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On functions close to homomorphisms between square symmetric structures

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Abstract. Let $\circ : S \times S \to S$ and $* : E \times E \to E$ be binary operations. Suppose $f : S \to E$, $\varphi : E \times E \to [0, \infty)$, and numbers $\omega, \varepsilon > 0$ are given. We provide conditions for (P) $\Rightarrow$ (Q) and for (Q) $\Rightarrow$ (P) to hold, where (P), (Q) have the following meanings:

(P) *There is a homomorphism* $h : S \to E$ *such that*

$$\varphi(f(x), h(x)) \leq \varepsilon \quad (x \in S).$$

(Q) *There are real numbers* δ, η *such that*

$$\varphi(f(x) * f(y), f(x \circ y)) \leq \delta, \ \varphi(f(x)^{2^n}, f(x^{2^n})) \leq \omega^n \varepsilon + \eta \quad (x, y \in S; \ n \in \mathbb{N}).$$

The 2^n-th powers in (Q) concern the operations $*$ and $\circ$, respectively. For the more important implication (Q) $\Rightarrow$ (P) we suppose $\circ$ and $*$ to be square symmetric operations (i.e., $(x \circ y) \circ (x \circ y) = (x \circ x) \circ (y \circ y)$ for $x, y \in S$, and similarly for $*$ in the set E). – We use our investigations to give a variant of a Forti's result on stability in the sense of Pólya, Szegő, Hyers, Ulam.

1. Introduction. By $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ we denote the system of natural numbers, integers, and reals, respectively; $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let $(S, \circ)$, $(E, *)$ be given sets with binary operations. A *homomorphism* $h : S \to E$ is a solution of the Cauchy functional equation

$$h(x \circ y) = h(x) * h(y) \quad (x, y \in S). \tag{1}$$

For $x \in S$ the powers x^{2^n} $(n \in \mathbb{N})$ are recursively defined by

$$x^2 = x \circ x, \ x^{2^{n+1}} = (x^{2^n})^2 \quad (n \geq 1), \tag{2}$$

and for $u \in E$ the powers u^{2^n} with respect to $*$ have a similar meaning. Then (1) implies

$$h(x^{2^n}) = h(x)^{2^n} \quad (x \in S, \ n \in \mathbb{N}). \tag{3}$$

Now let $f : S \to E,\ \varphi : E \times E \to [0, \infty)$ be given functions, let $\varepsilon > 0$, and consider the following requirement:

(P) *There is a homomorphism* $h : S \to E$ *such that*

$$\varphi(f(x), h(x)) \leq \varepsilon \ \ (x \in S). \tag{4}$$

(P) means that in some sense f is close to the homomorphism h. In the next paragraph we give conditions for the space $(E, *)$ and the function φ, in order to get from (P) the following properties (Q_1), (Q_2):

(Q_1) *There is a real number* δ *such that*

$$\varphi(f(x) * f(y), f(x \circ y)) \leq \delta \ \ (x, y \in S). \tag{5}$$

(Q_2) *There is a real number* η *such that*

$$\varphi(f(x)^{2^n}, f(x^{2^n})) \leq \omega^n \varepsilon + \eta \ \ (x \in S,\ n \in I\!N). \tag{6}$$

(Q_1) and (Q_2) together are sometimes simply called (Q), like in the abstract. In (6), ω is a given positive number, which later on will be linked to φ by the formula

$$\varphi(u^2, v^2) = \omega\varphi(u, v) \ \ (u, v \in E). \tag{A}$$

To get (Q_2) from (P) we rather use

$$\varphi(u^2, v^2) \leq \omega\varphi(u, v) \ \ (u, v \in E). \tag{$A_\leq$}$$

The inverse inequality

$$\varphi(u^2, v^2) \geq \omega\varphi(u, v) \ \ (u, v \in E) \tag{$A_\geq$}$$

is used in the third paragraph to get (P) from (Q): We construct the function h occuring in (4). To do so, we equip E with a complete metric $\rho \leq \varphi$, and we give conditions for obtaining h as the usual limit, which is known from Pólya and Szegő for $(S, \circ) = (I\!N, +), (E, *) = (I\!R, +)$ (cf. [10], Exercise I 99) and from Hyers [6] for Banach spaces S, E; cf. also Forti's survey paper [4]. To obtain the homomorphism property (1) for this function h, we suppose the operations $\circ$ in S and $*$ in E to be *square symmetric* (cf. [9]), i.e.

$$(x \circ y) \circ (x \circ y) = (x \circ x) \circ (y \circ y) \ \ (x, y \in S), \tag{V}$$

(W) $\qquad (u{*}v){*}(u{*}v) = (u{*}u){*}(v{*}v) \ \ (u, v \in E).$

Of course, these formulas also can be written as $(x{\circ}y)^2 = x^2{\circ}y^2$, $(u{*}v)^2 = u^2{*}v^2$. From Forti's paper [2] it is already clear that square symmetric operations provide a natural setting for studying stability of Cauchy functional equations (cf. also [1] by Borelli and Forti; the first paper using square symmetry in this context is due to Rätz [11]; for more recent results cf. Páles [8]).

In the fourth paragraph we discuss uniqueness of the homomorphism h in (P), and we summarize the hypotheses for the equivalence between (P) and (Q).

The fifth paragraph is devoted to stability. Concerning conditions (P), (Q_1), (A) we are less general than Forti [2]: He allows variable $\varepsilon = \varepsilon(x)$, $\delta = \delta(x, y)$, and instead of (A) he uses $\varphi(u^2, v^2) = k(\varphi(u, v))$, where $k : [0, \infty) \to [0, \infty)$ is an appropriate function. On the other hand, our function φ is not necessarily a metric on E, since $\varphi(v, u) = \varphi(u, v)$ $(u, v \in E)$ will not be required. Examples in the concluding sixth paragraph show the advantage of this.

A special case of our considerations is a square symmetric structure $(S, \circ)$ (i.e., (V) holds) and $(E, *) = (E, +)$ with an arbitrary Banach space E, where $\rho(u, v) = \varphi(u, v) = \|u - v\|$ $(u, v \in E)$ and $\omega = 2$. Then it is known from [16] (and it is easy to show) that (P), (Q) are equivalent; this result had been inspired by [5].

2. The implications (P) $\Rightarrow$ ($\mathbf{Q_1}$) and (P) $\Rightarrow$ ($\mathbf{Q_2}$). For the function $\varphi : E \times E \to [0, \infty)$ we deal with the following conditions:

(S) *There is a constant* $a \geq 0$ *such that*

$$\varphi(v, u) \leq a\varphi(u, v) \quad (u, v \in E).$$

(T) *There are constants* $b, c \geq 0$ *such that*

$$\varphi(u, w) \leq b\varphi(u, v) + c\varphi(v, w) \quad (u, v, w \in E).$$

(T_1) *There is a constant* $c \geq 0$ *such that*

$$\varphi(u, w) \leq \varphi(u, v) + c\varphi(v, w) \quad (u, v, w \in E).$$

(T_{11}) $\qquad \varphi(u,w) \le \varphi(u,v)+\varphi(v,w) \ (u,v,w \in E).$

Of course, $(\mathrm{T}_{11}) \Rightarrow (\mathrm{T}_1) \Rightarrow (\mathrm{T})$. The triangle inequality (T_{11}) will be used later, when discussing stability. At present we need a certain boundedness condition:

(B) *There is a real number β such that for $t,u,v,w \in E$ we have*

$$\varphi(t,v) \le \varepsilon,\ \varphi(u,w) \le \varepsilon \Rightarrow \varphi(t*u, v*w) \le \beta.$$

Proposition 1. *If* (S), (T), (B) *are satisfied, then* (P) $\Rightarrow$ (Q_1); *if* (S), (T_1), $(\mathrm{A}_{\le})$ *hold, then* (P) $\Rightarrow$ (Q_2).

Proof. To get (Q_1) from (P), consider $x,y \in S$ and use (S), (T), (B), (P), and (1) as follows:

$$\begin{aligned}
&\varphi(f(x)*f(y), f(x\circ y))\\
&\le b\varphi(f(x)*f(y), h(x)*h(y)) + c\varphi(h(x\circ y), f(x\circ y))\\
&\le b\beta + ca\varphi(f(x\circ y), h(x\circ y)) \le b\beta + ca\varepsilon.
\end{aligned}$$

This proves (Q_1) with $\delta = b\beta + ca\varepsilon$. To get (Q_2) from (P) we use (3). Then (S), (T_1), $(\mathrm{A}_{\le})$, (P) imply

$$\begin{aligned}
&\varphi(f(x)^{2^n}, f(x^{2^n})) \le \varphi(f(x)^{2^n}, h(x)^{2^n}) + c\varphi(h(x^{2^n}), f(x^{2^n}))\\
&\le \omega^n\varphi(f(x),h(x)) + ca\varphi(f(x^{2^n}), h(x^{2^n})) \le \omega^n\varepsilon + ca\varepsilon,
\end{aligned}$$

i.e., (Q_2) holds with $\eta = ca\varepsilon$.

3. The implication (Q) $\Rightarrow$ (P). Here we use the following property of $*$ in E:

(U) *To every $u \in E$ there is a unique $v \in E$ such that $v^2 = u$.*

We write $v = u^{1/2} = u^{2^{-1}}$, and we define recursively

$$u^{2^{-n-1}} = (u^{2^{-n}})^{2^{-1}} \quad (u \in E,\ n \in \mathbb{N}).$$

Together with $u^{2^0} = u^1 = u$ and with the analogue of (2) for the operation $*$ in E, the powers u^{2^m} are defined for all $m \in \mathbb{Z}$, and the rule $(u^{2^m})^{2^n} = u^{2^{m+n}}$ for $u \in E$ and $m,n \in \mathbb{Z}$ can easily be verified.

As mentioned in the introduction, ρ will be a metric on E; we suppose:

(R) (E,ρ) *is a complete metric space, and* $\rho \leq \varphi$.

All further topological (and metric) notions in E are understood with respect to ρ. In particular the function $h : S \to E$ in (P) will be given by the limit

$$h(x) = \lim_{n\to\infty} f(x^{2^n})^{2^{-n}} \quad (x \in S). \tag{7}$$

Proposition 2. *Suppose* (Q_2), (R), (U), $(\mathrm{A}_{\geq})$, *and*

$$\omega > 1. \tag{E}$$

Then (7) *defines a function* $h : S \to E$.

Proof. We fix $x \in S$. Because of (U) the expressions $f(x^{2^n})^{2^{-n}}$ have a meaning, and because of (R) it is sufficient to show that they form a Cauchy sequence: We put

$$\delta_{m,m+n} = \rho(f(x^{2^m})^{2^{-m}}, f(x^{2^{m+n}})^{2^{-m-n}}) \quad (m,n \in I\!N).$$

By $\rho \leq \varphi$ and $(\mathrm{A}_{\geq})$ we get

$$\delta_{m,m+n} \leq \frac{1}{\omega^{m+n}} \varphi(f(x^{2^m})^{2^n}, f((x^{2^m})^{2^n}))$$

((2) implies $x^{2^{m+n}} = (x^{2^m})^{2^n}$). Now (Q_2), (E) yield

$$\delta_{m,m+n} \leq \frac{1}{\omega^{m+n}}(\omega^n \varepsilon + \eta) \leq \frac{\varepsilon + |\eta|}{\omega^m},$$

and the last term tends to zero as $m \to \infty$.

The conditions (V), (W) will occur in the next proposition. From (V), (2) the formula $(x \circ y)^{2^n} = x^{2^n} \circ y^{2^n}$ $(x, y \in S;\ n \in I\!N)$ easily follows. From (W) we get a similar formula for the operation $*$ in E, and if also (U) holds, then we have more generally $(u * v)^{2^m} = u^{2^m} * v^{2^m}$ $(u, v \in E;\ m \in Z\!\!\!Z)$. Two further conditions will be used:

(C) $* : E \times E \to E$ *is continuous.*

(D) $\varphi : E \times E \to [0, \infty)$ *is continuous with respect to the second variable.*

In the next proposition we use again the definition of $h : S \to E$ from Proposition 2.

Proposition 3. *Assume* (Q_2), (R), (U), $(\mathrm{A}_{\geq})$, (E) *to hold and define* $h : S \to E$ *by* (7). *If* (D) *is satisfied, then* (4) *holds. If* (V), (W), (Q_1), (C) *are satisfied, then* $h : S \to E$ *is a homomorphism.*

Proof. Let (D) be satisfied: Dividing (6) by ω^n and using ($A_\geq$) yields

$$\varphi(f(x), f(x^{2^n})^{2^{-n}}) \leq \varepsilon + \frac{\eta}{\omega^n}.$$

By $n \to \infty$ we get (4).

Now let (V), (W), (Q_1), (C) be satisfied: For $x, y \in S$ and $n \in \mathbb{N}$ we get from (5) the inequality

$$\varphi(f(x^{2^n}) * f(y^{2^n}), f((x \circ y)^{2^n})) \leq \delta.$$

We divide by ω^n and we use ($A_\geq$) to obtain

$$\varphi(f(x^{2^n})^{2^{-n}} * f(y^{2^n})^{2^{-n}}, f((x \circ y)^{2^n})^{2^{-n}}) \leq \frac{\delta}{\omega^n}.$$

Because of $\rho \leq \varphi$ we can replace φ by ρ. Then, when using (C), $n \to \infty$ yields $h(x) * h(y) = h(x \circ y)$.

Observe that by the last reasoning we get $h(x) * h(x) = h(x \circ x)$, if (V), (W) are not required (cf. also Proposition 1 in Forti's paper [3]). But for this it is sufficient to have (5) only for $y = x$, and this point of view has been adopted in [18].

Observe furthermore that at the end of Proposition 3 we can replace (V) by a more general condition stemming from Józef Tabor [15] (cf. also [18]).

As an immediate consequence of Propositions 2, 3 we have:

Proposition 4. *Suppose* (R), (U), (V), (W), ($A_\geq$), (C), (D), (E) *to hold. Then* (Q) $\Rightarrow$ (P).

4. Uniqueness of the homomorphism h in (P) and the equivalence (P) $\Leftrightarrow$ (Q).

Proposition 5. *Assume* (S), (T), ($A_\geq$), (E), *and:*

(F) *For* $u, v \in E$, $\varphi(u, v) = 0$ *implies* $u = v$.

Then the homomorphism $h : S \to E$ *in* (P) *is unique.*

Proof. For homomorphisms $h_1, h_2 : S \to E$ satisfying

$$\varphi(f(x), h_1(x)) \leq \varepsilon, \quad \varphi(f(x), h_2(x)) \leq \varepsilon \quad (x \in S)$$

we have

$$\varphi(h_1(x), h_2(x)) \leq b\varphi(h_1(x), f(x)) + c\varphi(f(x), h_2(x))$$
$$\leq ba\varphi(f(x), h_1(x)) + c\varepsilon \leq (ba + c)\varepsilon =: \gamma,$$

hence, for $x \in S$ and $n \in I\!N$,

$$\begin{aligned}
\varphi(h_1(x^{2^n}), h_2(x^{2^n})) &\leq \gamma, \\
\varphi(h_1(x)^{2^n}, h_2(x)^{2^n}) &\leq \gamma, \\
\omega^n \varphi(h_1(x), h_2(x)) &\leq \gamma, \\
\varphi(h_1(x), h_2(x)) &\leq \gamma/\omega^n \to 0 \quad (n \to \infty).
\end{aligned}$$

Therefore, $\varphi(h_1(x), h_2(x)) = 0$ $(x \in S)$, and because of (F) we obtain $h_2 = h_1$.

Since (F) is a consequence of (R), we get from Propositions 1, 4, 5 the result:

Theorem 1. *Assume* (R), (S), (T_1), (U), (V), (W), (A), (B), (C), (D), (E) *to hold. Then* (P) $\Leftrightarrow$ (Q), *and the homomorphism* $h : S \to E$ *in* (P) *is uniquely determined; it is given by the limit* (7).

5. Stability. $(S, \circ)$ and $(E, *)$ being given, we understand stability of equation (1) by means of the function $\varphi : E \times E \to [0, \infty)$ in the following way:

Definition. *The homomorphism equation* (1) *is stable, if for each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for functions* $f : S \to E$ *satisfying* (5) *also* (P) *holds.*

In view of Proposition 4 it is now of interest to get for each $\varepsilon > 0$ some $\delta > 0$ such that the inequality (5) in (Q_1) implies (Q_2): In such a case one has stability, if also the hypotheses of Proposition 4 are satisfied.

Proposition 6. *Assume* ($A_{\leq}$), (E), *and the triangle inequality* (T_{11}) *to hold, and suppose* $0 < \delta \leq \varepsilon(\omega - 1)$. *Then* (5) *implies* ($Q_2$).

Proof. We use (5) only for $y = x$, i.e.,

$$\varphi(f(x)^2, f(x^2)) \leq \delta \;\; (x \in S). \tag{8}$$

For $x \in S$ and $n \in I\!N$, (T_{11}) implies

$$\varphi(f(x)^{2^n}, f(x^{2^n})) \leq \varphi(f(x)^{2^n}, f(x^2)^{2^{n-1}}) +$$
$$+\varphi(f(x^2)^{2^{n-1}}, f(x^4)^{2^{n-2}}) + \cdots + \varphi(f(x^{2^{n-1}})^2, f(x^{2^n})),$$

and by ($A_{\leq}$), (8) we get

$$\varphi(f(x)^{2^n}, f(x^{2^n})) \leq \omega^{n-1}\delta + \omega^{n-2}\delta + \cdots + \delta = \\ = \frac{\omega^n - 1}{\omega - 1}\delta = \omega^n \frac{\delta}{\omega - 1} - \frac{\delta}{\omega - 1} \leq \omega^n \varepsilon - \frac{\delta}{\omega - 1},$$

i.e., (6) holds with $\eta = -\delta/(\omega - 1)$.

As a consequence of Propositions 4, 6 we get:

Theorem 2. *Suppose* (R), (T_{11}), (U), (V), (W), (A), (C), (D), (E) *are fulfilled. Then equation* (1) *is stable: If* $\varepsilon > 0$ *is arbitrary and* $\delta = \varepsilon(\omega - 1)$, *then* (5) *implies* (P).

Remark. In the proof of Proposition 6 the inequality (5) was only needed for $y = x$. Therefore Theorem 2 can be strengthened in the following way: Suppose the hypotheses (R), ..., (E) of that theorem to hold. Let $\varepsilon > 0$ be given, suppose (5) to hold with some $\delta \geq 0$ (this δ not necessarily being linked to ε), and suppose

$$\varphi(f(x)^2, f(x^2)) \leq \varepsilon(\omega - 1) \quad (x \in S).$$

Then (P) is true.

In the simple case $(S, \circ) = (E, *) = (\mathbb{R}, +)$ (and $\varphi(x, y) = |x - y|$) this remark means that for $f : \mathbb{R} \to \mathbb{R}$ having the properties

$$|f(x) + f(y) - f(x + y)| \leq \delta, \quad |f(2x) - 2f(x)| \leq \varepsilon \quad (x, y \in \mathbb{R}),$$

there is an additive $h : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - h(x)| \leq \varepsilon$ $(x \in \mathbb{R})$.

6. Examples. 1. Let E be a Banach space. As square symmetric operation in this space we take the addition (and we write $+$, not $*$), as metric we take

(9) $$\rho(u, v) = \alpha\|u - v\| \quad (u, v \in E),$$

where $\alpha > 0$ will be specified in a moment. Let V be a closed, convex, bounded subset of E, having zero in its interior, and let $\mu : E \to [0, \infty)$ be the Minkowski functional of this set (cf., e.g., Rudin [13]), in particular we have

(10) $$V = \{u \mid u \in E,\ \mu(u) \leq 1\}.$$

We take

$$\varphi(u,v) = \mu(u-v) \ \ (u,v \in E), \tag{11}$$

and we choose α in (9) such that $\rho \le \varphi$. Then E, φ, ρ meet all the conditions (R), (S), (T_{11}), (U), (W), (A), (B), (C), (D), (E) in Theorems 1, 2 and we have $\omega = 2$ for this case. In condition (B) the dependence of β upon ε is given by $\beta = 2\varepsilon$.

Moreover, let $(S, \circ)$ be an arbitrary square symmetric structure (i.e., also (V) holds true); by Theorem 2 we get stability with $\delta = \varepsilon$, and because of (10), (11) this means for $\varepsilon = 1$ the following: If $f : S \to E$ satisfies

$$f(x)+f(y)-f(x\circ y) \in V \ \ (x,y \in S), \tag{12}$$

then there is $h : S \to E$ such that

$$h(x\circ y) = h(x)+h(y), f(x)-h(x) \in V \ \ (x,y \in S). \tag{13}$$

This result is already known for the more general case of bounded subsets V of E, which are ideally convex in the sense of Lifšic [7]; the proof in [17] is the same as the former proof by Jacek Tabor [14] for commutative semigroups $(S, \circ)$.

2. Suppose $n \in \mathbb{N}$, $n \ge 2$, and $0 < p < 1$. We take $E = \mathbb{R}^n$ with its addition $+$ as square symmetric operation, and we equip $\mathbb{R}^n$ with the F-norm

$$\|u\| = \sum_{\nu=1}^{n} |u_\nu|^p \ \ (u = (u_1, \dots, u_n) \in \mathbb{R}^n). \tag{14}$$

Then $\rho(u,v) = \|u - v\|$ $(u, v \in \mathbb{R}^n)$ defines a translation invariant metric, by which $\mathbb{R}^n$ becomes a complete metric linear space (cf. Rolewicz [12]). We take $\varphi = \rho$, and again E, φ, ρ meet all conditions (R), (S), (T_{11}), (U), (W), (A), (B), (C), (D), (E) in Theorems 1, 2; this time we have $\omega = 2^p$ in (A), hence $\omega < 2$.

In particular we get $\delta < \varepsilon$ in Theorem 2, and actually $\delta = \varepsilon$ is not possible: To see this, suppose the contrary and define

$$V = \{u \mid u \in \mathbb{R}^n, \ \|u\| \le 1\}. \tag{15}$$

As in the previous example, if $(S, \circ)$ is a square symmetric structure, then to each function $f : S \to E$ satisfying (12), there is an $h : S \to E$ such that

(13) holds. If we take $(S, \circ) = (I\!R, +)$, then a theorem of Jacek Tabor [14] forces V to be a convex subset of $I\!R^n$ (this space now being considered as a Banach space). But because of $0 < p < 1$ (and $n \geq 2$) in (14), the set (15) is not convex.

3. In the foregoing example φ is a metric ($\varphi = \rho$), and such cases are covered by the papers of Forti [2] and of Borelli and Forti [1]. Now we take $E = I\!R^2$, again with $+$ as operation, and we define

$$\mu(u) = \mu(u_1, u_2) = \begin{cases} \sqrt{2u_1} + \sqrt{|u_2|} & (u_1 \geq 0) \\ \sqrt{-u_1} + \sqrt{|u_2|} & (u_1 \leq 0) \end{cases} \qquad (u = (u_1, u_2) \in I\!R^2).$$

Then $\varphi(u, v) = \mu(u - v)$ $(u, v \in I\!R^2)$ is not symmetric, hence not a metric. Finally we put $\rho(u, v) = \|u - v\|$ $(u, v \in I\!R^2)$ where $\|\cdot\|$ is given by (14) with $n = 2,\ p = \frac{1}{2}$. Then E, φ, ρ meet all the conditions (R), (S), (T_{11}), (U), (W), (A), (B), (C), (D), (E) in Theorems 1, 2; here we have $\omega = \sqrt{2}$.

Let $(S, \circ)$ be an arbitrary square symmetric structure, and let us look at Theorem 2: If

$$W = \{u \mid u \in I\!R^2,\ \mu(u) \leq 1\},$$

$\varepsilon > 0$, and if $f : S \to E$ satisfies

$$f(x) + f(y) - f(x \circ y) \in \delta W$$

(where $\delta = \varepsilon(\sqrt{2} - 1)^2 = \varepsilon(3 - 2\sqrt{2})$), then there is $h : S \to E$ such that

$$\text{(16)} \qquad h(x \circ y) = h(x) + h(y),\ f(x) - h(x) \in \varepsilon W \ \ (x, y \in S).$$

The square in $\delta = \varepsilon(\sqrt{2} - 1)^2$ comes from the fact that for $r \geq 0$ we have $\mu(u) \leq r$ if and only if $u \in r^2 W$.

As Jacek Tabor has pointed out (oral communication), such type of stability result can be reduced to our first example: Take $E = I\!R^2$ and choose $\delta_1 \in (0, \varepsilon)$ according to

$$V := \delta_1 \cdot \operatorname{conv} W \subseteq \varepsilon W$$

(where $\operatorname{conv} W$ denotes the convex hull of W). Then, if a function $f : S \to E$ satisfies

$$f(x) + f(y) - f(x \circ y) \in \delta_1 W,$$

we get (12), hence also (13) for some $h : S \to E$, and therefore we have (16).

4. Let us conclude by an infinite-dimensional version of the foregoing example: We take the complete metric linear space

$$E = \{u \mid u = (u_1, u_2, \dots),\ \|u\| = \sum_{n=1}^{\infty} \sqrt{|u_n|} < \infty\}$$

with + as operation, and for $u = (u_1, u_2, \dots) \in E$ we define

$$\mu(u) = \begin{cases} \|(2u_1, u_2, u_3, u_4, \dots)\| & (u_1 \geq 0) \\ \|u\| & (u_1 \leq 0). \end{cases}$$

Again $\varphi(u, v) = \mu(u - v)$ $(u, v \in E)$ is not symmetric, hence not a metric, and again we take $\rho(u, v) = \|u - v\|$ $(u, v \in E)$.

Then E, φ, ρ meet all the conditions (R), (S), (T_{11}), (U), (W), (A), (B), (C), (D), (E) in Theorems 1, 2, where $\omega = \sqrt{2}$.

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