$ is contained in an open neighborhood of $w_* M_\Theta$ in K/M_Θ. Since the map κ_Θ is a diffeomorphism from N_Θ^+ onto a dense open submanifold of K/M_Θ which contains $w_* M_\Theta$, we have $\kappa_\Theta^{-1}(k_j M_\Theta) \in N_\Theta^+$ if $j > N_0$. Since $\eta_j \in G\cdot\eta$, we conclude $\eta_j \in \{(k, H) \mid kM_\Theta \in \kappa_\Theta(N_\Theta^+)\} = \mathrm{Vis}^\infty(\xi)$ by Lemma 2.13 and therefore $\xi \in \mathrm{Vis}^\infty(\eta_j)$ for any $j > N_0$. □

The last result of this section is a stronger version of the previous lemma and is proved in [E], page 322.

LEMMA 2.19 *Let* $\overline{\sigma}$ *be any unit speed geodesic in* X. *Then for any* $\varepsilon > 0$ *there exists a neighborhood* U *of the identity in* $G = Isom^o(X)$ *such that for arbitrary* $\eta \in U\cdot\overline{\sigma}(\infty)$ *and* $\xi \in U\cdot\overline{\sigma}(-\infty)$ *there exists a geodesic* $\sigma \in X$ *with*

$$\sigma(-\infty) = \eta\,, \quad \sigma(\infty) = \xi \quad \text{and} \quad d(\overline{\sigma}(0), \sigma) < \varepsilon\,.$$

Chapter 3

Directional distances and angles

In this chapter, we will introduce a family of (possibly nonsymmetric) pseudo distances for globally symmetric spaces of noncompact type which will play a fundamental role in our work. Since our construction relies on Buseman functions, we are going to recall their definition and properties following [Ba], chapter II.

3.1 Buseman functions

Let X be a globally symmetric space of noncompact type. For $z \in X$ we consider the continuous map

$$\begin{array}{rrcl} \mathcal{B}_z : & X \times X & \to & \mathbb{R} \\ & (x,y) & \mapsto & d(x,z) - d(y,z) \,. \end{array}$$

This map extends continuously to the boundary via

$$\mathcal{B}_\eta(x,y) := \lim_{s\to\infty} \big(d(x,\sigma(s)) - d(y,\sigma(s)) \big) \,,$$

where σ is an arbitrary geodesic ray asymptotic to $\eta \in \partial X$, and $x, y \in X$. The maps $\mathcal{B}_v$, $v \in \overline{X} = X \cup \partial X$ obviously satisfy the cocycle relation

$$\mathcal{B}_v(x,z) = \mathcal{B}_v(x,y) + \mathcal{B}_v(y,z) \quad \forall\, x,y,z \in X \,,$$

and the triangle inequality for the Riemannian distance yields

$$\mathcal{B}_v(x,y) \leq d(x,y) \quad \forall\, x,y \in X \,.$$

For $\xi \in \partial X$, $y \in X$, the function

$$\begin{array}{rrcl} \mathcal{B}_\xi(\cdot,y) : & X & \to & \mathbb{R} \\ & x & \mapsto & \mathcal{B}_\xi(x,y) \end{array}$$

is called the Busemann function centered at ξ based at y. It is independent of the chosen ray σ.

LEMMA 3.1 *([Ba], chapter II)*

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then the following statements are equivalent.

(i) x_n converges in the cone topology to $\eta \in \partial X$.

(ii) The functions $\mathcal{B}_{x_n}(\cdot, y)$ converge in the space of continuous functions $\mathrm{C}^0(\overline{X})$ to $\mathcal{B}_\eta(\cdot, y)$ with respect to the topology of uniform convergence on compact sets.

We remark that the above definition of Buseman functions and Lemma 3.1 are valid for every Hadamard manifold.

If X is a globally symmetric space of noncompact type and $G = \mathrm{Isom}^o(X)$, we obtain the following formulas for the Buseman function.

LEMMA 3.2 *Let $x \in X$ and $\xi \in \partial X$. Fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ with respect to x and ξ, and let H_ξ denote the Cartan projection of ξ. Then for any $H \in \overline{\mathfrak{a}^+}$ we have*

$$\mathcal{B}_\xi(x, e^H x) = \langle H_\xi, H\rangle .$$

Proof. Let $H \in \overline{\mathfrak{a}^+}$ arbitrary. We calculate

$$\begin{aligned}
\mathcal{B}_\xi(x, e^H x) &= \lim_{s\to\infty}\big(d(x, \sigma_{x,\xi}(s)) - d(e^H x, \sigma_{x,\xi}(s))\big) = \lim_{s\to\infty}\big(s - d(e^H x, e^{H_\xi s}x)\big) \\
&= \lim_{s\to\infty}\frac{s^2 - \|H_\xi s - H\|^2}{s + \|H_\xi s - H\|} = \lim_{s\to\infty}\frac{s^2 - (s^2 - 2s\langle H_\xi, H\rangle + \|H\|^2)}{s + \sqrt{s^2 - 2s\langle H_\xi, H\rangle + \|H\|^2}} \\
&= \langle H_\xi, H\rangle .
\end{aligned}$$

□

For Buseman functions centered at regular boundary points, we obtain a formula for any pair of points in X in terms of an appropriate Iwasawa decomposition for G.

LEMMA 3.3 *Let $x,\ y \in X$ and $\xi \in \partial X^{reg}$. If $G = N^+AK$ is an Iwasawa decomposition with respect to x and ξ, and H denotes the Iwasawa projection of y described in section 1.3, then*

$$\mathcal{B}_\xi(x, y) = \langle H_\xi, H\rangle .$$

Here $H_\xi \in \mathfrak{a}_1^+$ denotes the Cartan projection of ξ with respect to a Cartan decomposition determined by x and ξ.

Proof. Let $y \in X$ arbitrary, write $y = ne^H x_0$ in horospherical coordinates corresponding to the given Iwasawa decomposition. Then

$$\begin{aligned}
\mathcal{B}_\xi(x, y) &= \lim_{s\to\infty}\big(d(x, \sigma_{x,\xi}(s)) - d(y, \sigma_{x,\xi}(s))\big) \\
&\geq \lim_{s\to\infty}\big(s - d(y, n\sigma_{x,\xi}(s)) - d(ne^{H_\xi s}x, e^{H_\xi s}x)\big) , \\
\mathcal{B}_\xi(x, y) &\leq \lim_{s\to\infty}\big(s - d(y, n\sigma_{x,\xi}(s)) + d(ne^{H_\xi s}x, e^{H_\xi s}x)\big) .
\end{aligned}$$

The term $d(ne^{H_\xi s}x, e^{H_\xi s}x)$ tends to zero as $s \to \infty$, because $n \in N^+$ and therefore $e^{-H_\xi s}ne^{H_\xi s}$ converges to $\mathrm{id} \in N^+$. Furthermore, $d(y, n\sigma_{x,\xi}(s)) = d(e^H x, e^{H_\xi s}x)$, and the calculation in the proof of the previous lemma allows to conclude

$$\mathcal{B}_\xi(x, y) = \lim\nolimits_{s\to\infty}\big(s - d(e^H x, e^{H_\xi s}x)\big) = \langle H_\xi, H\rangle .$$

□

3.2 The directional distance

This section will introduce an important family of (possibly nonsymmetric) pseudo distances which we will need throughout the whole work.

Definition 3.4 *We define the* directional distance *of the ordered pair* $(x, y) \in X \times X$ *with respect to* $\xi \in \partial X$ *by*

$$\begin{array}{rcl} \mathcal{B}_{G\cdot\xi} : X \times X & \to & \mathbb{R} \\ (x,y) & \mapsto & \mathcal{B}_{G\cdot\xi}(x,y) = \sup\limits_{g\in G} \mathcal{B}_{g\cdot\xi}(x,y)\,. \end{array}$$

Note that the corresponding estimate for the Buseman functions implies

$$\mathcal{B}_{G\cdot\xi}(x,y) \le d(x,y) \qquad \forall\, \xi \in \partial X\,,\ \forall\, x,\, y \in X\,.$$

Lemma 3.5 *For any* $\xi \in \partial X$*, the directional distance* $\mathcal{B}_{G\cdot\xi}$ *is* G*-invariant.*

Proof. Fix $\xi \in \partial X$ and let $x, y \in X$. For any $g \in G$ we have

$$\begin{aligned} \mathcal{B}_{G\cdot\xi}(gx, gy) &= \sup_{h\in G} \Big(\lim_{t\to\infty} \big(d(gx, h\sigma(t)) - d(gy, h\sigma(t))\big)\Big) \\ &= \sup_{h\in G} \Big(\lim_{t\to\infty} \big(d(x, g^{-1}h\sigma(t)) - d(y, g^{-1}h\sigma(t))\big)\Big) = \mathcal{B}_{G\cdot\xi}(x,y)\,, \end{aligned}$$

which shows that $\mathcal{B}_{G\cdot\xi}$ is G-invariant. □

An easy consequence of the definition of $\mathcal{B}_{G\cdot\xi}$ and Corollary 1.15 is the following

Lemma 3.6 *Let* $x, y \in X$ *and* $\xi \in \partial X$*. Fix a Cartan decomposition* $G = Ke^{\overline{\mathfrak{a}^+}}K$ *with respect to* x *and* ξ*, and let* H_ξ *denote the Cartan projection of* ξ *and* k_y *an angular projection of* y*. Then*

$$\mathcal{B}_{G\cdot\xi}(x,y) = \mathcal{B}_{k_y\xi}(x,y)\,.$$

Proof. Let $H \in \overline{\mathfrak{a}^+}$ arbitrary. Using Corollary 1.15, we calculate for $k \in K$

$$\begin{aligned} \mathcal{B}_{k\xi}(x, e^H x) &= \lim_{s\to\infty} \big(d(x, \sigma_{x,k\xi}(s)) - d(e^H x, \sigma_{x,k\xi}(s))\big) \\ &= \lim_{s\to\infty} \big(d(x, ke^{H_\xi s}x) - d(e^H x, ke^{H_\xi s}x)\big) \\ &\le \lim_{s\to\infty} \big(d(x, e^{H_\xi s}x) - d(e^H x, e^{H_\xi s}x)\big) = \mathcal{B}_\xi(x, e^H x)\,. \end{aligned}$$

The compactness of K and $\mathcal{B}_{k\xi}(x, e^H x) = \mathcal{B}_\xi(x, e^H x)$ for $k = \mathrm{id}$ imply

$$\max_{k\in K} \mathcal{B}_{k\xi}(x, e^H x) = \mathcal{B}_\xi(x, e^H x)\,.$$

Since K acts transitively on $G\cdot\xi$, we further deduce

$$\mathcal{B}_{G\cdot\xi}(x, e^H x) = \max_{k\in K} \mathcal{B}_{k\xi}(x, e^H x) = \mathcal{B}_\xi(x, e^H x)\,.$$

Now let $y \in X$ arbitrary, $k_y \in K$ an angular projection and $H_y \in \overline{\mathfrak{a}^+}$ the Cartan projection of y. By G-invariance and the calculations above we obtain

$$\begin{aligned} \mathcal{B}_{G\cdot\xi}(x,y) &= \mathcal{B}_{G\cdot\xi}(x,k_y e^{H_y}x) = \mathcal{B}_{G\cdot\xi}(k_y^{-1}x, e^{H_y}x) = \mathcal{B}_{G\cdot\xi}(x, e^{H_y}x) \\ &= \mathcal{B}_{\xi}(x, e^{H_y}x) = \mathcal{B}_{k_y\xi}(k_y x, k_y e^{H_y}x) = \mathcal{B}_{k_y\xi}(x,y)\,, \end{aligned}$$

where the first equality in the second line follows directly from the definition of the Buseman function. □

Proposition 3.7 *For arbitrary $\xi \in \partial X$, $\mathcal{B}_{G\cdot\xi}$ is a (possibly nonsymmetric) G-invariant pseudo distance on X.*

Proof. Fix $\xi \in \partial X$ and let $x, y, z \in X$. The G-invariance was proved in Lemma 3.5.

The triangle inequality is an easy consequence of the cocycle relation

$$\begin{aligned} \mathcal{B}_{G\cdot\xi}(x,y) &= \sup_{g\in G} \mathcal{B}_{g\xi}(x,y) = \sup_{g\in G} \left(\mathcal{B}_{g\xi}(x,z) + \mathcal{B}_{g\xi}(z,y)\right) \\ &\le \sup_{g\in G} \mathcal{B}_{g\xi}(x,z) + \sup_{g\in G} \mathcal{B}_{g\xi}(z,y) = \mathcal{B}_{G\cdot\xi}(x,z) + \mathcal{B}_{G\cdot\xi}(z,y)\,. \end{aligned}$$

It remains to show that $\mathcal{B}_{G\cdot\xi}(x,y) \ge 0$. In order to do so, we fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ with respect to some base point $x_0 \in X$ and $\xi \in \partial X$ as above. Then for any $H \in \overline{\mathfrak{a}^+}$ we have by Lemma 3.6 and Lemma 3.2

$$\mathcal{B}_{G\cdot\xi}(x_0, e^H x_0) = \mathcal{B}_{\xi}(x_0, e^H x_0) = \langle H_\xi, H\rangle \ge 0\,,$$

because two vectors in the closure of the same Weyl chamber cannot have an angle larger than $\pi/2$. If $x, y \in X$ are arbitrary, then there exists $g \in G$ such that $gx = x_0$. Using the Cartan decomposition, we may write $gy = ke^H x_0$ and obtain by G-invariance

$$\mathcal{B}_{G\cdot\xi}(x,y) = \mathcal{B}_{G\cdot\xi}(gx,gy) = \mathcal{B}_{G\cdot\xi}(x_0, ke^H x_0) = \mathcal{B}_{G\cdot\xi}(x_0, e^H x_0) \ge 0\,.$$ □

Lemma 3.8 *For any $\xi \in \partial X$, $x, y \in X$ we have $\mathcal{B}_{G\cdot\xi}(y,x) = \mathcal{B}_{\iota(G\cdot\xi)}(x,y)$. In particular $\mathcal{B}_{G\cdot\xi}$ is symmetric if and only if either the opposition involution $\iota = \mathrm{id}$ or $G\cdot\xi$ is the barycentral boundary component.*

Proof. Let $\xi \in \partial X$ and fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ with respect to some base point $x_0 \in X$ and $\xi \in \partial X$. Since the directional distance is G-invariant, it suffices to prove the claim for $x = x_0$ and $y = e^H x_0$, where $H \in \overline{\mathfrak{a}^+}$. If $H_\xi \in \overline{\mathfrak{a}_1^+}$ denotes the Cartan projection of ξ, we obtain by G-invariance and Lemma 3.6

$$\begin{aligned} \mathcal{B}_{G\cdot\xi}(e^H x_0, x_0) &= \mathcal{B}_{G\cdot\xi}(x_0, e^{-H}x_0) = \mathcal{B}_{G\cdot\xi}(w_* x_0, e^{-\mathrm{Ad}(w_*)H}x_0) = \langle H_\xi, -\mathrm{Ad}(w_*)H\rangle \\ &= \langle -\mathrm{Ad}(w_*)H_\xi, H\rangle = \langle \iota(H_\xi), H\rangle = \mathcal{B}_{\iota(G\cdot\xi)}(x_0, e^H x_0)\,. \end{aligned}$$

In particular, the symmetry of $\mathcal{B}_{G\cdot\xi}$ is equivalent to $\langle H_\xi, H\rangle = \langle \iota(H_\xi), H\rangle\ \ \forall\, H \in \overline{\mathfrak{a}^+}$, which proves the last assertion. □

LEMMA 3.9 *For $\xi \in \partial X^{reg}$, $\mathcal{B}_{G\cdot\xi}$ is a (possibly nonsymmetric) distance.*

Proof. The same arguments as in the proof of the previous lemma show that $\mathcal{B}_{G\cdot\xi}$ is a distance if and only if for arbitrary $H \in \overline{\mathfrak{a}^+}$

$$\mathcal{B}_{G\cdot\xi}(x_0, e^H\cdot x_0) = 0 \quad \Longrightarrow \quad H = 0\,.$$

Now $\langle H_\xi, H\rangle = 0$ implies $H = 0$ or $H \perp H_\xi$. Since $\xi \in \partial X^{reg}$, the latter case cannot occur. □

3.3 Properties of the directional distance

The following lemma will characterize the directional distance in $X = G/K$ in terms of a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$. Let $x_0 \in X$ be the unique point stabilized by K.

LEMMA 3.10 *Let $\xi \in \partial X$, $x, y \in X$, and $g \in G$ such that $x = gx_0$. Then*

$$\mathcal{B}_{G\cdot\xi}(x,y) = d(x,y)\cdot \max_{k\in K}\ \cos\angle_x(y, gk\xi) = d(x,y)\cdot \sup_{h\in G}\ \cos\angle_x(y, h\xi)\,.$$

Proof. We first prove the claim for $x = x_0$, $y = e^H x_0$, where $H \in \overline{\mathfrak{a}^+}$. Let $H_\xi \in \overline{\mathfrak{a}_1^+}$ denote the Cartan projection of ξ. Using Lemma 3.6, Lemma 3.2 and Lemma 1.14 , we calculate

$$\begin{aligned}\mathcal{B}_{G\cdot\xi}(x_0, e^H x_0) &= \mathcal{B}_\xi(x_0, e^H x_0) = \langle H_\xi, H\rangle \\ &= \max_{k\in K}\langle \mathrm{Ad}(k)H_\xi, H\rangle = \|H\|\cdot \max_{k\in K}\ \cos\angle_{x_0}(e^H x_0, k\xi)\,.\end{aligned}$$

Let $x, y \in X$ and $g \in G$ such that $x = gx_0$. Then by the Cartan decomposition there exists $H \in \overline{\mathfrak{a}^+}$ such that $y = ge^H x_0$. The equality $\angle_x(y, gk\xi) = \angle_{gx_0}(ge^H x_0, gk\xi) = \angle_{x_0}(e^H x_0, k\xi)$ and G-invariance imply

$$\begin{aligned}\mathcal{B}_{G\cdot\xi}(x,y) &= \mathcal{B}_{G\cdot\xi}(x_0, e^H x_0) = \|H\|\cdot \max_{k\in K}\ \cos\angle_{x_0}(e^H x_0, k\xi) \\ &= d(x,y)\cdot \max_{k\in K}\ \cos\angle_x(y, gk\xi) = d(x,y)\cdot \sup_{h\in G}\ \cos\angle_x(y, h\xi)\,.\end{aligned}$$

□

Similarly as in the case of the Riemannian distance in $X = G/K$, we may define a Buseman function for the directional distance $\mathcal{B}_{G\cdot\xi}$, $\xi \in \partial X$. Before we state Proposition 3.12, we need a preliminary lemma.

LEMMA 3.11 *Let $\tilde\Theta \subset \Upsilon$, $\Theta \subseteq \tilde\Theta$, $\eta \in \partial X^\Theta$ and $\xi \in \partial X^{\tilde\Theta}$. Suppose there exists a Weyl chamber $\mathcal{C}_0 \subset X$ such that η and ξ belong to the closure $\overline{\mathcal{C}}_0$ of $\mathcal{C}_0$ in $\overline{X}$. Then for any Weyl chamber $\mathcal{C} \subset X$, $\eta \in \overline{\mathcal{C}}$ implies $\xi \in \overline{\mathcal{C}}$.*

Proof. Using the identification $\partial X^F \cong K/M$, we let $k_0 M \in K/M$ denote the asymptote class of $\mathcal{C}_0$ in ∂X^F. Then $\eta \in \overline{\mathcal{C}}_0 \cap \partial X^\Theta$ implies $\pi^B(\eta) = k_0 M_\Theta$, and $\xi \in \overline{\mathcal{C}}_0 \cap \partial X^{\tilde\Theta}$ implies $\pi^B(\xi) = k_0 M_{\tilde\Theta}$.

Now let $\mathcal{C} \subset X$ be a Weyl chamber such that $\eta \in \overline{\mathcal{C}} \cap \partial X^\Theta$. If $kM \in K/M$ denotes the asymptote class of $\mathcal{C}$ in ∂X^F, we obtain $\pi^B(\eta) = kM_\Theta$ by the assumption. Therefore $k^{-1}k_0 \in M_\Theta$ which implies $k^{-1}k_0 \in M_{\tilde\Theta}$, since M_Θ is a closed subgroup of $M_{\tilde\Theta}$. Hence $\pi^B(\xi) = k_0 M_\Theta = kM_\Theta$. □

PROPOSITION 3.12 *Let $(y_n) \subseteq X$ be a sequence converging to $\eta \in \partial X$ in the cone topology. Let $\Theta \subset \Upsilon$ such that $\eta \in \partial X^\Theta$, and $\tilde\Theta \supseteq \Theta$, $\tilde\Theta \subset \Upsilon$. Then for any $\xi \in \partial X^{\tilde\Theta}$, the functions*

$$\mathcal{B}_{G\cdot\xi}(x_0, y_n) - \mathcal{B}_{G\cdot\xi}(\cdot, y_n)$$

converge in the space of continuous functions $\mathrm{C}^0(\overline{X})$ endowed with the topology of uniform convergence on compact sets to $\mathcal{B}_{\xi_\eta}(x_0, \cdot)$, where $\xi_\eta \in \partial X$ is the unique element in $G \cdot \xi$ such that η and ξ_η are points in the closure of a common Weyl chamber at infinity..

Proof. Let $x \in X$ arbitrary, $(y_n) \subset X$ a sequence converging to η, and $(k_n) \subset K$ a corresponding sequence of angular projections. By Lemma 3.6, we have

$$\mathcal{B}_{G\cdot\xi}(x_0, y_n) = \mathcal{B}_{k_n\xi}(x_0, y_n) \quad \forall\, n \in \mathbb{N}.$$

If $k_0 \in K$ denotes an angular projection of η, Lemma 2.9 shows that k_n converges to k_0 in K/M_Θ. Since $M_\Theta \subseteq M_{\tilde\Theta}$ is a closed subgroup, this implies that k_n converges to k_0 in $K/M_{\tilde\Theta}$, and therefore $k_n\xi$ converges to a point $k_0\xi \in G\cdot\xi$. Since $k_0\xi$ and η have the same angular projection, the closure of every Weyl chamber in X asymptotic to $k_0 M$ contains both $k_0\xi$ and η.

Similarly, there exists a sequence $(\xi_n) \subset G\cdot\xi$ converging to a point $\xi_0 \in G\cdot\xi$ with the following properties.

$$\mathcal{B}_{G\cdot\xi}(x, y_n) = \mathcal{B}_{\xi_n}(x, y_n) \quad \forall\, n \in \mathbb{N},$$

and there exists a Weyl chamber $\mathcal{C} \subset X$ such that η and ξ_0 belong to the closure of $\mathcal{C}$ in $\overline{X}$. By the previous lemma, the closure of $\mathcal{C}$ also contains $k_0\xi$, hence $\xi_0 = k_0\xi = \xi_\eta$.

By definition of $\mathcal{B}_{G\cdot\xi}$ and the cocycle relation for the Buseman function we conclude

$$\begin{aligned} \mathcal{B}_{\xi_\eta}(x_0, x) &= \lim_{n\to\infty} \mathcal{B}_{k_n\xi}(x_0, x) = \lim_{n\to\infty} \left(\mathcal{B}_{k_n\xi}(x_0, y_n) - \mathcal{B}_{k_n\xi}(x, y_n)\right) \\ &\geq \lim_{n\to\infty} \left(\mathcal{B}_{G\cdot\xi}(x_0, y_n) - \mathcal{B}_{G\cdot\xi}(x, y_n)\right) \\ &\geq \lim_{n\to\infty} \left(\mathcal{B}_{\xi_n}(x_0, y_n) - \mathcal{B}_{\xi_n}(x, y_n)\right) = \lim_{n\to\infty} \mathcal{B}_{\xi_n}(x_0, x) = \mathcal{B}_{\xi_\eta}(x_0, x) \end{aligned}$$

and therefore $\lim_{n\to\infty} \left(\mathcal{B}_{G\cdot\xi}(x_0, y_n) - \mathcal{B}_{G\cdot\xi}(x, y_n)\right) = \mathcal{B}_{\xi_\eta}(x_0, x)$.

The uniform convergence on compact sets is a consequence of Lemma 3.1 and Lemma 2.9 applied to the sequence $z_n := k_n e^{\langle H_n, H_\xi\rangle H_\xi} x_0$, where $H_n \in \overline{\mathfrak{a}^+}$ denotes the Cartan projection of y_n. □

3.4 Maximal singular directions and roots

For this section, we fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ for $X = G/K$ and choose a fundamental set of roots Υ. Let $x_0 \in X$ denote the unique point stabilized by K. For $1 \le i \le r$ we put $\Theta^i := \Upsilon \setminus \{\alpha_i\}$. Then $\dim \mathfrak{a}^{\Theta^i} = 1$, and we give the following

DEFINITION 3.13 *The i-th* maximal singular direction $H_i \in \overline{\mathfrak{a}_1^+}$ *is the unique vector which spans $\mathfrak{a}^{\Theta^i}$. The boundary subset $\partial X^i := \partial X^{\Theta^i} \subset \partial X$ is called the i-th* maximal singular boundary component.

These maximal singular boundary components give rise to the definition of special directional distances which we will need in chapters 6 and 7.

DEFINITION 3.14 *If $\xi_i \in \partial X^i$, then $G\cdot\xi_i = \partial X^i$, and we define the i-th* singular distance

$$\begin{array}{rccl} d_i : & X \times X & \to & [0,\infty) \\ & (x,y) & \mapsto & \mathcal{B}_{G\cdot\xi_i}(x,y)\,. \end{array}$$

Since the set of maximal singular directions spans $\mathfrak{a}$ and belongs to $\overline{\mathfrak{a}^+}$, every element $H \in \overline{\mathfrak{a}^+}$ can be written as a linear combination of the H_i with nonnegative coefficients, which depend only on the root system of $(\mathfrak{g},\mathfrak{a})$. We therefore introduce for $1 \le i \le r$ the linear functionals

$$\begin{array}{rccl} c^i : & \overline{\mathfrak{a}^+} & \to & [0,\infty) \\ & H & \mapsto & c^i(H) \end{array}$$

uniquely determined by the condition $H = \sum_{i=1}^r c^i(H)H_i$ for $H \in \overline{\mathfrak{a}^+}$. For Riemannian products $X = X_1 \times X_2 \times \cdots \times X_r$ of rank one symmetric spaces X_i, the maximal singular directions form an orthonormal base and we have $c^i = \langle H_i, \cdot\rangle$. In this case the i-th singular distance $d_i(x,y)$ equals the distance in X_i of the projections of x and y to the factor X_i. This implies $d_i(x,y) = 0$ for every pair of points $x, y \in X$ which project to the same point in X_i.

For fixed $i \in \{1,2\ldots r\}$, we introduce the coefficients q_i^l, $1 \le l \le r$, uniquely determined by the equation $\langle H_i, \cdot\rangle = \sum_{l=1}^r q_i^l \alpha_l$. If $H \in \overline{\mathfrak{a}}_1^+$ denotes the Cartan projection of a point $y \in X$, we have the following formula for the maximal singular distances

$$d_i(x_0,y) = \left(\sum_{l=1}^r q_i^l\,\alpha_l(H)\right) d(x_0,y)\,, \quad 1 \le i \le r\,.$$

As an easy consequence of Proposition 3.12 concerning the Buseman functions for the directional distance we obtain the following

PROPOSITION 3.15 *Let $\Theta \subset \Upsilon$ arbitrary, and $(y_n) \subset X$ a sequence converging to $\eta \in \partial X^\Theta$ in the cone topology. Then for any $i \in \{1,2,\ldots,r\}$ with $\Theta \subseteq \Theta^i$, the functions $d_i(x_0,y_n) - d_i(\cdot,y_n)$ converge in the space of continuous functions $\mathrm{C}^0(\overline{X})$ to $\mathcal{B}_{\eta_i}(x,\cdot)$ with respect to the topology of uniform convergence on compact sets. Here $\eta_i \in \partial X^i$ denotes the unique element such that η and η_i are points in the closure of a common Weyl chamber at infinity.*

3.5 The case $SL(n,\mathbb{R})/SO(n)$

In this section, we illustrate the above notions for $SL(n,\mathbb{R})/SO(n)$, endowed with the left invariant metric described in section 1.6. We use the standard choice of positive Weyl chamber

$$\mathfrak{a}^+ = \{\mathrm{Diag}(t_1, t_2, \ldots, t_n) \mid t_1 > t_2 > \ldots > t_n, \sum_{i=1}^{n} t_i = 0\},$$

and, in order to get a more symmetric parametrization of $\mathfrak{a}^+$, introduce the endomorphism

$$\begin{array}{rccl} D: & \mathbb{R}^r & \to & \mathfrak{a} \\ & x & \mapsto & \mathrm{Diag}(d_1(x), d_2(x), \ldots, d_n(x)), \end{array}$$

where

$$d_i(x) = \sum_{j=1}^{r}(r-j+1)x_j - (r+1)\sum_{j=1}^{i-1} x_j = \sum_{j=i}^{r}(r+1)x_j - \sum_{j=1}^{r} jx_j, \quad 1 \le i \le r. \quad (3.1)$$

We remark that every element in $\overline{\mathfrak{a}^+}$ can be written as the image under D of a vector in $\mathbb{R}^r$ with nonnegative components. The barycenter $H_* \in \mathfrak{a}^+$ is given by

$$H_* = \frac{D(1,1,\ldots,1)}{\|D(1,1,\ldots,1)\|},$$

the i-th maximal singular direction equals the image of the i-th standard base vector e_i in $\mathbb{R}^r$ under the map D divided by its norm

$$H_i = \frac{D(e_i)}{\|D(e_i)\|}.$$

If $H = D(x) = \sum_{i=1}^{r} x_i D(e_i) \in \overline{\mathfrak{a}^+}$, we easily calculate

$$c^i(H) = x_i \|D(e_i)\| \ge 0 \quad \text{for} \quad 1 \le i \le r.$$

Lemma 3.16 *Let $H, H' \in \overline{\mathfrak{a}^+} \setminus \{0\}$. Then* $\langle H, H' \rangle > 0$.

Proof. Since $H, H' \in \overline{\mathfrak{a}^+}$, there exist vectors $x, y \in \mathbb{R}^r \setminus \{0\}$ with nonnegative components such that $H = D(x) = \sum_{i=1}^{r} x_i D(e_i)$ and $H' = D(y) = \sum_{i=1}^{r} y_i D(e_i)$. Then

$$\langle H, H' \rangle = \sum_{l=1}^{r}\sum_{k=1}^{r} x_l y_k \langle D(e_l), D(e_k) \rangle. \quad (3.2)$$

We calculate $\langle D(e_l), D(e_k) \rangle$ for $1 \le l < k \le r$. Using (3.1) and $n = r+1$, we obtain

$$\begin{aligned} \langle D(e_l), D(e_k) \rangle &= \sum_{i=1}^{n} d_i(e_l) d_i(e_k) = \sum_{i=1}^{l} (r+1-l)(r+1-k) + \sum_{i=l+1}^{k} (-l)(r+1-k) \\ &\quad + \sum_{i=k+1}^{n} (-l)(-k) = l(n-l)(n-k) - (k-l)l(n-k) + (n-k)lk \\ &= (n-k)(ln - l^2 - kl + l^2 + lk) = (n-k)ln \ge n > 0. \end{aligned}$$

For $1 \leq k < l \leq r$ we have $D(e_l), D(e_k)\rangle = \langle D(e_k), D(e_l)\rangle \geq 0$ by the above argument, if $1 \leq k = l \leq r$, then $D(e_l), D(e_k)\rangle = \|D(e_l)\|^2 > 0$.

Now assume $x_l y_k = 0$ for all $1 \leq l, k \leq r$. Since $H \neq 0$, there exists $l_0 \in \{1, 2, \ldots, r\}$ such that $x_{l_0} \neq 0$. Then $x_{l_0} y_k = 0$ for all $1 \leq k \leq r$ implies $y_k = 0$ for any $1 \leq k \leq r$ which is impossible by $H' \neq 0$. Hence there exists $k_0 \in \{1, 2, \ldots, r\}$ such that $x_{l_0} y_{k_0} > 0$. We conclude from (3.2) $\langle H, H'\rangle \geq x_{l_0} y_{k_0} \langle D(e_{l_0}), D(e_{k_0})\rangle > 0$. □

A consequence of the previous lemma and the proof of Lemma 3.9 is the following

COROLLARY 3.17 *If $X = SL(n,\mathbb{R})/SO(n)$, then for any $\xi \in \partial X$, $\mathcal{B}_{G\cdot\xi}$ is a (possibly nonsymmetric) distance.*

3.6 Angle estimates in symmetric spaces

The last result of this chapter will be needed in chapters 6 and 7. For $x, y \in X$, $x \neq y$, and $\xi \in \partial X$ we put

$$\angle_x(y, G\cdot\xi) := \inf_{g\in G} \angle_x(y, g\xi)\,.$$

LEMMA 3.18 *Let $x, y, \overline{x}, \overline{y} \in X$, $R := \min\{d(x,y), d(\overline{x},\overline{y})\}$, $c := d(x,\overline{x}) + d(y,\overline{y})$. Suppose $0 \leq \varphi_1 < \angle_{\overline{x}}(\overline{y}, G\cdot\xi) < \varphi_2 \leq \pi/4$. Then*

$$\begin{aligned} \angle_x(y, G\cdot\xi) &> \frac{1}{2}\varphi_1 \qquad \textit{if} \qquad R \geq \frac{12c}{\varphi_1^2}\,, \\ \angle_x(y, G\cdot\xi) &< 2\varphi_2 \qquad \textit{if} \qquad R \geq \frac{4c}{\varphi_2^2}\,. \end{aligned}$$

Proof. We are going to use the estimate for the cosine

$$1 - \frac{\varphi^2}{2} \leq \cos\varphi \leq 1 - \frac{\varphi^2}{2} + \frac{\varphi^4}{24}$$

which is valid for $\varphi \in [-5, 5]$. Using the triangle inequality and the inequality $\frac{1}{1-s} \leq 1+2s$ for $s \leq \frac{1}{2}$, we compute

$$\begin{aligned} \cos\angle_x(y, G\cdot\xi) &= \frac{\mathcal{B}_{G\cdot\xi}(x,y)}{d(x,y)} \leq \frac{\mathcal{B}_{G\cdot\xi}(\overline{x},\overline{y})}{d(\overline{x},\overline{y}) - d(x,\overline{x}) - d(y,\overline{y})} + \frac{\mathcal{B}_{G\cdot\xi}(x,\overline{x}) + \mathcal{B}_{G\cdot\xi}(\overline{y},y)}{d(x,y)} \\ &< \frac{\cos\varphi_1}{1 - \frac{C}{R}} + \frac{c}{R} \leq \cos\varphi_1\big(1 + \frac{2c}{R}\big) + \frac{c}{R}\,. \end{aligned}$$

If $R \geq \frac{12c}{\varphi_1^2}$ we further conclude

$$\begin{aligned} \cos\angle_x(y, G\cdot\xi) &\leq 1 - \frac{\varphi_1^2}{2} + \frac{\varphi_1^4}{24} + \frac{2c}{R} - \frac{c\varphi_1^2}{R} + \frac{c\varphi_1^4}{12R} + \frac{c}{R} \\ &\leq 1 - \frac{\varphi_1^2}{4} + \frac{\varphi_1^4}{24} + \frac{\varphi_1^6}{144} \leq 1 - \frac{\varphi_1^2}{8} \leq \cos\frac{\varphi_1}{2}. \end{aligned}$$

To prove the upper bound we estimate

$$\begin{aligned}\cos\angle_x(y, G\cdot\xi) &= \frac{\mathcal{B}_{G\cdot\xi}(x,y)}{d(x,y)} \geq \frac{\mathcal{B}_{G\cdot\xi}(\overline{x},\overline{y})}{d(\overline{x},\overline{y}) + d(x,\overline{x}) + d(y,\overline{y})} - \frac{\mathcal{B}_{G\cdot\xi}(x,\overline{x}) + \mathcal{B}_{G\cdot\xi}(\overline{y},y)}{d(x,y)} \\ &> \frac{\cos\varphi_2}{1+\frac{c}{R}} - \frac{c}{R} \geq \cos\varphi_2\big(1 - \frac{c}{R}\big) - \frac{c}{R}\,,\end{aligned}$$

since $\frac{1}{1+s} \geq 1 - s$ for $s > -1$. If $R \geq \frac{4c}{\varphi_2^2}$ we obtain

$$\begin{aligned}\cos\angle_x(y, G\cdot\xi) &\geq 1 - \frac{\varphi_2^2}{2} - \frac{c}{R} + \frac{c\varphi_2^2}{2R} - \frac{c}{R} \geq 1 - \frac{\varphi_2^2}{2} - \frac{2c}{R} \\ &\geq 1 - \varphi_2^2 \geq 1 - 2\varphi_2^2 + \frac{2}{3}\varphi_2^4 \geq \cos(2\varphi_2).\end{aligned}$$

□

Chapter 4

Dynamics of isometries

In this chapter, we define the limit set of discrete isometry groups of globally symmetric spaces X of noncompact type. We further introduce the radial limit set, an important subset of the limit set.

We then give a geometric classification and an algebraic characterization of individual isometries, describe some of their dynamical properties and relate their fixed points to the limit set.

4.1 The limit set

DEFINITION 4.1 *A subgroup Γ of the isometry group of a globally symmetric space X of noncompact type is called* discrete*, if the orbit $\Gamma\cdot x$ of a point $x \in X$ is discrete in X.*

The limit set L_Γ *of Γ is defined by* $L_\Gamma := \overline{\Gamma\cdot x} \cap \partial X\,, \quad x \in X\,.$

REMARK. If $\Gamma\cdot x$ is discrete in X for some point $x \in X$, then $\Gamma\cdot y$ is discrete in X for any $y \in X$. Furthermore, the limit set is independent of the choice of $x \in X$.

Note that the above definitions can be extended to isometry groups of any Hadamard manifold. For our purposes, we will need a precise description of the possible limit points of discrete groups of isometries $\Gamma \subset G = \mathrm{Isom}^o(X)$ in the case of a globally symmetric space X of noncompact type.

DEFINITION 4.2 *The equivalence class $[\mathcal{C}] \in \partial X^F$ is called a* radial limit Weyl chamber *for the action of Γ, if and only if there exists a sequence $(\gamma_j) \subset \Gamma$ such that $(\gamma_j x)$, $x \in X$, remains at a bounded distance of every Weyl chamber asymptotic to $\mathcal{C}$.*

We remark that the set of radial limit Weyl chambers equals the set L'_Γ defined by P. Albuquerque ([Al]). This notion, however, is too coarse for the study of the limit set in the geometric boundary.

Definition 4.3 *We call a boundary point* $\xi \in \partial X$ *a* radial limit point *for the action of* Γ*, if and only if there exists a sequence* $(\gamma_j) \subset \Gamma$ *and a Weyl chamber* $\mathcal{C} \subset X$ *with* $\xi \in \overline{\mathcal{C}} \cap \partial X$ *such that for any* $y \in X$ *the sequence* $(\gamma_j y)$ *remains at a bounded distance of* $\mathcal{C}$ *and converges to* ξ *in the cone topology. The set of radial limit points in* ∂X *is called the* radial limit set *of* Γ *and will be denoted by* L_Γ^{rad}.

Notation. If $x \in X$, $\eta \in \partial X^{reg}$, let $\mathcal{C}_{x,\eta} \subset X$ denote the unique open Weyl chamber with apex x which contains the geodesic ray $\sigma_{x,\eta}$, and $\overline{\mathcal{C}}_{x,\eta}$ its closure in $\overline{X}$.

We will now give equivalent definitions for radial limit points.

Lemma 4.4 *Let* $x, y \in X$ *arbitrary. A boundary point* $\xi \in \partial X$ *belongs to the radial limit set* L_Γ^{rad} *if and only if there exists a constant* $c > 0$ *and* $\eta \in \partial X^{reg}$ *such that* $\xi \in \overline{\mathcal{C}}_{x,\eta}$ *and if for any* $\varphi > 0$ *infinitely many* $\gamma \in \Gamma$ *satisfy*

$$d(\gamma y, \mathcal{C}_{x,\eta}) < c \quad \textit{and} \qquad \angle_x(\gamma y, G\cdot\xi) < \varphi\,.$$

Proof. Let $x, y \in X$ arbitrary and $\xi \in L_\Gamma^{rad}$. Then there exists a sequence $(\gamma_j) \subset \Gamma$ and a Weyl chamber $\mathcal{C} \subset X$ with $\xi \in \overline{\mathcal{C}}$ such that $d(\gamma_j y, \mathcal{C})$ remains bounded and $\gamma_j y$ converges to ξ in the cone topology. Choose $\eta \in \overline{\mathcal{C}} \cap \partial X^{reg}$. Then $\mathcal{C}$ is asymptotic to $\mathcal{C}_{x,\eta}$ and there exists a constant $c > 0$ such that $d(\gamma_j y, \mathcal{C}_{x,\eta}) < c$ for all $j \in \mathbb{N}$.

Let $\varphi \in (0, \pi/4)$ arbitrary. The convergence of $(\gamma_j y)$ to ξ implies that for any $R >> 1$, $\varepsilon < \frac{R}{2}\varphi$ there exists $N_0 \in \mathbb{N}$ such that for $j > N_0$ we have $d(x, \gamma_j y) > R$ and

$$d(\sigma_{x,\gamma_j y}(R), \sigma_{x,\xi}(R)) < \varepsilon < \frac{R}{2}\,\varphi\,.$$

Using comparison with Euclidean geometry, we estimate

$$\frac{R^2}{4}\varphi^2 > d(\sigma_{x,\gamma_j y}(R), \sigma_{x,\xi}(R))^2 \geq R^2 + R^2 - 2R^2 \cos\angle_x(\gamma_j y, \xi)\,,$$

which, together with $\angle_x(\gamma_j y, G\cdot\xi) = \inf_{g\in G}\angle_x(\gamma_j y, g\xi)$, implies

$$\begin{aligned}\cos\angle_x(\gamma_j y, G\cdot\xi) &\geq \cos\angle_x(\gamma_j y, \xi) > 1 - \frac{1}{8}\varphi^2\\ &\geq 1 - \frac{1}{2}\varphi^2 + \frac{1}{24}\varphi^4 > \cos\varphi\,.\end{aligned}$$

Conversely, let $\eta \in \partial X^{reg}$ such that $\xi \in \overline{\mathcal{C}}_{x,\eta} \cap \partial X$. By hypothesis, there exists a sequence $(\gamma_j) \subset \Gamma$ such that $d(\gamma_j y, \mathcal{C}_{x,\eta})$ is bounded and $\angle_x(\gamma_j y, G\cdot\xi) \to 0$ as $j \to \infty$. It remains to show that $\gamma_j y$ converges to ξ in the cone topology. Let $R >> 1$, $\varepsilon > 0$ arbitrary and put $d_j := d(x, \gamma_j y)$. For j sufficiently large, we have $d(x, \gamma_j y) > R$, because Γ is discrete. Choose a sequence $(\xi_j) \subset \overline{\mathcal{C}}_{x,\eta}$ such that $d(\gamma_j y, \mathcal{C}_{x,\eta}) = d(\gamma_j y, \sigma_{x,\xi_j})$. Using the convexity of the distance function and the fact, that the triangle $\Delta(x, \xi_j, \xi)$ is flat, we conclude

$$\begin{aligned}d(\sigma_{x,\gamma_j y}(R), \sigma_{x,\xi}(R)) &\leq d(\sigma_{x,\gamma_j y}(R), \sigma_{x,\xi_j}(R)) + d(\sigma_{x,\xi_j}(R), \sigma_{x,\xi}(R))\\ &\leq \frac{R}{d_j} d(\gamma_j y, \sigma_{x,\xi_j}(d_j)) + R\angle_x(\xi_j, \xi)\,. \qquad (4.1)\end{aligned}$$

Next let $t_j > 0$ such that $d(\gamma_j y, \sigma_{x,\xi_j}(t_j)) = d(\gamma_j y, \sigma_{x,\xi_j})$. Then by the triangle inequality

$$d_j = d(x, \gamma_j y) \le d(x, \sigma_{x,\xi_j}(t_j)) + d(\sigma_{x,\xi_j}(t_j), \gamma_j y) \le t_j + c\,,$$

and $d_j \ge t_j - c$, which gives

$$d(\gamma_j y, \sigma_{x,\xi_j}(d_j)) \le d(\gamma_j y, \sigma_{x,\xi_j}(t_j)) + d(\sigma_{x,\xi_j}(t_j), \sigma_{x,\xi_j}(d_j)) \le 2c\,.$$

Hence for j sufficiently large, we have $\frac{R}{d_j} d(\gamma_j y, \sigma_{x,\xi_j}(d_j)) \le \frac{R}{d_j} 2c < \frac{\varepsilon}{2}$ and $R\angle_x(\xi_j, \xi) < \frac{\varepsilon}{2}$, because $\angle_x(\gamma_j y, G\cdot\xi) = \angle_x(\xi_j, \xi)$ tends to zero as $j \to \infty$. Hence (4.1) and the fact that $d_j \to \infty$ as $j \to \infty$ prove that $\gamma_j y$ converges to ξ in the cone topology. □

The following corollary relates our definiton of the radial limit set to a common definition in the case of Hadamard manifolds with a negative upper bound for the sectional curvature.

COROLLARY 4.5 *Let $x, y \in X$ arbitrary. If rank(X)=1, then $\xi \in L_\Gamma^{rad}$ if and only if there exists a constant $c > 0$ such that infinitely many $\gamma \in \Gamma$ satisfy*

$$d(\gamma y, \sigma_{x,\xi}) < c\,.$$

Proof. Let $x, y \in X$ arbitrary and $\xi \in L_\Gamma^{rad}$. In rank one spaces, Weyl chambers reduce to geodesic rays and the first condition in Lemma 4.4 therefore implies the assertion. Conversely, if $d(\gamma y, \sigma_{x,\xi})$ is bounded for infinitely many $\gamma \in \Gamma$, there exists a sequence $(\gamma_j) \subset \Gamma$ such that $(\gamma_j y)$ remains at bounded distance of a geodesic ray asymptotic to ξ. In particular, $\gamma_j y$ converges to ξ in the cone topology. □

Given a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ for $X = G/K$ with respect to some point $x_0 \in X$, we have the following characterization of radial limit points.

COROLLARY 4.6 *Let $\Theta \subset \Upsilon$. The boundary point $\xi = (k_\xi, H_\xi) \in \partial X^\Theta$ is a radial limit point for the action of Γ if and only if there exists a constant $c > 0$ and $m \in M_\Theta$ such that for any $\varphi > 0$ infinitely many $\gamma \in \Gamma$ satisfy*

$$d(\gamma x_0, k_\xi m k_\xi^{-1} e^{\mathfrak{a}^+} x_0) < c \quad \textit{and} \quad \angle(H_\gamma, H_\xi) < \varphi\,.$$

Here $H_\gamma \in \overline{\mathfrak{a}_1^+}$ denotes the unit length Cartan projection of γx_0.

Proof. Let $\xi \in L_\Gamma^{rad}$, $\Theta \subset \Upsilon$ such that $\xi \in \partial X^\Theta$, and $\varphi \in (0, \pi/4)$ arbitrary. Lemma 4.4 implies the existence of $c > 0$, a point $\eta \in \partial X^{reg}$ such that $\xi \in \overline{\mathcal{C}}_{x_0,\eta}$ and infinitely many $\gamma \in \Gamma$ satisfy

$$d(\gamma x_0, \mathcal{C}_{x_0,\eta}) < c \quad \text{and} \qquad \angle_{x_0}(\gamma x_0, G\cdot\xi) < \varphi\,.$$

Furthermore, if $k_\eta \in K$ denotes an angular projection of η, then $\xi = (k_\xi, H_\xi) \in \overline{\mathcal{C}}_{x_0,\eta}$ implies that k_η belongs to the parabolic subgroup which stabilizes ξ. Hence $k_\eta = k_\xi m k_\xi^{-1}$ for some $m \in M_\Theta$. The first part of the statement now follows from $\mathcal{C}_{x_0,\eta} = k_\eta e^{\mathfrak{a}^+} x_0$. For the second assertion we simply remind of the fact $\cos\angle_{x_0}(\gamma x_0, G\cdot\xi) = \langle H_\gamma, H_\xi\rangle\,.$

Conversely, let $x, y \in X$, $\Theta \subset \Upsilon$ such that $\xi = (k_\xi, H_\xi) \in \partial X^\Theta$, and $m \in M_\Theta$. By the assumption, there exists a sequence $(\gamma_j) \subset \Gamma$ with Cartan projections $(H_j) \subset \overline{\mathfrak{a}^+}$ such that $d(\gamma_j x_0, k_\xi m k_\xi^{-1} e^{\mathfrak{a}^+} x_0) < c$ for some constant $c > 0$, and $\langle H_j, H_\xi \rangle \to 1$.

Choose $H_\eta \in \mathfrak{a}_1^+$, put $k_\eta := k_\xi m k_\xi^{-1}$ and $\eta := (k_\eta, H_\eta) \in \partial X^{reg}$. Then $\mathcal{C}_{x_0,\eta} = k_\eta e^{\mathfrak{a}^+} x_0$, and the Weyl chambers $\mathcal{C}_{x_0,\eta}$ and $\mathcal{C}_{x,\eta}$ have bounded Hausdorff distance $d \geq 0$ since they are asymptotic. Therefore

$$d(\gamma_j y, \mathcal{C}_{x,\eta}) \leq d(\gamma_j y, \gamma_j x_0) + d(\gamma_j x_0, \mathcal{C}_{x_0,\eta}) + d < d(y, x_0) + c + d =: c_0 \,.$$

Using the directional distance from section 3.2, we further conclude from the triangle inequality and the remark after Definition 3.4

$$\begin{aligned} \cos \angle_x(\gamma_j y, G{\cdot}\xi) &= \frac{\mathcal{B}_{G\cdot\xi}(x, \gamma_j y)}{d(x, \gamma_j y)} \geq \frac{\mathcal{B}_{G\cdot\xi}(x_0, \gamma_j x_0) - \mathcal{B}_{G\cdot\xi}(x_0, x) - \mathcal{B}_{G\cdot\xi}(\gamma_j y, \gamma_j x_0)}{d(x_0, \gamma_j x_0) + d(x_0, x) + d(x_0, y)} \\ &\geq \frac{\langle H_j, H_\xi \rangle - \frac{d(x_0,y)+d(x_0,x)}{d(x_0,\gamma_j x_0)}}{1 + \frac{d(x_0,y)+d(x_0,x)}{d(x_0,\gamma_j x_0)}} \to 1 \quad \text{as} \quad j \to \infty \,, \end{aligned}$$

hence the proof is complete. □

4.2 Convergence in horospherical coordinates

Fix a Cartan decomposition $G = K e^{\overline{\mathfrak{a}^+}} K$ of $X = G/K$ with respect to to the base point $x_0 \in X$ and some regular boundary point. Let $\Theta \subset \Upsilon$ denote a subset of some fundamental set of roots Υ. Lemma 2.9 and its proof show, that a sequence $(y_j) \subset X \cup \partial X$ converges to $\xi = (k_\xi, H_\xi) \in \partial X^\Theta$ in the cone topology, if and only if $d(x_0, y_j) \to \infty$, if every sequence of angular projections $(k_j) \subset K$ of (y_j) converges to k_ξ in K/M_Θ, and if the sequence of Cartan projections $(H_j) \subset \overline{\mathfrak{a}_1^+}$ converges to H_ξ.

Similarly, every sequence of angular projections $(k_j) \subset K$ of a sequence $(\gamma_j x) \subset X$ remaining at bounded distance of a Weyl chamber in the asymptote class $k_0 M \in K/M \cong \partial X^F$, converges to $k_0 M$ in K/M. We are now going to derive similiar properties for sequences converging towards a radial limit point in terms of horospherical coordinates.

Lemma 4.7 *Suppose $(\gamma_j y) \subset X$, $y \in X$, is a sequence converging towards a radial limit point $\xi \in \partial X$ at bounded distance of a Weyl chamber $\mathcal{C} \subset X$ with $\xi \in \overline{\mathcal{C}}$. Fix an Iwasawa decomposition $G = N^+ A K$ with respect to $x_0 \in X$ and the asymptote class of $\mathcal{C}$ in ∂X^F. Then the horospherical projections $(n_j) \subset N^+$ and the Iwasawa projections $(H_j) \subset \mathfrak{a}$ of $(\gamma_j y)$ satisfy*

$$d(e^{-H_j} n_j e^{H_j} x_0, x_0) \leq const\,, \qquad \|H_j\| \to \infty\,, \qquad \angle_{x_0}(e^{H_j} x_0, \xi) \to 0 \quad as \quad j \to \infty\,.$$

Proof. Let $G = N^+ A K$ be the Iwasawa decomposition as in the statement of the lemma. Since the Weyl chamber $\mathcal{C}_0 := e^{\mathfrak{a}^+} x_0 \subset X$ is asymptotic to $\mathcal{C}$, the sequence $(\gamma_j y)$ remains at a bounded distance of $\mathcal{C}_0$. Hence there exists a constant $c > 0$ and a sequence $(\overline{H}_j) \subset \mathfrak{a}^+$

such that $d(\gamma_j y, e^{\overline{H}_j}x_0) < c$. Writing $\gamma_j y := n_j e^{H_j}x_0$ in horospherical coordinates, we obtain by Lemma 1.11

$$d(e^{H_j}x_0, e^{\overline{H}_j}x_0) \le d(n_j e^{H_j}x_0, e^{\overline{H}_j}x_0) < c$$

and therefore

$$d(n_j e^{H_j}x_0, e^{H_j}x_0) \le d(n_j e^{H_j}x_0, e^{\overline{H}_j}x_0) + d(e^{\overline{H}_j}x_0, e^{H_j}x_0) < 2c\,, \tag{4.2}$$

which proves the first claim of the lemma.

We next show that $\|H_j\| \to \infty$. Suppose the contrary, and put $a := \sup_{j\in\mathbb{N}} \|H_j\|$. Then

$$\begin{aligned} d(x_0, n_j x_0) &\ge d(x_0, n_j e^{H_j}x_0) - d(n_j e^{H_j}x_0, n_j x_0) \\ &= d(x_0, \gamma_j y) - \|H_j\| \ge d(x_0,\gamma_j y) - a \;\to\; \infty \quad \text{as} \quad j\to\infty\,. \end{aligned} \tag{4.3}$$

Let $\varepsilon > 0$ arbitrary small. By (4.2), for any $j \in \mathbb{N}$ there exists a curve $c_j : [0,1] \to X$, $c_j(t) := n(t)e^{H(t)}x_0$, such that $c_j(0) = e^{H_j}x_0$, $c_j(1) = n_j e^{H_j}x_0$ and

$$\int_0^1 \|\dot c_j(t)\| dt < d(e^{H_j}x_0, n_j e^{H_j}x_0) + \varepsilon < 2c + \varepsilon\,.$$

In particular, we have $\|H_j - H(t)\| = d(e^{H_j}x_0, e^{H(t)}x_0) \le d(e^{H_j}x_0, c_j(t)) < 2c+\varepsilon$ for all $t\in[0,1]$, and therefore by the Cauchy Schwartz inequality

$$\begin{aligned} \alpha(H(t)) &= \alpha(H_j) + \alpha(H(t) - H_j) \le \|\alpha\|\left(\|H_j\| + \|H(t) - H_j\|\right) \\ &\le \|\alpha\|\,(a + 2c + \varepsilon) \qquad \forall\, \alpha\in\Sigma^+\,. \end{aligned}$$

Put $Z_j(t) := DL_{n(t)^{-1}} \frac{d}{ds}\big|_{s=t} n(s) \in \mathfrak{n}^+$, and $a_0 := \max_{\alpha\in\Sigma^+} \|\alpha\|\,(a+2c+\varepsilon)$. Using the pullback metric $\mu^* g$ from section 1.3 on N^+Ax_0, we estimate

$$\|\dot c_j(t)\|^2 \ge \frac{1}{2} e^{-2a_0}\langle Z_j(t), Z_j(t)\rangle$$

and obtain by (4.3)

$$2c + \varepsilon > d(n_j e^{H_j}x_0, e^{H_j}x_0) + \varepsilon \ge e^{-a_0} d(n_j x_0, x_0) \to \infty\,,$$

a contradiction. We conclude $\|H_j\| \to \infty$.

The third assertion follows as in Lemma 2.9 from the fact that $\gamma_j y$ converges to ξ in the cone topology. □

COROLLARY 4.8 *Let $(\gamma_j y) \subset X$, $y \in X$, be a sequence converging to a radial limit point $\xi\in\partial X$ at bounded distance of a Weyl chamber $\mathcal{C}\subset X$ such that $\xi\in\overline{\mathcal{C}}$. Fix an Iwasawa decomposition $G = N^+AK$ with respect to $x_0\in X$ and the asymptote class of $\mathcal{C}$ in ∂X^F, and write $\gamma_j y = n_j e^{H_j}x_0$ in horospherical coordinates. Then*

$$d_N(n_j, \mathrm{id}) \le const\cdot \max_{\alpha\in\Sigma^+} e^{\alpha(H_j)} \qquad \textit{as} \quad j\to\infty\,,$$

where d_N denotes the left invariant distance on N^+ as described in section 1.3.

Proof. Inequality (4.2) and the calculations in the proof of the previous lemma show that for $\alpha_j := \max_{\alpha\in\Sigma^+}\alpha(H_j)$ and $a := \max_{\alpha\in\Sigma^+}\|\alpha\|(2c+\varepsilon)$ we have

$$2c+\varepsilon \;>\; d(n_j e^{H_j}x, e^{H_j}x)+\varepsilon > \int_0^1 \|\dot c_j(t)\|dt \geq \frac{1}{\sqrt{2}}e^{-\alpha_j-a}\int_0^1 \sqrt{\langle Z_j(t), Z_j(t)\rangle}dt\,.$$

We conclude $d_N(n_j,\mathrm{id}) \leq \int_0^1 \sqrt{\langle Z_j(t), Z_j(t)\rangle}dt < \sqrt{2}e^{\alpha_j}e^{a}(2c+\varepsilon)$ and therefore, by the definition of α_j,

$$d_N(n_j,\mathrm{id}) \leq \mathit{const}\cdot \max_{\alpha\in\Sigma^+} e^{\alpha(H_j)} \text{ as } j\to\infty.$$ □

4.3 Classification of isometries

In this section, we classify geometrically individual isometries of a globally symmetric space X of noncompact type. As in [BGS], chapter 6, we introduce the displacement function of $\gamma\in\mathrm{Isom}(X)$

$$\begin{array}{rrcl} d_\gamma: & X & \to & \mathbb{R} \\ & x & \mapsto & d(x,\gamma x)\,. \end{array}$$

Definition 4.9 *Let γ be a nontrivial isometry of X. We call γ* elliptic*, if γ fixes a point in X,* axial*, if d_γ assumes the infimum and* $\min_{x\in X} d_\gamma(x) > 0$.

We call γ strictly parabolic*, if d_γ does not assume the infimum and* $\inf_{x\in X} d_\gamma(x) = 0$*,* mixed parabolic*, if d_γ does not assume the infimum and* $\inf_{x\in X} d_\gamma(x) > 0$.

A parabolic *isometry is a strictly parabolic or a mixed parabolic isometry.*

The following propositions summarize a few properties of individual isometries of a globally symmetric space X of noncompact type. The proofs can be found in [Ba], chapter II.

Proposition 4.10 *([Ba], chapter II, Proposition 3.2) An isometry $\gamma\in Isom(X)\setminus\{\mathrm{id}\}$ is elliptic if and only if γ has a bounded orbit.*

Proposition 4.11 *([Ba], chapter II, Proposition 3.3) An isometry $\gamma\in Isom(X)\setminus\{\mathrm{id}\}$ is axial if and only if there exists a unit speed geodesic σ and a number $l_\gamma > 0$ such that $\gamma(\sigma(t)) = \sigma(t+l_\gamma)$ for all $t\in\mathbb{R}$.*

Proposition 4.12 *([Ba], chapter II, Proposition 3.4) If $\gamma\in Isom(X)\setminus\{\mathrm{id}\}$ is parabolic, then there exists a point $\eta\in\partial X$ such that γ fixes η and $\mathcal{B}_\eta(x,\gamma x) = 0$ for all $x\in X$.*

We remark that the above definitions and propositions can be extended to isometries of Hadamard manifolds. As a matter of fact, the statements in [Ba], chapter II, are given in this more general context.

In the case of a globally symmetric space $X = G/K$ of noncompact type, an Iwasawa decomposition $G = N^+AK$ gives rise to a natural algebraic characterization of certain individual isometries.

DEFINITION 4.13 *An isometry $\gamma \in G \setminus \{\mathrm{id}\}$ is called* elliptic*, if it is conjugate to an element in K,* hyperbolic*, if it is conjugate to an element in A, and* unipotent*, if it is conjugate to an element in N^+.*

The following lemma relates these definitions to the geometric classification as above.

LEMMA 4.14 *$\gamma \in G \setminus \{\mathrm{id}\}$ is conjugate to an element in K if and only if γ fixes a point in X. Hyperbolic isometries are axial, unipotent isometries are strictly parabolic.*

Proof. The first assertion is trivial, because the stabilizer of any point in X is conjugate to K, the stabilizer of $x_0 \in X$. The remaining assertions will be corollaries of Propositions 4.17 and 4.21. □

4.4 Properties of parabolic isometries

In this section we recollect a few properties of parabolic isometries of a globally symmetric space $X = G/K$ of noncompact type with base point $x_0 \in X$ corresponding to K. Proposition 4.12 implies that every parabolic isometry possesses a fixed point in ∂X^Θ for some subset $\Theta \subset \Upsilon$.

DEFINITION 4.15 *We call γ* Θ-parabolic*, if γ fixes a point in ∂X^Θ, but not in $\partial X^{\tilde\Theta}$ for any $\tilde\Theta \subset \Theta$. If γ fixes a regular point, we call γ* regular parabolic*.*

We remark that unipotent isometries are regular parabolic. If γ is parabolic with fixed point $\eta \in \partial X$ as in Proposition 4.12, then the limit set of the cyclic group $\langle\gamma\rangle$ generated by γ is contained in the boundary of every horosphere centered at η.

Note that in rank one spaces X, a parabolic isometry γ possesses a unique fixed point η in the geometric boundary ∂X. Furthermore, for any $\xi \in \partial X$ the sequences $(\gamma^j\xi)$ and $(\gamma^{-j}\xi)$ converge to η (see [DaK]). Since the geometric boundary is a disjoint union of only two Bruhat cells $\partial X = \mathrm{Vis}^\infty(\eta) \cup \{\eta\}$, this is equivalent to the statement that the limits of $(\gamma^j\xi)$ and $(\gamma^{-j}\xi)$ do not belong to $\mathrm{Vis}^\infty(\eta)$. Unfortunately, this remains no longer true in higher rank symmetric spaces. For unipotent isometries, however, we have similar dynamics on the geometric boundary.

PROPOSITION 4.16 *Let γ be a unipotent isometry, and $\eta \in \partial X$ a fixed point of γ. Then either γ fixes $\sigma_{x_0,\eta}(-\infty)$, or, for any $\xi \in Vis^\infty(\iota\eta)$, the limits of the sequences $(\gamma^j\xi)$, $(\gamma^{-j}\xi) \subset \partial X$ do not belong to $Vis^\infty(\iota\eta)$.*

Proof. Fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ and an Iwasawa decompositon $G = N^+AK$ of $X = G/K$ with respect to x_0 and η. Let $\Theta \subset \Upsilon$ such that $\eta \in \partial X^\Theta$. Then the Cartan projection H_η of η belongs to $\mathfrak{a}_1^\Theta$ and $\gamma \in N^+ \setminus \{\mathrm{id}\}$. By Corollary 2.14, $\xi \in \mathrm{Vis}^\infty(\iota\eta)$ implies the existence of $n_0 \in N_\Theta^+$ such that $\xi = n_0\sigma_{x_0,\iota\eta}(-\infty)$. Write

$$\gamma = \exp\Big(\sum_{\alpha\in\Sigma^+} Y_\alpha\Big)\,, \quad n_0 = \exp\Big(\sum_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} Z_\alpha\Big)\,, \qquad Y_\alpha,\, Z_\alpha \in \mathfrak{g}_\alpha\,.$$

Suppose $Y_\alpha = 0$ for all $\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$. Then for all $H \in \mathfrak{a}^\Theta$ we have $e^H \gamma e^{-H} = \gamma$, because $\mathrm{ad}(H)Z = 0$ for every $Z \in \sum_{\alpha\in\langle\Theta\rangle^+} \mathfrak{g}_\alpha$. We conclude

$$d(\sigma_{x_0,\eta}(-t), \gamma\sigma_{x_0,\eta}(-t)) = d(x_0, e^{H_\eta t}\gamma e^{-H_\eta t}x_0) = d(x_0,\gamma x_0)$$

for any $t \in \mathbb{R}$. In particular, γ fixes $\sigma_{x_0,\eta}(-\infty)$.

If γ does not fix $\sigma_{x_0,\eta}(-\infty)$, we choose $\beta \in \Sigma^+ \setminus \langle\Theta\rangle^+$ such that $\|\beta\| \le \|\alpha\|$ for any $\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+$ with the property $Y_\alpha \neq 0$. For $j \in \mathbb{N}$, we write

$$\gamma^{\pm j} n_0 = \exp(\sum_{\alpha\in\Sigma^+} Y_\alpha^{(\pm j)})\,, \quad Y_\alpha^{(\pm j)} \in \mathfrak{g}_\alpha\,.$$

Then $[\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}] \subseteq \mathfrak{g}_{\alpha+\alpha'}$ $\forall\, \alpha,\alpha' \in \Sigma^+$ and the Campbell Hausdorff formula imply

$$Y_\beta^{(\pm j)} = \pm j Y_\beta + Z_\beta + f(n_0)\,,$$

where $f(n_0) \in \mathfrak{g}_\beta$ is a term consisting of successive Lie brackets of the Z_α. In particular, $\|f(n_0)\|$ is bounded, and therefore $\|Y_\beta^{(j)}\|$ and $\|Y_\beta^{(-j)}\|$ tend to infinity as $j \to \infty$. This implies $\lim_{j\to\infty} \gamma^j n_0 \notin N_\Theta^+$ and $\lim_{j\to\infty} \gamma^{-j} n_0 \notin N_\Theta^+$.

Suppose $\xi^+ := \lim_{j\to\infty}\gamma^j\xi \in \mathrm{Vis}^\infty(\iota\eta)$. Then by Corollary 2.14, there exists $n \in N_\Theta^+$ such that $\xi^+ = n\sigma_{x_0,\iota\eta}(-\infty)$. Since $\gamma^j\xi = \gamma^j n_0 \sigma_{x_0,\iota\eta}(-\infty)$ this is impossible because $\lim_{j\to\infty}\gamma^j n_0 \notin N_\Theta^+$. Analogously, we obtain $\lim_{j\to\infty}\gamma^{-j}\xi \notin \mathrm{Vis}^\infty(\iota\eta)$. □

Using Proposition 4.12 we will now give an equivalent definition for regular parabolic isometries in terms of an Iwasawa decomposition $G = N^+AK$.

PROPOSITION 4.17 *Let $G = N^+AK$ be an Iwasawa decomposition of G. An isometry $\gamma \in G\setminus\{\mathrm{id}\}$ is regular parabolic, if and only if γ is conjugate to nam, where $n \in N^+\setminus\{\mathrm{id}\}$, $m \in M$ and $a \in A$ such that $\langle \log a, H\rangle = 0$ for some $H \in \mathfrak{a}_1^+$. Here $\log a$ denotes the unique element in $\mathfrak{a}$ with the property $\exp(\log a) = a$.*

Proof. Let $\gamma \in G$ be regular parabolic. Then γ possesses a fixed point $\eta = (k,H) \in \partial X^{reg}$ by Lemma 4.14 and Proposition 4.12. This implies that γ belongs to the stabilizer of η, i.e. $k\gamma k^{-1} \in P = MAN^+$. Write $\gamma = knamk^{-1}$ with $n \in N^+$, $a \in A$ and $m \in M$. If $n = \mathrm{id}$, then Proposition 4.21 will show that γ is axial in contradiction to our assumption. Therefore $n \neq \mathrm{id}$.

We further conclude from $\mathcal{B}_\eta(x_0,\gamma x_0) = 0$ using Lemma 3.3

$$0 = \mathcal{B}_\eta(x_0, knamk^{-1}x_0) = \mathcal{B}_{k^{-1}\eta}(x_0, nax_0) = \langle H, \log a\rangle\,.$$

Conversely, let $\gamma = nam$, where $n \in N^+\setminus\{\mathrm{id}\}$, $m \in M$ and $a \in A$ such that $\langle \log a, H\rangle = 0$ for some $H \in \mathfrak{a}_1^+$. Suppose there exists $x \in X$ such that $d(x,\gamma x) = \inf_{x\in X} d_\gamma(x)$. Write $x = \tilde{n}\tilde{a}x_0$ in horospherical coordinates and consider the unit speed geodesic $\sigma(t) := \tilde{n}\tilde{a}e^{Ht}x_0$. We compute

$$\begin{aligned} d(\sigma(t),\gamma\sigma(t)) &= d(\tilde{n}\tilde{a}e^{Ht}x_0, nam\tilde{n}\tilde{a}e^{Ht}x_0) \\ &= d(x_0, e^{-Ht}\tilde{a}^{-1}\tilde{n}^{-1}nam\tilde{n}m^{-1}\tilde{a}e^{Ht}x_0) = d(x_0, \overline{n}(t)ax_0)\,, \end{aligned}$$

where $\overline{n}(t) := e^{-Ht}\tilde{a}^{-1}\tilde{n}^{-1}nam\tilde{n}m^{-1}a^{-1}\tilde{a}e^{Ht} \in N^+$ for any $t \in \mathbb{R}$.

Suppose $\overline{n}(0) = \text{id}$. Then $\overline{n}(t) = \text{id}$ for all $t \in \mathbb{R}$ and we have $d(\sigma(t), \gamma\sigma(t)) = d(x_0, ax_0)$ for all $t \in \mathbb{R}$. In particular, γ fixes $\sigma(-\infty) \in \partial X^{reg}$ which is impossible by $n \neq \text{id}$. Hence $\overline{n}(0) \in N^+ \setminus \{\text{id}\}$. Furthermore, we have $\overline{n}(t) \to \text{id}$ as $t \to \infty$, hence by Corollary 1.11,

$$\lim_{t\to\infty} d(\sigma(t), \gamma\sigma(t)) < d(\sigma(0), \gamma\sigma(0)) = d(x, \gamma x)\,.$$

The continuity of the function $t \mapsto d(\sigma(t), \gamma\sigma(t))$ now implies the existence of $t_0 > 0$ such that

$$d(\sigma(t_0), \gamma\sigma(t_0)) < d(x, \gamma x)\,,$$

a contradiction to the choice of $x \in X$. Hence $\inf_{x\in X} d_\gamma(x)$ is not assumed. □

4.5 Description of axial isometries

We begin this section with a few definitions concerning axial isometries of a globally symmetric space X of noncompact type.

Definition 4.18 *Let $\gamma \in Isom(X) \setminus \{\text{id}\}$ be axial. The number $l_\gamma := \min_{x\in X} d(x, \gamma x)$ is called the* translation length *of γ, the set $Ax(\gamma) := \{x \in X \mid d(x, \gamma x) = l_\gamma\}$ is called the* axis *of γ.*

$\text{Ax}(\gamma)$ is invariant under the action of the cyclic group $\langle\gamma\rangle$ and consists of the set of all geodesics translated by γ. The geometric boundary of $\text{Ax}(\gamma)$ equals the set of fixed points $\text{Fix}(\gamma)$ of γ in $\overline{X}$.

Since an axial isometry γ translates along some geodesic, $\text{Fix}(\gamma)$ consists of at least two points in the geometric boundary ∂X. We will see in the following chapter, that these two fixed points play an important role in the study of the dynamics of axial isometries.

Definition 4.19 *If $x \in X$, we call the limit of the sequence $(\gamma^j x)$ as $j \to \infty$ the* attractive fixed point *of γ, and the limit of $(\gamma^{-j}x)$ as $j \to \infty$ the* repulsive fixed point *of γ.*

It can be easily shown that this definition is again independent of the choice of $x \in X$. Since axial isometries translate along some geodesic, for $x \in \text{Ax}(\gamma)$ there exists an element $Y_\gamma \in T_xX$ such that

$$\gamma x = e^{Y_\gamma}x \in \text{Ax}(\gamma)\,.$$

As a consequence of the rich algebraic structure of symmetric spaces, we may further distinguish different kinds of axial isometries by means of a second characteristic besides the translation length. Let X be a globally symmetric space of noncompact type, fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ of $G = \text{Isom}^o(X)$ and let $x_0 \in X$ be the unique point stabilized by K. Then every unit speed geodesic $\sigma : \mathbb{R} \to X$ can be written in the following form with a unique vector $H \in \overline{\mathfrak{a}_1^+}$

$$\sigma(t) = ge^{Ht}x_0\,, \quad g \in G\,.$$

Hence there exists a unique element $L(\gamma) \in \overline{\mathfrak{a}_1^+}$ such that $\gamma x = g e^{L(\gamma)\, l_\gamma} x_0$ for some $g \in G$ with $g x_0 = x$.

DEFINITION 4.20 *We call* $L(\gamma) \in \overline{\mathfrak{a}_1^+}$ *the* translation direction *of* γ. *If* $L(\gamma) \in \mathfrak{a}_1^+$, *then* γ *is called* regular axial, *if* $L(\gamma) \in \mathfrak{a}_1^\Theta \setminus \bigcup_{\tilde\Theta \supsetneq \Theta} \mathfrak{a}^{\tilde\Theta}$, *then* γ *is called* Θ-axial.

We are now going to give an equivalent definition for axial isometries of a globally symmetric space $X = G/K$ of noncompact type with base point x_0 corresponding to K in terms of an Iwasawa decomposition $G = N^+AK$.

PROPOSITION 4.21 *An isometry* $\gamma \in G \setminus \{\mathrm{id}\}$ *is axial with translation length* l_γ *if and only if* γ *is conjugate to* $e^{Hl_\gamma} m$, *where* $H \in \overline{\mathfrak{a}_1^+}$ *and* $m \in \{k \in K \mid \mathrm{Ad}(k)H = H\}$.
Furthermore, $H \in \overline{\mathfrak{a}_1^+}$ *equals the translation direction of* γ.

Proof. (compare [E], Proposition 2.19.18 (3)) Let $\gamma \in G$ be axial, and let σ be a geodesic in X translated by γ. We first treat the case $\sigma(0) = x_0$ and $N^+AM\sigma(\infty) = \sigma(\infty)$. Then there exists $H \in \overline{\mathfrak{a}_1^+}$ such that $\sigma(t) = e^{Ht}x_0$ for all $t \in \mathbb{R}$. By hypothesis

$$(\gamma e^{Ht})x_0 = \gamma\sigma(t) = \sigma(t + l_\gamma) = e^{Ht}e^{Hl_\gamma}x_0$$

for all $t \in \mathbb{R}$. We obtain

$$\gamma e^{Ht} = e^{Ht}e^{Hl_\gamma}k(t)\,, \qquad k(t) \in K \quad \forall\, t \in \mathbb{R}\,. \tag{4.4}$$

Let Θ denote the smallest subset of Υ such that $H \in \mathfrak{a}^\Theta$. Since γ fixes ξ, the isometry $T_\Theta(\gamma) := \lim_{t\to\infty} e^{-Ht}\gamma e^{Ht}$ exists by Proposition 2.6. Hence (4.4) implies the existence of $m := \lim_{t\to\infty} k(t) \in K$ and therefore

$$T_\Theta(\gamma) = e^{Hl_\gamma}m\,. \tag{4.5}$$

Now e^{Hl_γ} commutes with e^{Hs} for any $s \in \mathbb{R}$, and, by definition of the map T_Θ, we obtain for any $s \in \mathbb{R}$

$$e^{-Hs}T_\Theta(\gamma)e^{Hs} = \lim_{t\to\infty} e^{-Hs}e^{-Ht}\gamma e^{Ht}e^{Hs} = \lim_{t'\to\infty} e^{-Ht'}\gamma e^{Ht'} = T_\Theta(\gamma)\,.$$

By (4.5) the element m commutes with e^{Hs} for all $s \in \mathbb{R}$. Proposition 2.7 further implies that $n := \gamma\,(T_\Theta(\gamma))^{-1} \in N_\Theta^+$, and from (4.5) we deduce

$$\gamma = ne^{Hl_\gamma}m\,.$$

Since γ, e^{Hl_γ} and m fix both ξ and $\sigma(-\infty)$, it follows that n fixes ξ and $\sigma(-\infty)$. The convex function $t \mapsto d(\sigma(t), n\sigma(t))$ is therefore bounded above on $\mathbb{R}$ and hence constant. From the fact that $n \in N_\Theta^+$, we deduce

$$d(\sigma(t), n\sigma(t)) = d(e^{Ht}x_0, ne^{Ht}x_0) = d(x_0, e^{-Ht}ne^{Ht}x_0) \to 0 \ \text{ as } \ t \to \infty\,.$$

Therefore n fixes every point of σ, including x_0. This implies $n = \mathrm{id}$, because $N^+_\Theta \cap M_\Theta = \{\mathrm{id}\}$ by the uniqueness of the generalized Iwasawa decomposition Proposition 2.7. We obtain

$$\gamma = e^{Hl_\gamma} m = m e^{Hl_\gamma}\,.$$

If the invariant geodesic σ of γ is arbitrary, there exists $g \in G$ such that $g\sigma(0) = x_0$ and $g\sigma(t) \in e^{\overline{\mathfrak{a}^+}} x_0$ for all $t > 0$. Furthermore, $\gamma\sigma(t) = \sigma(t + l_\gamma)$ implies that the isometry $g\gamma g^{-1}$ possesses an invariant geodesic as in the first part of the proof. Hence $\gamma = g^{-1} e^{Hl_\gamma} m g$ for some $H \in \overline{\mathfrak{a}_1^+}$ and $m \in \{k \in K \mid \mathrm{Ad}(k)H = H\}$.

Furthermore, for $x := \sigma(0) = g^{-1}x_0 \in \mathrm{Ax}(\gamma)$ we have

$$\gamma x = g^{-1} e^{Hl_\gamma} g g^{-1} x_0 = g^{-1} e^{Hl_\gamma} x_0\,,$$

hence, by definition, H equals the translation length of γ.

Conversely, let γ be conjugate to $e^{Hl_\gamma} m$ as above. The proof of Proposition 2.19.18 (3) in [E] shows that there exists a point $x \in X$ where the displacement functions of e^{Hl_γ} and m both assume their minimum. From $H \neq 0$ and $l_\gamma > 0$ it follows that d_γ assumes a positive minimum in X. □

Fix an Iwasawa decomposition $G = N^+AK$ of $G = \mathrm{Isom}^o(X)$. The final result of this chapter shows that the attractive and repulsive fixed points of a Θ-axial isometry belong to certain boundary components and are special radial limit points.

LEMMA 4.22 *Let $\Theta \subset \Upsilon$, and h be a Θ-axial isometry. Then the attractive fixed point h^+ belongs to ∂X^Θ, and the repulsive fixed point h^- belongs to $Vis^\infty(h^+) \subset \partial X^{\Theta^*}$. Furthermore, there exists a constant $c > 0$ such that for any $t > 0$ there exist isometries $\gamma_+, \gamma_- \in \langle h \rangle$ with the property*

$$d(\gamma_+ x_0, \sigma_{x_0,h^+}(t)) < c \qquad \text{and} \qquad d(\gamma_- x_0, \sigma_{x_0,h^-}(t)) < c\,.$$

Proof. Let $l > 0$ denote the translation length, and $L \in \mathfrak{a}_1^\Theta$ the translation direction of h. By Proposition 4.21 there exists $g \in G$ and $m \in M_\Theta$ such that $h = g e^{Ll} m g^{-1}$. We have $h^j = g e^{jLl} m^j g^{-1}$ and compute

$$\begin{aligned} d_j &:= d(x_0, h^j x_0) \geq d(gx_0, h^j g x_0) - d(x_0, gx_0) - d(h^j g x_0, h^j x_0) \\ &\geq d(x_0, e^{jLl} x_0) - 2d(x_0, gx_0) = jl - 2d(x_0, gx_0) \to \infty \ \text{ as } \ j \to \infty\,. \end{aligned}$$

Using the generalized Iwasawa decomposition with respect to Θ, we write $g = knb$ with $k \in K$, $n \in N^+_\Theta$ and $b \in \exp(\mathfrak{p}_\Theta)$. Then b commutes with e^{jLl}, and $e^{-jLl} n e^{jLl} \to \mathrm{id}$ as $j \to \infty$.

In order to prove that $h^j x_0$ converges to $(k,L) \in \partial X^\Theta$ in the cone topology, we let $R >> 1$ and $\varepsilon > 0$ arbitrary. Using the convexity of the distance function, we calculate

$$\begin{aligned} d(ke^{LR}x_0, \sigma_{x_0,h^j x_0}(R)) &\leq \frac{R}{d_j}\big(d(ke^{Ld_j}x_0, ke^{jLl}x_0) + d(ke^{jLl}x_0, ge^{jLl}x_0) + d(ge^{jLl}x_0, h^j g x_0) \\ &\quad + d(h^j g x_0, h^j x_0)\big) = \frac{R}{d_j}\big(|d_j - jl| + d(x_0, e^{-jLl}ne^{jLl}bx_0) \\ &\quad + d(e^{jLl}x_0, e^{jLl}m^j x_0) + d(gx_0, x_0)\big) \\ &\leq \frac{R}{d_j}\big(d(x_0, e^{-jLl}ne^{jLl}bx_0) + 3d(x_0, gx_0)\big)\,, \end{aligned}$$

because, by the triangle inequality, $jl \leq d_j \leq jl + 2d(x_0, gx_0)$ and $d(e^{jLl}x_0, e^{jLl}m^j x_0) = 0$. Since $d_j \to \infty$ and $d(x_0, e^{-jLl}ne^{jLl}bx_0) \to d(x_0, bx_0)$ as $j \to \infty$, we conclude

$$d(ke^{LR}x_0, \sigma_{x_0,h^j x_0}(R)) < \varepsilon$$

for sufficiently large j. Hence $h^+ := \lim_{j\to\infty} h^j x_0 = (k,L) \in \partial X^\Theta$.

We have

$$h^{-1} = (ge^{Ll}mg^{-1})^{-1} = gm^{-1}e^{-Ll}g^{-1} = gw_*^{-1}e^{\iota(L)l}w_* m^{-1}w_*^{-1}w_* g^{-1} = \tilde{g}e^{\iota(L)l}\tilde{m}\tilde{g}^{-1}\,,$$

where $\tilde{g} = gw_*^{-1} \in G$, $\tilde{m} = w_* m^{-1} w_*^{-1} \in M_{\Theta^*}$. Writing $\tilde{g} = \tilde{k}\tilde{n}\tilde{b}$ in generalized Iwasawa coordinates with respect to Θ^* and applying the first part of the proof to the Θ^*-axial isometry h^{-1}, we deduce that $h^- := \lim_{j\to\infty} h^{-j}x_0 = (\tilde{k}, \iota(L)) \in \partial X^{\Theta^*}$.

Next let σ be an invariant geodesic of h. Then $h^j\sigma(t) = \sigma(t + jl)$ for all $j \in \mathbb{Z}$, and therefore $h^j\sigma(0)$ converges to $\sigma(\infty)$, and $h^{-j}\sigma(0)$ converges to $\sigma(-\infty)$ in the cone topology as $j \to \infty$. Since $h^j\sigma(0)$ and $h^j x_0$ converge to the same point $h^+ \in \partial X^\Theta$, and $h^{-j}\sigma(0)$, $h^{-1}x_0$ both converge to $h^- \in \partial X^{\Theta^*}$, we conclude that $h^- \in \mathrm{Vis}^\infty(h^+)$.

For the last assertion, let $y \in \mathrm{Ax}(h)$ such that $d(x_0, \mathrm{Ax}(h)) = d(x_0, y) =: d$. For $t > 0$ let $j_t = [(2t + l)/2l]$, the largest integer smaller or equal than $(2t + l)/2l$. Then

$$\begin{aligned} d(h^{j_t}x_0, \sigma_{x_0,h^+}(t)) &\leq d(h^{j_t}x_0, h^{j_t}y) + d(h^{j_t}y, \sigma_{y,h^+}(t)) + d(\sigma_{y,h^+}(t), \sigma_{x_0,h^+}(t)) \\ &\leq d(x_0, y) + |j_t l - t| + d(y, x_0) \leq 2d + \frac{l}{2} =: c \end{aligned}$$

Since c is independent of $t > 0$, the proof is complete. □

Chapter 5

Geometry of the limit set

The goal of this section is to describe precisely the dynamics of axial isometries introduced in the previous chapter. The main result is Theorem 5.4, which allows to draw conclusions about the structure of the limit set and makes possible a natural construction of free groups.

The most satisfying results about the structure of the limit set can be obtained for a class of discrete isometry groups which we choose to call nonelementary. For Zariski dense discrete subgroups of G, similar results have been proved by Y. Guivarc'h ([G]) and by Y. Benoist [Be] using different, more algebraic methods. Our Theorems 5.15 and 5.16 below are valid for groups which are not Zariski dense, as for example the discrete subgroup of $SL(3,\mathbb{R})$ acting on $X = SL(3,\mathbb{R})/SO(3)$, generated by the elements

$$\begin{pmatrix} e^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-n} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cosh n & 0 & -\sinh n \\ 0 & 1 & 0 \\ -\sinh n & 0 & \cosh n \end{pmatrix}$$

for sufficiently large n.

Our proofs are inspired by ideas of F. Dal'bo who applied similar methods in the case of products of pinched Hadamard manifolds ([DaK]). We only use the geometry and dynamics of axial isometries.

In this chapter, X will again denote a globally symmetric space of noncompact type and $G = \mathrm{Isom}^o(X)$.

5.1 Dynamics of axial isometries

In order to describe the dynamics of axial isometries, we introduce an auxiliary distance for the Bruhat visibility sets $\mathrm{Vis}^B(\xi)$, $\xi \in \partial X$, defined in section 2.6.

Let $G = N^+AK$ be an Iwasawa decomposition with respect to $x \in X$ arbitrary and $\xi \in \partial X$. Then $\xi \in \partial X^{\Theta^*}$ for some subset $\Theta \subset \Upsilon$, and we may identify $\mathrm{Vis}^B(\xi)$ with the submanifold $N^+_\Theta x$ of X, where $N^+_\Theta = \exp(\mathfrak{n}^+_\Theta)$ is the nilpotent subgroup of N^+ defined in section 2.3. Proceeding as in section 1.3 for $\Theta = \emptyset$, let h_α denote the left invariant scalar product on N_Θ^+ which equals $\langle\cdot,\cdot\rangle$ from section 1.1 on $\mathfrak{g}_\alpha$ and is zero on $\mathfrak{g}_\beta$ if $\beta \neq \alpha$. We then obtain an N_Θ^+-invariant metric for the submanifold $N_\Theta^+ x$ of X

$$ds^2_{x,\xi} = \frac{1}{2} \sum_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} h_\alpha\,,$$

which defines a distance $d_{x,\xi}$ on $N_\Theta^+ x \cong \mathrm{Vis}^B(\xi)$. We remark that for $y \in X$, the distance $d_{y,\xi}$ is equivalent to the distance $d_{x,\xi}$ on $\mathrm{Vis}^\infty(\xi)$. Furthermore, since the map κ_Θ is a diffeomorphism from N_Θ^+ onto a dense open subset of K/M_Θ, we easily deduce that the topology induced by the distance $d_{x,\xi}$ on $\mathrm{Vis}^B(\xi) \subset K/M_\Theta$ is equivalent to the original topology on K/M_Θ.

The following lemma describes how this distance behaves under the action of an axial isometry which translates the geodesic $\sigma_{x,\xi}$.

Lemma 5.1 *Let $\Theta \subset \Upsilon$ and h a Θ-axial isometry with translation length $l > 0$ and translation direction $L \in \mathfrak{a}_1^\Theta$. Further denote by h^+ the attractive and by h^- the repulsive fixed point of h. Then for any $x \in Ax(h)$ we have*

$$\forall\,\xi \in Vis^\infty(h^+) \qquad d_{x,h^+}(h^{-1}\xi, h^-) \le e^{-\alpha_+ l}\cdot d_{x,h^+}(\xi, h^-)\,, \quad \alpha_+ := \min_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \alpha(L)$$

$$\forall\,\xi \in Vis^\infty(h^-) \qquad d_{x,h^-}(h\xi, h^+) \le e^{-\alpha_- l}\cdot d_{x,h^-}(\xi, h^+)\,, \quad \alpha_- := \min_{\alpha\in\Sigma^+\setminus\langle\Theta^*\rangle^+} \alpha(\iota(L))$$

Proof. We fix an Iwasawa decomposition $G = N^+AK$ with respect to $x \in \mathrm{Ax}(h)$ and $h^+ \in \partial X^\Theta$. Then there exists $m \in M_\Theta$ such that $h = e^{Ll}m$ and $\sigma:\ \mathbb{R} \to X$ defined by $\sigma(t) = e^{Lt}x$, $t \in \mathbb{R}$, is an invariant geodesic of h with $\sigma(\infty) = h^+$ and $\sigma(-\infty) = h^- \in \partial X^{\Theta^*}$. Now $\xi \in \mathrm{Vis}^\infty(h^+)$ implies the existence of $n \in N_\Theta^+$ such that $\xi = n\sigma(-\infty)$.

Let $\varepsilon > 0$ and $c:\ [0,1] \to N_\Theta^+ x$ a curve in the submanifold $N_\Theta^+ x$ with $c(0) = x$, $c(1) = nx$ and

$$\int_0^1 \|\dot c(t)\|\; dt < d_{x,h^+}(\xi, h^-) + \varepsilon\,.$$

For $t \in [0,1]$, we write $c(t) = n(t)x$ with $n(t) \in N_\Theta^+$ and put $Z(t) := DL_{n(t)^{-1}} \frac{d}{ds}\big|_{s=t} n(s) \in \mathfrak{n}_\Theta^+$. Then, by definition of the metric, $\|\dot c(t)\|^2 = ds^2_{x,\xi}(Z(t), Z(t))$.

Since h^{-1} fixes h^- and $h^{-1}\xi$ corresponds to the element $h^{-1}nhx$ in $N_\Theta^+ x$, the curve $c_h(t) = h^{-1}n(t)hx$ joins x to $h^{-1}nhx$, hence

$$d_{x,h^+}(h^{-1}\xi, h^-) \le \int_0^1 \|\dot{c_h}(t)\|\; dt\,.$$

Here $\|\dot{c_h}(t)\|^2 = ds^2_{x,\xi}(\mathrm{Ad}(h^{-1})Z(t), \mathrm{Ad}(h^{-1})Z(t))$. Since N_Θ^+ normalizes P_Θ and therefore M_Θ by Proposition 2.7, we conclude $\mathrm{Ad}(m)Z(t) \in \mathfrak{n}_\Theta^+$ for all $t \in [0,1]$. Furthermore, $ds^2_{x,\xi}(\mathrm{Ad}(m^{-1})Z(t), \mathrm{Ad}(m^{-1})Z(t)) = ds^2_{x,\xi}(Z(t), Z(t))$ because the scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ and hence h_α, $\alpha \in \Sigma^+$, is invariant by $\mathrm{Ad}(k)$. We conclude

$$\begin{aligned} ds^2_{x,\xi}(\mathrm{Ad}(h^{-1})Z(t), \mathrm{Ad}(h^{-1})Z(t)) &= ds^2_{x,\xi}(\mathrm{Ad}(e^{-Ll})Z(t), \mathrm{Ad}(e^{-Ll})Z(t)) \\ &\le \max_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} e^{-2\alpha(L)l} ds^2_{x,\xi}(Z(t), Z(t))\,. \end{aligned}$$

Putting $\alpha_0 := \min\{\alpha(L) \ > \ \alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+\} > 0$ we summarize

$$d_{x,h^+}(h^{-1}\xi, h^-) \le e^{-\alpha_0 l} \int_0^1 \|\dot{c}(t)\| \, dt < e^{-\alpha_0 l} \left(d_{x,h^+}(\xi, h^-) + \varepsilon \right) ,$$

and the first claim follows as ε tends to zero.

For the second claim, we remark that h^{-1} is Θ^*-axial with translation length $l > 0$ and translation direction $\iota(L) \in \mathfrak{a}^{\Theta^*}$. Furthermore, $\mathrm{Ax}(h) = \mathrm{Ax}(h^{-1})$ and hence the assertion follows from the first claim. □

COROLLARY 5.2 *Let h be an axial isometry and $x \in Ax(h)$. Then*

$$\begin{aligned} \forall \xi \in Vis^\infty(h^+) \qquad & \lim_{j\to\infty} d_{x,h^+}(h^{-j}\xi, h^-) = 0\,, \\ \forall \xi \in Vis^\infty(h^-) \qquad & \lim_{j\to\infty} d_{x,h^-}(h^{j}\xi, h^+) = 0\,. \end{aligned}$$

We further have the following equivalence for sequences of axial isometries.

LEMMA 5.3 *Fix $x \in X$ and let (h_j) be a sequence of axial isometries such that $d(x, Ax(h_j))$ remains bounded as $j \to \infty$. Then $(h_j x) \subset X$ converges to a boundary point $\xi \in \partial X$ in the cone topology if and only if the sequence of attractive fixed points $(h_j^+) \subset \partial X$ of (h_j) converges to ξ in the cone topology.*

Proof. Let (h_j) be a sequence of axial isometries with attractive fixed points $(h_j^+) \subset \partial X$, and $c \ge 0$ such that $d(x, \mathrm{Ax}(h_j)) \le c$. For any $j \in \mathbb{N}$ choose a point $x_j \in \mathrm{Ax}(h_j)$ with the property $d(x, x_j) \le c$, and put $d_j := d(x, h_j x) = d(x, h_j^{-1} x)$.

First suppose $h_j x \to \xi^+$ and let $R >> 1$, $\varepsilon > 0$. By hypothesis, there exists $N_0 \in \mathbb{N}$ such that

$$d(\sigma_{x,h_j x}(R), \sigma_{x,\xi^+}(R)) < \frac{\varepsilon}{2}$$

for $j > N_0$. Using the convexity of the distance function, we compute for $j > N_0$

$$\begin{aligned} d(\sigma_{x,h_j^+}(R), \sigma_{x,\xi^+}(R)) \ &\le \ d(\sigma_{x,h_j^+}(R), \sigma_{x,h_j x}(R)) + d(\sigma_{x,h_j x}(R), \sigma_{x,\xi^+}(R)) \\ \le \ \frac{R}{d_j} d(\sigma_{x,h_j^+}(d_j), h_j x) + \frac{\varepsilon}{2} \ &\le \ \frac{R}{d_j}\big(d(\sigma_{x,h_j^+}(d_j), \sigma_{x_j,h_j^+}(d_j)) + d(\sigma_{x_j,h_j^+}(d_j), h_j x_j) \\ + d(h_j x_j, h_j x)\big) + \frac{\varepsilon}{2} \ &\le \ \frac{R}{d_j} \left(d(x, x_j) + 0 + d(x_j, x) \right) + \frac{\varepsilon}{2} \ \le \ 2c\frac{R}{d_j} + \frac{\varepsilon}{2}\,. \end{aligned}$$

Since $d_j \to \infty$ as $j \to \infty$, this implies that h_j^+ converges to ξ^+ in the cone topology.

Conversely, suppose $h_j^+ \to \xi^+$ and let $R >> 1$, $\varepsilon > 0$. By hypothesis, we have

$$d(\sigma_{x,h_j^+}(R), \sigma_{x,\xi^+}(R)) < \frac{\varepsilon}{2}$$

for j sufficiently large. Again, the convexity of the distance function yields

$$\begin{aligned} d(\sigma_{x,h_jx}(R),\sigma_{x,\xi^+}(R)) &\leq d(\sigma_{x,h_jx}(R),\sigma_{x,h_j^+}(R)) + d(\sigma_{x,h_j^+}(R),\sigma_{x,\xi^+}(R)) \\ \leq \frac{R}{d_j} d(h_jx,\sigma_{x,h_j^+}(d_j)) + \frac{\varepsilon}{2} &\leq \frac{R}{d_j}\big(d(h_jx,h_jx_j) + d(h_jx_j,\sigma_{x_j,h_j^+}(d_j)) \\ +d(\sigma_{x_j,h_j^+}(d_j),\sigma_{x,h_j^+}(d_j)) + \frac{\varepsilon}{2} &\leq \frac{R}{d_j}\ (d(x,x_j)+0+d(x_j,x)) + \frac{\varepsilon}{2} \\ \leq 2c\frac{R}{d_j} + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

for j sufficiently large. This proves that $h_jx \to \xi^+$. □

For the remainder of this section, we fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ and let $x_0 \in X$ denote the unique point stabilized by K. Recall that for any $\Theta \subset \Upsilon$, the map $\pi^B : \partial X^\Theta \to K/M_\Theta$ denotes the projection introduced in section 2.5. Our main result of this section states that for certain sequences of axial isometries with attractive fixed points converging to $\xi \in \partial X^\Theta$, a dense and open subset of ∂X^Θ is mapped to a neighborhood of $\pi^B(\xi)$. We will see in section 5.4 that nonelementary groups contain many such sequences.

THEOREM 5.4 *Let $\Theta \subset \Upsilon$ and (γ_j) a sequence of axial isometries such that $\gamma_j x_0$ converges to a point $\xi^+ = (k^+, H_\xi) \in \partial X^\Theta$, and $\gamma_j^{-1} x_0$ converges to $\xi^- = (k^-, \iota(H_\xi)) \in Vis^\infty(\xi^+)$ in the cone topology. Suppose $d(x_0, Ax(\gamma_j))$ is bounded as $j \to \infty$. For $\tilde\Theta \supseteq \Theta$ and $H \in \mathfrak{a}_1^{\tilde\Theta}$, we put $\eta^+ := (k^+, H)$ and $\eta^- := (k^-, \iota(H))$. Then for any $\zeta \in Vis^\infty(\eta^-)$ there exist integers n_j, $j \in \mathbb{N}$, such that the sequence $(\gamma_j^{n_j}\zeta)$ converges to η^+ in $\partial X^{\tilde\Theta}$. In particular, if $\zeta \in Vis^\infty(\xi^-)$, then there exist integers n_j, $j \in \mathbb{N}$, such that $\gamma_j^{n_j}\zeta$ converges to ξ^+.*

Proof. Let $\Theta \subset \Upsilon$ and suppose $(\gamma_j)_{j\in\mathbb{N}}$ is a sequence of axial isometries with the properties stated in the theorem. Denote by $\gamma_j^+ = (k_j^+, H_j)$ the attractive fixed point and by $\gamma_j^- = (k_j^-, \iota(H_j))$ the repulsive fixed point of γ_j. Let $\tilde\Theta \supseteq \Theta$, $H \in \mathfrak{a}_1^{\tilde\Theta}$, and put $h_j^+ := (k_j^+, H)$, $h_j^- := (k_j^-, \iota(H))$, $\eta^+ := (k^+, H)$ and $\eta^- := (k^-, \iota(H)) \in \mathrm{Vis}^\infty(\eta^+)$.

By the previous lemma, (γ_j^+) converges to ξ^+ in the cone topoloy, hence by Lemma 2.9, $(k_j^+ M_\Theta) = (\pi^B(\gamma_j^+))$ converges to $k^+M_\Theta = \pi^B(\xi^+)$ in K/M_Θ, and $(h_j^+) \subset G{\cdot}\eta^+$ converges to η^+ in the cone topology. Therefore Lemma 2.18 implies that $h_j^+ \in \mathrm{Vis}^\infty(\eta^-)$ for j sufficiently large.

Since (γ_j^+) converges to $\xi^+ \in \partial X^\Theta$, by Lemma 2.9 we further have $\|H_j - H_\xi\| \to 0$. Then $H_\xi \in \mathfrak{a}_1^\Theta$ implies

$$\min_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \alpha(H_j) \geq \alpha_0 > 0$$

if j is sufficiently large. Let $U \subset G{\cdot}\eta^+$ be an open neighborhood of η^+. If $\zeta \in \mathrm{Vis}^\infty(\eta^-)$, then by Lemma 5.1 for any $j \in \mathbb{N}$ there exists $n_j \in \mathbb{N}$ such that $\gamma_j^{n_j}\zeta \in U$. We conclude that $\gamma_j^{n_j}\zeta$ converges to η^+ in the cone topology. □

5.2 Construction of free groups

We will now apply the results of the previous section to construct Schottky groups, an interesting kind of free and discrete isometry groups of infinite covolume. Their construction is based on the following

LEMMA 5.5 (KLEIN'S CRITERIUM) *(see [Ha])*
Let G be a group acting on a set S, Γ_1, Γ_2 two subgroups of G, where Γ_1 contains at least three elements, and let Γ be the subgroup they generate. Assume that there exist two nonempty subsets S_1, S_2 in S with S_2 not included in S_1 such that $\gamma(S_2) \subseteq S_1$ for all $\gamma \in \Gamma_1 \setminus \{\mathrm{id}\}$ and $\gamma(S_1) \subseteq S_2$ for all $\gamma \in \Gamma_2 \setminus \{\mathrm{id}\}$. Then Γ is isomorphic to the free product $\Gamma_1 * \Gamma_2$

Recall from Corollary 2.17 that a finite intersection of sets $\mathrm{Vis}^B(\xi_j) \subset K/M$, $\xi_j \in \partial X^{reg}$, is a dense and open subset of K/M. The following theorem therefore describes a very general construction of finitely generated free groups.

THEOREM 5.6 *Let $X = G/K$ be a globally symmetric space of noncompact type, and $\{\xi_1^+, \xi_1^-, \xi_2^+, \xi_2^-, \ldots, \xi_l^+, \xi_l^-\} \subset \partial X^{reg}$ a set of $2l$ points with the properties*

$$\xi_j^- \in Vis^\infty(\xi_j^+)\,, \quad \pi^B(\xi_j^+) \in \bigcap_{\substack{i=1\\ i\neq j}}^{l} \big(Vis^B(\xi_i^+) \cap Vis^B(\xi_i^-)\big) \qquad for \quad 1 \le j \le l\,.$$

Then there exist regular axial isometries γ_j ,$1 \le j \le l$, such that $\gamma_j^+ = \xi_j^+$ and $\gamma_j^- = \xi_j^-$, and pairwise disjoint open neighborhoods $U_j^+, U_j^- \subset K/M$ of $\pi^B(\xi_j^+), \pi^B(\xi_j^-)$ such that

$$\gamma_j(K/M \setminus \overline{U_j^-}) \subset U_j^+ \qquad and \qquad \gamma_j^{-1}(K/M \setminus \overline{U_j^+}) \subset U_j^-\,.$$

In particular, the finitely generated group $\langle \gamma_1, \gamma_2, \ldots, \gamma_l\rangle \subset Isom^o(X)$ is free and discrete.

Proof. Since for any $j \in \{1, 2, \ldots, l\}$ we have $\xi_j^+ \in \partial X^{reg}$ and $\xi_j^- \in \mathrm{Vis}^\infty(\xi_j^+) \subset \partial X^{reg}$, there exist regular unit speed geodesics $\sigma_j : \mathbb{R} \to X$ such that $\sigma_j(\infty) = \xi_j^+$ and $\sigma_j(-\infty) = \xi_j^-$ for $1 \le j \le l$. Let $x_0 \in X$ denote the base point of $X = G/K$ and fix an Iwasawa decomposition $G = N^+AK$ with respect to x_0 and $\xi_1^- \in \partial X^{reg}$. Then there exist $n \in N^+$, $a \in A$ and $H \in \mathfrak{a}_1^+$ such that $\sigma_1(t) = nae^{Ht}x_0$ for all $t \in \mathbb{R}$. The isometry $h_1 := nae^H a^{-1} n^{-1}$ is regular axial and satisfies $h_1\sigma_1(t) = \sigma_1(t+1)$ for any $t \in \mathbb{R}$. In particular, h_1 possesses the attractive and repulsive fixed points $h_1^+ = \sigma_1(\infty) = \xi_1^+$ and $h_1^- = \sigma_1(-\infty) = \xi_1^-$. Similarly, for all $j \in \{2, \ldots l\}$ there exists a regular axial isometry h_j such that $h_j\sigma_j(t) = \sigma_j(t+1)$ for all $t \in \mathbb{R}$, and $h_j^+ = \xi_j^+$, $h_j^- = \xi_j^-$.

We next choose open neighborhoods

$$U_1^+ \subset \bigcap_{i=2}^{l} \big(\mathrm{Vis}^B(\xi_i^+) \cap \mathrm{Vis}^B(\xi_i^-)\big) \text{ of } \pi^B(\xi_1^+)\,, \text{ and}$$

$$U_1^- \subset \bigcap_{i=2}^{l} \big(\mathrm{Vis}^B(\xi_i^+) \cap \mathrm{Vis}^B(\xi_i^-)\big) \text{ of } \pi^B(\xi_1^-)\,.$$

We remark that by the properties of the Furstenberg visibility sets $U_1^+ \subset \mathrm{Vis}^B(\xi_1^-)$ and $U_1^- \subset \mathrm{Vis}^B(\xi_1^+)$. By Corollary 5.2 there exists an integer $k_1 \in \mathbb{N}$ such that $h_1^{k_1}(K/M \setminus \overline{U_1^-}) \subset U_1^+$ and $h_1^{-k_1}(K/M \setminus \overline{U_1^+}) \subset U_1^-$. Inductively, we choose open neighborhoods

$$U_{j+1}^+, U_{j+1}^- \subset \bigcap_{i=1}^{j} \Big(\mathrm{Vis}^B(\xi_i^+) \setminus \overline{U_i^-} \cap \mathrm{Vis}^B(\xi_i^-) \setminus \overline{U_i^+}\Big) \bigcap_{i=j+2}^{l} \big(\mathrm{Vis}^B(\xi_i^+) \cap \mathrm{Vis}^B(\xi_i^-)\big)$$

of $\pi^B(h_{j+1}^+)$ and $\pi^B(h_{j+1}^-)$ respectively, and let $k_{j+1} \in \mathbb{N}$ such that $h_{j+1}^{k_{j+1}}(K/M \setminus \overline{U_{j+1}^-}) \subset U_{j+1}^+$ and $h_{j+1}^{-k_{j+1}}(K/M \setminus \overline{U_{j+1}^+}) \subset U_{j+1}^-$. Putting $\gamma_j := h_j^{k_j}$ for $1 \le j \le l$, we have the desired regular axial isometries and the corresponding open neighborhoods.

In order to apply Klein's Criterium, we put $S_1 := U_1^+ \cup U_1^-$ and $S_2 := U_2^+ \cup U_2^-$. Since the open neighborhoods U_2^+ and U_2^- are contained in $\mathrm{Vis}^B(\xi_1^+) \setminus \overline{U_1^-} \cap \mathrm{Vis}^B(\xi_1^-) \setminus \overline{U_1^+}$ we conclude that $\langle\gamma_1\rangle \cdot S_2 \subset S_1$. Similarly $U_1^\pm \subset \mathrm{Vis}^B(\xi_2^+) \setminus \overline{U_2^-} \cap \mathrm{Vis}^B(\xi_2^-) \setminus \overline{U_2^+}$ implies $\langle\gamma_2\rangle \cdot S_1 \subset S_2$. Hence the group generated by γ_1 and γ_2 is free by Klein's Criterium.

For $j \in \{2, \dots, l\}$, let Γ_j denote the group generated by the elements γ_i for $i \le j$. We put

$$S_j' := \bigcup_{i=1}^{j} \left(U_i^+ \cup U_i^-\right), \qquad S_{j+1} := U_{j+1}^+ \cup U_{j+1}^- .$$

Since $S_{j+1} \subset \bigcap_{i=1}^{j} \Big(\mathrm{Vis}^B(\xi_i^+) \setminus \overline{U_i^-} \cap \mathrm{Vis}^B(\xi_i^-) \setminus \overline{U_i^+}\Big)$ we have $\Gamma_2 \cdot S_{j+1} \subset S_j'$. From

$$S_j' \subset \mathrm{Vis}^B(\xi_{j+1}^+) \setminus \overline{U_{j+1}^-} \cap \mathrm{Vis}^B(\xi_{j+1}^-) \setminus \overline{U_{j+1}^+}$$

we further obtain $\langle\gamma_{j+1}\rangle \cdot S_j' \subset S_{j+1}$, and therefore the group Γ_{j+1} generated by the elements γ_i for $i \le j+1$, is free. We conclude inductively that $\langle\gamma_1, \gamma_2, \dots, \gamma_l\rangle$ is free.

Finally suppose $\Gamma := \langle\gamma_1, \gamma_2, \dots, \gamma_l\rangle$ is not discrete. Then there exists a sequence $(h_j) \subset \Gamma$ converging to the identity. For $j \in \mathbb{N}$ we write $h_j := a_1^{(j)} a_2^{(j)} \dots a_{k_j}^{(j)}$ as a reduced word, i.e. $a_i^{(j)} \in \{\gamma_1, \gamma_1^{-1}, \gamma_2, \gamma_2^{-1} \dots, \gamma_l, \gamma_l^{-1}\}$ and $a_{i+1}^{(j)} \neq (a_i^{(j)})^{-1}$ for $1 \le i \le k_j$. Passing to a subsequence if necessary, we may assume that $a_1^{(j)}$ is the same element for all $j \in \mathbb{N}$, say γ_i^ε, where $1 \le i \le l$ and $\varepsilon \in \{+1, -1\}$. Let $\eta \in \partial X$ such that $\pi^B(\eta) \in K/M \setminus \bigcup_{i=1}^{l} \left(U_i^+ \cup U_i^-\right)$. Then $\pi^B(h_j\eta)$ is contained in $U_i^\varepsilon \subset K/M$ for all $j \in \mathbb{N}$, and $(\pi^B(h_j\eta))$ converges to $\pi^B(\eta) \notin \overline{U_i^\varepsilon}$, a contradiction. □

5.3 Nonelementary groups

We are now going to generalize to symmetric spaces $X = G/K$ of higher rank the notion of "nonelementary groups" familiar in the context of isometry groups of real hyperbolic spaces. We denote by $x_0 \in X$ the unique point stabilized by the maximal compact subgroup $K \subset G$.

DEFINITION 5.7 *A discrete subgroup* Γ *of the isometry group* $Isom^o(X)$ *is called* nonelementary *if and only if* $L_\Gamma \neq \emptyset$ *and if for any* $\xi \in L_\Gamma$, $\eta \in G{\cdot}\xi \subseteq \partial X$ *we have*

$$\Gamma{\cdot}\xi \cap Vis^\infty(\iota\eta) \neq \emptyset\,.$$

Otherwise Γ *is called* elementary.

Note that an abelian discrete group $\Gamma \subset \mathrm{Isom}^o(X)$ of axial isometries is elementary, because its limit set is contained in the boundary of the invariant maximal flats. Hence $\Gamma{\cdot}\xi = \xi$ for every $\xi \in L_\Gamma$ which implies $\Gamma{\cdot}\xi = \xi \notin \mathrm{Vis}^\infty(\iota\xi)$. The same argument shows, that a discrete group $\Gamma \subset \mathrm{Isom}^o(X)$ is elementary, if it is contained in the stabilizer of a limit point. Nevertheless, there are many examples of nonelementary groups.

EXAMPLE 5.8 *If* $rank(X) = 1$, *then a discrete isometry group* $\Gamma \subset Isom^o(X)$ *is nonelementary if it possesses infinitely many limit points.*

Proof. Since $\mathrm{rank}(X) = 1$, we have $\iota = \mathrm{id}$ and $G{\cdot}\zeta = \partial X$, $\partial X = \mathrm{Vis}^\infty(\zeta) \cup \{\zeta\}$ for any point ζ in the geometric boundary. Suppose $\Gamma \subset G = \mathrm{Isom}^o(X)$ possesses infinitely many limit points, and assume there exists $\xi \in L_\Gamma$ and $\eta \in \partial X$ such that $\Gamma{\cdot}\xi \cap \mathrm{Vis}^\infty(\eta) = \emptyset$. Then $\gamma\xi = \eta$ for all $\gamma \in \Gamma$, in particular $\xi = \eta$. This implies that every element in Γ fixes ξ. Let $\Gamma' \subseteq \Gamma$ be a torsion free subgroup of finite index which exists by Selberg's Lemma. Since Γ' does not contain elliptic elements, Γ' contains only parabolic and axial isometries which all fix ξ. By discreteness, the set of axial elements in Γ' must all have the same axis. We conclude that Γ' possesses at most two limit points, hence Γ possesses only finitely many limit points, a contradiction. $\square$

EXAMPLE 5.9 *Free groups generated by regular axial isometries as in Theorem 5.6 are nonelementary.*

Proof. Let $\gamma_1, \gamma_2, \ldots, \gamma_l$ be the generators of Γ and $U_1^+, U_1^-, U_2^+, U_2^-, \ldots U_l^+, U_l^- \subset K/M$ pairwise disjoint open sets as in the proof of Theorem 5.6. Let $\xi \in L_\Gamma$, $\eta \in G{\cdot}\xi$ and choose a generator γ_k^ε with attractive fixed point ζ_k^ε, where $k \in \{1, 2, \ldots, l\}$ and $\varepsilon \in \{+1, -1\}$, such that $\xi_k^\varepsilon := \overline{\mathcal{C}}_{x_0,\zeta_k^\varepsilon} \cap G{\cdot}\xi$ satisfies $\xi_k^\varepsilon \in \mathrm{Vis}^\infty(\iota\eta)$. By Lemma 2.18 there exists an open neighborhood $V \subset G{\cdot}\xi$ of ξ_k^ε such that $\zeta \in \mathrm{Vis}^\infty(\iota\eta)$ for all $\zeta \in V$, and therefore $V \subseteq \mathrm{Vis}^\infty(\iota\eta)$. If $\xi \in \mathrm{Vis}^\infty(\xi_k^{-\varepsilon})$, then by Lemma 5.1, $(\gamma_k^\varepsilon)^n\xi$ converges to ξ_k^ε as $n \to \infty$. Hence there exists $n_0 \in \mathbb{N}$ such that $(\gamma_k^\varepsilon)^{n_0}\xi \in V \subset \mathrm{Vis}^\infty(\iota\eta)$. If $\xi \notin \mathrm{Vis}^\infty(\xi_k^{-\varepsilon})$, there exists a generator γ_i, $i \neq k$, and $l \in \mathbb{N}$ such that $(\gamma_i)^l\xi \in \mathrm{Vis}^\infty(\iota\eta)$. $\square$

The following lemma will be useful in the proof of Lemma 7.2.

LEMMA 5.10 *Let* $\Gamma \subset Isom^o(X)$ *be a nonelementary discrete group, and* $\Theta \subset \Upsilon$ *such that* $L_\Gamma \cap \partial X^\Theta \neq \emptyset$. *Then for any* $\xi \in \partial X^\Theta$ *and for all* $\eta \in G{\cdot}\xi \subseteq \partial X^\Theta$ *we have*

$$L_\Gamma \cap G{\cdot}\xi \subseteq \bigcup_{\gamma\in\Gamma} Vis^\infty(\iota\gamma\eta)\,.$$

Proof. Let $\Theta \subset \Upsilon$ with $L_\Gamma \cap \partial X^\Theta \neq \emptyset$ and fix $\xi \in \partial X^\Theta$. Let $\eta \in G{\cdot}\xi \subseteq \partial X^\Theta$ and suppose $\zeta \in L_\Gamma \cap G{\cdot}\xi$. Since Γ is nonelementary, there exists $\gamma \in \Gamma$ such that $\gamma^{-1}\zeta \in \mathrm{Vis}^\infty(\iota\eta)$, hence $\zeta \in \mathrm{Vis}^\infty(\iota\gamma\eta)$. □

Since we do not know much about the dynamics of parabolic isometries, the description of the limit set of discrete isometry groups, which in general contain parabolic isometries, is difficult. The following proposition, however, states that every regular limit point of a nonelementary group can be obtained from a sequence of axial isometries. This allows to use the dynamics of axial isometries developped in section 5.1 in order to describe the structure of the regular limit set. The proof is a slight modification of the proof of Proposition 4.5.14 in [E].

PROPOSITION 5.11 *Let $\Gamma \subset G = Isom^o(X)$ be a nonelementary discrete group. Then for every $\xi \in L_\Gamma \cap \partial X^{reg}$ there exists a sequence of axial isometries $(\gamma_j) \subset \Gamma$ such that $\gamma_j x_0$ converges to ξ and $\gamma_j^{-1}x_0$ converges to a point in $Vis^\infty(\xi)$. Furthermore, $d(x_0, Ax(\gamma_j))$ is bounded as $j \to \infty$.*

Proof. Let $\xi \in L_\Gamma \cap \partial X^{reg}$ and fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ with respect to x_0 and ξ. Let $(h_j) \subset \Gamma$ be a sequence such that $h_j x_0$ converges to $\xi = (\mathrm{id}, H)$. Then $h_j^{-1}x_0$ converges to a point $\zeta = (k, \iota(H))$. If $\pi^B(\zeta) \notin \mathrm{Vis}^B(\xi)$, there exists an element $\gamma \in \Gamma$ such that $\pi^B(\gamma^{-1}\zeta) \in \mathrm{Vis}^B(\xi)$, since Γ is nonelementary. The sequence $(\gamma_j) := (h_j\gamma) \subset \Gamma$ then satisfies

$$\begin{aligned}\lim_{j\to\infty} \gamma_j x_0 &= \lim_{j\to\infty} h_j\gamma x_0 = \xi\,,\\ \lim_{j\to\infty} \gamma_j^{-1} x_0 &= \lim_{j\to\infty} (h_j\gamma)^{-1}x_0 = \lim_{j\to\infty} \gamma^{-1}h_j^{-1}\gamma x_0 = \gamma^{-1}\zeta \in \mathrm{Vis}^\infty(\xi)\,.\end{aligned}$$

This implies the existence of a unit speed geodesic σ joining ξ to $\eta := \gamma^{-1}\zeta$.

Lemma 2.19 allows to choose a sequence (U_k) of neighborhoods in G of the identity such that $\overline{U_{k+1}} \subseteq U_k$ for all $k \in \mathbb{N}$, $\cap_{k\in\mathbb{N}} U_k = \{\mathrm{id}\}$, and for any points $\xi_k \in U_k{\cdot}\xi$ and $\eta_k \in U_k{\cdot}\eta$ there exists a geodesic σ_k in X that joins ξ_k to η_k and satisfies $d(\sigma(0), \sigma_k) < 1/k$ for any $k \in \mathbb{N}$. We show that for any $k \in \mathbb{N}$ there exists an integer N_k such that for all $j \geq N_k$

$$\gamma_j(\overline{U_k{\cdot}\xi}) \subseteq U_k{\cdot}\xi\,, \quad \text{and} \qquad \gamma_j^{-1}(\overline{U_k{\cdot}\eta}) \subseteq U_k{\cdot}\eta\,.$$

Put $x := \sigma(0)$ and fix $k \in \mathbb{N}$. Since the set $U_k{\cdot}\xi$ is an open subset of $G{\cdot}\xi$, we find a number $\varphi > 0$ so that any $\zeta \in G{\cdot}\xi$ with $\angle_x(\xi, \zeta) < \varphi$ is contained in $U_k{\cdot}\xi$. We choose $N \in \mathbb{N}$ such that for any $j \geq N$ and for every $y \in B_x(2)$

$$\max\{\angle_x(\gamma_j y, \xi), \angle_y(\gamma_j^{-1}x, \eta)\} < \varphi/2\,. \tag{5.1}$$

Finally let $\zeta \in \overline{U_k{\cdot}\xi}$ be given and let y be a point on a geodesic joining η to ζ with $d(y, x) \leq 1/k$. It follows that for $j \geq N$ we have

$$\angle_x(\gamma_j\zeta, \xi) \leq \angle_x(\gamma_j\zeta, \gamma_j y) + \angle_x(\gamma_j y, \xi) < \angle_{\gamma_j^{-1}x}(\zeta, y) + \varphi/2$$

by (5.1). Considering the triangle $\Delta(\gamma_j^{-1}x, y, \zeta)$, we obtain $\angle_{\gamma_j^{-1}x}(\zeta, y) + \angle_y(\gamma_j^{-1}x, \zeta) \le \pi$. From the fact that y is a point on the geodesic joining η to ζ we further conclude $\angle_y(\gamma_j^{-1}x, \zeta) = \pi - \angle_y(\gamma_j^{-1}x, \eta)$ and with (5.1)

$$\angle_x(\gamma_j\zeta, \xi) \le \angle_y(\gamma_j^{-1}x, \eta) + \varphi/2 < \varphi .$$

This proves $\gamma_j(\overline{U_k\cdot\xi}) \subseteq U_k\cdot\xi$ for any $j \ge N$ by the choice of φ. The proof of $\gamma_j^{-1}(\overline{U_k\cdot\eta}) \subseteq U_k\cdot\eta$ is analogous.

We next show the existence of sequences $(\xi_j) \subset G\cdot\xi$ converging to ξ, and $(\eta_j) \subset G\cdot\eta$ converging to η such that $\gamma_j\xi_j = \xi_j$ and $\gamma_j\eta_j = \eta_j$ for all $j \in \mathbb{N}$. If U_k is chosen such that $\overline{U_k\cdot\xi}$ is homeomorphic to a closed disk, the Brouwer fixed point theorem together with the fact

$$\gamma_j(\overline{U_k\cdot\xi}) \subseteq U_k\cdot\xi\,, \quad \text{and} \qquad \gamma_j^{-1}(\overline{U_k\cdot\eta}) \subseteq U_k\cdot\eta$$

shows that for any $k \in \mathbb{N}$ the isometry γ_j has fixed points in $U_k\cdot\xi$ and $U_k\cdot\eta$ for all sufficiently large j. For any such j we choose a fixed point ξ_j of γ_j with the property

$$\angle_x(\xi_j, \xi) \le \angle_x(\zeta, \xi) \qquad \forall\,\zeta \in G\cdot\xi \ \text{ fixed by } \gamma_j .$$

Similarly we choose a fixed point η_j such that

$$\angle_x(\eta_j, \eta) \le \angle_x(\zeta, \eta) \qquad \forall\,\zeta \in G\cdot\eta \ \text{ fixed by } \gamma_j .$$

We claim that the sequences (ξ_j) and (η_j) constructed as above converge to ξ and η respectively. Given $\varepsilon > 0$ we choose a neighborhood U in G of the identity such that $\angle_x(\zeta, \xi) < \varepsilon$ for all $\zeta \in U\cdot\xi$. The argument above shows that for sufficiently large j the isometry γ_j has a fixed point ζ_j in $U\cdot\xi$ and hence $\angle_x(\xi_j, \xi) \le \angle_x(\zeta_j, \xi) < \varepsilon$. Therefore $\xi_j \to \xi$ and a similar argument gives $\eta_j \to \eta$.

Since $\xi, \eta \in \partial X^{reg}$, and $\xi_j \in U_k\cdot\xi$, $\eta_j \in U_k\cdot\eta$ for all sufficiently large j, there exist regular geodesics σ_j joining ξ_j to η_j with the property $d(\sigma(0), \sigma_j) < 1/k$. Since γ_j fixes both ξ_j and η_j for j sufficiently large, we conclude that γ_j fixes both $\sigma_j(\infty)$ and $\sigma_j(-\infty)$. If $F_j \subset X$ denotes the unique maximal flat which contains σ_j, then by Lemma 4.2.1a in [E], $\gamma_j F_j = F_j$ and the displacement function d_{γ_j} is constant on F_j. Hence γ_j translates every geodesic joining some point $x_j \in F_j$ to $\gamma_j x_j \in F_j$, and therefore γ_j is axial for sufficiently large j.

Furthermore, we have

$$d(x_0, \mathrm{Ax}(\gamma_j)) \le d(x_0, \sigma(0)) + 1\,,$$

because $\sigma_j \subset F_j = \mathrm{Ax}(\gamma_j)$. □

As a consequence of the previous proof, we obtain the following

THEOREM 5.12 *If $\Gamma \subset G = Isom^o(X)$ is a nonelementary discrete group which possesses a regular limit point, then the set of fixed points of axial isometries is a dense subset of the limit set L_Γ.*

Proof. Let $\xi \in L_\Gamma \cap \partial X^{reg}$ arbitrary. Then the proof of Proposition 5.11 shows that there exists a sequence $(\gamma_j) \subset \Gamma$ of axial isometries such that γ_j has fixed points ξ_j and η_j as above. In particular, ξ_j converges to ξ as $j \to \infty$. □

5.4 The structure of the limit set

We are finally able to describe precisely the limit set of nonelementary discrete groups acting on a globally symmetric space $X = G/K$ of noncompact type. For this section, we fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ and let $x_0 \in X$ denote the unique point stabilized by K.

DEFINITION 5.13 *The* limit cone $P_\Gamma \subseteq \overline{\mathfrak{a}_1^+}$ *of* Γ *is defined as the set of Cartan projections of all elements in the geometric limit set* L_Γ. *The projection* $K_\Gamma := \pi^B(L_\Gamma \cap \partial X^{reg}) \subseteq K/M$ *is called the* transversal limit set.

THEOREM 5.14 *Let* $\Gamma \subset G = Isom^o(X)$ *be a nonelementary discrete group of isometries. Then the transversal limit set* K_Γ *is a minimal closed set under the action of* Γ.

Proof. If $L_\Gamma \cap \partial X^{reg} = \emptyset$ there is nothing to prove. We therefore assume $L_\Gamma \cap \partial X^{reg} \neq \emptyset$ which implies $K_\Gamma \neq \emptyset$.

Fix $k_0 M \in K_\Gamma$, and let $kM \in K_\Gamma$ be arbitrary. Let $\xi^+ \in L_\Gamma \cap \partial X^{reg}$ be a preimage $(\pi^B)^{-1}(kM)$ and denote by $H \in \mathfrak{a}_1^+$ the Cartan projection of ξ^+. Due to Proposition 5.11 there exists a sequence $(\gamma_j) \subset \Gamma$ of axial isometries such that $\gamma_j x_0$ converges to ξ^+ and $\gamma_j^{-1} x_0$ converges to a point $\xi^- \in \mathrm{Vis}^\infty(\xi^+) \subset \partial X^{reg}$. Put $\xi_0 := (k_0, H)$ with $H \in \mathfrak{a}_1^+$ as above. If $k_0 M \in \mathrm{Vis}^B(\xi^-)$, then $\xi_0 \in \mathrm{Vis}^\infty(\xi^-)$ and, by Theorem 5.4, there exist integers n_j, $j \in \mathbb{N}$ such that

$$\lim_{j\to\infty} \gamma_j^{n_j} \xi_0 = \xi^+ .$$

If $k_0 M = \pi^B(\xi_0) \notin \mathrm{Vis}^B(\xi^-)$, there exists $\gamma \in \Gamma$ such that $\pi^B(\gamma\xi_0) \in \mathrm{Vis}^B(\xi^-)$ because Γ is nonelementary. Therefore $\gamma\xi_0 \in \mathrm{Vis}^\infty(\xi^-)$ and Theorem 5.4 garantees the existence of integers n_j, $j \in \mathbb{N}$ such that

$$\lim_{j\to\infty} \gamma_j^{n_j} (\gamma\xi_0) = \xi^+ .$$

Using the natural G-action (2.1) on K/M, this proves that $\Gamma(k_0 M) = \pi^B(\Gamma\xi_0)$ is dense in K_Γ. Since $\Gamma(k_0 M)$ is the smallest Γ-invariant set in K/M, the closure $\overline{\Gamma(k_0 M)} = K_\Gamma$ is a minimal closed set under the action of Γ. □

THEOREM 5.15 *Let* $\Gamma \subset G = Isom^o(X)$ *be a nonelementary discrete group of isometries. Then the regular geometric limit set is isomorphic to the product* $K_\Gamma \times (P_\Gamma \cap \mathfrak{a}_1^+)$.

Proof. If $L_\Gamma \cap \partial X^{reg} = \emptyset$ there is nothing to prove. We therefore assume $L_\Gamma \cap \partial X^{reg} \neq \emptyset$. If $\xi \in L_\Gamma \cap \partial X^{reg}$, then $\pi^B(\xi) \in K_\Gamma$ and the Cartanprojection of ξ belongs to $P_\Gamma \cap \mathfrak{a}_1^+$.

Conversely, let $kM \in K_\Gamma$ and $H \in P_\Gamma \cap \mathfrak{a}_1^+$. By definition of P_Γ, there exists a sequence $(\gamma_j) \subset \Gamma$ such that the Cartan projections $(H_j) \subset \overline{\mathfrak{a}^+}$ of $\gamma_j x_0$ satisfy $\angle(H_j, H) \to 0$ as $j \to \infty$. Furthermore, $\xi_0 := \lim_{j\to\infty} \gamma_j x_0$ belongs to $L_\Gamma \cap \partial X^{reg}$ and we may write $\xi_0 = (k_0, H)$ where $k_0 M := \pi^B(\xi_0) \in K_\Gamma$.

By Theorem 5.14, $K_\Gamma = \overline{\Gamma(k_0M)}$ is a minimal closed set under the action of Γ, hence

$$kM \in \overline{\Gamma(k_0M)} = \pi^B(\overline{\Gamma\cdot\xi_0})$$

by (2.1). Since the action of $\mathrm{Isom}^o(X)$ on the geometric boundary does not change the Cartan projections, and there exists exactly one preimage $\xi := (\pi^B)^{-1}(kM)$ with the same Cartan projection $H \in \mathfrak{a}_1^+$ as ξ_0, the closure of $\Gamma\cdot\xi_0$ contains ξ. This proves $\xi \in \overline{\Gamma\cdot\xi_0} \subseteq L_\Gamma \cap \partial X^{reg}$. □

If γ is an axial isometry, let $L(\gamma) \in \overline{\mathfrak{a}_1^+}$ denote the translation direction of γ from Definition 4.20.

THEOREM 5.16 *Let $\Gamma \subset G = \mathrm{Isom}^o(X)$ be a nonelementary discrete group of isometries, and $\mathcal{L}_\Gamma := \{L(\gamma) \mid \gamma \in \Gamma\,,\ \gamma \text{ axial}\} \subseteq \overline{\mathfrak{a}_1^+}$. Then $P_\Gamma = \overline{\mathcal{L}_\Gamma}$.*

Proof. In order to prove $P_\Gamma \supseteq \overline{\mathcal{L}_\Gamma}$, we let $H \in \overline{\mathcal{L}_\Gamma}$ arbitrary. Then there exists a sequence $(h_j) \subset \Gamma$ of axial isometries with translation lengths l_j and translation directions $L(h_j)$ satisfying $\angle L(h_j), H) \to 0$ as $j \to \infty$.

Suppose $H \notin P_\Gamma$, and, for $\gamma \in \Gamma$, let H_γ denote the Cartan projection of γx_0. Then there exists $\varepsilon > 0$ such that $\angle(H_\gamma, H) > \varepsilon$ for all but finitely many $\gamma \in \Gamma$.

Put $\xi = (\mathrm{id}, H) \in \partial X$. For $j \in \mathbb{N}$, we let $g_j \in G$ such that $h_j = g_j e^{L(h_j)l_j} g_j^{-1}$, put $x_j := g_j x_0 \in \mathrm{Ax}(h_j)$, and let n_j be an integer greater than $2j\cdot d(x_0, x_j)/l_j$. We abbreviate $\gamma_j := h_j^{n_j} = g_j e^{L(h_j)l_j n_j} g_j^{-1}$ and $H_j := H_{\gamma_j}$. By G-invariance of the directional distance, Lemma 3.6 and Lemma 3.2 we obtain

$$\begin{aligned}\mathcal{B}_{G\cdot\xi}(x_j, \gamma_j x_j) &= \mathcal{B}_{G\cdot\xi}(g_j^{-1}x_j, g_j^{-1}\gamma_j x_j) = \mathcal{B}_{G\cdot\xi}(x_0, e^{L(h_j)l_j n_j}x_0)\\ &= \mathcal{B}_\xi(x_0, e^{L(h_j)l_j n_j}x_0) = \langle L(h_j)l_j n_j, H\rangle = l_j n_j \langle L(h_j), H\rangle\,.\end{aligned}$$

Since $\angle(L(h_j), H) \to 0$ by hypothesis, we conclude

$$\frac{\mathcal{B}_{G\cdot\xi}(x_j, \gamma_j x_j)}{d(x_j, \gamma_j x_j)} = \frac{l_j n_j \langle L(h_j), H\rangle}{l_j n_j} = \langle L(h_j), H\rangle \ \to\ 1\,.$$

Using again Lemma 3.2, Lemma 3.6 and the triangle inequality, we obtain

$$\begin{aligned}\cos\angle(H_j, H) &= \frac{\langle H_j, H\rangle}{\|H_j\|} = \frac{\mathcal{B}_\xi(x_0, e^{H_j}x_0)}{d(x_0, \gamma_j x_0)} = \frac{\mathcal{B}_{G\cdot\xi}(x_0, e^{H_j}x_0)}{d(x_0, \gamma_j x_0)}\\ &= \frac{\mathcal{B}_{G\cdot\xi}(x_0, \gamma_j x_0)}{d(x_0, \gamma_j x_0)} \ge \frac{\mathcal{B}_{G\cdot\xi}(x_j, \gamma_j x_j) - 2d(x_0, x_j)}{d(x_j, \gamma_j x_j) + 2d(x_0, x_j)}\\ &\ge \frac{\mathcal{B}_{G\cdot\xi}(x_j, \gamma_j x_j)}{n_j\cdot l_j}\cdot\frac{1-\frac{1}{j}}{1+\frac{1}{j}} \ \to\ 1 \quad \text{as } j \to \infty\,,\end{aligned}$$

a contradiction to our assumption.

Conversely, we first prove $P_\Gamma \cap \mathfrak{a}_1^+ \subseteq \overline{\mathcal{L}_\Gamma}$. Let $H \in P_\Gamma \cap \mathfrak{a}_1^+$ and put $\xi = (\mathrm{id}, H) \in \partial X^{reg}$. By the assumption and the same arguments as above, there exists a sequence $(h_j) \subset \Gamma$ such that

$$\frac{\mathcal{B}_{G\cdot\xi}(x_0, h_j x_0)}{d(x_0, h_j x_0)} \to 1 \quad \text{as } j \to \infty\,.$$

Set $\xi^+ := \lim_{j\to\infty} h_j x_0 \in L_\Gamma \cap \partial X^{reg}$ and choose a sequence of axial isometries $(\gamma_j) \subset \Gamma$ as in Proposition 5.11 with the properties $\gamma_j x_0 \to \xi^+$, $\gamma_j^{-1} x_0 \to \xi^- \in \mathrm{Vis}^\infty(\xi^+)$ and $d(x_0, \mathrm{Ax}(\gamma_j)) \le c$ for some constant $c > 0$. For $j \in \mathbb{N}$, we choose $x_j \in \mathrm{Ax}(\gamma_j)$ such that $d(x_0, x_j) \le c$. Then the translation length $l_j := l(\gamma_j) = d(x_j, \gamma_j x_j)$ of γ_j satisfies

$$\begin{aligned} l_j \le d(x_0, \gamma_j x_0) &\le d(x_0, x_j) + d(x_j, \gamma_j x_j) + d(\gamma_j x_j, \gamma_j x_0) \\ &= l_j + 2d(x_0, x_j) \le l_j + 2c\,. \end{aligned}$$

Hence the sequence of translation directions $(L(\gamma_j)) \subset \overline{\mathfrak{a}_1^+}$ of $(\gamma_j x_j)$ satisfies

$$\begin{aligned} \langle L(\gamma_j), H\rangle &= \frac{1}{d(x_j, \gamma_j x_j)} \sup_{g\in G} \cos \angle_{x_j}(\gamma_j x_j, g\xi) = \frac{1}{l_j} \mathcal{B}_{G\cdot\xi}(x_j, \gamma_j x_j) \\ &\le \frac{\mathcal{B}_{G\cdot\xi}(x_0, \gamma_j x_0) + 2d(x_0, \gamma_j x_0)}{d(x_0, \gamma_j x_0) - 2d(x_0, x_j)} \le \frac{\frac{\mathcal{B}_{G\cdot\xi}(x_0,\gamma_j x_0)}{d(x_0,\gamma_j x_0)} + \frac{2c}{d(x_0,\gamma_j x_0)}}{1 - \frac{2c}{d(x_0,\gamma_j x_0)}} \ \to\ 1\,, \\ \langle L(\gamma_j), H\rangle &\ge \frac{\mathcal{B}_{G\cdot\xi}(x_0, \gamma_j x_0) - 2d(x_0, x_j)}{d(x_0, \gamma_j x_0)} \ge \frac{\mathcal{B}_{G\cdot\xi}(x_0, \gamma_j x_0)}{d(x_0, \gamma_j x_0)} - \frac{2c}{d(x_0, \gamma_j x_0)} \ \to\ 1 \end{aligned}$$

as $j \to \infty$. This yields $\angle(L(\gamma_j), H) \to 0$ as $j \to \infty$ and therefore $H \in \overline{\mathcal{L}_\Gamma}$.

Since the closure of $P_\Gamma \cap \mathfrak{a}_1^+$ equals P_Γ and $\overline{\mathcal{L}}_\Gamma$ is a closed set in $\overline{\mathfrak{a}_1^+}$, we conclude $P_\Gamma = P_\Gamma \cap \overline{\mathfrak{a}_1^+} \subseteq \overline{\mathcal{L}}_\Gamma\,.$ □

Chapter 6

Generalized Poincaré series

In order to relate the critical exponent of the Poincaré series to the Hausdorff dimension of the limit set of Fuchsian groups, Patterson ([P]) and Sullivan ([S]) developped a theory of conformal densities for real hyperbolic spaces. An extensive description of their work is given in [N]. In 1996, P. Albuquerque extended part of this theory to arbitrary symmetric spaces $X = G/K$ of noncompact type. He showed that for Zariski dense subgroups Γ of G, the support of any $\delta(\Gamma)$-dimensional conformal density either lies in ∂X^{sing} or is contained in a unique G-invariant subset $G\cdot\xi \subseteq \partial X^{reg}$ (Theorem A in [Al]).

Inspired by the paper [Bu] of M. Burger, we are going to construct families of Γ-equivariant measures on every G-invariant subset of the limit set. We will use the Patterson Sullivan construction to obtain orbital measures with many degrees of freedom on the geometric limit set. An important role plays the directional distribution of the number of orbit points, which allows to single out those measures with support in a certain subset $G\xi \subseteq \partial X$ of the geometric boundary. We remark that similar measures have been constructed independently by J. F. Quint ([Q]) using different methods. His measures, however, are all supported on the Furstenberg boundary and therefore lack an essential piece of information concerning the geometry of Γ-orbits.

As usual, $X = G/K$ will be a globally symmetric space of noncompact type with geometric boundary ∂X, $x_0 \in X$ the unique point stabilized by K, and $\mathcal{M}^+(\overline{X})$ the cone of positive finite Borel measures on $X \cup \partial X$. $\Gamma \subset G$ will denote a discrete group of isometries of X.

6.1 Exponential growth in direction $G\cdot\xi$

Let $x, y \in X$, $\xi \in \partial X$ and $\gamma \in \Gamma$. Recall from section 3.6 that

$$\angle_x(\gamma y, G\cdot\xi) = \inf_{g\in G} \angle_x(\gamma y, g\xi) \quad \text{for} \quad \gamma y \neq x\,.$$

If $\gamma y = x$, which is true for only finitely many $\gamma \in \Gamma$ by the discreteness of Γ, we put

$$\angle_x(\gamma y, G\cdot\xi) = 0\,.$$

Using this convention, for $\varphi > 0$ and $s \geq 0$ the series

$$Q^{s,\varphi}_{G\cdot\xi}(x,y) = \sum_{\substack{\gamma\in\Gamma \\ \angle_x(\gamma y, G\cdot\xi)<\varphi}} e^{-sd(x,\gamma y)}$$

is well defined. Its critical exponent $\delta^{\varphi}_{G\cdot\xi}(x,y)$ is the unique real number such that $Q^{s,\varphi}_{G\cdot\xi}(x,y)$ converges if $s > \delta^{\varphi}_{G\cdot\xi}(x,y)$ and diverges if $s < \delta^{\varphi}_{G\cdot\xi}(x,y)$. Note that by the triangle inequality, the series

$$\sum_{\substack{\gamma\in\Gamma \\ \angle_x(\gamma y, G\cdot\xi)<\varphi}} e^{-sd(x_0,\gamma x_0)}$$

possesses the same criticial exponent as $Q^{s,\varphi}_{G\cdot\xi}(x,y)$.

This critical exponent can also be interpreted as an exponential growth rate of the number of orbit points close in direction to $G\cdot\xi$

$$\Delta N^{\varphi}_{G\cdot\xi}(x,y;R) = \#\{\gamma\in\Gamma \mid R-1 \leq d(x,\gamma y) < R\,,\ \angle_x(\gamma y, G\cdot\xi) < \varphi\}\,,$$

because an easy calculation shows that

$$\delta^{\varphi}_{G\cdot\xi}(x,y) = \limsup_{R\to\infty} \frac{\log \Delta N^{\varphi}_{G\cdot\xi}(x,y;R)}{R}\,.$$

Definition 6.1 *The number* $\delta_{G\cdot\xi}(\Gamma) := \liminf_{\varphi\to 0} \delta^{\varphi}_{G\cdot\xi}(x_0,x_0)$ *is called the* exponent of growth of Γ in direction $G\cdot\xi$.

Lemma 6.2 *For any* $x,y\in X$ *we have* $\delta_{G\cdot\xi}(\Gamma) = \liminf_{\varphi\to 0} \delta^{\varphi}_{G\cdot\xi}(x,y)$.

Proof. Choose $x,y\in X$ arbitrary and put $c := d(x_0,x) + d(x_0,y)$. Let $(\varphi_j) \searrow 0$ be a sequence of positive numbers, $\varphi_j \leq \pi/4$. Suppose $\gamma\in\Gamma$ satisfies $d(x_0,\gamma x_0) \geq \frac{4c}{\varphi_j^2}$ and $d(x,\gamma y) \geq \frac{4c}{\varphi_j^2}$ for sufficiently large j. Then applying Lemma 3.18 twice gives

$$\angle_{x_0}(\gamma x_0, G\cdot\xi) < \varphi_j/2 \quad\Longrightarrow\quad \angle_x(\gamma y, G\cdot\xi) < \varphi_j \quad\Longrightarrow\quad \angle_{x_0}(\gamma x_0, G\cdot\xi) < 2\varphi_j\,,$$

and we conclude $\delta^{\varphi_j/2}_{G\cdot\xi}(x_0,x_0) \leq \delta^{\varphi_j}_{G\cdot\xi}(x,y) \leq \delta^{2\varphi_j}_{G\cdot\xi}(x_0,x_0)$. Taking the limit inferior as $j\to\infty$ finishes the proof. □

Lemma 6.3 *If* $L_\Gamma \cap G\cdot\xi \neq \emptyset$, *then* $\delta_{G\cdot\xi}(\Gamma) \geq 0$.

Proof. Suppose $L_\Gamma \cap G\cdot\xi \neq \emptyset$. Then for any $\varphi > 0$, there exist infinitely many $\gamma\in\Gamma$ with the property $\angle_{x_0}(\gamma x_0, G\cdot\xi) < \varphi$. In particular

$$\sum_{\substack{\gamma\in\Gamma \\ \angle_{x_0}(\gamma x_0, G\cdot\xi)<\varphi}} 1 = Q^{0,\varphi}_{G\cdot\xi}(x_0,x_0)$$

diverges, hence $\delta^{\varphi}_{G\cdot\xi}(x_0,x_0) \geq 0$. We conclude $\delta_{G\cdot\xi}(\Gamma) = \liminf_{\varphi\to 0} \delta^{\varphi}_{G\cdot\xi}(x_0,x_0) \geq 0$. □

PROPOSITION 6.4 *Let $(\xi_j) \subset \partial X$ be a sequence converging to $\xi \in \partial X$. Then*

$$\limsup_{j\to\infty} \delta_{G\cdot\xi_j}(\Gamma) \le \delta_{G\cdot\xi}(\Gamma) .$$

Proof. Let $\varphi_0 \in (0,\pi/2)$. Then $\xi_j \to \xi$ implies $\angle_{x_0}(\xi_j, G\cdot\xi) < \varphi_0/2$ for j sufficiently large. Let $\varphi \in (0,\varphi_0/2)$ and $\gamma \in \Gamma$ such that $\angle_{x_0}(\gamma x_0, G\cdot\xi_j) < \varphi$. Then

$$\angle_{x_0}(\gamma x_0, G\cdot\xi) < \varphi + \varphi_0/2 < \varphi_0 ,$$

which proves $\delta^{\varphi}_{G\cdot\xi_j}(x_0,x_0) \le \delta^{\varphi_0}_{G\cdot\xi}(x_0,x_0)$, and therefore

$$\delta_{G\cdot\xi_j}(\Gamma) = \liminf_{\varphi\to 0} \delta^{\varphi}_{G\cdot\xi_j}(x_0,x_0) \le \delta^{\varphi_0}_{G\cdot\xi}(x_0,x_0) .$$

We conclude

$$\begin{aligned} \limsup_{j\to\infty} \delta_{G\cdot\xi_j}(\Gamma) &\le \delta^{\varphi_0}_{G\cdot\xi}(x_0,x_0) , \quad \text{hence} \\ \limsup_{j\to\infty} \delta_{G\cdot\xi_j}(\Gamma) &= \liminf_{\varphi_0\to 0}\left(\limsup_{j\to\infty} \delta_{G\cdot\xi_j}(\Gamma)\right) \le \liminf_{\varphi_0\to 0} \delta^{\varphi_0}_{G\cdot\xi}(x_0,x_0) = \delta_{G\cdot\xi}(\Gamma) . \end{aligned}$$ □

REMARK. If $\mathrm{rank}(X) = 1$ or $\varphi > \pi/2$, then the series $Q^{s,\varphi}_{G\cdot\xi}(x,y)$, $\xi \in \partial X$, reduces to the familiar Poincaré series

$$\sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)} .$$

The critical exponent of this series is called the critical exponent of Γ and will be denoted by $\delta(\Gamma)$. In particular, we have $\delta_{G\cdot\xi}(\Gamma) \le \delta(\Gamma)$ for any $\xi \in \partial X$.

6.2 The region of convergence

Let d denote the Riemannian distance, $\mathcal{B}_{G\cdot\xi}$, $\xi \in \partial X$, and d_i, $1 \le i \le r$, the directional distances introduced in sections 3.2 and 3.4. We observe that for any r-tuple $b = (b^1, b^2, \dots, b^r) \in \mathbb{R}^r$, $G\cdot\xi \subseteq \partial X$ and $\tau \ge 0$ fixed, the series

$$P^{s,b,\tau}_{G\cdot\xi}(x,y) = \sum_{\gamma\in\Gamma} e^{-s\left(\sum_{i=1}^r b^i d_i(x,\gamma y) + \tau\left(d(x,\gamma y) - \mathcal{B}_{G\cdot\xi}(x,\gamma y)\right)\right)}$$

possesses a critical exponent which is independent of $x, y \in X$ by the triangle inequalities for d, $\mathcal{B}_{G\cdot\xi}$ and d_1, $d_2, \dots d_r$.

For any subset $G\cdot\xi \subseteq \partial X$ and $\tau \ge 0$, we may therefore define a region of convergence

$$\mathcal{R}^{\tau}_{G\cdot\xi} := \{b = (b^1, b^2, \dots b^r) \mid P^{s,b,\tau}_{G\cdot\xi}(x_0,x_0) \text{ has critical exponent } s \le 1\} \subseteq \mathbb{R}^r .$$

This region possesses the following properties.

LEMMA 6.5 *If $\tau \le \tau'$, then $\mathcal{R}^{\tau}_{G\cdot\xi} \subseteq \mathcal{R}^{\tau'}_{G\cdot\xi}$.*

Proof. Let $\tau \le \tau'$, $b \in \mathcal{R}^{\tau}_{G\cdot\xi}$. Then for any $\gamma \in \Gamma$

$$e^{-s\left(\sum_{i=1}^r b^i d_i(x_0,\gamma x_0)+\tau'\left(d(x_0,\gamma x_0)-\mathcal{B}_{G\cdot\xi}(x_0,\gamma x_0)\right)\right)} \le e^{-s\left(\sum_{i=1}^r b^i d_i(x_0,\gamma x_0)+\tau\left(d(x_0,\gamma x_0)-\mathcal{B}_{G\cdot\xi}(x_0,\gamma x_0)\right)\right)}$$

and therefore $P^{s,b,\tau'}_{G\cdot\xi}(x_0,x_0) \le P^{s,b,\tau}_{G\cdot\xi}(x_0,x_0)$. Hence $P^{s,b,\tau'}_{G\cdot\xi}(x_0,x_0)$ converges if $s>1$. In particular, $P^{s,b,\tau'}_{G\cdot\xi}(x_0,x_0)$ has critical exponent less than or equal to 1. □

LEMMA 6.6 *For any $\tau \ge 0$, the region $\mathcal{R}^{\tau}_{G\cdot\xi}$ is convex.*

Proof. Let $\tau \ge 0$, $a,b \in \mathcal{R}^{\tau}_{G\cdot\xi}$ and $t\in[0,1]$. For $\gamma\in\Gamma$, we abbreviate
$(ta+(1-t)b)_\gamma := \sum_{i=1}^r (ta^i+(1-t)b^i)d_i(x_0,\gamma x_0) + \tau\left(d(x_0,\gamma x_0) - \mathcal{B}_{G\cdot\xi}(x_0,\gamma x_0)\right)$.
Then by Hölder's inequality

$$\sum_{\gamma\in\Gamma} e^{-s(ta^i+(1-t)b^i)_\gamma} = \sum_{\gamma\in\Gamma} e^{-sta_\gamma} e^{-s(1-t)b_\gamma} \le \left(\sum_{\gamma\in\Gamma} e^{-sa_\gamma}\right)^t \left(\sum_{\gamma\in\Gamma} e^{-sb_\gamma}\right)^{1-t}.$$

The latter sum converges if $s>1$, hence $ta+(1-t)b \in \mathcal{R}^{\tau}_{G\cdot\xi}$. □

In order to describe the region of convergence more precisely, we prove the following proposition concerning convergence and divergence of certain series.

PROPOSITION 6.7 *Let $x,y\in X$, $\xi\in\partial X$ and $D\subseteq\partial X$ an open set with respect to the cone topology. Put $\Gamma_D := \{\gamma\in\Gamma \mid \angle_x(\gamma y, G\cdot D) := \inf_{g\in G}\inf_{\eta\in D}\angle_x(\gamma y, g\eta) = 0\}$.*
Then for all $s,\tau\in\mathbb{R}$, and $(b^1,b^2,\ldots,b^r)\in\mathbb{R}^r$ we have the following implications:

(1) If there exists $\eta_0\in D$ such that
$s\left(\sum_{i=1}^r b^i\cos\angle_x(\eta_0,\partial X^i) + \tau\left(1-\cos\angle_x(\eta_0,G\cdot\xi)\right)\right) < \delta_{G\cdot\eta_0}(\Gamma)$, then the series

$$\sum_{\gamma\in\Gamma_D} e^{-s\left(\sum_{i=1}^r b^i d_i(x,\gamma y)+\tau\left(d(x,\gamma y)-\mathcal{B}_{G\cdot\xi}(x,\gamma y)\right)\right)} \qquad \textit{diverges}.$$

(2) If $s\left(\sum_{i=1}^r b^i\cos\angle_x(\eta,\partial X^i) + \tau\left(1-\cos\angle_x(\eta,G\cdot\xi)\right)\right) > \delta_{G\cdot\eta}(\Gamma)$ for all $\eta\in\overline{D}$, then

$$\sum_{\gamma\in\Gamma_D} e^{-s\left(\sum_{i=1}^r b^i d_i(x,\gamma y)+\tau\left(d(x,\gamma y)-\mathcal{B}_{G\cdot\xi}(x,\gamma y)\right)\right)} \qquad \textit{converges}.$$

Proof.

(1) Let $\eta_0 \in D$ such that $s\big(\sum_{i=1}^r b^i \cos\angle_x(\eta_0, \partial X^i) + \tau\,(1-\cos\angle_x(\eta_0, G\cdot\xi))\big) < \delta_{G\cdot\eta_0}(\Gamma)$. Since $\delta_{G\cdot\eta_0}(\Gamma) = \liminf_{\varphi\to 0}\delta^{\varphi}_{G\cdot\eta_0}(x,y)$, there exists $\varphi \in (0,\pi/4)$ and $s_0 \in \mathbb{R}$ such that for any $\gamma \in \Gamma_D$ with $\angle_x(\gamma y, G\cdot\eta_0) < \varphi$ we have

$$s\big(\sum_{i=1}^r b^i \cos\angle_x(\gamma y, \partial X^i) + \tau\,(1-\cos\angle_x(\gamma y, G\cdot\xi))\big) < s_0 < \delta^{\varphi}_{G\cdot\eta_0}(x,y)\,.$$

By Lemma 3.10 we have

$$\begin{aligned} d_i(x,\gamma y) &= d(x,\gamma y)\cos\angle_x(\gamma y, \partial X^i)\,, \quad 1\le i\le r\,,\\ \mathcal{B}_{G\cdot\xi}(x,y) &= d(x,\gamma y)\cos\angle_x(\gamma y, G\cdot\xi)\,. \end{aligned} \tag{6.1}$$

Therefore

$$\begin{aligned} &\sum_{\gamma\in\Gamma_D} e^{-s\left(\sum_{i=1}^r b^i d_i(x,\gamma y) + \tau\left(d(x,\gamma y) - \mathcal{B}_{G\cdot\xi}(x,\gamma y)\right)\right)}\\ &\quad\ge \sum_{\substack{\gamma\in\Gamma_D\\ \angle_x(\gamma y, G\cdot\xi)<\varphi}} e^{-s\left(\sum_{i=1}^r b^i \cos\angle_x(\gamma y,\partial X^i) + \tau(1-\cos\angle_x(\gamma y, G\xi))\right)d(x,\gamma y)}\\ &\quad> \sum_{\substack{\gamma\in\Gamma_D\\ \angle_x(\gamma y, G\cdot\xi)<\varphi}} e^{-s_0 d(x,\gamma y)}\,, \end{aligned}$$

and the latter sum diverges since $s_0 < \delta^{\varphi}_{G\cdot\eta_0}$.

(2) Let $\eta_0 \in \overline{D}$ and fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ with respect to x and ξ. Let $H_\xi, H_0 \in \overline{\mathfrak{a}_1^+}$ denote the Cartan projections of ξ and η_0. The condition

$$\delta_{G\cdot\eta_0}(\Gamma) < s\big(\sum_{i=1}^r b^i \cos\angle_x(\eta_0, \partial X^i) + \tau\,(1-\cos\angle_x(\eta_0, G\cdot\xi))\big)$$

is equivalent to $\delta_{G\cdot\eta_0}(\Gamma) < s\big(\sum_{i=1}^r b^i\langle H_i, H_0\rangle + \frac{\tau}{2}\|H_\xi - H_0\|^2\big) =: s(H_0)$.
Since $\delta_{G\cdot\eta_0}(\Gamma) = \liminf_{\varphi\to 0}\delta^{\varphi}_{G\cdot\eta_0}(x,y)$, there exists $\varphi_0' \in (0,\pi/4)$ and $s_0 < s(H_0)$ such that

$$\delta^{\varphi_0'}_{G\cdot\eta_0}(x,y) < s_0 < s(H_0)\,. \tag{6.2}$$

For $\eta \in \partial X$ and $\varphi > 0$ we put $S_{G\cdot\eta}(\varphi) := \{\zeta \in \partial X \mid \angle_x(\zeta, G\cdot\eta) < \varphi\}$. The continuity of the function

$$s:\quad \mathfrak{a}_1 \;\to\; \mathbb{R}\,,\qquad H \mapsto s\left(\sum_{i=1}^r b^i\langle H_i, H\rangle + \frac{\tau}{2}\|H_\xi - H\|^2\right)$$

and inequality (6.2) imply the existence of $\varphi_0 < \varphi_0'$ such that for any $\eta \in S_{G\cdot\eta_0}(\varphi_0)$ with Cartan projection $H_\eta \in \overline{\mathfrak{a}_1^+}$, we have $s_0 < s(H_\eta)$. Hence

$$\delta^{\varphi_0}_{G\cdot\eta_0} \le \delta^{\varphi_0'}_{G\cdot\eta_0} < s_0 < s(H_\eta) \qquad \forall\,\eta \in S_{G\cdot\eta_0}(\varphi_0)\,.$$

We now choose a sequence $(\eta_j) \subset \overline{D}$ and corresponding sequences $(s_j) \subset \mathbb{R}^+$ and $(\varphi_j) \subset \mathbb{R}^+$ such that for any $\eta \in S_{G\cdot\eta_j}(\varphi_j)$ with Cartan projection $H_\eta \in \overline{\mathfrak{a}_1^+}$ we have

$$\delta^{\varphi_j}_{G\cdot\eta_j}(x,y) < s_j < s(H_\eta)\,,$$

$$\text{and} \qquad G\cdot D \subset G\cdot\overline{D} \subseteq \bigcup_{j\in\mathbb{N}} S_{G\cdot\eta_j}(\varphi_j)\,.$$

Since $G\cdot\overline{D}$ is a compact subset of the geometric boundary, we may extract a finite covering $\bigcup_{j=1}^{l} S_{G\cdot\eta_j}(\varphi_j)$, and conclude using equations (6.1)

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_D} e^{-s\left(\sum_{i=1}^r b^i d_i(x,\gamma y)+\tau\left(d(x,\gamma y)-\mathcal{B}_{G\cdot\xi}(x,\gamma y)\right)\right)}\\
&\qquad \le \sum_{j=1}^{l} \sum_{\substack{\gamma\in\Gamma\\ \angle_x(\gamma y, G\cdot\eta_j)<\varphi_j}} e^{-sd(x,\gamma y)\left(\sum_{i=1}^r b^i \cos\angle_x(\gamma y,\partial X^i)+\tau(1-\cos\angle_x(\gamma y, G\cdot\xi))\right)}\\
&\qquad \le \sum_{j=1}^{l} \sum_{\substack{\gamma\in\Gamma\\ \angle_x(\gamma y, G\cdot\eta_j)<\varphi_j}} e^{-s_j d(x,\gamma y)} < \infty\,,
\end{aligned}$$

because $s_j > \delta^{\varphi_j}_{G\cdot\eta_j}$ for $1 \le j \le l$. □

For the remainder of this section we fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ with respect to $x_0 \in X$ and some regular boundary point. The proof of the previous proposition allows to deduce

COROLLARY 6.8 *Let $D \subseteq \partial X$ be an open set with respect to the cone topology, and $\Gamma_D := \{\gamma \in \Gamma \mid \angle_x(\gamma y, G\cdot D) := \inf_{g\in G}\inf_{\eta\in D}\angle_x(\gamma y, g\eta) = 0\}$ as before. For $\gamma\in\Gamma$ let $H_\gamma \in \overline{\mathfrak{a}_1^+}$ denote the unit length Cartan projection of γx_0.*

If $s : \overline{\mathfrak{a}_1^+} \to \mathbb{R}$ is a continuous function with the property $s(H_\eta) > \delta_{G\cdot\eta}(\Gamma)$ for all $\eta \in \overline{D}$ with Cartan projection $H_\eta \in \overline{\mathfrak{a}_1^+}$, then the series

$$\sum_{\gamma\in\Gamma_D} e^{-s(H_\gamma)d(x_0,\gamma x_0)} \qquad \textit{converges}\,.$$

The following two results relate the region of convergence $\mathcal{R}^\tau_{G\cdot\xi}$ to the exponent of growth in direction $G\cdot\xi$.

LEMMA 6.9 *Let $\xi \in \partial X$ and $\tau \ge 0$. If $H_\xi \in \overline{\mathfrak{a}_1^+}$ denotes the Cartan projection of ξ and $(b^1, b^2, \ldots b^r) \in \mathcal{R}^\tau_{G\cdot\xi}$, then $\sum_{i=1}^r b^i\langle H_i, H_\xi\rangle \ge \delta_{G\cdot\xi}(\Gamma)$.*

Proof. Recall that $\langle H_i, H_\xi\rangle = \cos\angle_{x_0}(\xi, \partial X^i)$ for $1 \le i \le r$, and $\cos\angle_{x_0}(\xi, G\cdot\xi) = 1$. If $\sum_{i=1}^r b^i\langle H_i, H_\xi\rangle < \delta_{G\cdot\xi}(\Gamma)$, then there exists $s > 1$ such that

$$s\left(\sum_{i=1}^r b^i\cos\angle_{x_0}(\xi,\partial X^i) + \tau\big(1-\cos\angle_{x_0}(\xi, G\cdot\xi)\big)\right) < \delta_{G\cdot\xi}(\Gamma)\,.$$

Applying Proposition 6.7 (1) to $D = \partial X$, we conclude that

$$\sum_{\gamma\in\Gamma} e^{-s\left(\sum_{i=1}^r b^i d_i(x_0,\gamma x_0)+\tau\left(d(x_0,\gamma x_0)-\mathcal{B}_{G\cdot\xi}(x_0,\gamma x_0)\right)\right)}$$

diverges, in contradiction to $(b^1, b^2, \dots b^r) \in \mathcal{R}^\tau_{G\cdot\xi}$. □

Notation. For $\xi \in \partial X$ and $\varphi > 0$ we put $S_{G\cdot\xi}(\varphi) := \{\eta \in \partial X \mid \angle_{x_0}(\eta, G\cdot\xi) < \varphi\}$.

Proposition 6.10 *Let $\xi \in \partial X$, $\tau \geq 0$ and $b = (b^1, b^2, \dots b^r) \in \partial\mathcal{R}^\tau_{G\cdot\xi}$ with $b^i \geq 0$ for $1 \leq i \leq r$. If $H_\xi \in \overline{\mathfrak{a}_1^+}$ denotes the Cartan projection of ξ, then*

(1) For any $\eta = (k, H) \in \partial X$ we have $\sum_{i=1}^r b^i\langle H_i, H\rangle + \frac{\tau}{2}\|H_\xi - H\|^2 \geq \delta_{G\cdot\eta}(\Gamma)$.

(2) There exists $\eta_\tau \in S_{G\cdot\xi}(\sqrt{3\delta(\Gamma)/\tau})$ with Cartan projection $H_\tau \in \overline{\mathfrak{a}_1^+}$ such that

$$\sum_{i=1}^r b^i\langle H_i, H_\tau\rangle + \frac{\tau}{2}\|H_\xi - H_\tau\|^2 \leq \delta_{G\cdot\eta_\tau}(\Gamma)\,.$$

Proof. The first claim of the statement follows as in the previous lemma from Proposition 6.7 (1).

For $\xi \in \partial X$ and $\tau \geq 0$, $b \in \partial\mathcal{R}^\tau_{G\cdot\xi}$ implies that for any $s < 1$ the series

$$\sum_{\gamma\in\Gamma} e^{-s\left(\sum_{i=1}^r b^i d_i(x,\gamma y)+\tau\left(d(x,\gamma y)-\mathcal{B}_{G\cdot\xi}(x,\gamma y)\right)\right)}$$

diverges. By Proposition 6.7 (2), there exists $\eta_\tau \in \partial X$ with Cartan projection $H_\tau \in \overline{\mathfrak{a}_1^+}$ so that

$$s\left(\sum_{i=1}^r b^i\langle H_i, H_\tau\rangle + \frac{\tau}{2}\|H_\xi - H_\tau\|^2\right) \leq \delta_{G\cdot\eta_\tau}(\Gamma)\,.$$

Taking the limit as $s \nearrow 1$, we obtain $\sum_{i=1}^r b^i\langle H_i, H_\tau\rangle + \frac{\tau}{2}\|H_\xi - H_\tau\|^2 \leq \delta_{G\cdot\eta_\tau}(\Gamma)$, and since $\sum_{i=1}^r b^i\langle H_i, H_\tau\rangle \geq 0$, we deduce

$$\tau\left(1 - \langle H_\xi, H_\tau\rangle\right) = \frac{\tau}{2}\|H_\xi - H_\tau\|^2 \leq \delta_{G\cdot\eta_\tau}(\Gamma)\,.$$

Writing $\varphi := \angle_{x_0}(\eta_\tau, G\cdot\xi) \in [0, \pi/2]$ and using the fact that $\langle H_\xi, H_\tau\rangle = \cos\varphi < 1 - \varphi^2/3$, we conclude $\tau\varphi^2/3 < \delta_{G\cdot\eta_\tau}(\Gamma) \leq \delta(\Gamma)$. Hence $\varphi = \angle_{x_0}(\eta_\tau, G\cdot\xi) < \sqrt{3\delta(\Gamma)/\tau}$. □

6.3 The Patterson Sullivan construction

Let $\xi \in \partial X$ such that $G\cdot\xi \cap L_\Gamma \neq \emptyset$, $\tau \geq 0$ and $b = (b^1, b^2, \dots b^r) \in \partial\mathcal{R}^\tau_{G\cdot\xi}$. Recall that $\mathcal{B}_{G\cdot\xi}$ and d_i, $1 \leq i \leq r$, are the directional and maximal singular distances introduced in sections 3.2 and 3.4. For $\gamma \in \Gamma$, we abbreviate

$$b_\gamma := \sum_{i=1}^r b^i d_i(x_0, \gamma x_0) + \tau\left(d(x_0, \gamma x_0) - \mathcal{B}_{G\cdot\xi}(x_0, \gamma x_0)\right)\,.$$

Lemma 6.11 *(Patterson, [P])*

There exists a positive increasing function h on $[0,\infty)$ such that

(i) $\Psi^s = \sum_{\gamma\in\Gamma} e^{-sb_\gamma} h(b_\gamma)$ *has exponent of convergence $s=1$ and diverges at $s=1$.*

(ii) For any $\varepsilon>0$ there exists $r_0>0$ such that for $r\ge r_0$ and $t>1$

$$h(rt)\le t^\varepsilon h(r)\,.$$

Following the original idea of Patterson ([P]), we construct a family of orbital measures on $\overline{X}$ in the following way. If D denotes the unit Dirac point measure, we put for $x\in X$ and $s>1$

$$\mu_x^s := \frac{1}{\Psi^s}\sum_{\gamma\in\Gamma} e^{-s\left(\sum_{i=1}^r b^i d_i(x,\gamma x_0)+\tau(d(x,\gamma x_0)-\mathcal{B}_{G\cdot\xi}(x,\gamma x_0))\right)} h(b_\gamma) D(\gamma x_0)\,.$$

These measures are Γ-equivariant by construction and absolutely continuous with respect to each other. Note that they also depend on $G\cdot\xi\subseteq\partial X$, $\tau\ge 0$ and $b=(b^1,b^2,\dots,b^r)\in\partial\mathcal{R}^\tau_{G\cdot\xi}$. For $\gamma\in\Gamma$ and $x,y\in X$ we put

$$\begin{aligned} q_\gamma(y,x) \;:=\; & \sum_{i=1}^r b^i\big(d_i(y,\gamma x_0)-d_i(x,\gamma x_0)\big) \\ & +\tau\big(d(y,\gamma x_0)-d(x,\gamma x_0)-\mathcal{B}_{G\cdot\xi}(y,\gamma x_0)+\mathcal{B}_{G\cdot\xi}(x,\gamma x_0)\big)\,. \end{aligned} \tag{6.3}$$

Then for $s>1$, the Radon-Nikodym derivative is given by

$$\begin{aligned} \frac{d\mu_x^s}{d\mu_y^s}\;:\qquad \Gamma\cdot x_0 &\;\to\; \mathbb{R} \\ \gamma x_0 &\;\mapsto\; e^{s q_\gamma(y,x)}\,. \end{aligned}$$

Let $(\mathrm{C}^0(\overline{X}),\|\cdot\|_\infty)$ denote the space of real valued continuous functions on $\overline{X}$ with norm $\|f\|_\infty=\max\{|f(x)|\;\big|\;x\in\overline{X}\}$, $f\in\mathrm{C}^0(\overline{X})$. We endow the cone $\mathcal{M}^+(\overline{X})$ of positive finite Borel measures on $\overline{X}$ with the pseudo metric

$$\rho(\mu_1,\mu_2):=\sup\{\Big|\int_{\overline{X}} f\,d\mu_1-\int_{\overline{X}} f\,d\mu_2\Big|\;\Big|\;f\in\mathrm{C}^0(\overline{X})\,,\|f\|_\infty=1\}\,,\qquad \mu_1,\,\mu_2\in\mathcal{M}^+(\overline{X})\,,$$

and obtain the following

Lemma 6.12 *Let $\xi\in\partial X$ with $G\cdot\xi\cap L_\Gamma\neq\emptyset$, $\tau\ge 0$ and $b=(b^1,b^2,\dots,b^r)\in\partial\mathcal{R}^\tau_{G\cdot\xi}$. Then the family of maps $\mathcal{F}(G\cdot\xi,\tau,b):=\{x\mapsto\mu_x^s\mid 1<s\le 2\}$ from X to $\mathcal{M}^+(\overline{X})$ is equicontinuous.*

Proof. Let $x,y\in X$. For $\gamma\in\Gamma$ we use the abbreviation $q_\gamma(y,x)$ from (6.3), $\|b\|_1:=\sum_{i=1}^r|b^i|$ and estimate

$$|q_\gamma(y,x)|\le\sum_{i=1}^r b^i d_i(y,x)+2\tau d(y,x)\le d(x,y)\,(||b||_1+2\tau)\,. \tag{6.4}$$

If $s \in (1,2]$ and $f \in \mathrm{C}^0(\overline{X})$, the inequality $|1-e^{-t}| \le e^{|t|}-1$, $t \in \mathbb{R}$, yields

$$\begin{aligned} &\Big| \int_{\overline{X}} f\, d\mu_x^s - \int_{\overline{X}} f\, d\mu_y^s \Big| \\ &\quad \le \frac{1}{\Psi^s} \sum_{\gamma\in\Gamma} e^{-s\left(\sum_{i=1}^r b^i d_i(x,\gamma x_0) + \tau(d(x,\gamma x_0) - \mathcal{B}_{G\cdot\xi}(x,\gamma x_0))\right)} h(b_\gamma) |f(\gamma x_0)| \cdot \left|1 - e^{-s q_\gamma(y,x)}\right| \\ &\quad \le \frac{\|f\|_\infty}{\Psi^s} \sum_{\gamma\in\Gamma} e^{-s b_\gamma} e^{-s q_\gamma(x,x_0)} h(b_\gamma) \left(e^{s|q_\gamma(y,x)|} - 1\right) . \end{aligned}$$

Since $f \in \mathrm{C}^0(\overline{X})$ was arbitrary, $s \le 2$ and $\sum_{\gamma\in\Gamma} e^{-s b_\gamma} h(b_\gamma) = \Psi^s$, we conclude using (6.4)

$$\rho(\mu_x^s, \mu_y^s) \le e^{2d(x_0,x)(||b||_1+2\tau)} \cdot \left(e^{2d(x,y)(||b||_1+2\tau)} - 1\right) .$$

This proves that $\mathcal{F}(G\cdot\xi, \tau, b)$ is equicontinuous. □

LEMMA 6.13 *Let $\xi \in \partial X$ with $G\xi \cap L_\Gamma \neq \emptyset$, $\tau \ge 0$ and $b = (b^1, b^2, \dots, b^r) \in \partial\mathcal{R}^\tau_{G\cdot\xi}$. Then for any $x \in X$ there exists a sequence $(s_n) \searrow 1$ such that the measures $\mu_x^{s_n} \subset \mathcal{M}^+(\overline{X})$ converge weakly to a measure $\mu_x := \mu_x(G\cdot\xi, \tau, b)$ as $n \to \infty$.*

Proof. The compactness of the space $\overline{X}$ implies that every sequence of measures in $\mathcal{M}^+(\overline{X})$ possesses a weakly convergent subsequence. □

The theorem of Arzelà Ascoli ([K], Theorem 7.17, p. 233) now allows to conclude that $\mathcal{F}(G\cdot\xi, \tau, b)$ is relatively compact in the space of continuous maps $\mathrm{C}(X, \mathcal{M}^+(\overline{X}))$ endowed with the topology of uniform convergence on compact sets. From the definition of $(\mu_x^s)_{x\in X}$ it follows, that every accumulation point $\mu = \mu(G\cdot\xi, \tau, b) = (\mu_x)_{x\in X}$ of $\mathcal{F}(G\cdot\xi, \tau, b)$ as $s \searrow 1$ takes its values in $\mathcal{M}^+(\partial X)$.

The following proposition will provide the key ingredient in the construction of orbital measures with support in a single orbit $G\cdot\xi \subseteq \partial X$ in section 6.4. Recall that for $\varphi > 0$

$$S_{G\cdot\xi}(\varphi) := \{\eta \in \partial X \mid \angle_{x_0}(\eta, G\cdot\xi) < \varphi\} .$$

PROPOSITION 6.14 *Fix $\xi \in \partial X$ and suppose there exists $b = (b^1, b^2, \dots, b^r) \in \mathbb{R}^r$ and $\varphi_0 \in (0, \pi/4)$ such that*

$$\begin{aligned} \sum_{i=1}^r b^i \langle H_i, H_\xi \rangle &= \delta_{G\cdot\xi}(\Gamma) , \quad \textit{and} \\ \sum_{i=1}^r b^i \langle H_i, H_\eta \rangle &\ge \delta_{G\cdot\eta}(\Gamma) \quad \forall\, \eta \in S_{G\cdot\xi}(\varphi_0) \ \textit{with Cartan projection}\ H_\eta \in \overline{\mathfrak{a}_1^+} . \end{aligned}$$

Then there exists $\tau_0 = \tau_0(b, \varphi_0) \ge 0$ such that for all $\tau \ge \tau_0$ and for all $\varphi > 0$

$$\sum_{\substack{\gamma\in\Gamma \\ \angle_{x_0}(\gamma x_0, G\cdot\xi) > \varphi}} e^{-b_\gamma} h(b_\gamma) < \infty ,$$

where b_γ and h are as in Proposition 6.11.

Proof. Let $\|b\|_1 := \sum_{i=1}^r |b^i|$, $\varphi \in (0, \varphi_0]$, and put

$$\begin{aligned} D_1 &:= \{\eta \in \partial X \mid \varphi/2 < \angle_{x_0}(\eta, G{\cdot}\xi) < \varphi_0\}\,, \\ D_2 &:= \{\eta \in \partial X \mid \angle_{x_0}(\eta, G{\cdot}\xi) > \varphi_0/2\}\,. \end{aligned}$$

By property (ii) of the function h in Lemma 6.11, there exists $r_0 = r_0(\varphi) > 0$ such that for $r \geq r_0$ and $t > 1$ we have $h(rt) \leq (t)^{\varphi^2} h(r)$. Let $R = R(\varphi) > 0$ such that $d(x_0, \gamma x_0) > R$ implies $b_\gamma > r_0$ and

$$d(x_0, \gamma x_0)\big(\|b\|_1 + 2\varphi^2 + 2\delta(\Gamma)\big) < \min\{e^{d(x_0,\gamma x_0)}, e^{\delta(\Gamma) d(x_0,\gamma x_0)}\}\,. \tag{6.5}$$

Then for $\gamma \in \Gamma$ with $d(x_0, \gamma x_0) > R$ we have

$$h(b_\gamma) = h\big(\frac{b_\gamma}{r_0} r_0\big) \leq \Big(\frac{b_\gamma}{r_0}\Big)^{\varphi^2} h(r_0) = \frac{h(r_0)}{(r_0)^{\varphi^2}}\, e^{\varphi^2 \log b_\gamma}\,.$$

For $\gamma \in \Gamma$ we put $\eta_\gamma := \sigma_{x_0,\gamma x_0}(\infty) \in \partial X$ and let $H_\gamma \in \overline{\mathfrak{a}_1^+}$ denote the unit length Cartan projection of γx_0.

If $\gamma \in \Gamma_{D_1}$, then $\langle H_\xi, H_\gamma\rangle < \cos\frac{\varphi}{2} < 1 - \varphi^2/12$, and we obtain using equations (6.1)

$$\begin{aligned} b_\gamma &= d(x_0,\gamma x_0)\Big(\sum_{i=1}^r b^i\langle H_i, H_\gamma\rangle + \tau\big(1 - \langle H_\xi, H_\gamma\rangle\big)\Big) \\ &> d(x_0,\gamma x_0)\big(\sum_{i=1}^r b^i\langle H_i, H_\gamma\rangle + \frac{\tau\varphi^2}{12}\big) > d(x_0,\gamma x_0)\big(\sum_{i=1}^r b^i\langle H_i, H_\gamma\rangle + 2\varphi^2\big) \end{aligned}$$

if $\tau > 24$. Since the function $f(t) := t - \varphi^2 \log t$, $t > 0$, is monotone increasing, we conclude for $\gamma \in \Gamma_{D_1}$ with $d(x_0, \gamma x_0) > R$

$$\begin{aligned} f(b_\gamma) &> d(x_0,\gamma x_0)\big(\sum_{i=1}^r b^i\langle H_i, H_\gamma\rangle + 2\varphi^2\big) - \varphi^2 \log\Big(d(x_0,\gamma x_0)\big(\sum_{i=1}^r b^i\langle H_i, H_\gamma\rangle + 2\varphi^2\big)\Big) \\ &> d(x_0,\gamma x_0)\big(\sum_{i=1}^r b^i\langle H_i, H_\gamma\rangle + 2\varphi^2 - \varphi^2\big) = d(x_0,\gamma x_0)\big(\sum_{i=1}^r b^i\langle H_i, H_\gamma\rangle + \varphi^2\big)\,, \end{aligned}$$

where we used inequality (6.5) in the second step. For $\eta \in \partial X$ with Cartan projection $H \in \overline{\mathfrak{a}_1^+}$ we put $s(H) := \sum_{i=1}^r b^i\langle H_i, H\rangle + \varphi^2$. We estimate

$$\begin{aligned} \sum_{\substack{\gamma\in\Gamma_{D_1}\\ d(x_0,\gamma x_0)>R}} e^{-b_\gamma} h(b_\gamma) &\leq \frac{h(r_0)}{(r_0)^{\varphi^2}} \sum_{\substack{\gamma\in\Gamma_{D_1}\\ d(x_0,\gamma x_0)>R}} e^{-b_\gamma + \varphi^2 \log b_\gamma} = \frac{h(r_0)}{(r_0)^{\varphi^2}} \sum_{\substack{\gamma\in\Gamma_{D_1}\\ d(x_0,\gamma x_0)>R}} e^{-f(b_\gamma)} \\ &< \frac{h(r_0)}{(r_0)^{\varphi^2}} \sum_{\substack{\gamma\in\Gamma_{D_1}\\ d(x_0,\gamma x_0)>R}} e^{-s(H_\gamma) d(x_0,\gamma x_0)} < \frac{h(r_0)}{(r_0)^{\varphi^2}} \sum_{\gamma\in D_1} e^{-s(H_\gamma) d(x_0,\gamma x_0)}\,. \end{aligned}$$

This sum converges by Corollary 6.8 applied to D_1 and the continous function s on $\overline{\mathfrak{a}_1^+}$.

If $\gamma \in \Gamma_{D_2}$, then $\langle H_\xi, H_\gamma\rangle < \cos\frac{\varphi_0}{2} < 1-\varphi_0^2/12$. Using the Cauchy Schwarz inequality, $\|H_\xi - H_\gamma\| \le 2$, we further obtain

$$\begin{aligned}\sum_{i=1}^r b^i\langle H_i, H_\gamma\rangle &= \sum_{i=1}^r b^i\langle H_i, H_\xi\rangle + \sum_{i=1}^r b^i\langle H_i, H_\gamma - H_\xi\rangle \\ &\ge \delta_{G\cdot\xi}(\Gamma) - \|\sum_{i=1}^r b^i H_i\|\cdot\|H_\xi - H_\gamma\| \ge \delta_{G\cdot\xi}(\Gamma) - 2\|b\|_1\end{aligned}$$

from the assumption $\sum_{i=1}^r b^i\langle H_i, H_\xi\rangle = \delta_{G\cdot\xi}(\Gamma)$. If $\tau \ge 12\left(2\delta(\Gamma) - \delta_{G\cdot\xi}(\Gamma) + 2\|b\|_1\right)/\varphi_0^2$, we have

$$\begin{aligned}b_\gamma &= d(x_0,\gamma x_0)\big(\sum_{i=1}^r b^i\langle H_i, H_\gamma\rangle + \tau(1-\langle H_\xi, H_\gamma\rangle)\big) \\ &> d(x_0,\gamma x_0)\big(\delta_{G\cdot\xi}(\Gamma) - 2\|b\|_1 + \frac{\tau\varphi_0^2}{12}\big) > 2\delta(\Gamma)d(x_0,\gamma x_0)\,.\end{aligned}$$

As above we conclude for $\gamma \in \Gamma_{D_2}$ with $d(x_0,\gamma x_0) > R$

$$\begin{aligned}f(b_\gamma) &> 2\delta(\Gamma)d(x_0,\gamma x_0) - \varphi^2\log\big(2\delta(\Gamma)d(x_0,\gamma x_0)\big) \\ &> d(x_0,\gamma x_0)\big(2\delta(\Gamma) - \varphi^2\delta(\Gamma)\big)\end{aligned}$$

by inequality (6.5). Hence

$$\sum_{\substack{\gamma\in\Gamma_{D_2}\\ d(x_0,\gamma x_0)>R}} e^{-b_\gamma}h(b_\gamma) \le \frac{h(r_0)}{(r_0)^{\varphi^2}}\sum_{\substack{\gamma\in\Gamma_{D_2}\\ d(x_0,\gamma x_0)>R}} e^{-f(b_\gamma)} < \frac{h(r_0)}{(r_0)^{\varphi^2}}\sum_{\gamma\in\Gamma} e^{-(2-\varphi^2)\delta(\Gamma)d(x_0,\gamma x_0)}\,,$$

which converges since $\varphi^2 \le \varphi_0^2 < 1$. From the fact that

$$\sum_{\substack{\gamma\in\Gamma\\ d(x_0,\gamma x_0)\le R}} e^{-b_\gamma}h(b_\gamma) < \infty\,,$$

we conclude that for any $\tau \ge \tau_0 := \max\{12\left(2\delta(\Gamma) - \delta_{G\cdot\xi}(\Gamma) + 2\|b\|_1\right)/\varphi_0^2\,,\, 24\}$

$$\sum_{\substack{\gamma\in\Gamma\\ \angle_{x_0}(\gamma x_0, G\cdot\xi)>\varphi}} e^{-b_\gamma}h(b_\gamma) < \sum_{\gamma\in\Gamma_{D_1}} e^{-b_\gamma}h(b_\gamma) + \sum_{\gamma\in\Gamma_{D_2}} e^{-b_\gamma}h(b_\gamma) < \infty\,.$$

If $\varphi \ge \varphi_0$, then $\angle_{x_0}(\gamma x_0, G\cdot\xi) > \varphi$ implies $\angle_{x_0}(\gamma x_0, G\cdot\xi) > \varphi_0$ and the claim follows. □

The proof of the following proposition is based on a result of J. F. Quint ([Q]).

Proposition 6.15 *If $\Gamma \subset G$ is a Zariski dense subgroup, then the conditions of Proposition 6.14 are satisfied for every $\xi \in L_\Gamma \cap \partial X$ with $\varphi_0 > 0$ arbitrary.*

Proof. Let $\Gamma \subset G$ be Zariski dense, fix $\xi \in \partial X$ and let $H_\xi \in \overline{\mathfrak{a}_1^+}$ denote the Cartan projection of ξ. By a result of J. F. Quint ([Q]), the function

$$\Psi:\quad \mathfrak{a}\to\mathbb{R},\qquad H\mapsto \|H\|\cdot\delta_{G\cdot\eta}(\Gamma)\,,\quad \text{where}\quad \eta = (\mathrm{id}, H)\in\partial X\,,$$

is concave. Hence there exists a linear functional $\Phi \in \mathfrak{a}^*$ with the properties

$$\Phi(H_\xi) = \Psi(H_\xi)\,, \quad \text{and} \quad \Phi(H) \geq \Psi(H) \quad \forall\, H \in \mathfrak{a}\,.$$

Since the maximal singular directions $H_1, H_2, \ldots, H_r \in \overline{\mathfrak{a}_1^+}$ provide a basis for $\mathfrak{a}$, there exist coefficients $b^1, b^2, \ldots, b^r \in \mathbb{R}$ such that $\Phi = \langle \sum_{i=1}^r b^i H_i, \cdot\rangle$. We conclude

$$\sum_{i=1}^r b^i \langle H_i, H_\xi\rangle = \delta_{G\cdot\xi}(\Gamma)\,, \quad \text{and} \quad \sum_{i=1}^r b^i \langle H_i, H_\eta\rangle \geq \Psi(H_\eta) = \delta_{G\cdot\eta}(\Gamma)$$

for any $\eta \in \partial X$ with Cartan projection $H_\eta \in \overline{\mathfrak{a}_1^+}$. □

6.4 $(b, \Gamma\cdot\xi)$-densities

Recall from section 3.4 that for any $\eta \in \partial X$ the point $\eta_i \in \partial X^i$ denotes the unique element in the i-th maximal singular boundary component ∂X^i such that η and η_i are points in the closure of a common Weyl chamber at infinity.

Definition 6.16 *Let $b = (b^1, b^2, \ldots b^r) \in \mathbb{R}^r$. A* Γ-invariant b-density *is a continuous map*

$$\begin{aligned} \mu :\quad X &\to \mathcal{M}^+(\partial X) \\ x &\mapsto \mu_x \end{aligned}$$

with the properties

$$\begin{array}{ll} (i) & supp(\mu_{x_0}) \subseteq L_\Gamma\,, \\ (ii) & \gamma * \mu_x = \mu_{\gamma^{-1}x} \;\; \text{for any } \gamma \in \Gamma\,,\; x \in X\,, \\ (iii) & \dfrac{d\mu_x}{d\mu_{x_0}}(\eta) = e^{\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x_0,x)} \;\; \text{for any } x \in X\,,\; \eta \in supp(\mu_{x_0})\,. \end{array}$$

A $(b, \Gamma\cdot\xi)$-density *μ is a Γ-invariant b-density with $supp(\mu_{x_0}) \subseteq L_\Gamma \cap G\cdot\xi$.*

Remark. Let H_ξ denote the Cartan projection of ξ, and c^i, $1 \leq i \leq r$, the linear functionals defined in section 3.4. A $(b, \Gamma\cdot\xi)$-density is an α-dimensional conformal density with support in $G\cdot\xi$ (see [Al]) if and only if

$$b^i = \alpha\cdot c^i(H_\xi) \quad \text{for} \quad 1 \leq i \leq r\,.$$

We now fix $\xi \in \partial X$ such that $G\cdot\xi \cap L_\Gamma \neq \emptyset$. In order to obtain a $(b, \Gamma\cdot\xi)$-density, we require that $b = (b^1, b^2, \ldots b^r) \in \mathbb{R}^r$ and $\varphi_0 \in (0, \pi/4)$ satisfy the conditions necessary for Proposition 6.14. We fix $\tau = \tau_0(b, \varphi_0)$ and consider the corresponding family $\mathcal{F}(G\cdot\xi, \tau, b)$ as in Lemma 6.12. Then $\mathcal{F}(G\cdot\xi, \tau, b)$ is relatively compact in the space of continuous maps $\mathrm{C}(X, \mathcal{M}^+(\overline{X}))$ endowed with the topology of uniform convergence on compact sets by the argument at the end of the previous section. The following proposition characterizes the possible accumulation points of this family $\mathcal{F}(G\cdot\xi, \tau, b)$.

Proposition 6.17 *Every accumulation point $\mu=\mu(G\cdot\xi,\tau,b)$ of the family $\mathcal{F}(G\cdot\xi,\tau,b)$ in $\mathrm{C}(X,\mathcal{M}^+(\overline{X}))$ is a $(b,\Gamma\cdot\xi)$-density.*

Proof. Let $(\mu_x)_{x\in X}$ be an accumulation point of $\mathcal{F}(G\cdot\xi,\tau,b)$ in $\mathrm{C}(X,\mathcal{M}^+(\overline{X}))$. By construction, the measures μ_x, $x\in X$, are Γ-equivariant and supported on the limit set L_Γ. Proposition 6.14 further implies $\mathrm{supp}(\mu_{x_0})\subseteq L_\Gamma\cap G\cdot\xi$. It therefore suffices to prove

$$\frac{d\mu_x}{d\mu_{x_0}}(\eta)=e^{\sum_{i=1}^r b^i\mathcal{B}_{\eta_i}(x_0,x)}\qquad\text{for any } x\in X\,,\ \eta\in\mathrm{supp}(\mu_{x_0})\,.$$

Lemma 3.12 shows that if a sequence $(y_n)\subset X$ converges to a point $\eta\in G\cdot\xi\subseteq\partial X$ in the cone topology, then $d(x,y_n)-d(\cdot,y_n)\to\mathcal{B}_\eta(x,\cdot)$, $\mathcal{B}_{G\cdot\xi}(x,y_n)-\mathcal{B}_{G\cdot\xi}(\cdot,y_n)\to\mathcal{B}_\eta(x,\cdot)$ and $d_i(x,y_n)-d_i(\cdot,y_n)\to\mathcal{B}_{\eta_i}(x,\cdot)\,,\ i=1,2,\ldots r$, uniformly on compact sets.

Thus for any constant $c\geq 0$ and for arbitrary $\varepsilon>0$, there exist $R>0$, $\varphi>0$ such that for any $\gamma\in\Gamma$ with $d(x_0,\gamma x_0)>R$ and $\angle_{x_0}(\gamma x_0,\eta)<\varphi$, and for any $x\in\overline{B_{x_0}(c)}$

$$|\mathcal{B}_\eta(x_0,x)-d(x_0,\gamma x_0)+d(x,\gamma x_0)|<\varepsilon\,,\quad|\mathcal{B}_\eta(x_0,x)-\mathcal{B}_{G\cdot\xi}(x_0,\gamma x_0)+\mathcal{B}_{G\cdot\xi}(x,\gamma x_0)|<\varepsilon\,,$$

$$\text{and}\quad|\mathcal{B}_{\eta_i}(x_0,x)-d_i(x_0,\gamma x_0)+d_i(x,\gamma x_0)|<\varepsilon\,,\qquad i=1,2,\ldots r\,. \tag{6.6}$$

Let $\varepsilon>0$ arbitrary, fix $x\in X$, put $c:=d(x_0,x)$ and choose $R>0$ and $\varphi>0$ as above. Put

$$\begin{aligned}\Gamma_1&:=\{\gamma\in\Gamma\mid d(x_0,\gamma x_0)\leq R\}\,,\\ \Gamma_2&:=\{\gamma\in\Gamma\mid\angle_{x_0}(\gamma x_0,G\cdot\xi)>\varphi/2\}\,,\\ \Gamma_3&:=\{\gamma\in\Gamma\mid d(x_0,\gamma x_0)>R_0\text{ and }\angle_{x_0}(\gamma x_0,G\cdot\xi)<\varphi\}\,.\end{aligned}$$

Note that if $\gamma\in\Gamma_3$, then there exists an element $\eta_\gamma\in G\cdot\xi$ such that $\sigma_{x_0,\gamma x_0}(\infty)$ and η_γ are points in the closure of a common Weyl chamber at infinity, and $\angle_{x_0}(\gamma x_0,\eta_\gamma)<\varphi$. Using q_γ from (6.3) and inequalities (6.6), we estimate for $\gamma\in\Gamma_3$

$$\begin{aligned}q_\gamma(x_0,x)&\leq\sum_{i=1}^r b^i\big(\mathcal{B}_{(\eta_\gamma)_i}(x_0,x)+\varepsilon\big)+\tau\big(\mathcal{B}_{\eta_\gamma}(x_0,x)+\varepsilon\\ &\qquad-\mathcal{B}_{\eta_\gamma}(x_0,x)+\varepsilon\big)\leq\sum_{i=1}^r b^i\mathcal{B}_{(\eta_\gamma)_i}(x_0,x)+(\|b\|_1+2\tau)\varepsilon\,,\\ q_\gamma(x_0,x)&\geq\sum_{i=1}^r b^i\mathcal{B}_{(\eta_\gamma)_i}(x_0,x)+(\|b\|_1-2\tau)\varepsilon\,.\end{aligned}\tag{6.7}$$

For any $f\in\mathrm{C}^0(\overline{X})$, $s\in(1,2]$, we have

$$\begin{aligned}\Big|\int_{\overline{X}}f(\eta)\,d\mu^s_{x_0}(\eta)\;&-\;\int_{\overline{X}}f(\eta)\,e^{\sum_{i=1}^r b^i\mathcal{B}_{\eta_i}(x_0,x)}\,d\mu^s_x(\eta)\Big|\\ &\leq\frac{1}{\Psi^s}\sum_{\gamma\in\Gamma}|f(\gamma x_0)|\cdot e^{-sb_\gamma}h(b_\gamma)\cdot\big|1-e^{\sum_{i=1}^r b^i(d_i(x_0,\gamma x_0)-d_i(x,\gamma x_0))+sq_\gamma(x_0,x)}\big|\,.\end{aligned}$$

The triangle inequality and inequality (6.4) imply that for any $\gamma \in \Gamma$

$$\Big|\sum_{i=1}^{r} b^i\big(d_i(x_0,\gamma x_0) - d_i(x,\gamma x_0)\big) + sq_\gamma(x_0,x)\Big| \le \|b\|_1 d(x_0,x) + sd(x_0,x)\,(||b||_1 + 2\tau)\ .$$

This proves that for $x \in \overline{B_{x_0}(c)}$ and $s \le 2$, the term

$$\begin{aligned}&\Big|1 - e^{\sum_{i=1}^r b^i(d_i(x_0,\gamma x_0) - d_i(x,\gamma x_0)) + sq_\gamma(x_0,x)}\Big| \\ &\quad\le\ e^{|\sum_{i=1}^r b^i(d_i(x_0,\gamma x_0) - d_i(x,\gamma x_0)) + sq_\gamma(x_0,x)|} - 1 \le e^{c(3||b||_1+4\tau)} - 1\end{aligned}$$

is bounded above by a constant $A = A(c,b,\tau)$.

If $\gamma \in \Gamma_3$ and $\eta_\gamma \in G{\cdot}\xi$ such that $\angle_{x_0}(\gamma x_0, \eta_\gamma) < \varphi$, we deduce from (6.6) and (6.7)

$$\begin{aligned}\sum_{i=1}^r b^i \mathcal{B}_{(\eta_\gamma)_i}(x_0,x) + sq_\gamma(x_0,x) &\le (1-s)\sum_{i=1}^r b^i \mathcal{B}_{(\eta_\gamma)_i}(x_0,x) + s(\|b\|_1 + 2\tau)\varepsilon \\ &\le (1-s)\|b\|_1 c + s(\|b\|_1+2\tau)\varepsilon\ , \\ \sum_{i=1}^r b^i \mathcal{B}_{(\eta_\gamma)_i}(x_0,x) + sq_\gamma(x_0,x) &\ge (1-s)\|b\|_1 c - s(\|b\|_1+2\tau)\varepsilon\ ,\end{aligned}$$

and therefore $\lim_{s\searrow 1}\big|1 - e^{\sum_{i=1}^r b^i \mathcal{B}_{(\eta_\gamma)_i}(x_0,x) + sq_\gamma(x_0,x)}\big| \le e^{\varepsilon(\|b\|_1+2\tau)} - 1$. We conclude

$$\begin{aligned}\Big|\int_{\overline{X}} f(\eta)\, d\mu^s_{x_0}(\eta) - \int_{\overline{X}} f(\eta)\, e^{\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x_0,x)}\, d\mu^s_x(\eta)\Big| &\le \frac{\|f\|_\infty}{\Psi^s}\Big(\sum_{\gamma\in\Gamma_1} e^{-b_\gamma} h(b_\gamma) A\ + \\ + \sum_{\gamma\in\Gamma_2} e^{-sb_\gamma} h(b_\gamma) A + \sum_{\gamma\in\Gamma_3} e^{-sb_\gamma} h(b_\gamma)\big|1 - e^{\sum_{i=1}^r b^i \mathcal{B}_{(\eta_\gamma)_i}(x_0,x) + sq_\gamma(x_0,x)}\big|\Big)\ .\end{aligned}$$

Now the first term tends to zero as $s \searrow 1$ since $\sum_{\gamma\in\Gamma_1} e^{-sb_\gamma} h(b_\gamma)$ converges for any $s \ge 0$ by the finiteness of Γ_1. By Proposition 6.14, $\sum_{\gamma\in\Gamma_2} e^{-b_\gamma} h(b_\gamma)$ converges, hence the second term tends to zero as $s \searrow 1$. For the last term, we have $\sum_{\gamma\in\Gamma_3} e^{-sb_\gamma} h(b_\gamma) \le \Psi^s$ for any $s > 1$, therefore

$$\begin{aligned}&\lim_{s\searrow 1}\Big|\int_{\overline{X}} f(\eta)\, d\mu^s_{x_0}(\eta) - \int_{\overline{X}} f(\eta)\, e^{\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x_0,x)}\, d\mu^s_x(\eta)\Big| \\ &\qquad = 0 + 0 + \|f\|_\infty\left(e^{\varepsilon(\|b\|_1+2\tau)} - 1\right) = \|f\|_\infty\left(e^{\varepsilon(\|b\|_1+2\tau)} - 1\right)\ .\end{aligned}$$

The claim follows taking the limit as $\varepsilon \searrow 0$. □

6.5 Illustrating examples

EXAMPLE 1. The first important kind of example we consider in this section are lattices in $SL(n,\mathbb{R})$ acting on the symmetric space $X = SL(n,\mathbb{R})/SO(n)$ described in section 1.6.

The calculation in [A] shows, that for these finite covolume subgroups the exponent of growth in a direction $G\cdot\xi$ is equal to ρ, evaluated on the Cartan projection H_ξ of ξ,

$$\delta_{G\cdot\xi}(\Gamma) = \rho(H_\xi)\,.$$

Since ρ is a linear functional on $\mathfrak{a}$ and the maximal singular directions $H_1, H_2, \ldots, H_r$ form a basis of $\mathfrak{a}$, there exist parameters $b^1, b^2, \ldots, b^r \in \mathbb{R}$ such that $\sum_{i=1}^r b^i\langle H_i, H\rangle = \rho(H)$ for any $H \in \mathfrak{a}$. This implies that for any $\xi \in \partial X$ the conditions necessary for Proposition 6.14 are satisfied for arbitrary $\varphi_0 > 0$ with the same tuple $b = (b^1, b^2, \ldots, b^r)$. Using $\tau \geq \tau_0(b)$ according to that proposition, we are able to construct a $(b, \Gamma\cdot\xi)$-density for every $\xi \in \partial X$.

We are going to calculate the parameters $b^1, b^2, \ldots b^r \in \mathbb{R}$ in the case $G = SL(3,\mathbb{R})$ and $G = SL(4,\mathbb{R})$. In $SL(3,\mathbb{R})/SO(3)$, the maximal singular directions are given by

$$H_1 = \frac{D(e_1)}{\|D(e_1)\|} = \frac{\mathrm{Diag}(2,-1,-1)}{\sqrt{6}}\,, \qquad H_2 = \frac{D(e_2)}{\|D(e_2)\|} = \frac{\mathrm{Diag}(1,1,-2)}{\sqrt{6}}\,.$$

Identifying $\mathfrak{a}$ with its dual space $\mathfrak{a}^*$, we may write $\rho = \alpha_1 + \alpha_2 = \mathrm{Diag}(1,0,-1)$. We solve the system of linear equations

$$b^1 H_1 + b^2 H_2 = \rho$$

and obtain the unique solution $b^1 = \sqrt{\frac{2}{3}}$, $b^2 = \sqrt{\frac{2}{3}}$. For the barycenter

$$H_* = \frac{D(1,1)}{\|D(1,1)\|} = \frac{D(1,1)}{3\sqrt{2}} = \frac{\mathrm{Diag}(1,0,-1)}{\sqrt{2}} \in \mathfrak{a}_1^+$$

we have $c^1(H_*) = 1/\sqrt{3} = c^2(H_*)$, hence $b^i = \sqrt{2}\,c^i(H_*)$ for $i = 1,2$. This shows that for $\xi = (\mathrm{id}, H_*) \in \partial X$ our $(b, \Gamma\cdot\xi)$-density is a conformal density of dimension $\sqrt{2} = \rho(H_*)$. Since the critical exponent of a lattice equals $\rho(H_*)$, this conformal density is exactly the $\delta(\Gamma)$-dimensional conformal density constructed by P. Albuquerque ([Al]).

In $SL(4,\mathbb{R})/SO(4)$ we have the three maximal singular directions

$$H_1 = \frac{\mathrm{Diag}(3,-1,-1,-1)}{2\sqrt{3}}\,, \quad H_2 = \frac{\mathrm{Diag}(1,1,-1,-1)}{2}\,, \quad H_3 = \frac{\mathrm{Diag}(1,1,1,-3)}{2\sqrt{3}}\,.$$

In this case, we may write $\rho = (3\alpha_1 + 4\alpha_2 + 3\alpha_3)/2 = \mathrm{Diag}(3,1,-1,-3)/2$, and the system of linear equations

$$b^1 H_1 + b^2 H_2 + b^3 H_3 = \rho$$

possesses the unique solution $b^1 = \frac{\sqrt{3}}{2}$, $b^2 = 1$, $b^3 = \frac{\sqrt{3}}{2}$. The barycenter

$$H_* = \frac{D(1,1,1)}{\|D(1,1,1)\|} = \frac{D(1,1,1)}{4\sqrt{5}} = \frac{\mathrm{Diag}(3,1,-1,-3)}{2\sqrt{5}} \in \mathfrak{a}_1^+$$

satisfies

$$c^1(H_*) = 2\sqrt{3}\frac{1}{4\sqrt{5}} = \frac{\sqrt{3}}{2\sqrt{5}} = c^3(H_*)\,, \quad c^2(H_*) = \frac{1}{\sqrt{5}}\,,$$

hence $b^i = \sqrt{5}\,c^i(H_*)$, $i = 1,2,3$. Again, for $\xi = (\mathrm{id}, H_*) \in \partial X$, our $(b, \Gamma\cdot\xi)$-density is a conformal density of dimension $\sqrt{5} = \rho(H_*) = \delta(\Gamma)$.

EXAMPLE 2. Another class of examples are lattices in $SL(m,\mathbb{R})$ imbedded in the upper left corner of $SL(n,\mathbb{R})$ for $n > m$. Such groups are not Zariski dense and there exist many subsets $G\cdot\xi \subseteq \partial X$ which do not contain limit points. If we imbed for example a lattice in $SL(3,\mathbb{R})$ in $SL(4,\mathbb{R})$, the Cartan projections H of the subsets $G\cdot\xi \subset \partial X$ in $X = SL(4,\mathbb{R})/SO(4)$ which contain limit points belong to either one of the sets

$$\begin{aligned} \mathfrak{a}_+ : &= \left\{ \frac{\mathrm{Diag}(2\beta_1+\beta_2, -\beta_1+\beta_2, 0, -\beta_1-2\beta_2)}{\sqrt{6}\sqrt{\beta_1^2+\beta_1\beta_2+\beta_2^2}} \mid \beta_1 \geq 0,\ \beta_2 \geq \beta_1 \,, \right\}, \\ \mathfrak{a}_- : &= \left\{ \frac{\mathrm{Diag}(2\beta_1+\beta_2, 0, -\beta_1+\beta_2, -\beta_1-2\beta_2)}{\sqrt{6}\sqrt{\beta_1^2+\beta_1\beta_2+\beta_2^2}} \mid \beta_2 \geq 0,\ \beta_1 > \beta_2 \right\}. \end{aligned}$$

In terms of the simple roots α_1, α_2, α_3 in $SL(4,\mathbb{R})/SO(4)$, we have

$$\begin{aligned} \mathfrak{a}_+ &= \{H \in \overline{\mathfrak{a}_1^+} \mid \alpha_1(H) + 2\alpha_2(H) = \alpha_3(H)\}\,, \\ \mathfrak{a}_- &= \{H \in \overline{\mathfrak{a}_1^+} \mid \alpha_1(H) - 2\alpha_2(H) = \alpha_3(H)\}\,. \end{aligned}$$

We obtain

$$\delta_{G\cdot\xi}(\Gamma) = \frac{4\beta_1 + 5\beta_2}{\sqrt{6}\sqrt{\beta_1^2+\beta_1\beta_2+\beta_2^2}}\,, \quad 0 \leq \beta_1 \leq \beta_2\,,$$

if the Cartan projection of ξ belongs to $\mathfrak{a}_+$,

$$\delta_{G\cdot\xi}(\Gamma) = \frac{5\beta_1 + 4\beta_2}{\sqrt{6}\sqrt{\beta_1^2+\beta_1\beta_2+\beta_2^2}}\,, \quad \beta_1 > \beta_2 \geq 0\,,$$

if the Cartan projection of ξ belongs to $\mathfrak{a}_-$, and $\delta_{G\cdot\xi}(\Gamma) \leq 0$ otherwise. The conditions for b^1, b^2, b^3 in a subset $G\cdot\xi \subset \partial X$ with Cartan projection $H \in \mathfrak{a}_+ \cup \mathfrak{a}_+$ read

$$b^1\langle H_1, H\rangle + b^2\langle H_2, H\rangle + b^3\langle H_3, H\rangle = \delta_{G\cdot\xi}(\Gamma)\,.$$

This leads to the following systems of linear equations

$$\begin{aligned} &\frac{8}{2\sqrt{3}}b^1 + \frac{2}{2}b^2 + \frac{4}{2\sqrt{3}}b^3 = 4 \quad \text{and} \quad \frac{4}{2\sqrt{3}}b^1 + \frac{4}{2}b^2 + \frac{8}{2\sqrt{3}}b^3 = 5\,, \quad \text{if } H \in \mathfrak{a}_+\,, \\ &\frac{8}{2\sqrt{3}}b^1 + \frac{4}{2}b^2 + \frac{4}{2\sqrt{3}}b^3 = 5 \quad \text{and} \quad \frac{4}{2\sqrt{3}}b^1 + \frac{2}{2}b^2 + \frac{8}{2\sqrt{3}}b^3 = 4\,, \quad \text{if } H \in \mathfrak{a}_-\,. \end{aligned}$$

Their solutions are given by

$$b^1 = \frac{\sqrt{3}}{2}\,,\ b^2 + \frac{2}{\sqrt{3}}b^3 = 2\,,\ \text{if } H \in \mathfrak{a}_+\,, \qquad \frac{2}{\sqrt{3}}b^1 + b^2 = 2\,,\ b^3 = \frac{\sqrt{3}}{2}\,,\ \text{if } H \in \mathfrak{a}_-\,.$$

In both cases, there exists a one-dimensional vector space of solutions for the parameters b^1, b^2, b^3 which satisfy the conditions necessary for Proposition 6.14 for some $\varphi_0 > 0$. Using $\tau \geq \tau_0(b,\varphi_0)$, we may therefore construct $(b, \Gamma\cdot\xi)$-densities for every subset $G\cdot\xi \subset \partial X$ which contains limit points.

EXAMPLE 3. Let $X = G/K$ be a symmetric space of noncompact type, and $G = Ke^{\overline{\mathfrak{a}^+}}K$ a Cartan decomposition. For free groups generated by regular axial elements, we know from Theorem 5.16 that the limit cone P_Γ is contained in a subset of $\mathfrak{a}_1^+$. Using the fact that such groups are Zariski dense and applying Proposition 6.15, we conclude that for every subset $G\cdot\xi \subseteq \partial X$ with Cartan projection $H_\xi \in P_\Gamma$, there exist parameters $b^1, b^2, \ldots, b^r$ such that the conditions of Proposition 6.14 are satisfied for any $\varphi_0 > 0$. The construction of a $(b, \Gamma\cdot\xi)$-density for every $\xi \in \partial X$ with Cartan projection $H_\xi \in P_\Gamma$ is therefore possible.

EXAMPLE 4. Our last example considers products $X = X_1 \times X_2$ of rank one symmetric spaces. If $\Gamma_1 \subset \mathrm{Isom}^o(X_1)$, $\Gamma_2 \subset \mathrm{Isom}^o(X_2)$ are convex cocompact groups with critical exponents δ_1, δ_2, we know from Theorem 6.2.5 in [Y] that there exists a constant $C > 1$ such that

$$\frac{1}{C}e^{\delta_i R} \le \#\{\gamma_i \in \Gamma_i \mid R-1 \le d(x_i, \gamma_i x_i) < R\} \le Ce^{\delta_i R}, \quad i = 1, 2. \tag{6.8}$$

We are now going to examine the action of the product group $\Gamma = \Gamma_1 \times \Gamma_2 \subseteq \mathrm{Isom}^o(X)$ on the product manifold X. In this case, $\overline{\mathfrak{a}^+}$ is isomorphic to the first quadrant in $\mathbb{R}^2$, and we may identify the i-th maximal singular direction with the i-th standard basis vector in $\mathbb{R}^2$. Given a subset $G\cdot\xi \subset \partial X$, we may therefore write $H_\xi = (\cos\theta, \sin\theta)$ with $\theta \in [0, \pi/2]$. Using the estimates (6.8) and putting $x_0 := (x_1, x_2)$, $\theta_\gamma := \arctan\left(d(x_2, \gamma_2 x_2)/d(x_1, \gamma_1 x_1)\right)$ for $\gamma \in \Gamma$, we estimate the number of orbit points

$$\begin{aligned}
\Delta N^{\varphi}_{G\cdot\xi}(x_0, x_0; R) &= \#\{\gamma \in \Gamma \mid R-1 \le d(x_0, \gamma x_0) < R,\ \angle_{x_0}(\gamma x_0, G\cdot\xi) < \varphi\} \\
&= \#\{(\gamma_1, \gamma_2) \in \Gamma \mid R-1 \le \sqrt{d(x_1, \gamma_1 x_1)^2 + d(x_2, \gamma_2 x_2)^2} < R, \\
&\qquad |\theta_\gamma - \theta| < \varphi\} \\
&\le \#\{(\gamma_1, \gamma_2) \in \Gamma \mid R-1 \le d(x_1, \gamma_1 x_1)/\cos\theta_\gamma < R, \\
&\qquad R-1 \le d(x_2, \gamma_2 x_2)/\sin\theta_\gamma < R,\ |\theta_\gamma - \theta| < \varphi\} \\
&\le C^2 \cdot R \exp\left(\delta_1 R\cos(\theta+\varphi)\right) \exp\left(\delta_2 R\sin(\theta+\varphi)\right).
\end{aligned}$$

As a lower bound, we obtain

$$\begin{aligned}
\Delta N^{\varphi}_{G\cdot\xi}(x_0, x_0; R) &\ge \#\{(\gamma_1, \gamma_2) \in \Gamma \mid R-1 \le d(x_1, \gamma_1 x_1)/\cos\theta < R, \\
&\qquad R-1 \le d(x_2, \gamma_2 x_2)/\sin\theta < R\} \\
&\ge \frac{1}{C^2} \cdot \exp\left(\delta_1 R\cos\theta\right) \exp\left(\delta_2 R\sin\theta\right)
\end{aligned}$$

and therefore conclude $\delta_{G\cdot\xi}(\Gamma) = \delta_1\cos\theta + \delta_2\sin\theta$.

In order to construct a $(b, \Gamma\cdot\xi)$-density, we solve the linear equation

$$b^1\cos\theta + b^2\sin\theta = \delta_{G\cdot\xi}(\Gamma) = \delta_1\cos\theta + \delta_2\sin\theta$$

and obtain $b^1 = \delta_1$, $b^2 = \delta_2$ as a solution. As above, the conditions necessary for Proposition 6.14 are satisfied for any $\varphi_0 > 0$ with the same tuple $b = (b^1, b^2) = (\delta_1, \delta_2)$ for every subset $G\cdot\xi \subset \partial X$. Using $\tau \ge \tau_0(b)$ according to that proposition, we are able to construct a $(b, \Gamma\cdot\xi)$-density for every $\xi \in \partial X$.

In order to find a subset $G\cdot\xi \subset \partial X$ with Cartan projection $H \in \overline{\mathfrak{a}^+}$ which supports a conformal density, we require $H := (\cos\phi, \sin\phi)$ to satisfy $b^i = \alpha c^i(H)$, $i = 1, 2$, for some $\alpha > 0$. This is equivalent to $\delta_1 = \alpha\cos\phi$, $\delta_2 = \alpha\sin\phi$, which implies $\alpha^2 = \delta_1^2 + \delta_2^2$, hence $H = (\delta_1, \delta_2)/\sqrt{\delta_1^2 + \delta_2^2}$. From Example 4.6 in [Al] we know that $\delta(\Gamma) = \sqrt{\delta_1^2 + \delta_2^2}$, which shows that our $(b, \Gamma\cdot\xi)$-density restricted to the subset $G\cdot\xi \subset \partial X$ with Cartan projection $H = (\delta_1, \delta_2)/\delta(\Gamma)$ is the $\delta(\Gamma)$-dimensional conformal density constructed in that example.

Chapter 7

Measures on the limit set

Let X be a globally symmetric space of noncompact type and $G = \mathrm{Isom}^o(X)$. In this chapter, we derive properties of $(b, \Gamma\cdot\xi)$-densities invariant by a nonelementary discrete group $\Gamma \subset G$. Our main tool will be Theorem 7.6, the shadow lemma, a generalization of Theorem 3.3 in [Al] valid for conformal densities invariant by a Zariski dense discrete isometry group. For any $\xi \in \partial X$, this theorem yields a relation between the parameters of a $(b, \Gamma\cdot\xi)$-density and the exponent of growth in direction $G\cdot\xi$.

We then deal with the atomic part of the measure and prove that the radial limit set does not contain any atoms. We further address the question of ergodicity and give a general argument following [Al] and [N], provided that every subset of the radial limit set with positive measure contains a density point. The final section of this chapter introduces an appropriate notion of Hausdorff measure and Hausdorff dimension on each G-invariant subset $G\cdot\xi \subseteq \partial X$ in order to estimate the size of the radial limit set in $G\cdot\xi$. Our results are most precise for a class of groups which we call radially cocompact. In this case, the Hausdorff dimension of the radial limit set in a given subset $G\cdot\xi \subseteq \partial X$ equals the exponent of growth in direction $G\cdot\xi$.

7.1 The shadow lemma

Let X be a globally symmetric space of noncompact type and $G = \mathrm{Isom}^o(X)$. The goal of this section is to generalize Theorem 3.3 in [Al] to $(b, \Gamma\cdot\xi)$-densities invariant by a nonelementary discrete group $\Gamma \subset G = \mathrm{Isom}^o(X)$.

Definition 7.1 *For a subset $B \subset X$ and a point $x \in X \setminus B$ the* Furstenberg shadow *$S(x : B) \subseteq \partial X$ is defined as the set of those points in the geometric boundary which belong to the closure of all Weyl chambers with apex x which intersect B.*

The shadow at infinity *of B viewed from $x \notin B$ is defined by*

$$sh_x(B) = \{\eta \in \partial X \mid \exists\, t > 0 : \ \sigma_{x,\eta}(t) \in B\}\,.$$

*If $\varphi > 0$ is an angle, then the φ-*shadow at infinity *of B viewed from x is defined by*

$$sh_x^{\varphi}(B) = \{\eta \in \partial X \mid \eta \in S(x:B) \ \ \text{and} \ \ \angle_x(\eta, G{\cdot}\xi) < \varphi\}\,.$$

The following lemma extends Lemma 3.4 in [Al] valid for "generic" groups to the larger class of nonelementary groups introduced in chapter 4.1.

LEMMA 7.2 *Let $\Gamma \subset G = Isom^o(X)$ be a nonelementary discrete subgroup and $\xi \in \partial X$. Further let ν be a measure supported on $L_\Gamma \cap G{\cdot}\xi \subseteq \partial X$ and $F \subseteq \partial X$ a Γ-invariant measurable set with $0 < \nu(F) < \infty$. Then there exist constants $\varepsilon_0 > 0$ and $q > 0$ such that for every Borel set E contained in the ε_0-neighborhood of $G{\cdot}\xi \setminus Vis^\infty(\iota\eta)$, $\eta \in G{\cdot}\xi$, we have*

$$\nu(E \cap F) \le q < \nu(F) \le mass(\nu)\,.$$

Proof. Let $d := \sup\{\nu(F \cap G{\cdot}\xi \setminus \mathrm{Vis}^\infty(\iota\eta)) \mid \eta \in G{\cdot}\xi\} \le \nu(F)$ be the supremum of the ν-measure of complements of big cells in $G{\cdot}\xi$ intersected with F. This supremum is in fact a maximum, because a sequence $G{\cdot}\xi \setminus \mathrm{Vis}^\infty(\iota\eta_j)$, $\eta_j \in G{\cdot}\xi$, has an accumulation point with respect to the Hausdorff topology which is itself a set $G{\cdot}\xi \setminus \mathrm{Vis}^\infty(\iota\eta_0)$, where $\eta_0 := \lim_{j\to\infty} \eta_j \in G{\cdot}\xi$.

Suppose $d = \nu(F)$. Then there exists $\eta \in G{\cdot}\xi$ such that $\nu(F \cap (G{\cdot}\xi \setminus \mathrm{Vis}^\infty(\iota\eta))) = \nu(F)$. Since F is Γ-invariant, we have

$$\nu\big(\gamma(F \cap (G{\cdot}\xi \setminus \mathrm{Vis}^\infty(\iota\eta)))\big) = \nu\big(F \cap (G{\cdot}\xi \setminus \mathrm{Vis}^\infty(\iota\gamma\eta))\big) = \nu(F) \quad \forall\, \gamma \in \Gamma\,,$$

hence $\nu(F \cap \mathrm{Vis}^\infty(\iota\gamma\eta)) = 0$ for all $\gamma \in \Gamma$, because $G{\cdot}\xi$ equals the disjoint union of $G{\cdot}\xi \setminus \mathrm{Vis}^\infty(\iota\gamma\eta)$ and $\mathrm{Vis}^\infty(\iota\gamma\eta)$. By the assumption that Γ is nonelementary we obtain from Lemma 5.10

$$L_\Gamma \cap G{\cdot}\xi \subseteq \bigcup_{\gamma\in\Gamma} \mathrm{Vis}^\infty(\iota\gamma\eta)\,,$$

and, since ν is supported on $L_\Gamma \cap G{\cdot}\xi$,

$$\nu(F) = \nu(F \cap L_\Gamma) \le \nu\big(F \cap \bigcup_{\gamma\in\Gamma} \mathrm{Vis}^\infty(\iota\gamma\eta)\big) \le \sum_{\gamma\in\Gamma} \nu(F \cap \mathrm{Vis}^\infty(\iota\gamma\eta)) = 0\,.$$

We conclude $d < \nu(F)$ and put $q := \frac{1}{2}(\nu(F)+d) > 0$. Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be a sequence of positive numbers with limit zero, and suppose there exists a sequence $(E_j)_{j\in\mathbb{N}}$ of ε_j-neighborhoods of sets $G{\cdot}\xi \setminus \mathrm{Vis}^\infty(\iota\eta_j)$ such that $\nu(F \cap E_j) > q$. Then a subsequence $(E_{j_l})_{l\in\mathbb{N}}$ converges in the Hausdorff topology to a set $G{\cdot}\xi \setminus \mathrm{Vis}^\infty(\iota\eta_0)$, $\eta_0 \in G{\cdot}\xi$, with

$$\nu(F \cap G{\cdot}\xi \setminus \mathrm{Vis}^\infty(\iota\eta_0)) \ge q > d\,,$$

in contradiction to the definition of d. □

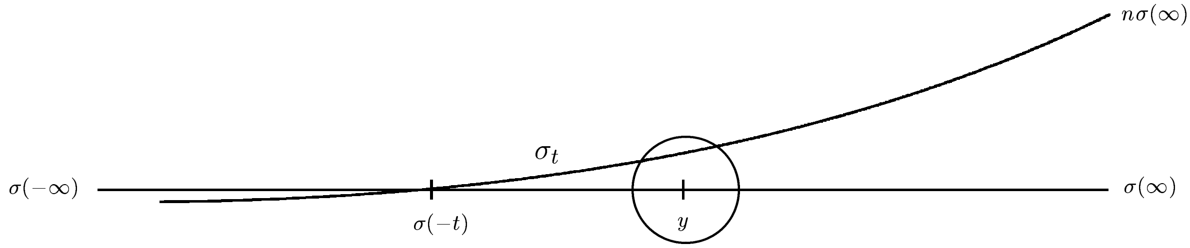
We are now going to give a generalization of Lemma 3.5 in [Al] which is crucial for the proof of Theorem 7.6.

LEMMA 7.3 *Fix $y \in X$ and $\xi \in \partial X$. Then for any $\varepsilon_0 > 0$ there exists a constant $c_0 > 0$ with the following property:*

For every $c > c_0$ and for any $x \in X \setminus B_y(c)$, the set $G{\cdot}\xi \setminus (S(x : B_y(c)) \cap G{\cdot}\xi)$ is contained in the ε_0-neighborhood of a set $G{\cdot}\xi \setminus Vis^\infty(\iota\eta)$, $\eta \in G{\cdot}\xi$.

Proof. Let σ be a geodesic in X with $\sigma(\infty) = \zeta \in G{\cdot}\xi$ and fix an Iwasawa decomposition $G = N^+AK$ with respect to $\sigma(0)$, $\sigma(-\infty)$. Then $G{\cdot}\zeta = G{\cdot}\xi \subset \partial X^\Theta$ for some subset $\Theta \subset \Upsilon$, and the subgroup $N_\sigma := \exp(\sum_{\alpha\in\langle\Theta\rangle} \mathfrak{g}_\alpha) \subset N^+$ stabilizes $\sigma(\infty)$. Note that if $\Theta = \emptyset$, then $N_\sigma = \{\mathrm{id}\}$.

For $r > 0$ we put $N(r) := \{n \in N^+ \mid \exists\, t \geq 0 \ \text{ such that } \ nN_\sigma\sigma(t) \in B_y(r)\}$.



Since $\cup_{r>0} N(r) = N^+$, for any $\varepsilon > 0$ there exists a number $r_0 > 0$ such that the set $G{\cdot}\zeta \setminus N(r_0){\cdot}\zeta$ is contained in an ε-neighborhood of $G{\cdot}\zeta \setminus N^+\zeta$. For $n \in N(r_0)$ and $t > 0$ we consider the geodesic ray σ_t emanating from $\sigma(-t)$ and asymptotic to $n\sigma(\infty)$. Since nN_σ stabilizes $\sigma(-\infty)$, there exists $t_0 = t_0(\varepsilon)$ such that $d(\sigma_t, nN_\sigma\sigma) < \varepsilon$ for $t > t_0$. This implies that for $t > t_0$, the ray σ_t intersects $B_y(r_0+\varepsilon)$, and therefore

$$N(r_0){\cdot}\zeta \subseteq S(\sigma(-t) : B_y(r_0+\varepsilon)) \,.$$

By Corollary 2.14 we have $N^+{\cdot}\zeta = \mathrm{Vis}^\infty(\sigma(-\infty))$. Choosing $c_0 := \max\{t_0, r_0+\varepsilon\}$ and using the fact that $S(x : B_y(c)) \subset S(x : B_y(c'))$ for $c' > c$, we conclude that for any $c > c_0$ and for arbitrary $x \in X$ with $d(y,x) \geq c$

$$N(r_0){\cdot}\zeta \subset S(x : B_y(c)) \cap G{\cdot}\xi \,.$$

The assertion now follows from the density of the open set $N^+w_*P_\Theta$ in G/P_Θ. $\square$

LEMMA 7.4 *Let $c > 0$, $x, z \in X$ with $d(x,z) > c$, and $\xi \in \partial X$. Then*

$$\forall\, \eta \in G{\cdot}\xi \cap S(x : B_z(c)) : \qquad 0 \leq \mathcal{B}_{G\cdot\xi}(x,z) - \mathcal{B}_\eta(x,z) < 2c \,.$$

Proof. Let $\eta \in G{\cdot}\xi \cap S(x : B_z(c))$ and fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ and an Iwasawa decompositon $G = N^+AK$ with respect to x and η. Then for any $k \in K$ we have $k\eta \in G{\cdot}\xi \cap S(x : B_{kz}(c))$ and

$$\mathcal{B}_{k\eta}(x,z) = \mathcal{B}_\eta(x, k^{-1}z) \,, \qquad \mathcal{B}_{G\cdot\xi}(x,z) = \mathcal{B}_{G\cdot\xi}(x, k^{-1}z) \,.$$

It therefore suffices to prove the claim for $z = e^H x$, $H \in \overline{\mathfrak{a}^+}$. If $H_\eta \in \overline{\mathfrak{a}_1^+}$ denotes the Cartan projection of η, we obtain from Lemma 3.2

$$\mathcal{B}_\eta(x,z) = \mathcal{B}_\eta(x, e^H x) = \langle H, H_\eta\rangle \,.$$

Let $y \in B_z(c)$ and write $y = ke^{H'}x$, where $k \in K$ is an angular projection and $H' \in \overline{\mathfrak{a}_1^+}$ the Cartan projection of y. Let $H_\xi \in \overline{\mathfrak{a}_1^+}$ denote the Cartan projection of ξ. Using the equality

$$\mathcal{B}_{k\eta}(x,y) = \mathcal{B}_\eta(x, e^{H'}x) = \mathcal{B}_\eta(x,z) + \mathcal{B}_\eta(z, e^{H'}x)$$

we conclude

$$\begin{aligned} 0 &\le \mathcal{B}_{G\cdot\xi}(x,z) - \mathcal{B}_\eta(x,z) = \mathcal{B}_{G\cdot\xi}(x,z) - \mathcal{B}_{k\eta}(x,y) + \mathcal{B}_{k\eta}(x,y) - \mathcal{B}_\eta(x,z) \\ &= \langle H, H_\xi\rangle - \mathcal{B}_\eta(x, e^{H'}x) + \mathcal{B}_\eta(x,z) + \mathcal{B}_\eta(z, e^{H'}x) - \mathcal{B}_\eta(x,z) \\ &\le \langle H_\xi, H - H'\rangle + d(z, e^{H'}x) \overset{CSU}{\le} \|H - H'\| + d(z,y) \\ &= d(e^H x, e^{H'}x) + d(z,y) \le 2d(z,y) < 2c \end{aligned}$$

□

Corollary 7.5 *Let $c > 0$, $x, z \in X$ with $d(x,z) > c$. Then*

$$\forall\, \eta_i \in \partial X^i \cap S(x : B_z(c)) : \qquad 0 \le d_i(x,z) - \mathcal{B}_{\eta_i}(x,z) < 2c\,.$$

Let $X = G/K$ be a globally symmetric space of noncompact type with base point $x_0 \in X$ corresponding to K. We are now going to give an extended version of Theorem 3.3 in [Al], valid for $(b, \Gamma\cdot\xi)$-densities and measurable sets invariant by a discrete nonelementary group of isometries.

Theorem 7.6 (Shadow Lemma) *Let $\Gamma \subset Isom^o(X)$ be a discrete nonelementary subgroup, $\xi \in \partial X$, μ a $(b, \Gamma\cdot\xi)$-density. Then there exists a constant $c_0 > 0$ such that for any $c > c_0$ and for every Γ-invariant measurable set $F \subseteq \partial X$ with $\mu_{x_0}(F) > 0$ there exists a constant $D(c) > 1$ with the property*

$$\frac{1}{D(c)} e^{-\sum_{i=1}^r b^i d_i(x_0,\gamma x_0)} \le \mu_{x_0}\left(S(x_0 : B_{\gamma x_0}(c)) \cap F\right) \le D(c) e^{-\sum_{i=1}^r b^i d_i(x_0,\gamma x_0)}\,.$$

Proof. Let $c_0 > 0$ as in Lemma 7.3, corresponding to $\varepsilon_0 > 0$ in Lemma 7.2. For $c > c_0$ and $\gamma \in \Gamma$ such that $d(x_0, \gamma x_0) > c$ we have

$$\mu_{x_0}(F) \ge \mu_{x_0}(S(\gamma^{-1}x_0 : B_{x_0}(c)) \cap F) \ge \mu_{x_0}(F) - q > 0\,. \tag{7.1}$$

The properties (ii) and (iii) of a $(b, \Gamma\cdot\xi)$-density and the Γ-invariance of F imply

$$\begin{aligned} &\mu_{x_0}(S(\gamma^{-1}x_0 : B_{x_0}(c)) \cap F) = \mu_{x_0}(\gamma^{-1}S(x_0 : B_{\gamma x_0}(c)) \cap F) \\ &\quad = \gamma^{-1} * \mu_{x_0}(S(x_0 : B_{\gamma x_0}(c)) \cap F) = \mu_{\gamma x_0}(S(x_0 : B_{\gamma x_0}(c)) \cap F) \\ &\quad = \int_{S(x_0:B_{\gamma x_0}(c))\cap F} d\mu_{\gamma x_0}(\eta)\cdot \frac{d\mu_{x_0}}{d\mu_{\gamma x_0}}(\eta)\cdot e^{\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x_0,\gamma x_0)} \\ &\quad = \int_{S(x_0:B_{\gamma x_0}(c))\cap F} e^{\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x_0,\gamma x_0)} d\mu_{x_0}(\eta)\,. \end{aligned}$$

By Corollary 7.5, $\quad e^{-2rc}\exp\left(\sum_{i=1}^r b^i d_i(x_0,\gamma x_0)\right)\mu_{x_0}(S(x_0 : B_{\gamma x_0}(c)) \cap F)$

$$< \mu_{x_0}(S(\gamma^{-1}x_0 : B_{x_0}(c)) \cap F) \le \exp\left(\sum_{i=1}^r b^i d_i(x_0,\gamma x_0)\right)\mu_{x_0}(S(x_0 : B_{\gamma x_0}(c)) \cap F)\,.$$

Equation (7.1) now allows to conclude

$$\begin{aligned} &\exp\left(-\sum_{i=1}^r b^i d_i(x_0,\gamma x_0)\right)(\mu_{x_0}(F) - q) \le \mu_{x_0}(S(x_0 : B_{\gamma x_0}(c)) \cap F) \\ &\qquad \le \exp\left(-\sum_{i=1}^r b^i d_i(x_0,\gamma x_0)\right)e^{2rc}\cdot\mu_{x_0}(F)\,. \end{aligned}$$

□

7.2 Relation to the exponent of growth

For this section, we fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ for $X = G/K$ and let x_0 denote the unique point in X stabilized by K. The following applications of Theorem 7.6, the shadow lemma, yield relations between the exponent of growth in a direction $G{\cdot}\xi$ and the parameters of a $(b, \Gamma{\cdot}\xi)$-density.

THEOREM 7.7 *Let $\xi \in \partial X$ and $H_\xi \in \overline{\mathfrak{a}_1^+}$ the Cartan projection of ξ. If a $(b, \Gamma{\cdot}\xi)$-density exists, then $\delta_{G\cdot\xi}(\Gamma) \le \sum_{i=1}^r b^i \langle H_i, H_\xi\rangle$.*

Proof. Suppose μ is a $(b, \Gamma{\cdot}\xi)$-density with support in $G{\cdot}\xi \subseteq \partial X$. Let $c > c_0 + 1$ with c_0 as in Theorem 7.6, $\varphi > 0$ and $R >> 1$ arbitrary. We only need $N(\varphi)R^{r-1}$ balls of radius 1 in $e^{\mathfrak{a}}x_0 \subset X$ to cover the set $\{e^{RH}x_0 \mid H \in \mathfrak{a}_1 \ , \angle(H, H_\xi) < \varphi\}$, and $N(\varphi)$ is independent of R. Since Γ is discrete, a $2c$-neighborhood of any of these balls contains a uniformly bounded number M_c of elements of $\Gamma{\cdot}x_0$. The compactness of the group K implies the existence of a constant $A > 0$ such that every point in $G{\cdot}\xi$ is contained in at most $AM_cN(\varphi)R^{r-1}$ sets $\mathrm{sh}^\varphi_{x_0}(B_{\gamma x_0}(c))$, $\gamma \in \Gamma_0$, where

$\Gamma_0 := \{\gamma \in \Gamma \mid \angle_{x_0}(\gamma x_0, G\cdot\xi) < \varphi,\ R - 1 \le d(x_0, \gamma x_0) < R\}$, and therefore

$$\sum_{\gamma\in\Gamma_0} \mu_{x_0}(\mathrm{sh}^\varphi_{x_0}(B_{\gamma x_0}(c))) \le M_cAN(\varphi)R^{r-1}\mu_{x_0}(G{\cdot}\xi)\,.$$

Furthermore, for any $\gamma \in \Gamma_0$ the unit length Cartan projection $H_\gamma \in \overline{\mathfrak{a}_1^+}$ of γx_0 satisfies $\langle H_i, H_\gamma - H_\xi\rangle \le \|H_\gamma - H_\xi\| = \sqrt{2 - 2\cos\varphi} \le \varphi$, which implies

$$\sum_{i=1}^r b^i d_i(x_0, \gamma x_0) \le d(x_0, \gamma x_0)\big(\sum_{i=1}^r b^i\langle H_i, H_\xi\rangle + \|b\|_1\varphi\big)\,.$$

Using Theorem 7.6 and $\Delta N^\varphi_{G\cdot\xi}$ from section 6.1, we conclude

$$\begin{aligned}\Delta N^\varphi_{G\cdot\xi}(x_0, x_0; R)\frac{1}{D(c)}e^{-\sum_{i=1}^r b^i\langle H_i,H_\xi\rangle R} \ &\le\ \sum_{\gamma\in\Gamma_0}\frac{1}{D(c)}e^{-\sum_{i=1}^r b^i d_i(x_0,\gamma x_0)+\varphi\|b\|_1 d(x_0,\gamma x_0)}\\ \le e^{\varphi\|b\|_1 R}\sum_{\gamma\in\Gamma_0}\mu_{x_0}(\mathrm{sh}^\varphi_{x_0}(B_{\gamma x_0}(c))) \ &\le\ e^{\varphi\|b\|_1 R}M_cAN(\varphi)R^{r-1}\mu_{x_0}(G{\cdot}\xi)\,.\end{aligned}$$

Hence
$$\begin{aligned}\delta^\varphi_{G\cdot\xi}(x_0, x_0) \ \le\ &\limsup_{R\to\infty}\frac{1}{R}\log\Big(D(c)M_cAN(\varphi)\mu_{x_0}(G{\cdot}\xi)R^{r-1}\,\cdot\\ &\cdot\ \exp(\sum_{i=1}^r b^i\langle H_i, H_\xi\rangle R + \varphi\|b\|_1 R)\Big) = \sum_{i=1}^r b^i\langle H_i, H_\xi\rangle + \varphi\|b\|_1\end{aligned}$$

and the claim follows as $\varphi \searrow 0$. □

Furthermore, if a $(b, \Gamma{\cdot}\xi)$-density gives positive measure to the radial limit set, then the exponent of growth in direction $G{\cdot}\xi$ is completely determined by its parameters.

THEOREM 7.8 *Let $\xi \in \partial X$ and $H_\xi \in \overline{\mathfrak{a}_1^+}$ the Cartan projection of ξ. If a $(b, \Gamma{\cdot}\xi)$-density gives positive measure to L^{rad}_Γ, then $\delta_{G\cdot\xi}(\Gamma) = \sum_{i=1}^r b^i\langle H_i, H_\xi\rangle$.*

Proof. Suppose μ is a $(b,\Gamma{\cdot}\xi)$-density with support $G{\cdot}\xi \subseteq \partial X$ such that $\mu_{x_0}(L_\Gamma^{rad}) > 0$, and let $c > c_0$ with c_0 as in Theorem 7.6. By definition of the radial limit set,

$$L_\Gamma^{rad} \cap G{\cdot}\xi \subseteq \bigcap_{R>0} \bigcap_{\varphi>0} \bigcup_{\substack{\gamma\in\Gamma \\ d(x_0,\gamma x_0)>R \\ \angle_{x_0}(\gamma x_0, G\xi)<\varphi}} \mathrm{sh}_{x_0}^{\varphi}(B_{\gamma x_0}(c)) \,.$$

Let $\varphi > 0$ and $R > c$ arbitrary, put $\Gamma' := \{\gamma \in \Gamma \mid d(x_0,\gamma x_0) > R,\ \angle_{x_0}(\gamma x_0, G{\cdot}\xi) < \varphi\}$. Hence

$$L_\Gamma^{rad} \cap G{\cdot}\xi \subseteq \bigcup_{\gamma\in\Gamma'} \mathrm{sh}_{x_0}^{\varphi}(B_{\gamma x_0}(c))\,,$$

and we estimate

$$\begin{aligned} 0 &< \mu_{x_0}(L_\Gamma^{rad}) = \mu_{x_0}(L_\Gamma^{rad} \cap G{\cdot}\xi) \\ &\le \sum_{\gamma\in\Gamma'} \mu_{x_0}(\mathrm{sh}_{x_0}^{\varphi}(B_{\gamma x_0}(c))) \le D(c) \sum_{\gamma\in\Gamma'} e^{-\sum_{i=1}^r b^i d_i(x_0,\gamma x_0)} \,. \end{aligned}$$

This implies that for any $\varphi > 0$ the tail of the series

$$\sum_{\substack{\gamma\in\Gamma \\ \angle_{x_0}(\gamma x_0, G\cdot\xi)<\varphi}} e^{-\sum_{i=1}^r b^i d_i(x_0,\gamma x_0)}$$

does not tend to zero. Therefore the sum above diverges and by Proposition 6.7 (2), there exists $\eta \in S_{G\cdot\xi}(\varphi)$ with Cartan projection $H_\eta \in \overline{\mathfrak{a}_1^+}$, such that $\sum_{i=1}^r b^i \langle H_i, H_\eta\rangle \le \delta_{G\cdot\eta}(\Gamma)$. Taking the limit as $\varphi \searrow 0$, we conclude $\sum_{i=1}^r b^i\langle H_i, H_\xi\rangle \le \delta_{G\cdot\xi}(\Gamma)$. The assertion now follows from the previous theorem. □

7.3 The atomic part of the measure

LEMMA 7.9 *Let $\Gamma \subset G = Isom^o(X)$ be a nonelementary discrete group, $\xi \in \partial X$ and μ a $(b,\Gamma{\cdot}\xi)$-density. If $\eta \in G{\cdot}\xi$ is a point mass and Γ_η its stabilizer, then for any $\gamma \in \Gamma_\eta$ and $x \in X$ we have*

$$\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x,\gamma x) = 0\,.$$

In particular, if $\gamma_1, \gamma_2 \in \Gamma$ are representatives of the same coset in Γ/Γ_η, then

$$\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x,\gamma_1^{-1}x) = \sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x,\gamma_2^{-1}x)\,.$$

Proof. If $\gamma \in \Gamma_\eta$, then for $x \in X$ we have by Γ-equivariance

$$\mu_x(\eta) = \mu_x(\gamma^{-1}\eta) = \mu_{\gamma x}(\eta)\,.$$

From the assumption that η is a point mass and property (iii) in Definition 6.16 we conclude

$$1 = \frac{\mu_{\gamma x}(\eta)}{\mu_x(\eta)} = e^{\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x,\gamma x)}\,,$$

hence $\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x,\gamma x) = 0$ for any $\gamma \in \Gamma_\eta$.

Let $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1\Gamma_\eta = \gamma_2\Gamma_\eta \in \Gamma/\Gamma_\eta$. Then $\gamma_2^{-1}\gamma_1 \in \Gamma_\eta$ and we obtain from above $\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x,\gamma_1^{-1}x) = \sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x,\gamma_2^{-1}\gamma_1\gamma_1^{-1}x) = \sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x,\gamma_2^{-1}x)$. □

Let $x_0 \in X$ denote the base point of $X = G/K$ corresponding to K.

LEMMA 7.10 *Let $\Gamma \subset G = Isom^o(X)$ be a nonelementary discrete subgroup, and $\eta \in G{\cdot}\xi$ a point mass for a $(b, \Gamma{\cdot}\xi)$-density μ. Then the sum*

$$\sum e^{\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x_0,\gamma^{-1}x_0)}$$

taken over a system of coset representatives of Γ/Γ_η converges.

Proof. If γ_1 and γ_2 are representatives of different cosets in Γ/Γ_η, then $\gamma_1\eta \neq \gamma_2\eta$ and so, by Γ-equivariance and since μ_{x_0} is a finite measure,

$$\sum \mu_{\gamma^{-1}x_0}(\eta) = \sum \mu_{x_0}(\gamma\eta) \le \mu_{x_0}(G{\cdot}\xi) < \infty\,.$$

By property (iii) in Definition 6.16 and the assumption that η is a point mass we conclude that the sum

$$\sum e^{\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x_0,\gamma^{-1}x_0)} = \sum \frac{\mu_{\gamma^{-1}x_0}(\eta)}{\mu_{x_0}(\eta)} = \frac{1}{\mu_{x_0}(\eta)} \sum \mu_{\gamma^{-1}x_0}(\eta)$$

taken over a system of coset representatives of Γ/Γ_η converges. □

THEOREM 7.11 *If $\Gamma \subset G = Isom^o(X)$ is a nonelementary discrete group and $\xi \in \partial X$ such that $\delta_{G\cdot\xi}(\Gamma) > 0$, then a radial limit point $\eta \in G{\cdot}\xi \subseteq \partial X$ is not a point mass for any $(b, \Gamma{\cdot}\xi)$-density.*

Proof. Suppose μ is a $(b, \Gamma{\cdot}\xi)$-density, and $\eta \in L_\Gamma^{rad} \cap G{\cdot}\xi$. Fix a Cartan decomposition $G = K e^{\overline{\mathfrak{a}^+}} K$ and an Iwasawa decomposition $G = N^+AK$ with respect to x_0 and η, and let $H_\xi \in \overline{\mathfrak{a}_1^+}$ denote the Cartan projection of ξ. By Theorem 7.7 we have $\sum_{i=1}^r b^i\langle H_i, H_\xi\rangle \ge \delta_{G\cdot\xi}(\Gamma) > 0$.

Put $\varepsilon := \delta_{G\cdot\xi}(\Gamma)/\|b\|_1 > 0$. Since $\eta \in L_\Gamma^{rad} \cap G\xi$, there exists a sequence $(\gamma_j) \subset \Gamma$ such that the Iwasawa projections H_{γ_j} of $\gamma_j x_0$ satisfy $\angle(H_{\gamma_j}, H_\xi) < \varepsilon/2$, hence $\|\frac{H_{\gamma_j}}{\|H_{\gamma_j}\|} - H_\xi\| < \varepsilon/2$. Then, using the Cauchy Schwartz inequality, we compute

$$\begin{aligned}\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x_0,\gamma_j x_0) &= \sum_{i=1}^r b^i \langle H_i, H_{\gamma_j}\rangle = \|H_{\gamma_j}\|\Big(\sum_{i=1}^r b^i\langle H_i, H_\xi\rangle + \sum_{i=1}^r b^i\langle H_i, \frac{H_{\gamma_j}}{\|H_{\gamma_j}\|} - H_\xi\rangle\Big)\\ &> \|H_{\gamma_j}\|\,(\delta_{G\cdot\xi}(\Gamma) - \|b\|_1\varepsilon/2) = \|H_{\gamma_j}\|\delta_{G\cdot\xi}(\Gamma)/2 \ \to\ \infty,\end{aligned}$$

because $\delta_{G\cdot\xi}(\Gamma) > 0$ and $\|H_{\gamma_j}\| \to \infty$ as $j \to \infty$. We may therefore extract a subsequence $(\gamma_k) := (\gamma_{j_k}) \subset \Gamma$ such that $\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x_0,\gamma_k x_0)$ is strictly increasing to infinity as $j \to \infty$.

Assume now η is a point mass for μ_{x_0} and suppose there exist $l, j \in \mathbb{N}$, $l \neq j$ such that $\gamma_l^{-1}\Gamma_\eta = \gamma_j^{-1}\Gamma_\eta$. Then by Lemma 7.9 we have

$$\sum_{i=1}^{r} b^i \mathcal{B}_{\eta_i}(x_0, \gamma_j x_0) = \sum_{i=1}^{r} b^i \mathcal{B}_{\eta_i}(x_0, \gamma_l x_0)\,,$$

in contradiction to the choice of the subsequence (γ_k). Hence $\gamma_l^{-1}\Gamma_\eta \neq \gamma_j^{-1}\Gamma_\eta$ for all $l \neq j$, and the sum over a system of coset representatives of Γ/Γ_η is bounded below by

$$\sum_{k\in\mathbb{N}} e^{\sum_{i=1}^{r} b^i \mathcal{B}_{\eta_i}(x_0, \gamma_k x_0)}$$

and therefore diverges in contradiction to Lemma 7.10. We conclude that η cannot be a point mass for μ_{x_0}. □

7.4 Ergodicity discussion

The main result Theorem 7.15 of this section generalizes Theorem 5.6 in [Al]. Suppose every Γ-invariant subset of the radial limit set in $G{\cdot}\xi \subset \partial X$ with positive measure contains a density point with respect to a covering by shadows. Then our theorem implies that Γ acts ergodically on L_Γ^{rad} with respect to the measure class defined by μ. It is not clear, however, if this condition is satisfied, because it seems impossible to construct a Vitali cover from shadows in the general case. The following definition is again a generalization of Zariski dense groups.

Definition 7.12 *A discrete subgroup $\Gamma \subset Isom^o(X)$ is called* strongly nonelementary *if $L_\Gamma \neq \emptyset$ and if for any $\Theta \subset \Upsilon$ with $L_\Gamma \cap \partial X^\Theta \neq \emptyset$ there exists a Θ-axial isometry $h \in \Gamma$ with attractive and repulsive fixed points h^+, h^- and the following property:*

For every $\eta \in \partial X^\Theta$, $Y := K/M_\Theta \setminus Vis^B(\iota\eta)$, there exists $\gamma \in \Gamma$ such that

$$\gamma Y \cap \pi^B(L_\Gamma \cap \partial X^\Theta) \subseteq Vis^B(\iota h^+) \cap Vis^B(h^-)\,.$$

Lemma 7.13 *If Γ is strongly nonelementary, then for any $\Theta \subset \Upsilon$ with $L_\Gamma \cap \partial X^\Theta \neq \emptyset$ and for every $\eta \in \partial X^\Theta$, $Y := K/M_\Theta \setminus Vis^B(\iota\eta)$, there exists a sequence $(\gamma_j) \subset \Gamma$ such that the sets $\gamma_j Y$ are pairwise disjoint.*

Proof. Suppose Γ is strongly nonelementary, let $\Theta \subset \Upsilon$ such that $L_\Gamma \cap \partial X^\Theta \neq \emptyset$, and h a Θ-axial isometry as in the definition. Let $\eta \in G{\cdot}h^- \subseteq \partial X^{\Theta^*}$ arbitrary, put $V := G{\cdot}h^+ \setminus \mathrm{Vis}^\infty(\eta)$ and choose $\gamma \in \Gamma$ such that

$$\gamma(\pi^B(V)) \cap \pi^B(L_\Gamma \cap \partial X^\Theta) \subseteq \mathrm{Vis}^B(\iota h^+) \cap \mathrm{Vis}^B(h^-)\,.$$

Then $\gamma V \cap L_\Gamma \subseteq \mathrm{Vis}^\infty(\iota h^+) \cap \mathrm{Vis}^\infty(h^-)$, and since $h^+ \notin \mathrm{Vis}^\infty(\iota h^+)$ we have $h^+ \notin \gamma V$. Consider the distance d_{x_0,h^-} on $\mathrm{Vis}^\infty(h^-)$ from section 5.1 and put

$$t_0 := \max\{d_{x_0,h^-}(\xi, h^+) \mid \xi \in \gamma V\}\,, \quad t_1 := \min\{d_{x_0,h^-}(\xi, h^+) \mid \xi \in \gamma V\}\,.$$

Let $l > 0$ denote the translation length and $L \in \mathfrak{a}_1^\Theta$ the translation direction of h. By Lemma 5.1

$$d_{x_0,h^-}(h\xi, h^+) \le e^{-\alpha_0 l}\cdot d_{x,h^-}(\xi, h^+)\,, \quad \alpha_0 := \min_{\alpha\in\Sigma^+\setminus\langle\Theta^*\rangle^+} \alpha(\iota(L))\,.$$

Let $j_1 \in \mathbb{N}$ be the smallest integer greater than $\log(t_0/t_1)/(\alpha_0 l)$. Then for any $\xi \in V$ we have

$$d_{x_0,h^-}(h^{j_1}\xi, h^+) \le e^{-j_1\alpha_0 l}\cdot d_{x,h^-}(\xi, h^+) < \frac{t_1}{t_0}\cdot t_0 = t_1\,,$$

which proves $h^{j_1}\gamma V \cap \gamma V = \emptyset$. Next put $t_2 := \min\{d_{x_0,h^-}(h^{j_1}\xi, h^+)\ |\, \xi \in \gamma V\}$ and let $j_2 \in \mathbb{N}$ be the smallest integer greater than $\log(t_0/t_2)/(\alpha_0 l)$. We conclude

$$d_{x_0,h^-}(h^{j_2}\xi, h^+) \le e^{-j_2\alpha_0 l}\cdot d_{x,h^-}(\xi, h^+) < \frac{t_2}{t_0}\cdot t_0 = t_2$$

for any $\xi \in \gamma V$, hence $h^{j_2}\gamma V \cap h^{j_1}\gamma V = \emptyset$ and $h^{j_2}\gamma V \cap \gamma V = \emptyset$. Then inductively, if $t_k > 0$ and $j_k \in \mathbb{N}$ have been chosen, we put $t_{k+1} := \min\{d_{x_0,h^-}(h^{j_k}\xi, h^+)\ |\, \xi \in \gamma V\}$ and let $j_{k+1} \in \mathbb{N}$ be the smallest integer greater than $\log(t_0/t_k)/(\alpha_0 l)$. This process does not terminate after finitely many steps, because for $\xi \ne h^+$ we have $h^j\xi \ne h^+$ for all $j \in \mathbb{N}$. We obtain a decreasing sequence $(t_k) \searrow 0$, an increasing sequence $(j_k) \subset \mathbb{N}$ and a sequence $(\gamma_k) := (\gamma^{-1}h^{j_k}\gamma) \subset \Gamma$ with the property $\gamma_k V \cap \gamma_l V = \emptyset$ for $k, l \in \mathbb{N}$, $k \ne l$. □

The following statement will be essential in the proof of Theorem 7.15.

LEMMA 7.14 *Let $\Gamma \subset G = Isom^o(X)$ be a strongly nonelementary discrete group, $\Theta \subset \Upsilon$ and $\xi \in \partial X^\Theta$. If ν is a measure of finite total mass such that $supp(\nu) \subseteq L_\Gamma \cap G\cdot\xi$ is Γ-invariant, then for any $\varepsilon > 0$ and every $\eta \in G\cdot\xi$ we have*

$$\nu(\gamma Y) < \varepsilon\,, \quad \textit{where}\ \ Y = G\cdot\xi \setminus Vis^\infty(\iota\eta)\,.$$

Proof. Suppose there exists $\varepsilon > 0$ and $\eta \in G\cdot\xi$ such that $Y := G\cdot\xi \setminus \mathrm{Vis}^\infty(\iota\eta)$ satisfies $\nu(\gamma\cdot Y) > \varepsilon\ \ \forall\, \gamma \in \Gamma$. In particular, $\mathrm{id} \in \Gamma$ implies $\nu(Y) \ge \varepsilon$ and therefore $Y \cap L_\Gamma \ne \emptyset$.

Since Γ is strongly nonelementary, by the previous lemma there exists a sequence $(\gamma_j) \subset \Gamma$ such that $\gamma_k Y \cap \gamma_l Y = \emptyset$ for all $k \ne l$. Hence

$$\nu(G\cdot\xi) \ge \sum_{j\in\mathbb{N}} \nu(\gamma_j Y) \to \infty\,,$$

a contradiction to the finiteness of ν. □

THEOREM 7.15 *Let $\Gamma \subset G = Isom^o(X)$ be a strongly nonelementary discrete group of isometries, $\xi \in \partial X^\Theta$ such that $\delta_{G\cdot\xi}(\Gamma) > 0$, and μ a $(b, \Gamma\cdot\xi)$-density. Then for every Γ-invariant subset $A \subseteq L_\Gamma^{rad}$ which possesses a point $\eta \in A$ with the property*

$$\lim_{j\to\infty} \frac{\mu_{x_0}(S(x_0 : B_{\gamma_j x_0}(c)) \cap A)}{\mu_{x_0}(S(x_0 : B_{\gamma_j x_0}(c)) \cap L_\Gamma^{rad})} = 1$$

for some sequence $(\gamma_j x_0) \subset X$ converging to η, and $c > c_0$ as in Theorem 7.6, either $\mu_{x_0}(A) = 0$ or $\mu_{x_0}(A) = \mu_{x_0}(L_\Gamma^{rad})$.

Proof. Suppose $\mu_{x_0}(A) > 0$, let $\eta \in A$, $c > c_0$, $(\gamma_j) \subset \Gamma$ such that $\gamma_j x_0$ converges to η and

$$\lim_{j\to\infty} \frac{\mu_{x_0}(S(x_0 : B_{\gamma_j x_0}(c)) \cap L_\Gamma^{rad} \setminus A)}{\mu_{x_0}(S(x_0 : B_{\gamma_j x_0}(c)) \cap L_\Gamma^{rad})} = 0\,. \tag{7.2}$$

Passing to a subsequence if necessary, $(\gamma_j^{-1}x_0) \subset X$ converges to a point $\zeta \in \iota(G\cdot\xi)$, put $Y := G\cdot\xi \setminus \mathrm{Vis}^\infty(\zeta)$. Given $\varepsilon > 0$, Lemma 7.14 implies the existence of $\gamma \in \Gamma$ such that $\mu_{x_0}(\gamma Y) < \varepsilon$. Since μ_{x_0} is nonatomic in L_Γ^{rad} by Theorem 7.11, there exists an open neighborhood U of γY such that $\mu_{x_0}(U) < \mu_{x_0}(\gamma Y) + \varepsilon < 2\varepsilon$.

Furthermore, $\gamma_j^{-1}x_0 \to \zeta$ and Lemma 7.3 imply the existence of $c_0' > c$, such that for $c' > c_0'$ and j sufficiently large, the set $G\cdot\xi \setminus (S(\gamma_j^{-1}x_0 : B_{x_0}(c')) \cap G\cdot\xi)$ is contained in the open neighborhood $\gamma^{-1}U$ of $Y = G\cdot\xi \setminus \mathrm{Vis}^\infty(\zeta)$. Hence $G\cdot\xi \setminus \gamma S(\gamma_j^{-1}x_0 : B_{x_0}(c')) \subset U$, and therefore

$$\mu_{x_0}(\gamma S(\gamma_j^{-1}x_0 : B_{x_0}(c')) \cap L_\Gamma^{rad}) > \mu_{x_0}(L_\Gamma^{rad}) - 2\varepsilon\,. \tag{7.3}$$

We compute

$$\begin{aligned}
&\frac{\mu_{x_0}(\gamma S(\gamma_j^{-1}x_0 : B_{x_0}(c')) \cap L_\Gamma^{rad} \setminus A))}{\mu_{x_0}(\gamma(S(\gamma_j x_0 : B_{x_0}(c')) \cap L_\Gamma^{rad})} = \frac{\mu_{x_0}(\gamma\gamma_j^{-1}(S(x_0 : B_{\gamma_j x_0}(c')) \cap L_\Gamma^{rad} \setminus A))}{\mu_{x_0}(\gamma\gamma_j^{-1}(S(x_0 : B_{\gamma_j x_0}(c')) \cap L_\Gamma^{rad})}\\
&= \frac{\mu_{(\gamma\gamma_j^{-1})^{-1}x_0}(S(x_0 : B_{\gamma_j x_0}(c')) \cap L_\Gamma^{rad} \setminus A)}{\mu_{(\gamma\gamma_j^{-1})^{-1}x_0}(S(x_0 : B_{\gamma_j x_0}(c')) \cap L_\Gamma^{rad})} = \frac{\mu_{\gamma_j\gamma^{-1}x_0}(S(x_0 : B_{\gamma_j x_0}(c')) \cap L_\Gamma^{rad} \setminus A)}{\mu_{\gamma_j\gamma^{-1}x_0}(S(x_0 : B_{\gamma_j x_0}(c')) \cap L_\Gamma^{rad})}\\
&= \frac{\int_{S(x_0:B_{\gamma_j x_0}(c'))\cap L_\Gamma^{rad}\setminus A} d\mu_{\gamma_j\gamma^{-1}x_0}(\eta)}{\int_{S(x_0:B_{\gamma_j x_0}(c'))\cap L_\Gamma^{rad}} d\mu_{\gamma_j\gamma^{-1}x_0}(\eta)} = \frac{\int_{S(x_0:B_{\gamma_j x_0}(c'))\cap L_\Gamma^{rad}\setminus A} e^{\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x_0,\gamma_j\gamma^{-1}x_0)} d\mu_{x_0}(\eta)}{\int_{S(x_0:B_{\gamma_j x_0}(c'))\cap L_\Gamma^{rad}} e^{\sum_{i=1}^r b^i \mathcal{B}_{\eta_i}(x_0,\gamma_j\gamma^{-1}x_0)} d\mu_{x_0}(\eta)}\\
&\le \frac{e^{\sum_{i=1}^r b^i d_i(x_0,\gamma_j\gamma^{-1}x_0)} \mu_{x_0}(S(x_0 : B_{\gamma_j x_0}(c')) \cap L_\Gamma^{rad} \setminus A)}{e^{\sum_{i=1}^r b^i d_i(x_0,\gamma_j\gamma^{-1}x_0) - 2rc'} \mu_{x_0}(S(x_0 : B_{\gamma_j x_0}(c')) \cap L_\Gamma^{rad})}\\
&= \frac{\mu_{x_0}(S(x_0 : B_{\gamma_j x_0}(c')) \cap L_\Gamma^{rad} \setminus A)}{e^{-2rc'} \mu_{x_0}(S(x_0 : B_{\gamma_j x_0}(c')) \cap L_\Gamma^{rad})}\,.
\end{aligned}$$

Now $c' > c$ implies $\mu_{x_0}(S(x_0 : B_{\gamma_j x_0}(c')) \cap L_\Gamma^{rad}) \ge \mu_{x_0}(S(x_0 : B_{\gamma_j x_0}(c)) \cap L_\Gamma^{rad})$, and since A is Γ-invariant, $L_\Gamma^{rad} \setminus A \subseteq \partial X$ is Γ-invariant and either $\mu_{x_0}(A) = \mu_{x_0}(L_\Gamma^{rad})$ or $\mu_{x_0}(L_\Gamma^{rad} \setminus A) > 0$. In the first case we are done, otherwise the shadow lemma Theorem 7.6 with $F := L_\Gamma^{rad} \setminus A$ yields

$$\begin{aligned}
\mu_{x_0}(S(x_0 : B_{\gamma_j x_0}(c')) \cap L_\Gamma^{rad} \setminus A) &\le D(c') e^{-\sum_{i=1}^r b^i d_i(x_0,\gamma_j x_0)}\\
&\le D(c')D(c)\mu_{x_0}(S(x_0 : B_{\gamma_j x_0}(c)) \cap L_\Gamma^{rad} \setminus A)\,.
\end{aligned}$$

For sufficiently large $j \in \mathbb{N}$ we therefore obtain from (7.2)

$$\frac{\mu_{x_0}(S(x_0 : B_{\gamma_j x_0}(c')) \cap L_\Gamma^{rad} \setminus A)}{e^{-2rc'} \mu_{x_0}(S(x_0 : B_{\gamma_j x_0}(c')) \cap L_\Gamma^{rad})} \le \frac{D(c')D(c)\mu_{x_0}(S(x_0 : B_{\gamma_j x_0}(c)) \cap L_\Gamma^{rad} \setminus A)}{e^{-2rc'} \mu_{x_0}(S(x_0 : B_{\gamma_j x_0}(c)) \cap L_\Gamma^{rad})} < \varepsilon$$

and conclude

$$\begin{aligned}\mu_{x_0}(A) &\geq \mu_{x_0}(\gamma S(\gamma_j^{-1}x_0 : B_{x_0}(c')) \cap A) \\ &= \mu_{x_0}(\gamma S(\gamma_j^{-1}x_0 : B_{x_0}(c')) \cap L_\Gamma^{rad}) - \mu_{x_0}(\gamma S(\gamma_j^{-1}x_0 : B_{x_0}(c')) \cap L_\Gamma^{rad} \setminus A) \\ &> (1-\varepsilon)\, \mu_{x_0}(\gamma S(\gamma_j^{-1}x_0 : B_{x_0}(c')) \cap L_\Gamma^{rad}) \overset{(7.3)}{>} (1-\varepsilon)\left(\mu_{x_0}(L_\Gamma^{rad}) - 2\varepsilon\right).\end{aligned}$$

Letting $\varepsilon \searrow 0$ we obtain $\mu_{x_0}(A) \geq \mu_{x_0}(L_\Gamma^{rad})$. □

7.5 Hausdorff measure

We will follow the idea of G. Knieper ([Kn], chapter 4) in order to define Hausdorff measure on the G-invariant subsets of the limit set.

For $\xi \in \partial X$, $c > 0$ and $0 < r < e^{-c}$ we call the set

$$B_r^c(\xi) := \{\eta \in \partial X \mid d(\sigma_{x_0,\eta}(-\log r), \sigma_{x_0,\xi}(-\log r)) < c\}$$

a c-ball of radius r centered at ξ. Using this conformal structure, we define as in the case of metric spaces a Hausdorff measure and Hausdorff dimension on the geometric boundary.

DEFINITION 7.16 *Let E be a Borel subset of ∂X,*

$$\mathrm{Hd}_\varepsilon^\alpha(E) = \inf\{\sum r_i^\alpha \mid E \subseteq \bigcup B_{\xi_i}^c(r_i),\, r_i < \varepsilon\}\,.$$

The α-dimensional Hausdorff measure *of E is defined by* $\mathrm{Hd}^\alpha(E) = \lim_{\varepsilon\to 0} \mathrm{Hd}_\varepsilon^\alpha(E)$,

the Hausdorff dimension *of E is the number* $\dim_{\mathrm{Hd}}(E) = \inf\{\alpha \geq 0 \mid \mathrm{Hd}^\alpha(E) < \infty\}$.

For the remainder of this chapter, we fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ for $X = G/K$, and let $x_0 \in X$ denote the unique point stabilized by K. Lemma 1.16 allows to give a relation between shadows at infinity and c-balls. For $y \in Ke^{\mathfrak{a}^+}x_0 \subset X$, let $\mathcal{C}_{x_0,y} \subset X$ denote the unique Weyl chamber with apex x_0 which contains y.

LEMMA 7.17 *Let $c > 0$, $\xi \in \partial X^{reg}$ with Cartan projection $H_\xi \in \mathfrak{a}_1^+$. Then there exists $\varphi_0 \in (0, \pi/4)$ and $R_0 > 0$ such that with $A_0 := \max\{\|\alpha\|/\alpha(H_\xi) \mid \alpha \in \Sigma^+\}$ the following holds:*

If $\varphi \leq \varphi_0$, $y \in X$ with $d(x_0, y) \geq R_0$, $r := \exp\left(-d(x_0,y)(\cos\varphi - A_0 \sin\varphi) + 2A_0 c\right)$ and $\eta := \overline{\mathcal{C}}_{x_0,y} \cap G\cdot\xi$, then

$$sh_{x_0}^\varphi(B_y(c)) \cap G\cdot\xi \subseteq B_r^c(\eta)\,.$$

Proof. For $c > 0$ and $H_\xi \in \mathfrak{a}_1^+$ let $\varphi_0 \in (0, \pi/4)$ and $R_0 > 0$ be the constants as in Lemma 1.16. Let $\varphi \in (0, \varphi_0]$, choose $y \in X$ with $t := d(x_0, y) \geq R_0$ and let $k_y \in K$ denote an angular projection of y. If $\angle_{x_0}(y, G\cdot\xi) \geq \varphi$, then $\mathrm{sh}_{x_0}^\varphi(B_y(c)) \cap G\cdot\xi = \emptyset$ and the claim is trivial. If $\angle_{x_0}(y, G\cdot\xi) < \varphi$, then the unit length Cartan projection $H_y \in \overline{\mathfrak{a}_1^+}$

of y satisfies $\angle H_y, H_\xi) < \varphi$ and belongs to $\mathfrak{a}_1^+$ by equation (1.2). Hence $\eta = \overline{\mathcal{C}}_{x_0,y} \cap G{\cdot}\xi$ is well defined, and $\eta = (k_y, H_\xi)$. If $\zeta := (k, H_\xi) \in \mathrm{sh}_{x_0}^{\varphi}(B_y(c)) \cap G{\cdot}\xi$, then

$$d(k^{-1}k_y e^{H_y t}x_0, e^{\mathfrak{a}^+}x_0) = d(k_y e^{H_y t}x_0, k e^{\mathfrak{a}^+}x_0) = d(y, \mathcal{C}_{x_0,\zeta}) < c\,.$$

Put $t_0 := t(\cos\varphi - A_0 \sin\varphi) - 2A_0 c$. We obtain from Lemma 1.16

$$d(\sigma_{x_0,\eta}(t_0), \sigma_{x_0,\zeta}(t_0)) = d(k^{-1}k_y e^{H_\xi t_0}x_0, e^{H_\xi t_0}x_0) < c\,,$$

and conclude $\zeta \in B_r^c(\eta)$ for $r = e^{-t_0}$. □

The above inclusions now yield an upper bound for the Hausdorff dimension of the radial limit set.

THEOREM 7.18 *If $\Gamma \subset Isom^o(X)$ is a discrete nonelementary group and $\xi \in \partial X^{reg}$, then the Hausdorff dimension of the radial limit set in $G{\cdot}\xi \subseteq \partial X$ is bounded above by $\delta_{G\cdot\xi}(\Gamma)$.*

Proof. Fix $\xi \in \partial X^{reg}$ with Cartan projection $H_\xi \in \mathfrak{a}_1^+$, and $c > 3c_0$ with $c_0 > 0$ as in Theorem 7.6. By definition of the radial limit set,

$$L_\Gamma^{rad} \cap G{\cdot}\xi \subseteq \bigcap_{R>0} \bigcap_{\varphi>0} \bigcup_{\substack{\gamma\in\Gamma \\ d(x_0,\gamma x_0)>R \\ \angle_{x_0}(\gamma x_0, G\xi)<\varphi}} \mathrm{sh}_{x_0}^{\varphi}(B_{\gamma x_0}(c))\,.$$

Fix $\varphi_0 \in (0, \pi/4)$ and $R_0 > 0$ as in the assertion of the previous lemma and abbreviate $A_0 := \max\{\|\alpha\|/\alpha(H_\xi) \mid \alpha \in \Sigma^+\}$. Let $\varepsilon \in (0, e^{-R_0})$ and $\varphi \in (0, \varphi_0]$ arbitrary, and put $\Gamma' := \{\gamma \in \Gamma \mid d(x_0, \gamma x_0) > -\log\varepsilon, \angle_{x_0}(\gamma x_0, G{\cdot}\xi) < \varphi\}$. For $\gamma \in \Gamma'$, let $\xi_\gamma := \overline{\mathcal{C}}_{x_0,\gamma x_0} \cap G{\cdot}\xi$, and

$$r_\gamma := \exp\big(-d(x_0,\gamma x_0)(\cos\varphi - A_0\sin\varphi) + 2A_0 c\big)\,.$$

Then by the previous lemma we have $\mathrm{sh}_{x_0}^{\varphi}(B_{\gamma x_0}(c) \cap G{\cdot}\xi \subseteq B_{r_\gamma}(\xi_\gamma)$ for all $\gamma \in \Gamma'$, hence

$$L_\Gamma^{rad} \cap G{\cdot}\xi \subseteq \bigcup_{\gamma\in\Gamma'} B_{r_\gamma}(\xi_\gamma)\,.$$

Using the definition of $\mathrm{Hd}_\varepsilon^\alpha$ we estimate

$$\begin{aligned}\mathrm{Hd}_\varepsilon^\alpha(L_\Gamma^{rad} \cap G{\cdot}\xi) &\le \sum_{\gamma\in\Gamma'} r_\gamma^\alpha = \sum_{\gamma\in\Gamma'} e^{-\alpha\left(d(x_0,\gamma x_0)(\cos\varphi - A_0\sin\varphi) - 2A_0 c\right)} \\ &\le e^{2\alpha A_0 c} \sum_{\gamma\in\Gamma'} e^{-\alpha(\cos\varphi - A_0\sin\varphi)d(x_0,\gamma_j x_0)}\,.\end{aligned}$$

Recall from section 6.1 that
$$Q_{G\cdot\xi}^{s,\varphi}(x_0,x_0) = \sum_{\substack{\gamma\in\Gamma \\ \angle_{x_0}(\gamma x_0, G\cdot\xi)<\varphi}} e^{-sd(x_0,\gamma x_0)}$$

converges for $s > \delta_{G\cdot\xi}^{\varphi}(x_0,x_0)$. If $s_0 := \alpha(\cos\varphi - A_0\sin\varphi) > \delta_{G\cdot\xi}^{\varphi}(x_0,x_0)$, we have

$$\mathrm{Hd}_\varepsilon^\alpha(L_\Gamma^{rad} \cap G{\cdot}\xi) \le e^{2\alpha A_0 c} Q_{G\cdot\xi}^{s_0,\varphi}(x_0,x_0)\,.$$

This shows that for $\alpha > \delta^{\varphi}_{G\cdot\xi}(x_0,x_0)/(\cos\varphi - A_0\sin\varphi)$, $\mathrm{Hd}^{\alpha}_{\varepsilon}(L^{rad}_{\Gamma}\cap G\cdot\xi)$ is finite. Taking the limit as $\varphi \searrow 0$ shows that the same is true for $\alpha > \delta_{G\cdot\xi}(\Gamma)$. Letting $\varepsilon\searrow 0$, we obtain $\mathrm{Hd}^{\alpha}(L^{rad}_{\Gamma}\cap G\cdot\xi) < \infty$ if $\alpha > \delta_{G\cdot\xi}(\Gamma)$, hence $\dim_{\mathrm{Hd}}(L^{rad}_{\Gamma}\cap G\cdot\xi) \le \delta_{G\cdot\xi}(\Gamma)$. □

For certain discrete subgroups of $\mathrm{Isom}^o(X)$, the existence of a $(b,\Gamma\cdot\xi)$-density μ together with Theorem 7.6 allows to also obtain a lower bound for the Hausdorff dimension of the radial limit set.

DEFINITION 7.19 *A nonelementary discrete group $\Gamma\subset Isom^o(X)$ is called* radially cocompact, *if and only if there exists a constant $c>0$ such that for any $\eta\in L^{rad}_{\Gamma}$ and for all $t>0$ there exists an element $\gamma\in\Gamma$ with*

$$d(\gamma x_0, \sigma_{x_0,\eta}(t)) < c\,.$$

The most familiar radially cocompact groups are convex cocompact and geometrically finite isometry groups of real hyperbolic spaces, as well as uniform lattices acting on symmetric spaces of higher rank. A further example is given by products of convex cocompact groups acting on the Riemannian product of rank one symmetric spaces of noncompact type.

THEOREM 7.20 *Let $\Gamma\subset G = Isom^o(X)$ be radially cocompact, $\xi\in\partial X$ with Cartan projection $H_\xi\in\overline{\mathfrak{a}^+_1}$ and μ a $(b,\Gamma\cdot\xi)$-density. Then there exists a constant $C_0>0$ such that for any Borel subset $E\subseteq\partial X$*

$$\mathrm{Hd}^{\alpha}(E)\ge C_0\cdot\mu_{x_0}(E)\,,\quad \alpha=\sum_{i=1}^{r} b^i\langle H_i,H_\xi\rangle\,.$$

Proof. Fix $c>c_0$, let $\varepsilon>0$, $s>0$ arbitrary, and choose a cover of E by balls $B^c_{r_j}(\eta_j)$, $r_j<\varepsilon$, such that with $\alpha:=\sum_{i=1}^r b^i\langle H_i,H_\xi\rangle$ $\qquad \mathrm{Hd}^{\alpha}_{\varepsilon}(E)\ge \sum_{j\in\mathbb{N}} r_j^{\alpha} - s\,.$

If $B^c_{r_j}(\eta_j)\cap L^{rad}_{\Gamma}=\emptyset$, we do not need $B^c_{r_j}(\eta_j)$ to cover $E\subset L^{rad}_{\Gamma}$, otherwise we choose $\xi_j\in B^c_{r_j}(\eta_j)\cap L^{rad}_{\Gamma}$. Since Γ is radially cocompact, there exists $\gamma_j\in\Gamma$ such that $d(\gamma_j x_0,\sigma_{x_0,\xi_j}(-\log r_j))\le c$. This implies $d(x_0,\gamma_j x_0)\ge -\log r_j - c$, and for any $\varphi>0$ $\mathrm{sh}^{\varphi}_{x_0}(B_{\gamma_j x_0}(3c))\supseteq B^c_{r_j}(\eta_j)$. We conclude using $\alpha=\sum_{i=1}^r b^i\langle H_i,H_\xi\rangle$

$$E\subseteq\bigcup_{j\in\mathbb{N}}\mathrm{sh}^{\varphi}_{x_0}(B_{\gamma_j x_0}(3c))\,,$$

$$\begin{aligned}
\mu_{x_0}(E) &\le \mu_{x_0}\Big(\bigcup_{j\in\mathbb{N}}\mathrm{sh}^{\varphi}_{x_0}(B_{\gamma_j x_0}(3c)\Big)\le\sum_{j\in\mathbb{N}}\mu_{x_0}(\mathrm{sh}^{\varphi}_{x_0}(B_{\gamma_j x_0}(3c))\\
&\le D(3c)\sum_{j\in\mathbb{N}} e^{-\sum_{i=1}^r b^i d_i(x_0,\gamma_j x_0)}\le D(3c)\sum_{j\in\mathbb{N}} e^{-d(x_0,\gamma_j x_0)\alpha}\\
&\le D(3c)\sum_{j\in\mathbb{N}} e^{\alpha(\log r_j + c)}\le D(3c)e^{\alpha c}\sum_{j\in\mathbb{N}} r_j^{\alpha}\\
&\le D(3c)e^{\alpha c}\left(\mathrm{Hd}^{\alpha}_{\varepsilon}(E)+s\right)\,.
\end{aligned}$$

The claim now follows as $s\searrow 0$ and $\varepsilon\searrow 0$. □

THEOREM 7.21 *Let $\Gamma \subset G = Isom^o(X)$ be radially cocompact, $\xi \in \partial X$ and μ a $(b, \Gamma\cdot\xi)$-density. Then*

$$\dim_{\mathrm{Hd}}(L_\Gamma^{rad} \cap G\cdot\xi) = \delta_{G\cdot\xi}(\Gamma)\,.$$

Proof. Let $\xi \in \partial X$ with Cartan projection $H_\xi \in \overline{\mathfrak{a}_1^+}$ and μ a $(b, \Gamma\cdot\xi)$-density. From the previous theorem we deduce that for $\alpha := \sum_{i=1}^r b^i \langle H_i, H_\xi\rangle$

$$\mathrm{Hd}^\alpha(L_\Gamma^{rad} \cap G\cdot\xi) \geq C_0 \mu_{x_0}(L_\Gamma^{rad}) \leq 0\,,$$

hence $\dim_{\mathrm{Hd}}(L_\Gamma^{rad} \cap G\cdot\xi) \geq \alpha = \sum_{i=1}^r b^i\langle H_i, H_\xi\rangle \geq \delta_{G\cdot\xi}(\Gamma)$ by Theorem 7.7.

If $\xi \in \partial X^{reg}$, then the assertion follows directly from Theorem 7.18. Suppose $\xi \notin \partial X^{reg}$, fix $c > 2c_0$ with $c_0 > 0$ as in Theorem 7.6, let $\varepsilon > 0$ arbitrary and

$$\Gamma' := \{\gamma \in \Gamma \mid d(x_0, \gamma x_0) > -\log\varepsilon,\ d(\gamma x_0, \sigma_{x_0,\eta}) \leq c/2 \text{ for some } \eta \in G\cdot\xi\}\,.$$

For $\gamma \in \Gamma'$, we put $\xi_\gamma := \overline{\mathcal{C}}_{x_0,\gamma x_0} \cap G\cdot\xi$ and $r_\gamma := \exp\big(-d(x_0,\gamma x_0)\big)$. Then

$$L_\Gamma^{rad} \cap G\cdot\xi \subseteq \bigcup_{\gamma\in\Gamma'} B_{r_\gamma}(\xi_\gamma)\,,$$

and we estimate $$\mathrm{Hd}_\varepsilon^\alpha(L_\Gamma^{rad} \cap G\cdot\xi) \leq \sum_{\gamma\in\Gamma'} r_\gamma^\alpha = \sum_{\gamma\in\Gamma'} e^{-\alpha d(x_0,\gamma x_0)} \leq Q_{G\cdot\xi}^{\alpha,\varphi}(x_0,x_0)\,.$$

This number is finite if $\alpha > \delta_{G\cdot\xi}^\varphi(x_0,x_0)$. Taking the limit as $\varphi \searrow 0$ shows that the same is true for $\alpha > \delta_{G\cdot\xi}(\Gamma)$, and letting $\varepsilon \searrow 0$, we conclude that $\mathrm{Hd}^\alpha(L_\Gamma^{rad} \cap G\cdot\xi)$ is bounded for $\alpha > \delta_{G\cdot\xi}(\Gamma)$, hence $\dim_{\mathrm{Hd}}(L_\Gamma^{rad} \cap G\cdot\xi) \leq \delta_{G\cdot\xi}(\Gamma)$. □

Using the results of section 6.5, we deduce the following two corollaries.

COROLLARY 7.22 *Let $X = SL(n,\mathbb{R})/SO(n)$, and $\Gamma \subset SL(n,\mathbb{R})$ a cocompact lattice. Then for any $\xi \in \partial X$ with Cartan projection $H_\xi \in \overline{\mathfrak{a}_1^+}$ we have*

$$\dim_{\mathrm{Hd}}(L_\Gamma^{rad} \cap G\cdot\xi) = \rho(H_\xi)\,.$$

COROLLARY 7.23 *Let $X = X_1 \times X_2$ be the Riemannian product of rank one symmetric spaces, $\Gamma_1 \subset Isom^o(X_1)$, $\Gamma_2 \subset Isom^o(X_2)$ convex cocompact groups with critical exponents δ_1, δ_2, and $\Gamma = \Gamma_1 \times \Gamma_2 \subset G = Isom^o(X)$. Then for any $\xi \in \partial X$ with Cartan projection $H_\xi = \cos\theta H_1 + \sin\theta H_2$, $\theta \in [0,\pi/2]$, we have*

$$\dim_{\mathrm{Hd}}(L_\Gamma^{rad} \cap G\cdot\xi) = \delta_1\cos\theta + \delta_2\sin\theta\,.$$

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