On Second Order Homogeneous Linear Differential Equations with Liouvillian Solutions

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We determine all minimal polynomials for second order homogeneous linear differential equations with algebraic solutions decomposed into invariants and we show how easily one can recover the known conditions on differential Galois groups [12,19,25] using invariant theory. Applying these conditions and the differential invariants of a differential equation we deduce an alternative method to the algorithms given in [12,20,25] for computing Liouvillian solutions. For irreducible second order equations our method determines solutions by formulas in all but three cases.

1 Introduction

Algorithms computing algebraic solutions of second order differential equations are well-known since last century. Already in 1839, J. Liouville published such a procedure. However, the degree of the minimal polynomial of a solution must be known. Among other renowned mathematicians, L. Fuchs [5,6] developed from 1875 to 1877 a method for computing algebraic solutions, which is based only on binary forms. He wanted to clear up the question of when a second order linear differential equation has algebraic solutions and he solved it by determining the possible orders of symmetric powers associated with the given differential equation for which at least one needs to have a root of a rational function as a solution (see e.g. [5, No. 22, Satz]). Thereby, he gave a method – presumably without taking note of it – that remains valid for determining Liouvillian solutions of irreducible linear differential equations of second order.

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Modern algorithms for computing Liouvillian solutions are based on differential Galois theory. These algorithms determine a minimal polynomial of the logarithmic derivative of a Liouvillian solution since one knows that these derivatives are algebraic of bounded degree (see Singer [16] Theorem 2.4). This approach for second order equations stems from Kovacic [12] and has been implemented in Maple and some other computer algebra systems. A more accessible version of this algorithm was given by Ulmer and Weil [25] and is implemented in Maple, too.
Even when the solutions are algebraic, one can determine the minimal polynomial of a solution. In Singer and Ulmer [20] this is used to solve equations with a finite primitive unimodular Galois group by extending the Fuchsian method to arbitrary order. For this, one first has to compute a minimal polynomial decomposed into invariants for every possible Galois group.

In this paper we take up the ideas from Fuchs once again. Applying invariant theory we reformulate these ideas and state them more precisely. From that we obtain an alternative method for computing Liouvillian solutions. Unlike the known algorithms [12,20,25] we compute for irreducible second order equations – except for three cases – all Liouvillian solutions directly by formulas and not via their minimal polynomials (Theorem 11).
In the three exceptional cases we get a minimal polynomial of a solution using *exclusively* absolute invariants and their syzygies by computing – depending on the case – one rational solution of the 6^{th}, 8^{th} or 12^{th} symmetric power of the differential equation and determining its corresponding constant (Theorem 16). There is no need for a Gröbner basis computation in these cases. In Fuchs [5, p. 100] and Singer and Ulmer [20, p. 67] one needs in these cases to substitute a minimal polynomial decomposed into invariants in the differential equation. But this is very expensive.
We note, that it is possible to extend the algorithm presented here at least to all linear differential equations of prime power order.

The paper is organized as follows. In the rest of this section we briefly introduce differential Galois theory and the concept of invariants. In section 2 we summarize important properties of linear differential equations with algebraic solutions, which we use in section 3 to compute minimal polynomials decomposed into invariants. In section 4 we show, how easily one can obtain the known criteria for differential Galois groups [12,19,25] using invariant theory. These criteria result in an algorithm for computing Liouvillian solutions of a second order linear differential equation which is presented in section 5. Finally we give for every (irreducible) case an example.

The rest of this section and the following one contains nothing new, but are included to complete the picture.

For the exact definitions of the following concepts we refer to Kaplansky [10], Kolchin [11] and Singer [17].

Functions, which one gets from the rational functions by successive adjunctions of nested integrals, exponentials of integrals and algebraic functions, are the *Liouvillian functions.*

A *differential field* $(k,\ ')$ is a field k together with a derivation $'$ in k. The set of all constants $\mathcal{C} = \{a \in k \mid a' = 0\}$ is a subfield of $(k,\ ')$.
Let $\mathcal{C}$ be algebraically closed and k be of characteristic 0. Consider the following ordinary homogeneous linear differential equation

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ \dots\ + a_1y' + a_0y = 0 \qquad (a_i \in k) \qquad (1)$$

over k with a system $\{y_1, \dots, y_n\}$ of fundamental solutions.
By extending the derivation $'$ to a system of fundamental solutions and by adjunction of these solutions and their derivatives to k in a way the field of constants does not change, one gets $K = k\langle y_1, \dots, y_n\rangle$, the so-called *Picard-Vessiot extension* (PVE) of $L(y) = 0$. With the above assumptions, the PVE of $L(y) = 0$ always exists and is unique up to differential isomorphisms. This extension plays the same role for a differential equation as a splitting field for a polynomial equation.

The set of all automorphisms of K, which fix k elementwise and commute with the derivation in K, is a group, the *differential Galois group* $\mathcal{G}(K/k) = \mathcal{G}(L)$ of $L(y) = 0$. Since the automorphisms must commute with the derivation, they map a solution to a solution. Therefore $\mathcal{G}(L)$ operates on the $\mathcal{C}$-vector space of the fundamental solutions and from that one gets a faithful matrix representation of $\mathcal{G}(L)$, hence $\mathcal{G}(L)$ is isomorphic to a linear subgroup of $\mathrm{GL}(n, \mathcal{C})$. Moreover, it is isomorphic to a linear algebraic group. Furthermore, there is a (differential) Galois correspondence between the linear algebraic subgroups of $\mathcal{G}(L)$ and the differential subfields of K/k (see Kaplansky [10], Theorems 5.5 and 5.9).
The choice of another system of fundamental solutions leads to an equivalent representation. Hence, for every differential equation $L(y) = 0$, there is exactly one representation of $\mathcal{G}(L)$ up to equivalence.

Many properties of $L(y) = 0$ and its solutions can be found in the structure of $\mathcal{G}(L)$. Such an important property is: The component of the identity of $\mathcal{G}(L)^\circ$ of $\mathcal{G}(L)$ in the Zariski topology is solvable, if and only if K is a Liouvillian extension of k (see Kolchin [11], §25, Theorem). By this, we have a criterion to decide whether a linear differential equation $L(y) = 0$ has Liouvillian solutions.

An ordinary homogeneous linear differential polynomial $L(y)$ is called *reducible* over k, if there are two homogeneous linear differential polynomials $L_1(y)$ and $L_2(y)$ of positive order over k with $L(y) = L_2(L_1(y))$, otherwise $L(y)$ is called *irreducible.* $L(y) = 0$ is reducible, if and only if the corresponding representation of $\mathcal{G}(L)$ is reducible (see Kolchin [11], §22, Theorem 1). If an irreducible linear differential equation $L(y) = 0$ has a Liouvillian solution over k, then all solutions of $L(y) = 0$ are Liouvillian (see Singer [16], Theorem 2.4). However, if $L(y) = 0$ is reducible then Liouvillian solutions only possibly exist.
Against this, a second order linear differential equation has either only Liouvillian solutions or no Liouvillian solutions (see e.g. Ulmer and Weil [25], section 1.2).

1.2 Invariants

In this section we introduce informally some concepts of invariant theory. For the exact definitions we refer the reader to Sturmfels [21], Springer [18] or Schur [15].
Let V be a finite dimensional $\mathcal{C}$-vector space and G a linear subgroup of $\mathrm{GL}(V)$. An *(absolute) invariant* is a polynomial function $f \in \mathcal{C}[V]$ which remains unchanged under the group action, i.e. $f = f \circ g$ for all $g \in G$. If, for some $g \in G$, f and $f \circ g$ differ from each other only by a constant factor then the polynomial function f is called a *relative invariant.* The set of all invariants of G forms the *ring of invariants* $\mathcal{C}[V]^G$. For irreducible groups $G \in \mathrm{GL}(V)$, the rings of invariants $\mathcal{C}[V]^G$ are finitely generated by Hilbert's finiteness theorem (see e.g. Sturmfels [21]).
For finite groups $G \in \mathrm{GL}(V)$ the *Reynolds operator* $R_G(f) = \frac{1}{|G|}\sum_{g\in G} f \circ g$ maps a polynomial function $f \in \mathcal{C}[V]$ to the invariant $R_G(f) \in \mathcal{C}[V]^G$. With the *Hessian* $H(I_1) = \det\left(\frac{\partial^2 I_1}{\partial v_i \partial v_j}\right)$ and the *Jacobian* $J(I_1, \ldots, I_n) = \det\left(\frac{\partial I_i}{\partial v_j}\right)$ it is possible to generate new invariants from the invariants $I_1(\mathbf{v}), \ldots, I_n(\mathbf{v})$ (see e.g. [21,18,15]).
Molien and *Hilbert series* (see e.g. Sturmfels [21]) of a ring of invariants allow us to decide whether a set of invariants already generates the whole ring.

Let V be the $\mathcal{C}$-vector space of a system of fundamental solutions of $L(y) = 0$ and let $I(\mathbf{v}) \in \mathcal{C}[V]^{\mathcal{G}(L)}$ be an invariant of $\mathcal{G}(L)$. If one evaluates the invariant $I(\mathbf{v})$ with the fundamental solutions and takes into account that exactly the elements $a \in k$ are invariant under the Galois group $\mathcal{G}(L)$ then $I(y_1, \ldots, y_n)$ must be an element of k. An important tool for computing such an element are the symmetric powers of $L(y) = 0$.
The m^{th} *symmetric power* $L^{\circledS m}(y) = 0$ of $L(y) = 0$ is the differential equation whose solution space consists exactly of all m^{th} power products of solutions of $L(y) = 0$. There is an efficient algorithm to construct symmetric powers

described e.g. in Singer and Ulmer [19], pp. 20 or Fakler [3], pp. 14.

2 Algebraic Solutions

In this section we briefly give some important properties of linear differential equations with algebraic solutions.

Theorem 1 *([23], Theorem 2.2; [16], Theorem 2.4)*
Let k be a differential field of characteristic 0 with an algebraically closed field of constants. If an irreducible linear differential equation $L(y) = 0$ has an algebraic solution, then

- *all solutions are algebraic*
- *$\mathcal{G}(L)$ is finite*
- *the PVE of $L(y) = 0$ is a normal extension and coincides with the splitting field $k(y_1, \ldots, y_n)$.*

For many statements on differential equations it is assumed that the Galois group corresponding to $L(y) = 0$ is unimodular (i.e. $\subseteq \mathrm{SL}(n, \mathcal{C})$).

Theorem 2 *([10], p. 41; [20], Theorem 1.2)*
Let $L(y)$ be the linear differential equation (1), then $\mathcal{G}(L)$ is unimodular, if and only if there is a $W \in k$ such that $W'/W = a_{n-1}$.

Using the variable transformation $y = z \cdot \exp\left(-\frac{\int a_{n-1}}{n}\right)$, it is always possible to transform the equation $L(y) = 0$ into the equation

$$L_{\mathrm{SL}}(z) = z^{(n)} + b_{n-2} z^{(n-2)} + \ \ldots\ + b_1 z' + b_0 z = 0 \qquad (b_i \in k).$$

According to Theorem 2 $\mathcal{G}(L_{\mathrm{SL}})$ is unimodular. For second order equations we get $L_{\mathrm{SL}}(z) = z'' + \left(a_0 - \frac{a_1^2}{4} - \frac{a_1'}{2}\right) z = 0$.
Under such a transformation it is clear that $L(y) = 0$ has Liouvillian solutions if and only if $L_{\mathrm{SL}}(z) = 0$ has Liouvillian solutions. Furthermore, if $L(y) = 0$ has only algebraic solutions, then $L_{\mathrm{SL}}(z) = 0$ has only algebraic solutions (cf. [23], p. 184).

Theorem 3 *([20], Corollary 1.4)*
Let $k \subset K$ be a differential field of characteristic 0 and let the common field of constants of k and K be algebraically closed. If $y \in K$ is algebraic over k and y'/y is algebraic of degree m over k, then the minimal polynomial $P(\mathbf{Y}) = 0$ of y over k can be written in the following way

$$P(\mathbf{Y}) = \mathbf{Y}^{d \cdot m} + a_{m-1} \mathbf{Y}^{d \cdot (m-1)} + \ldots + a_0 = \prod_{\sigma \in \mathcal{T}} \left(\mathbf{Y}^d - (\sigma(y))^d\right), \qquad (2)$$

where $[k(y) : k(y'/y)] = d = |H/N|$, H/N *is cyclic,* $a_j \in k$, $H = \mathcal{G}(K/k(y'/y))$ *is a 1-reducible subgroup of* $G = \mathcal{G}(K/k)$ *and* $\mathcal{T}$ *is a set of left coset representatives of* H *in* G *of minimal index* m.

3 Minimal polynomials decomposed into invariants

Theorem 3 and Theorem 1 imply that any irreducible linear differential equation $L(y) = 0$ with algebraic solutions has a minimal polynomial $P(\mathbf{Y})$ of the form (2). Therefore, it remains to compute for any finite differential Galois group such a minimal polynomial.
In this section, we compute for any finite unimodular group a minimal polynomial written in terms of invariants. The restriction to unimodular groups is necessary, since only these groups are all known. However, Theorem 2 secures that we can construct a linear differential equation with unimodular Galois group from any linear differential equation $L(y) = 0$.

3.1 Imprimitive unimodular groups of degree 2

The finite imprimitive algebraic subgroups of $\mathrm{SL}(2, \mathcal{C})$ are the binary dihedral groups $D_n^{\mathrm{SL}_2}$ of order $4n$ [25]. These are central extensions of the dihedral groups D_n. They are generated by (Springer [18], p. 89)

$$u_n = \begin{pmatrix} e^{\frac{\pi i}{n}} & 0 \\ 0 & e^{-\frac{\pi i}{n}} \end{pmatrix} \qquad \text{and} \qquad v = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

A simple calculation shows that these representations are irreducible. The invariants of the binary dihedral groups are generated by

$$I_4 = y_1^2 y_2^2, \quad I_{2n} = y_1^{2n} + (-1)^n y_2^{2n}, \quad I_{2n+2} = y_1 y_2 (y_1^{2n} - (-1)^n y_2^{2n})$$

and they satisfy the relation

$$I_{2n+2}^2 - I_4 I_{2n}^2 + (-1)^n 4 I_4^{n+1} = 0, \tag{3}$$

see Springer [18], p. 95.
Let $\{y_1, y_2\}$ be a set of fundamental solutions of an equation $L(y) = 0$ of second order.

Theorem 4
Let $L(y) = 0$ *be an irreducible second order linear differential equation over* k

with a finite unimodular Galois group $\mathcal{G}(L) \cong D_n^{\mathrm{SL}_2}$. Then

$$P(\mathbf{Y}) = \mathbf{Y}^{4n} - I_{2n}\mathbf{Y}^{2n} + (-1)^n I_4^n$$

is a minimal polynomial decomposed into invariants for a solution of $L(y) = 0$.

PROOF. The degree of a minimal polynomial for a solution of $L(y) = 0$ of order 2 equals the order of the group $\mathcal{G}(L)$, see e.g. Singer and Ulmer [20], p. 55. Comparing this with $P(\mathbf{Y})$ from Theorem 3 shows that $d \cdot m = |\mathcal{G}(L)|$. $H = \langle u_n \rangle$ with $|H| = 2n$ is a maximal subgroup of $\mathcal{G}(L)$. H is a cyclic group and hence Abelian and 1-reducible and the elements of H have the common eigenvector $z = y_1$ (z is a solution of $L(y) = 0$). $\mathcal{T} = \{u_n^n, vu_n^n\}$ is a set of left coset representatives of H in $\mathcal{G}(L)$.
Together with $m = [\mathcal{G}(L) : H] = 2$ and thus $d = 2n$ one can calculate the minimal polynomial in the following way:

$$\begin{aligned}
P(\mathbf{Y}) &= \prod_{\sigma \in \mathcal{T}} \left(\mathbf{Y}^{2n} - \sigma(z)^{2n}\right) \\
&= \left(\mathbf{Y}^{2n} - (-y_1)^{2n}\right)\left(\mathbf{Y}^{2n} - (-iy_2)^{2n}\right) \\
&= \mathbf{Y}^{4n} - (y_1^{2n} + (-1)^n y_2^{2n})\mathbf{Y}^{2n} + (-1)^n y_1^{2n} y_2^{2n}.
\end{aligned}$$

Decomposing this expression into the above mentioned invariants completes the proof.

□

3.2 Primitive unimodular groups of degree 2

Up to isomorphisms, there are three finite primitive unimodular linear algebraic groups of degree 2. These groups are the tetrahedral group ($A_4^{\mathrm{SL}_2}$), the octahedral group ($S_4^{\mathrm{SL}_2}$) and the icosahedral group ($A_5^{\mathrm{SL}_2}$), see e.g. Ulmer and Weil [25].
In contrast to Fuchs, the minimal polynomials in this section are determined using exclusively absolute invariants. The definitions of the matrix groups stem from Miller, Blichfeldt and Dickson [1] pp. 221, while the necessary 1-reducible subgroups, left coset representatives and eigenvectors are found in Singer and Ulmer [20]. All the fundamental invariants are computed with the algorithms and implementations given in Fakler [3,4] (see also the relative invariants given in [1] pp. 225).

3.2.1 The tetrahedral group

$\mathbf{Y}^{24}+48I_1\mathbf{Y}^{18}+\left(90I_3+228{I_1}^2\right)\mathbf{Y}^{12}+\left(288I_1I_3+2368{I_1}^3\right)\mathbf{Y}^6-3{I_3}^2+36{I_1}^2I_3-108{I_1}^4$

is a minimal polynomial decomposed into invariants for the tetrahedral group. The invariants of this group are generated by

$$\begin{aligned} I_1 &= \frac{1}{2}R_{A_4^{\mathrm{SL}_2}}(y_1^5y_2)(\mathbf{y}) = y_1{y_2}^5 - {y_1}^5y_2 \\ I_2 &= -\frac{1}{25}H(I_1) = {y_2}^8 + 14{y_1}^4{y_2}^4 + {y_1}^8 \\ I_3 &= \frac{1}{8}J(I_1, I_2) = {y_2}^{12} - 33{y_1}^4{y_2}^8 - 33{y_1}^8{y_2}^4 + {y_1}^{12}. \end{aligned}$$

and they satisfy the relation ${I_3}^2 - {I_2}^3 + 108{I_1}^4 = 0$.
Using Molien and Hilbert series one can show that the ring of invariants can be written as the direct sum of graded $\mathcal{C}$-vector spaces

$$\mathcal{C}[y_1, y_2]^{A_4^{\mathrm{SL}_2}} = \mathcal{C}[I_1, I_2, I_3] = \mathcal{C}[I_1, I_2] \oplus I_3 \cdot \mathcal{C}[I_1, I_2].$$

In this expression for the minimal polynomial I_1 was multiplied by $-\mu^3$ and I_3 by the factor $-\frac{26}{3}\mu^2 + \frac{26}{3}\mu - \frac{7}{3}$, where $\mu^4 - 2\mu^3 + 5\mu^2 - 4\mu + 1 = 0$ [2] and $i = \sqrt{-1} = 2\mu^3 - 3\mu^2 + 9\mu - 4$.

The above representation needs an algebraic extension. It can be an advantage to choose a representation which is less sparse but does not require an algebraic extension. One obtains such a representation e.g. by computing a lexicographical Gröbner basis from the three equations of the fundamental invariants for $y_2 \succ y_1 \succ I_3 \succ I_2 \succ I_1$:

$$\mathbf{Y}^{24} + 10I_2\mathbf{Y}^{16} + 5I_3\mathbf{Y}^{12} - 15I_2^2\mathbf{Y}^8 - I_2I_3\mathbf{Y}^4 + I_1^4. \tag{4}$$

In this expression for a minimal polynomial decomposed into invariants for the tetrahedral group I_1 was multiplied by $\frac{1}{4}$, I_2 by $-\frac{5}{80}$ and I_3 by the factor $-\frac{1}{16}$.

3.2.2 The octahedral group

$\mathbf{Y}^{48}+20I_1\mathbf{Y}^{40}+70{I_1}^2\mathbf{Y}^{32}+\left(2702{I_2}^2+100{I_1}^3\right)\mathbf{Y}^{24}+\left(-1060I_1{I_2}^2+65{I_1}^4\right)\mathbf{Y}^{16} + \left(78{I_1}^2{I_2}^2+16{I_1}^5\right)\mathbf{Y}^8+{I_2}^4$

[2] This algebraic extension becomes necessary for computing an eigenvector.

is a minimal polynomial decomposed into invariants for the octahedral group. The ring of invariants of this group is generated by

$$
\begin{aligned}
I_1 &= \frac{1}{24} R_{S_4^{\mathrm{SL}_2}}(y_1^4 y_2^4)(\mathbf{y}) = {y_2}^8 + 14{y_1}^4{y_2}^4 + {y_1}^8 \\
I_2 &= \frac{1}{9408} H(I_1) = {y_1}^2{y_2}^{10} - 2{y_1}^6{y_2}^6 + {y_1}^{10}{y_2}^2 \\
I_3 &= -\frac{1}{16} J(I_1, I_2) = y_1{y_2}^{17} - 34{y_1}^5{y_2}^{13} + 34{y_1}^{13}{y_2}^5 - {y_1}^{17}\, y_2.
\end{aligned}
$$

These three invariants satisfy the sysygy $\quad {I_3}^2 + 108{I_2}^3 - {I_1}^3 I_2 = 0$.
That this syzygy is the only relation among the fundamental invariants is confirmed by the Molien and the Hilbert series. They also show, that the ring of invariants decomposes as the direct sum of graded $\mathcal{C}$-vector spaces

$$
\mathcal{C}[y_1, y_2]^{S_4^{\mathrm{SL}_2}} = \mathcal{C}[I_1, I_2, I_3] = \mathcal{C}[I_1, I_2] \oplus I_3 \cdot \mathcal{C}[I_1, I_2].
$$

In the above-mentioned expression for the minimal polynomial I_1 was multiplied by $-\frac{1}{16}$ and I_2 by the factor $\frac{1}{16}$.

3.2.3 The icosahedral group

$$
\begin{aligned}
&\mathbf{Y}^{120} + 20570 I_2 \mathbf{Y}^{100} + 91 I_3 \mathbf{Y}^{90} - 86135665 {I_2}^2 \mathbf{Y}^{80} - 78254 I_2 I_3 \mathbf{Y}^{70} + \\
&\left(14993701690 {I_2}^3 + 11137761250 {I_1}^5\right) \mathbf{Y}^{60} + 897941 {I_2}^2 I_3 \mathbf{Y}^{50} + \\
&\left(-11602919295 {I_2}^4 + 273542733750 {I_1}^5 I_2\right) \mathbf{Y}^{40} + \\
&\left(-151734 {I_2}^3 - 6953000 {I_1}^5\right) I_3 \mathbf{Y}^{30} + \left(503123324 {I_2}^5 - 7854563750 {I_1}^5 {I_2}^2\right) \mathbf{Y}^{20} + \\
&\left(1331 {I_2}^4 + 500 {I_1}^5 I_2\right) I_3 \mathbf{Y}^{10} + 3125 {I_1}^{10}
\end{aligned}
$$

is a minimal polynomial decomposed into invariants for the icosahedral group. The three invariants

$$
\begin{aligned}
I_1 &= -\frac{1}{25} R_{A_5^{\mathrm{SL}_2}}(y_1^6 y_2^6)(\mathbf{y}) = y_1{y_2}^{11} - 11{y_1}^6{y_2}^6 - {y_1}^{11}y_2 \\
I_2 &= -\frac{1}{121} H(I_1) = {y_2}^{20} + 228{y_1}^5{y_2}^{15} + 494{y_1}^{10}{y_2}^{10} - 228{y_1}^{15}{y_2}^5 + {y_1}^{20} \\
I_3 &= \frac{1}{20} J(I_1, I_2) = {y_2}^{30} - 522{y_1}^5{y_2}^{25} - 10005{y_1}^{10}{y_2}^{20} \\
&\qquad\qquad - 10005{y_1}^{20}{y_2}^{10} + 522{y_1}^{25}{y_2}^5 + {y_1}^{30}
\end{aligned}
$$

are the fundamental invariants of the icosahedral group and satisfy the algebraic relation $\quad {I_3}^2 - {I_2}^3 + 1728{I_1}^5 = 0$.
Molien and Hilbert series verify that this relation is the only syzygy and show,

that the ring of invariants decomposes as the direct sum of graded $\mathcal{C}$-vector spaces

$$\mathcal{C}[y_1, y_2]^{A_5^{\mathrm{SL}_2}} = \mathcal{C}[I_1, I_2, I_3] = \mathcal{C}[I_1, I_2] \oplus I_3 \cdot \mathcal{C}[I_1, I_2].$$

In the above-mentioned expression for the minimal polynomial I_1 was multiplied by $\frac{1}{125}$, I_2 by $-\frac{1}{275 \cdot 125}$ and I_3 by the factor $-\frac{11}{25 \cdot 125}$.

4 Criteria for differential Galois groups

The numbers and degrees of the invariants of all finite unimodular linear algebraic groups determined in the previous section yield conditions for the Galois group of a second order differential equation. In this section, we show how easily one can recover the known results (see Kovacic [12], Singer and Ulmer [19] and Ulmer and Weil [25]) using invariant theory.

If the Galois group $\mathcal{G}(L)$ is an imprimitive group, it is not easy to distinguish between a finite and an infinite group (see Singer and Ulmer [19], p. 25). The only infinite imprimitive unimodular Galois group of degree 2 is

$$D_\infty = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -a \\ a^{-1} & 0 \end{pmatrix} \right\} \quad \text{where} \quad a \in \mathbb{C}^*.$$

This group has only one fundamental invariant $I_4 = y_1^2 y_2^2$ (see Ulmer and Weil [25], section 3.2).
The following Lemma allows a simple method to distinguish all Galois groups $\mathcal{G}(L)$ for which an irreducible second order linear differential equation $L(y) = 0$ has Liouvillian solutions. This is no longer true in higher order.

Lemma 5 *(cf. [21], Lemma 3.6.3; [15], p. 47)*
A binary form of positive degree over k cannot vanish identically. In particular, this holds for homogeneous invariants in two independant variables.

Rational solutions of the m-th symmetric power $L^{ⓢ m}(y) = 0$ correspond to homogeneous invariants of degree m of $\mathcal{G}(L)$ (cf. Fakler [3], Singer and Ulmer [20]). Hence, as a consequence of Lemma 5, any invariant of degree m corresponds bijectively to a non-trivial rational solution of the m-th symmetric power of $L(y) = 0$ (see Singer and Ulmer [20], Lemma 3.5 (iii)).

Corollary 6 *(see [25], Lemma 3.2)*
Let $L(y) = 0$ be an irreducible second order linear differential equation over k with $\mathcal{G}(L) \cong D_n^{\mathrm{SL}_2}$. Then $L^{ⓢ 4}(y) = 0$ has a non-trivial rational solution.
In particular

(1) $L^{\circledS 4}(y) = 0$ *has two non-trivial rational solutions, if and only if* $\mathcal{G}(L) \cong D_2^{\mathrm{SL}_2}$.
(2) Otherwise, $L^{\circledS 4}(y) = 0$ *has exactly one non-trivial rational solution.*

PROOF. $D_2^{\mathrm{SL}_2}$ has two fundamental invariants of degree 4 (see section 3.1). All the other binary dihedral groups $D_n^{\mathrm{SL}_2}$ have exactly one fundamental invariant of fourth degree. □

The determination of the fundamental invariants of all finite unimodular groups in the last section allows the following result.

Proposition 7
Let $L(y) = 0$ *be a second order linear differential equation over* k *with an unimodular Galois group* $\mathcal{G}(L)$. *If* $L^{\circledS m}(y) = 0$ *has a non trivial rational solution for* $m = 2$ *or odd* $m \in \mathbb{N}$, *then* $L(y) = 0$ *is reducible.*

PROOF. If $L(y) = 0$ is irreducible, $L^{\circledS m}(y) = 0$ has at most non-trivial rational solutions for even $m \geq 4$. □

It ought to be clear, that the practical use of such a statement is restricted. However, the following proposition allows effective computations.

Proposition 8 *(see [25], Lemmata 3.2 and 3.3)*
Let $L(y) = 0$ *be an irreducible second order linear differential equation over* k *with an unimodular Galois group* $\mathcal{G}(L)$. *Then the following holds*

(1) $\mathcal{G}(L)$ *is imprimitive, if and only if* $L^{\circledS 4}(y) = 0$ *has a non-trivial rational solution.*
(2) $\mathcal{G}(L) \cong D_\infty$, *if and only if* $L^{\circledS 4m}(y) = 0$ *has exactly one non-trivial rational solution for any* $m \in \mathbb{N}$.
(3) $\mathcal{G}(L) \cong D_n^{\mathrm{SL}_2}$, *if and only if* $L^{\circledS 4}(y) = 0$ *has one and* $L^{\circledS 2n}(y) = 0$ *has two or exactly one non-trivial rational solution depending on whether* $4|2n$ *or not.*
(4) $\mathcal{G}(L)$ *is primitive and finite, if and only if* $L^{\circledS 4}(y) = 0$ *has none and* $L^{\circledS 12}(y) = 0$ *has at least one non-trivial rational solution.*
(5) $\mathcal{G}(L) \cong A_4^{\mathrm{SL}_2}$ *(tetrahedral group), if and only if* $L^{\circledS 4}(y) = 0$ *has none and* $L^{\circledS 6}(y) = 0$ *has a non-trivial rational solution.*
(6) $\mathcal{G}(L) \cong S_4^{\mathrm{SL}_2}$ *(octahedral group), if and only if* $L^{\circledS m}(y) = 0$ *for* $m \in \{4, 6\}$ *has none and* $L^{\circledS 8}(y) = 0$ *has a non-trivial rational solution.*

(7) $\mathcal{G}(L) \cong A_5^{\mathrm{SL}_2}$ *(icosahedral group), if and only if* $L^{ⓢ m}(y) = 0$ *for* $m \in \{4, 6, 8\}$ *has none and* $L^{ⓢ 12}(y) = 0$ *has a non-trivial rational solution.*
(8) $\mathcal{G}(L) \cong \mathrm{SL}(2, \mathcal{C})$*, if none of the above cases hold.*

PROOF. From Corollary 6 and the above remarks on the infinite imprimitive group D_∞ one gets immediately (1)-(3).
The Galois group of an irreducible linear differential equation $L(y) = 0$ is irreducible (see Kolchin [11] §22, Theorem 1). An irreducible group is either imprimitive or primitive. Comparing the degrees of the fundamental invariants of the three finite primitive unimodular linear algebraic groups of degree 2 and the fact that there is no infinite primitive algebraic subgroup of $\mathrm{SL}(2, \mathcal{C})$ (see Singer and Ulmer [19] p. 13) together with Lemma 5 yields (4).
(5)-(7) are simple consequences of Lemma 5 and the invariants computed in the previous section.
If none of the above cases hold, then $\mathcal{G}(L)$ is primitive and infinite and thus, as above stated, equals $\mathrm{SL}(2, \mathcal{C})$. □

As a consequence, we get a nice criterion to decide, whether an irreducible second order linear differential equation has Liouvillian solutions (cf. Singer and Ulmer [19] Proposition 4.4, Kovacic [12], Fuchs [5] Satz II, No. 17 and Satz I & II, No. 20).

Corollary 9
Let $L(y) = 0$ *be an irreducible second order linear differential equation over* k *with an unimodular Galois group* $\mathcal{G}(L)$*. Then* $L(y) = 0$ *has a Liouvillian solution, if and only if* $L^{ⓢ 12}(y) = 0$ *has a non trivial rational solution.*
In particular, $L(y) = 0$ *has a Liouvillian solution, if and only if* $L^{ⓢ m}(y) = 0$ *has a non trivial rational solution for at least one* $m \in \{4, 6, 8, 12\}$*.*

PROOF. $L(y) = 0$ has a Liouvillian solution, if and only if the corresponding Galois group is either imprimitive, or primitive and finite. Now, the result follows from Proposition 8. □

5 An alternative algorithm

In this section we derive a direct method to compute Liouvillian solutions of irreducible second order linear differential equations with an imprimitive unimodular Galois group. Computing a minimal polynomial is no longer necessary, but to compute it is still possible. When the differential equation has a primitive unimodular Galois group, we show how one can determine a minimal polynomial of a solution by knowing the group explicitly and using all the fundamental invariants. There is no longer a need to substitute a minimal polynomial decomposed into invariants in the differential equation as it is in Fuchs [5, p. 100] and in Singer and Ulmer [20, p. 67].

Let $\{y_1, \ldots, y_n\}$ be a system of fundamental solutions of $L(y) = 0$ and

$$\Delta = \begin{vmatrix} y_1 & \cdots & y_n & y \\ y_1' & \cdots & y_n' & y' \\ \vdots & \ddots & \vdots & \vdots \\ y_1^{(n)} & \cdots & y_n^{(n)} & y^{(n)} \end{vmatrix}.$$

Further let $W_i = \frac{\partial \Delta}{\partial y^{(i)}}$ $(i = 0, \ldots, n)$, and let $W = W_n$, the Wronskian, and $W' = W_{n-1}$ its first derivative. With this, the differential equation $L(y) = 0$ is uniquely determined by

$$L(y) = \frac{\Delta}{W} = y^{(n)} - \frac{W'}{W} y^{(n-1)} + \frac{W_{n-2}}{W} y^{(n-2)} + \ \ldots \ + (-1)^n \frac{W_0}{W} y = 0$$

or

$$a_i = (-1)^{n-i} \frac{W_i}{W_n} \quad (i = 0, \ldots, n-1).$$

Transforming a fundamental system into another system of fundamental solutions of $L(y) = 0$ does not change $L(y) = 0$, e.g. the coefficients are differentially invariant under the general linear group $\mathrm{GL}(n, \mathcal{C})$. Because these transformations depend on $L(y) = 0$, we will denote their group with $\mathrm{G}(L)$. The coefficients a_k are n^{th} order differential invariants. They form a basis for the differential invariants of $\mathrm{G}(L)$, see Schlesinger [14], p. 16. Hence, one can represent any differential invariant of $\mathrm{G}(L)$ as a rational function in the $a_0, \ldots, a_{n-1}$ and their derivatives.

Definition 10
Let $L(y) = 0$ be a linear differential equation with Galois group $\mathcal{G}(L)$ and I an invariant of degree m of $\mathcal{G}(L)$. The rational solution R of the m^{th} symmetric power $L^{ⓢ m}(y) = 0$ corresponding to I, is called the **rationalvariant** *of I. An algebraic equation, which determines the constant c $(c \in \mathcal{C},\ c \neq 0)$ for $I \mapsto c \cdot R$, $R \neq 0$ is the* **determining equation** *for the rationalvariant R.*

All imprimitive Galois groups possess the common invariant $I_4 = y_1^2 y_2^2$ (see sections 3.1 and 4), which consists of a single monomial. This common invariant allows to compute Liouvillian solutions with ease.

Theorem 11
Let $L(y) = 0$ be an irreducible second order linear differential equation with an imprimitive unimodular Galois group $\mathcal{G}(L)$. Then $L(y) = 0$ has a fundamental system in the following two solutions

$$y_1 = \sqrt[4]{r}\, e^{-\frac{C}{2}\int \frac{W}{\sqrt{r}}} \quad \text{and} \quad y_2 = \sqrt[4]{r}\, e^{\frac{C}{2}\int \frac{W}{\sqrt{r}}}.$$

Thereby, W is the Wronskian, r is the rationalvariant of the invariant $I_4 = \frac{1}{C^2} \cdot r \quad (C \in \mathcal{C},\ C \neq 0)$ and

$$\frac{4r''r - 3(r')^2}{16r^2} + \frac{W^2}{4r}C^2 + \frac{r'}{4r}a_1 + a_0 = 0 \tag{5}$$

its determining equation.
In particular (cf. Fuchs [5], p. 118), if $a_1 = 0$ then

$$y_1 = \sqrt[4]{r}\, e^{-\frac{\bar{C}}{2}\int \frac{1}{\sqrt{r}}} \quad \text{and} \quad y_2 = \sqrt[4]{r}\, e^{\frac{\bar{C}}{2}\int \frac{1}{\sqrt{r}}} \qquad (\bar{C} = CW)$$

form a system of fundamental solutions, where $\bar{C}$ is determined by equation (5).

PROOF. Let r be a rational solution of $L^{ⓢ 4}(y) = 0$ with $I_4 = y_1^2 y_2^2 = c \cdot r$ $(c \in \mathcal{C},\ c \neq 0)$. Hence, it is $y_2 = \frac{\sqrt{c \cdot r}}{y_1}$. If we substitute this expression for y_2 and for y_2' its derivative in the Wronskian $W = y_1 y_2' - y_1' y_2$, we have

$$\frac{y_1'}{y_1} = \frac{r'}{4r} - \frac{W}{2\sqrt{c \cdot r}} \tag{6}$$

or

$$y_1 = \sqrt[4]{r} e^{-\frac{1}{2\sqrt{c}}\int \frac{W}{\sqrt{r}}}$$

respectively. Substituting y_1 in the differential equation $L(y) = 0$ we obtain the determining equation (5) for the constant $c = \frac{1}{C^2}$.

If $a_1 = 0$ e.g. W is constant, then y_1 is simplified to $\sqrt[4]{r}e^{-\frac{W}{2\sqrt{c}}\int\frac{1}{\sqrt{r}}}$ and we get with $\bar{C} = \frac{W}{\sqrt{c}}$ for equation (5)

$$\frac{4r''r - 3(r')^2}{16r^2} + \frac{1}{4r}\bar{C}^2 + a_0 = 0.$$

□

Remark 12
Equation (6) is already the solved minimal polynomial of the logarithmic derivative of a solution, which is computed in the second case of Kovacic's algorithm [12]. Indeed, Kovacic has used the invariant I_4 to prove the second case of his algorithm ([12], p. 10).

In the case of an imprimitive unimodular Galois group, $L^{ⓢ4}(y) = 0$ has exactly one non-trivial rational solution except for $D_2^{\mathrm{SL}_2}$ by Proposition 8. Now, suppose $L^{ⓢ4}(y) = 0$ has exactly one non-trivial rational solution. Then, using Theorem 11, we can directly compute both Liouvillian solutions of $L(y) = 0$. Since the determining equation for the constant C must be valid for all regular points of $L(y) = 0$, we only have to evaluate this equation for an arbitrary regular point.

When $L^{ⓢ4}(y) = 0$ has two linearly independent non-trivial rational solutions r_1 and r_2 (e.g. $\mathcal{G}(L) \cong D_2^{\mathrm{SL}_2}$) then we have two ways to compute Liouvillian solutions. In the first way we only set $r = c_1r_1 + c_2r_2$ and $C = 1$ and get the solutions by solving the determining equation (5).
The second possibility is to compute a further non-trivial rational solution r_3 of $L^{ⓢ6}(y) = 0$. With this rational solutions one makes the Ansatz

$$I_{4a} = c_1r_1 + c_2r_2, \qquad I_{4b} = c_3r_1 + c_4r_2, \qquad I_6 = c_5r_3$$

and substitute into the syzygy

$$I_6^2 - I_{4a}I_{4b}^2 + 4I_{4a}^3 = 0.$$

From the numerator of the thereby obtained rational function we get a system of polynomial equations for the constants $c_1, \ldots, c_5$. Solving this system can be done by computing a lexicographical Gröbner basis (cf. Sturmfels [21]). This gives a necessary condition for the previous invariants. It can be made sufficient by choosing the constants in a way that makes I_{4a}, I_{4b} and I_6 non-trivial and furthermore I_{4a} and I_{4b} linear independent. Since there are infinite many solutions for the invariants this is always possible. Using Theorem 11 we now can compute the Liouvillian solutions from the just constructed invariant I_{4a}.

Another way to compute the solutions is to solve the minimal polynomial of Theorem 4 explicitly.

The condition, that a linear differential equation in the imprimitive case has algebraic solutions is based on a Theorem of Abel, see Fuchs [5] p. 118. One can state this condition more precisely as follows.

Lemma 13
Let $L(y) = 0$ be a second order linear differential equation with a finite imprimitive unimodular Galois group $\mathcal{G}(L)$. Then the following equation holds

$$\int \frac{W}{\sqrt{I_4}} = \frac{1}{2n} \log \frac{I_{2n+2} + I_{2n}\sqrt{I_4}}{I_{2n+2} - I_{2n}\sqrt{I_4}}. \tag{7}$$

PROOF. Theorem 4 implies that the solutions of $L(y) = 0$ are of the form

$$y_{1,2} = \sqrt[2n]{\frac{1}{2}\left(I_{2n} \pm \sqrt{I_{2n}^2 - (-1)^n 4 I_4^n}\right)}. \tag{8}$$

Substituting I_{2n}^2 by syzygy (3) together with further manipulations give

$$y_{1,2} = \sqrt[4]{I_4}\ \sqrt[2n]{\frac{\pm I_{2n+2} + I_{2n}\sqrt{I_4}}{2\sqrt{I_4^{n+1}}}}.$$

Once more applying syzygy (3) on I_4^{n+1} and manipulations we get by Theorem 11

$$y_{1,2} = \sqrt[4]{I_4}\ \sqrt[4n]{(-1)^{n+1} \frac{\pm I_{2n+2} + I_{2n}\sqrt{I_4}}{\pm I_{2n+2} - I_{2n}\sqrt{I_4}}} = \sqrt[4]{I_4}\ e^{\pm\frac{1}{2}\int \frac{W}{\sqrt{I_4}}}$$

and therefore

$$\pm\frac{1}{2}\int \frac{W}{\sqrt{I_4}} = \pm\frac{1}{4n} \log \frac{I_{2n+2} + I_{2n}\sqrt{I_4}}{I_{2n+2} - I_{2n}\sqrt{I_4}}.$$

□

The solutions of $L(y) = 0$ are algebraic, if and only if one can write the integral $\int \frac{W}{\sqrt{I_4}}$ in the form (7).

Remark 14
It seems Lemma 13 allows us to determine explicitly the (imprimitive) Galois group of $L(y) = 0$. We will study this in a seperate paper.

This section present the tools for determining the rationalvariant of an invariant of degree m. The idea stems from Fuchs [6], p. 22.

Lemma 15
Let y_1, y_2 be independent functions in x, and let $f(y_1, y_2)$ and $g(y_1, y_2)$ be binary forms of degree m and n respectively. Then the following identities hold:

(1) for the Hessian of $f(y_1, y_2)$

$$H(f) = \frac{m-1}{W^2}\left[\left(\frac{f'}{f}\right)^2 + m\left(\frac{f'}{f}\right)' + ma_1\left(\frac{f'}{f}\right) + m^2 a_0\right] f^2$$

(for $a_1 = 0$, cf. Fuchs [6] p. 22) and
(2) for the Jacobian of $f(y_1, y_2)$ and $g(y_1, y_2)$

$$J(f,g) = \frac{mfg' - nf'g}{W}.$$

Thereby, W is the Wronskian of y_1 and y_2 and further $a_0 = \frac{W_0}{W}$ and $a_1 = -\frac{W_1}{W}$ are differential invariants of second order.

PROOF. For an arbitrary binary form $f(y_1, y_2) = \sum_{i=0}^{m} b_i y_1^{m-i} y_2^i$ the following identity holds

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \cdot \begin{pmatrix} f_{y_1} \\ f_{y_2} \end{pmatrix} = \begin{pmatrix} mf \\ f' \end{pmatrix} \quad \text{resp.} \quad \frac{1}{W}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \cdot \begin{pmatrix} mf \\ f' \end{pmatrix} = \begin{pmatrix} f_{y_1} \\ f_{y_2} \end{pmatrix}.$$

In particular, this is valid for the forms $\frac{\partial f}{\partial y_1} = f_{y_1}$ and $\frac{\partial f}{\partial y_2} = f_{y_2}$ of degree $m-1$:

$$\frac{1}{W}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} (m-1)f_{y_1} \\ f_{y_1}' \end{pmatrix} = \begin{pmatrix} f_{y_1 y_1} \\ f_{y_1 y_2} \end{pmatrix},$$

$$\frac{1}{W}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} (m-1)f_{y_2} \\ f_{y_2}' \end{pmatrix} = \begin{pmatrix} f_{y_2 y_1} \\ f_{y_2 y_2} \end{pmatrix}.$$

From this one gets the identities by reverse substitution in $H(f) = f_{y_1 y_1} f_{y_2 y_2} - f_{y_1 y_2} f_{y_2 y_1}$ and $J(f,g) = f_{y_1} g_{y_2} - f_{y_2} g_{y_1}$ if one takes the Wronskian and the differential equation $\frac{\Delta}{W} = 0$ for $n = 2$ into account. $\square$

Thus, it suffices to compute the non-trivial rational solution of the smallest possible symmetric power of $L(y) = 0$. The two remaining fundamental rationalvariants can be determined with Lemma 15. If the rationalvariants are known, one gets the constants from the sygyzies.

Theorem 16
Let $L(y) = 0$ be an irreducible second order linear differential equation over k with finite primitive unimodular Galois group $\mathcal{G}(L)$ and let r be the smallest rationalvariant (e.g. $I_1 = c \cdot r \quad (c \in \mathcal{C},\ c \neq 0)$). If one sets the Wronskian $W = 1$ in the case of $a_1 = 0$, then a determining equation for the rationalvariant r for each case is given by

$$\mathcal{G}(L) \cong A_4^{\mathrm{SL}_2} : (25J(r, H(r))^2 + 64H(r)^3)\, c^2 + 10^6 \cdot 108r^4 = 0$$

$$\mathcal{G}(L) \cong S_4^{\mathrm{SL}_2} : (49J(r, H(r))^2 + 144H(r)^3)\, c - 118013952r^3 H(r) = 0$$

$$\mathcal{G}(L) \cong A_5^{\mathrm{SL}_2} : (121J(r, H(r))^2 + 400H(r)^3)\, c + 708624400 \cdot 1728r^5 = 0.$$

PROOF. Let denote $H(f) = \frac{1}{W^2}\tilde{H}(f)$, $J(f, g) = \frac{1}{W}\tilde{J}(f, g)$ and for constant W let $J(f, H(f)) = \frac{1}{W^3}\tilde{J}(f, \tilde{H}(f))$. Then

$$\begin{aligned} H(c \cdot r) &= c^2 H(r) = \frac{c^2}{W^2}\tilde{H}(r) \\ J(c \cdot r, H(c \cdot r)) &= c^3 J(r, H(r)) \end{aligned}$$

and for constant W (e.g. $a_1 = 0$)

$$J(c \cdot r, H(c \cdot r)) = \frac{c^3}{W^3}\tilde{J}(r, \tilde{H}(r)).$$

Furthermore, let $I_1 = c \cdot r$. Substituting respectively the expressions for the fundamental invariants in the corresponding syzygies, see section 3.2, one obtains in the case of $a_1 = 0$

$$\mathcal{G}(L) \cong A_4^{\mathrm{SL}_2} : \left(\left(\tfrac{\tilde{J}(r, \tilde{H}(r))}{8 \cdot 25} \right)^2 + \left(\tfrac{\tilde{H}(r)}{25} \right)^3 \right) c^2 + 108r^4 W^6 = 0$$

$$\mathcal{G}(L) \cong S_4^{\mathrm{SL}_2} : \left(\left(\tfrac{\tilde{J}(r, \tilde{H}(r))}{16 \cdot 9408} \right)^2 + 108 \left(\tfrac{\tilde{H}(r)}{9408} \right)^3 \right) c - \tfrac{r^3 \tilde{H}(r)}{9408} W^4 = 0$$

$$\mathcal{G}(L) \cong A_5^{\mathrm{SL}_2} : \left(\left(\tfrac{\tilde{J}(r, \tilde{H}(r))}{20 \cdot 121} \right)^2 + \left(\tfrac{\tilde{H}(r)}{121} \right)^3 \right) c + 1728r^5 W^6 = 0.$$

For satisfying these equations one can arbitrary choose one of the two nonzero constants c and W, respectively. The assertion follows from the previous relations by setting $W = 1$ in each of them. In a similiar way one gets for $a_1 \neq 0$ the equations

$$\mathcal{G}(L) \cong A_4^{\mathrm{SL}_2} : \left(\left(\tfrac{J(r,H(r))}{8 \cdot 25} \right)^2 + \left(\tfrac{H(r)}{25} \right)^3 \right) c^2 + 108 r^4 = 0$$

$$\mathcal{G}(L) \cong S_4^{\mathrm{SL}_2} : \left(\left(\tfrac{J(r,H(r))}{16 \cdot 9408} \right)^2 + 108 \left(\tfrac{H(r)}{9408} \right)^3 \right) c - \tfrac{r^3 H(r)}{9408} = 0$$

$$\mathcal{G}(L) \cong A_5^{\mathrm{SL}_2} : \left(\left(\tfrac{J(r,H(r))}{20 \cdot 121} \right)^2 + \left(\tfrac{H(r)}{121} \right)^3 \right) c + 1728 r^5 = 0.$$

□

It is possible to solve the determining equation for the smallest rationalvariant through evaluation of an arbitrary regular point of $L(y) = 0$, since it must hold for all regular points.
Consequently, Theorem 16 allows to determine for second order linear differential equations with primitive unimodular Galois group a minimal polynomial of a solution without a Gröbner basis computation.

5.3 The algorithm

Based on the results of the previous two sections, we propose the following method as an alternative to the already known algorithms of Kovacic [12], Singer and Ulmer [20] and Ulmer and Weil [25]. Thereby, for solving a reducible differential equation we refer to one of these procedures. Computing rational solutions can be done e.g. with the algorithm described in Bronstein [2]. Moreover, rationalvariants can be determined by the method of van Hoeij and Weil [8] without computing any symmetric power.

Algorithm 1

Input: a linear differential equation $L(y) = 0$ with $\mathcal{G}(L) \subseteq \mathrm{SL}(2, \mathcal{C})$

Output: fundamental system of solutions $\{y_1, y_2\}$ of $L(y) = 0$
or minimal polynomial of a solution

(i) Test, if $L(y) = 0$ is reducible. If yes, then compute an exponential and a further Liouvillian solution by applying e.g. one of the previous algorithms.

(ii) Test, if $L^{ⓢ 4}(y) = 0$ has a non-trivial rational solution.
  (a) If the rational solution space is one-dimensional: Apply Theorem 11.
  (b) If the rational solution space is two-dimensional:
  Either set $r = c_1 r_1 + c_2 r_2$, $C = 1$ and apply Theorem 11,
  or compute the rational solution of $L^{ⓢ 6}(y) = 0$ and determine the

three rationalvariants I_{4a}, I_{4b} and I_6 (with a Gröbner basis computation) from syzygy (3) for $n = 2$. Subsequently[3]: substitute the rationalvariants in equation (8).

(iii) Test successively, for $m \in \{6, 8, 12\}$, if $L^{ⓢ m}(y) = 0$ has a non-trivial rational solution. If yes, then: compute both remaining rationalvariants with Lemma 15 and determine their constants (Proposition 8) by Theorem 16. Substituting the rationalvariants in the matching minimal polynomial decomposed into invariants from section 3.2 gives the minimal polynomial of a solution.

(iv) $L(y) = 0$ has no Liouvillian solution.

In the following we solve for each of the cases 2(a), 2(b) and 3 of Algorithm 1 an example with the computer algebra system AXIOM 1.2 (see Jenks and Sutor [9]).

Example 17 *(see Ulmer and Weil [25] pp. 193, Weil [26], pp. 93)*
The differential equation

$$L(y) = y'' - \frac{2}{2x-1}y' + \frac{(27x^4 - 54x^3 + 5x^2 + 22x + 27)(2x-1)^2}{144x^2(x-1)^2(x^2-x-1)^2}y = 0$$

is irreducible and has an unimodular Galois group, since $\frac{W'}{W} = \frac{2}{2x-1}$ and $W \in k$. Its fourth symmetric power $L^{ⓢ 4}(y) = 0$ has an one-dimensional rational solution space generated by $r = x(x-1)(x^2-x-1)^2$.
The constant C is determined by

$$\frac{(36C^2-4)x^2 + (-36C^2+4)x + 9C^2 - 1}{36x^6 - 108x^5 + 36x^4 + 108x^3 - 36x^2 - 36x} = 0$$

or e.g. for the regular point $x_0 = 2$ by

$$9C^2 - 1 = 0.$$

For the integral $\int \frac{W}{\sqrt{9r}}$ one gets

$$\int \frac{2x-1}{\sqrt{9x(x-1)(x^2-x-1)^2}} = \frac{1}{3}\log\frac{(-2x-1)\sqrt{x(x-1)} + 2x^2 - 1}{(-2x+3)\sqrt{x(x-1)} + 2x^2 - 4x + 1}.$$

[3] or apply Theorem 11 to the rationalvariant of I_{4a}

Therefore $L(y) = 0$ has a fundamental system in the solutions

$$y_{1,2} = \sqrt[4]{x(x-1)(x^2-x-1)^2}\left(\frac{(-2x+3)\sqrt{x(x-1)}+2x^2-4x+1}{(-2x-1)\sqrt{x(x-1)}+2x^2-1}\right)^{\pm\frac{1}{6}}.$$

To this fundamental system corresponds the invariant $I_4 = x(x-1)(x^2-x-1)^2$. Substituting both solutions in I_{2n} for $n = 3$ we get

$$I_6 = 4x^2(x-1)^2(x^2-x-1)^2.$$

Hence, $\mathcal{G}(L) \cong D_3^{\mathrm{SL}_2}$. By the relation (3) we obtain the remaining fundamental invariant

$$I_8 = \sqrt{I_4 I_6^2 + 4I_4^4} = 2x^2(x^2-x+1)(x-1)^2(x^2-x-1)^3.$$

$\diamond$

Example 18 *(see Ulmer [24] pp. 396, [27])*
Consider the irreducible differential equation

$$L(y) = y'' + \frac{27x}{8(x^3-2)^2}y = 0$$

constructed from Hendriks. Its fourth symmetric power $L^{\circledS 4}(y) = 0$ has a two-dimensional rational solution space, generated by $r_1 = x^3 - 2$ and $r_2 = x(x^3-2)$. Corollary 6 implies that $\mathcal{G}(L) \cong D_2^{\mathrm{SL}_2}$ is the corresponding Galois group of $L(y) = 0$. The rational solution space of $L^{\circledS 6}(y) = 0$ is generated by $r_3 = (x^3-2)^2$.
Substituting the Ansatz

$$I_{4a} = c_1(x^3-2) + c_2 x(x^3-2), \quad I_{4b} = c_3(x^3-2) + c_4 x(x^3-2),$$
$$I_6 = c_5(x^3-2)^2$$

in the relation (3) for $n = 2$ gives the necessary condition:

$$\begin{aligned}
&({c_5}^2 - c_2{c_4}^2 + 4{c_2}^3)x^{12} + (-c_1{c_4}^2 - 2c_2c_3c_4 + 12\ c_1{c_2}^2)x^{11}\\
&+ (-2c_1c_3c_4 - c_2{c_3}^2 + 12{c_1}^2c_2)x^{10} + (-8{c_5}^2 + 6c_2{c_4}^2 - c_1{c_3}^2 - 24{c_2}^3 + 4{c_1}^3)x^9\\
&+ (6c_1{c_4}^2 + 12c_2c_3c_4 - 72c_1{c_2}^2)x^8 + (12c_1c_3c_4 + 6c_2{c_3}^2 - 72{c_1}^2c_2)x^7\\
&+ (24{c_5}^2 - 12c_2{c_4}^2 + 6c_1{c_3}^2 + 48{c_2}^3 - 24{c_1}^3)x^6 +\\
&(-12c_1{c_4}^2 - 24c_2c_3c_4 + 144c_1{c_2}^2)x^5 + (-24c_1c_3c_4 - 12c_2{c_3}^2 + 144{c_1}^2c_2)x^4 +\\
&(-32{c_5}^2 + 8c_2{c_4}^2 - 12c_1{c_3}^2 - 32{c_2}^3 + 48{c_1}^3)x^3\\
&+ (8c_1{c_4}^2 + 16c_2c_3c_4 - 96c_1{c_2}^2)x^2 +\\
&(16c_1c_3c_4 + 8c_2{c_3}^2 - 96{c_1}^2c_2)x + 16{c_5}^2 + 8c_1{c_3}^2 - 32{c_1}^3 = 0.
\end{aligned}$$

In order to satisfying this condition all the coefficients must vanish identically. For instance, we can add $c_4 - \lambda = 0$ to the coefficient equations and compute for this system a lexicographical Gröbner basis for $c_1 \succ c_2 \succ c_3 \succ c_4 \succ c_5$. If one computes an ideal decomposition from this result with the algorithm **groebnerFactorize** and take therein the secondary condition $c_5 \neq 0$ into account, one gets the (parametrized) ideal $(\lambda \neq 0)$

$$\{\lambda^3 c_1 + \frac{3}{4} c_3 c_5^2,\ \lambda^2 c_2 - \frac{3}{4} c_5^2,\ c_3^3 - 2\lambda^3,\ c_4 - \lambda,\ c_5^4 + \frac{4}{27}\lambda^6\},$$

or the variety

$$\mathcal{P} = \left\{ \begin{array}{c} \left\{c_1 = -\frac{\frac{3}{4} c_3 {c_5}^2}{\lambda^3}\right\}, \left\{c_2 = \frac{\frac{3}{4} {c_5}^2}{\lambda^2}\right\}, \\ \left\{c_3 = \sqrt[3]{2\lambda^3},\ c_3 = \left(\pm\frac{1}{2}\sqrt{-1}\sqrt{3} - \frac{1}{2}\right)\sqrt[3]{2\lambda^3}\right\}, \{c_4 = \lambda\}, \\ \left\{c_5 = \pm\sqrt[4]{-\frac{4}{27}\lambda^6},\ c_5 = \pm\sqrt{-1}\sqrt[4]{-\frac{4}{27}\lambda^6}\right\} \end{array} \right\}.$$

$\mathcal{P}$ contains all possible choices for the constants of the fundamental invariants. For instance, the points $(c_1,\ c_2,\ c_3,\ c_4) = (\frac{1}{6}\sqrt{-3}\sqrt[3]{2}\lambda,\ -\frac{1}{6}\sqrt{-3}\lambda,\ \sqrt[3]{2}\lambda,\ \lambda)$ satisfy the sufficient condition for the rationalvariants. Substituting these points in equation (8) for $n = 2$, we get the two solutions

$$y_{1,2} = \sqrt[4]{\frac{1}{6}\lambda(x^3 - 2)\left(3x + 3\sqrt[3]{2} \pm 2\sqrt{3\left(x^2 + \sqrt[3]{2}\,x + \sqrt[3]{2}^2\right)}\right)}\ .$$

$\diamond$

Example 19 *(see Singer and Ulmer [20] p. 68, Kovacic [12] p. 23, [25], [7])* In order to illustrate the given method in the primitive case, we consider the irreducible differential equation (Kovacic [12])

$$L(y) = y'' + \left(\frac{3}{16x^2} + \frac{2}{9(x-1)^2} - \frac{3}{16x(x-1)}\right) y = 0.$$

Its fourth symmetric power $L^{ⓢ4}(y) = 0$ has no non-trivial rational solutions. While $L^{ⓢ6}(y) = 0$ has the rationalvariant $r = x^2(x-1)^2$ which generates its one-dimensonal rational solution space. Therefore, by Proposition 8 $\mathcal{G}(L) \cong A_4^{\mathrm{SL}_2}$ is the corresponding Galois group of $L(y) = 0$ (cf. Kovacic [12]).
For $W = 1$, the further two rationalvariants are computed with

$$H(r) = \frac{25}{4} x^2 (x-1)^3$$

and

$$J(r,H(r)) = -\frac{25}{2}x^3(x-1)^4(x-2).$$

From these rationalvariants one gets the determining equation of r

$$\begin{aligned}\Big(c^2+27648\Big)x^{16}+\Big(-8c^2-221184\Big)x^{15}+\Big(28c^2+774144\Big)x^{14}+\\ \Big(-56c^2-1548288\Big)x^{13}+\Big(70c^2+1935360\Big)x^{12}+\\ \Big(-56c^2-1548288\Big)x^{11}+\Big(28c^2+774144\Big)x^{10}+\\ \Big(-8c^2-221184\Big)x^{9}+\Big(c^2+27648\Big)x^{8}=0,\end{aligned}$$

respectively e.g. for the regular point $x_0 = 2$ the equation

$$c^2 + 27648 = 0.$$

Hence, $c = \pm 96\sqrt{-3}$. Substituting

$$\begin{aligned}I_1 &= \frac{1}{4}\cdot c\cdot r = 24\sqrt{-3}\ x^2(x-1)^2\\ I_2 &= -\frac{5}{80}\cdot\frac{-1}{25}c^2H(r) = 432x^2(x-1)^3\\ I_3 &= -\frac{1}{16}\cdot\frac{1}{8}\cdot\frac{-1}{25}c^3J(r,H(r)) = 10368\sqrt{-3}\ x^3(x-1)^4(x-2)\end{aligned}$$

in the minimal decomposed into invariants (4), we obtain the minimal polynomial of a solution:

$$\begin{aligned}&\mathbf{Y}^{24}-4320x^2(x-1)^3\mathbf{Y}^{16}+51840\sqrt{-3}\ x^3(x-1)^4(x-2)\mathbf{Y}^{12}\\ &-2799360x^4(x-1)^6\mathbf{Y}^8+4478976\sqrt{-3}\ x^5(x-1)^7(x-2)\mathbf{Y}^4+2985984x^8(x-1)^8.\end{aligned}$$

$\diamond$

6 Conclusion

The work of Fuchs is difficult to read. The author has first developed the algorithm presented here by himself and noticed afterwards that it is basically a reformulation and improvement of the Fuchsian method. Nevertheless, our method is essentially more efficient. The reason for this lies in using *all* absolute fundamental invariants of the Galois group associated with the differential equation; this enables us to compute the constants from the syzygies.

But in principle our algorithm cannot be more efficient than the algorithm given by Ulmer and Weil [25]. Indeed, both methods have the same time complexity. The algorithm from Ulmer and Weil computes a minimal polynomial of the logarithmic derivative of a solution via a recursion for the coefficients in all cases, while our method tries to determine the solutions explicity as much as possible. If the associated Galois group is the tetrahedral or the octahedral group one can represent both algebraic solutions in radicals. [4]

We feel, that this paper shows the connection between determining the Galois group, the rationalvariants and the Liouvillian solutions of a given (irreducible) second order differential equation very clearly. For instance, in the imprimitive case it is easier to compute first the Liouvillian solutions and determine from them the (possibly) missing rationalvariants and the Galois group. Against it, in the primitive case the better way is to compute first the Galois group and to determine from it the remaining rationalvariants and the minimal polynomial of a solution. The behaviour in the case of $D_2^{\mathrm{SL}_2}$ is somehow special (cf. Ulmer [24]). Also it becomes clear, that a Liouvillian solution or a minimal polynomial of a solution always contains *all* fundamental rationalvariants.

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